Distribution of Primes

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Prime Numbers

Prime numbers are positive numbers whose only integer divisors are 1 and itself

They are essentially the building blocks of integers: every positive integer has a **unique** prime factorization

Infinitude of Primes

Assume for contradiction that there are finitely many primes, all of which can be written in the set $\{p_1, p_2, p_3, \ldots, p_k\}$. Let

$$n = p_1 p_2 p_3 \ldots p_k + 1$$

Because n is not prime, It has a prime factor not in the set

Concerns

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$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{a_1, a_2, a_3, a_k \geq 0} \frac{1}{(p_1^{a_1} p_2^{a_2} p_3^{a_3} ... p_k^{a_k})^s}$$

$$= \left(\sum_{a_1=0}^{\infty} \frac{1}{(p_1^{a_1})^s} \right) \left(\sum_{a_2=0}^{\infty} \frac{1}{(p_2^{a_2})^s} \right) \left(\sum_{a_3=0}^{\infty} \frac{1}{(p_3^{a_3})^s} \right) ... \left(\sum_{a_k=0}^{\infty} \frac{1}{(p_k^{a_k})^s} \right)$$

$$= \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^s} \right)^{-1}$$

$$\lim_{s \to 1^+} \sum_{n=1}^{\infty} \frac{1}{n^s} \;=\; \lim_{s \to 1^+} \prod_{i=1}^{k} \left(1 - \frac{1}{p_i^s}\right)^{-1}$$

Contradiction

$$\lim_{s \to 1^+} \log \left(\prod_p \left(1 - \frac{1}{p^s} \right)^{-1} \right) = - \lim_{s \to 1^+} \sum_p \log \left(1 - \frac{1}{p^s} \right)$$

$$= \lim_{s \to 1^+} \left(\sum_p \frac{1}{p} \right) + \text{ negligable terms}$$

$$\sum_p \frac{1}{p} \text{ diverges}$$

What does this reveal?

- Primes are more "numerous" than squares
- Primes are more "numerous" than numbers in the form $n \log(n)$
- There are infinitely many integers x such that

$$\pi(x) > \frac{x}{\log^2(x)}$$

Chebyshev Estimates

- Three estimates: $\pi(x), \theta(x), \psi(x)$

$$\pi(x) = \sum_{p \leq x} 1$$

$$\theta(x) = \sum_{p \leq x} \log(p)$$

$$\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{p^m \leq x} \log(p)$$

$$e^{\theta(x)} = e^{\sum_{p \leq x} \log(p)} = \prod_{p \leq x} e^{\log(p)} = \prod_{p \leq x} p$$

What are the next steps?

- If a divides b, then $a \leq b$

- To show that $\theta(x)$ is linear, all we must show is that $\theta(2n) - \theta(n) = O(n)$ for all n

All primes between n and $2n$ for any integer n divide $\binom{2n}{n}$

Therefore

$$\prod_{n \leq p \leq 2n} p \leq \binom{2n}{n} \leq \sum_{i=0}^{2n} \binom{2n}{i} = 2^{2n}$$

$$\frac{e^{2\theta(n)}}{e^{\theta(n)}} \le 2^{2n}$$

$$\theta(2n) - \theta(n) \le 2n \log 2$$

$$\theta(x) = O(x)$$

In fact

$$\theta(x) \asymp x$$

$$\psi(x) - \theta(x) = \sum_{p^m \leq x} \log(p) - \sum_{p \leq x} \log(p)$$

$$\leq \sum_{p \leq \sqrt{x}} \log(p) \lfloor \log_p(x) \rfloor \leq \sum_{p \leq \sqrt{x}} \log(p) \cdot \frac{\log(x)}{\log(p)}$$

