The Universe

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Outline

Section 1

Some basic axioms of Z^- as a subsystem of ZFC

Our Axioms (so far)

Axiom (Existence)

$$\exists x(x = x)$$

Axiom (Extensionality)

$$\forall x \forall y(\forall z(z \in x \iff z \in y) \implies x = y)$$

Axiom (Comprehension Scheme)

For each formula ϕ without y free,

$$\forall z \exists y \forall x(x \in y \iff x \in z \land \phi)$$

Axioms (cont.)

Axiom (Pairing)

$$\forall x \forall y \exists z (x \in z \land y \in z)$$

Axiom (Union)

$$\forall F \exists A \forall Y \forall x (x \in Y \land Y \in F \implies x \in A)$$

Axiom (Power Set)

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

Results

Using just the axioms we have so far, we can create lots of sets and operations! The following are some examples.

- $\emptyset$ (Comprehension and Existence)
- Arbitrary unions over families indexed by a set $\cup$ (Union and Comprehension)
- Pairs $\{x, y\}$ (Pairing and Comprehension).
- Ordered pairs $\{x, \{x, y\}\}$ abbreviated as (x, y) (Pairing and Comprehension).
- Cartesian product $X \times Y$, the set of all ordered pairs between two sets (Power Set three times and Comprehension).
- Functions and relations which are subsets of cartesian products (Power Set three times and Comprehension).
- Quotients (Power set and Comprehension).

Section 2

Numbers

Natural Numbers

Definition 1

If x is a set, then the **successor** of x, denoted $S(x)$, is $x \cup \{x\}$

Definition 2

1. $\emptyset$ is 0.
2. $1 = S(0) = \{\emptyset\}$, $2 = S(1) = \{\emptyset, \{\emptyset\}\}$, ..., $n + 1 = S(n)$.

Idea

Each natural number n has n "layers" of brackets. These somehow store information about order, together with the relation ϵ.

Infinity

Axiom (Infinity)

$$\exists x(\emptyset \in x \wedge \forall y \in x(S(y) \in x))$$

Definition 3

Using Infinity and Comprehension, we may define $\omega = \{0, 1, 2, \ldots\}$

The "real" Natural Numbers?

$$(\omega, \epsilon) \cong (\mathbb{N}, <).$$

We can keep going!

$$\omega + 1 = S(\omega)$$
$$\omega + 2 = S(\omega + 1)$$
$$\ldots \omega + \omega?$$

$\mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$

Definition 4

$$\mathbb{Z} = (\mathbb{N} \times \mathbb{N})/\sim$$

Definition 5

$$\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\}))/\sim$$

Definition 6

$$\mathbb{R} = \{X \in \mathcal{P}(\mathbb{Q}) \mid X \neq \emptyset \wedge X \neq \mathbb{Q} \wedge \\ \forall x \in X \forall y \in \mathbb{Q}(y < x \implies y \in X)\}$$

Definition 7

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}$$

Section 3

The universe of mathematical objects

Foundations

We can begin to form sets differentiated by their complexity, i.e., how many "layers" of brackets there are in each.

Definition 8

$$V_0 = \emptyset$$
$$V_1 = \mathcal{P}(V_0)$$
$$\dots$$
$$V_{n+1} = \mathcal{P}(V_n)$$
$$\dots$$
$$V_\omega = \bigcup_{n\in\omega} V_n$$

Fact: Everything in V_ω is **hereditarily finite**.

It doesn't stop here

We have to keep going! $V_{\omega+1} = \mathcal{P}(V_\omega)$. In fact, $\omega \notin V_\omega$, but $\omega \in V_{\omega+1}$.

- $\mathbb{Z} \in V_{\omega+4}$
- $\mathbb{Q} \in V_{\omega+7}$
- $\mathbb{R} \in V_{\omega+9}$
- $\mathbb{C} \in V_{\omega+12}$
- $(\mathbb{Q}, +, 0) \in V_{\omega+22}$

$\omega + \omega$

- This is the limit of $\omega + n$ for $n = 1, 2, \ldots$.
- $V_{\omega+\omega}$ is the closure of the power set operation on $\omega + n$ for $n = 1, 2, \ldots$
- *Fact*: $V_{\omega+\omega}$ is *a* universe for all regular mathematics, and is the smallest such.

It (still) doesn't stop here

- $\omega + \omega + 1 = S(\omega + \omega)$,
- $\omega + \omega + \omega = 3\omega$,
- $\omega \cdot \omega = \omega^2$,
- $\omega^2 \cdot \omega^2 = \omega^4$,
- ω^ω,
- ω^{ω^ω},
- $\ldots\mathcal{E}_0\ldots$

Do these large numbers exist?

Idea

Every "image" of a function defined by a formula on a set is still a set.

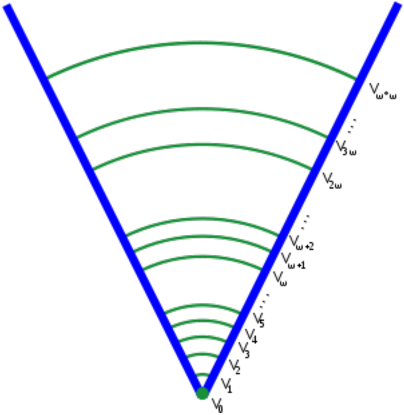
Axiom (Replacement)

For each formula ϕ without Y free,

$$\forall A \forall x \in A \exists! y \phi(x,y) \implies \exists Y \forall x \in A \exists y \in Y \phi(x,y)$$

Define $\phi : \omega \to \mathbf{V}$ by $n \mapsto \omega + n$. Thus, $\phi(\omega) = \omega + \omega$ is a definable set. So, all these sets *do* exist.

The universe of mathematical objects



Axiom (Foundation)

$$\forall x[\exists y(y \in x) \implies \exists y(y \in x \wedge \neg\exists z(z \in x \wedge z \in y))]]$$

References

[Set Theory, 1980] Kenneth Kunen