$$= \sum_{p \leq \sqrt{x}} \log(x) \leq \sqrt{x} \log(x)$$

$$\psi(x) - \theta(x) = O(\sqrt{x}\log(x))$$

$$\psi(x) = \theta(x) + O(\sqrt{x}\log(x))$$

$$\psi(x) \asymp x$$

Abel's Summation by Parts

$$\sum_{y \leq n \leq x} c_n f(n) = C(x)f(x) - C(y)f(y) - \int_y^x C(t)f'(t)dt$$

$$\text{where } C_n = \sum_{y \leq n \leq x} c_n$$

$$\sum_{y \le n \le x} c_n f(n) = C(x)f(x) - C(y)f(y) - \int_y^x C(t)f'(t)dt$$

$$\pi(x) = \sum_{p \le x} 1 = \sum_{n=1}^x c_n \cdot \frac{1}{\log(n)}$$

where $c_n = \log(p)$ if $n = p$ and 0 otherwise

$$= \frac{\theta(x)}{\log(x)} + \int_1^x \frac{\theta(t)dt}{t \log^2(t)}$$

$$\int_1^x \frac{\theta(t)dt}{t\log^2(t)} \asymp \int_1^x \frac{dt}{\log^2(t)} = O\left(\frac{x}{\log^2(x)}\right)$$

$$\pi(x) = \frac{\theta(x)}{\log(x)} + O\left(\frac{x}{\log^2(x)}\right)$$

Equivalents to the Prime Number Theorem

- The Prime Number Theorem states that $\pi(x) \sim \dfrac{x}{\log(x)}$

- These are all equivalent

$$\pi(x) \sim \frac{x}{\log(x)}$$

$$\theta(x) \sim x$$

$$\psi(x) \sim x$$

Brief Sketch Prime Number Theorem

- $\psi(x)$ is equal to the sum of the Von Mangoldt function

$$\psi(x) = \sum_{n \le x} \Lambda(n)$$

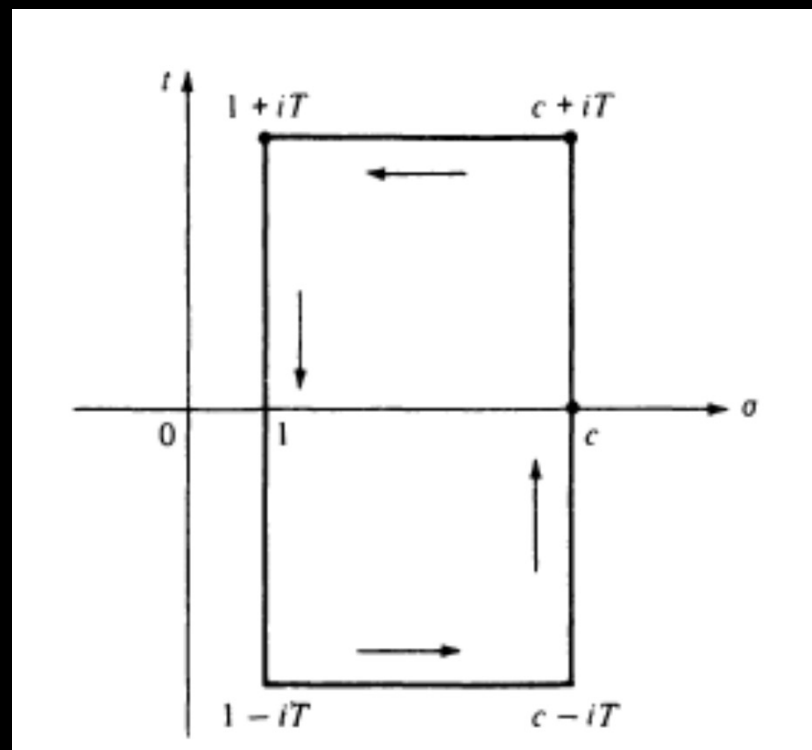$$\sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s} = -\frac{\zeta'(s)}{\zeta(s)}$$

Analytic Tools

- Mellin Transformations

- Perron's Formula

- Contour Integration

$$\psi(x) \sim x$$

Consequences of the Prime Number Theorem

- If you were to pick a square-free number at random, the probability that it will have an even number of prime factors is equal to the probability that it will have an odd number of prime factors

- The Riemann Hypothesis is equivalent to the following

$$\psi(x) - x = O_\epsilon(x^{1/2+\epsilon})$$