Some basic axioms of Z ⁻ as a subsystem of ZFC	Numbers	The universe of mathematical objects	References

The Universe

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Section 1

Some basic axioms of $\overline{Z^-}$ as a subsystem of ZFC

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Our Axioms (so far)

Axiom (Existence)

$$\exists x(x = x)$$

Axiom (Extensionality)

$$\forall x \forall y (\forall z (z \in x \iff z \in y) \implies x = y)$$

Axiom (Comprehension Scheme)

For each formula ϕ without y free,

$$\forall z \exists y \forall x (x \in y \iff x \in z \land \phi)$$

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Axioms (cont.)

Axiom (Pairing)

$\forall x \forall y \exists z (x \in z \land y \in z)$

Axiom (Union)

$$\forall F \exists A \forall Y \forall x (x \in Y \land Y \in F \implies x \in A)$$

Axiom (Power Set)

$$\forall x \exists y \forall z (z \subseteq x \implies z \in y)$$

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Results			

Using just the axioms we have so far, we can create lots of sets and operations! The following are some examples.

- Ø (Comprehension and Existence)
- Arbitrary unions over families indexed by a set ∪ (Union and Comprehension)
- Pairs $\{x, y\}$ (Pairing and Comprehension).
- Ordered pairs {x, {x, y}} abbreviated as (x, y) (Pairing and Comprehension).
- Cartesian product $X \times Y$, the set of all ordered pairs between two sets (Power Set three times and Comprehension).
- Functions and relations which are subsets of cartesian products (Power Set three times and Comprehension).
- Quotients (Power set and Comprehension).

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Section 2

Numbers

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Natural Numbers

Definition 1

If x is a set, then the **successor** of x, denoted S(x), is $x \cup \{x\}$

Definition 2

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$$1 = S(0) = \{\emptyset\}, 2 = S(1) = \{\emptyset, \{\emptyset\}\}, ..., n+1 = S(n).$$

Idea

Each natural number *n* has *n* "layers" of brackets. These somehow store information about order, together with the relation ϵ .

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Infinity

Axiom (Infinity)

$$\exists x (\emptyset \in x \land \forall y \in x (S(y) \in x))$$

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Definition 3

Using Infinity and Comprehension, we may define $\omega = \{0, 1, 2, ...\}$

The "real" Natural Numbers?

$$(\omega, \epsilon) \cong (\mathbb{N}, <).$$

We can keep going!

$$\begin{split} \omega + 1 &= S(\omega) \\ \omega + 2 &= S(\omega + 1) \\ \dots \ \omega + \omega? \end{split}$$

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$\mathbb{Z},\mathbb{Q},\mathbb{R},$ and \mathbb{C}			

Definition 4

$$\mathbb{Z} = (\mathbb{N} imes \mathbb{N}) / \sim$$

Definition 5

$$\mathbb{Q} = (\mathbb{Z} \times (\mathbb{Z} \setminus \{0\})) / \sim$$

Definition 6

$$\mathbb{R} = \{ X \in \mathcal{P}(\mathbb{Q}) \mid X \neq \emptyset \land X \neq \mathbb{Q} \land \\ \forall x \in X \forall y \in \mathbb{Q} (y < x \implies y \in X) \}$$

Definition 7

$$\mathbb{C} = \mathbb{R} \times \mathbb{R}$$

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Section 3

The universe of mathematical objects

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Foundations			

We can begin to form sets differentiated by their complexity, i.e., how many "layers" of brackets there are in each.

Definition 8

$$egin{aligned} & V_0 = \emptyset \ & V_1 = \mathcal{P}(V_0) \ & \dots \ & V_{n+1} = \mathcal{P}(V_n) \ & \dots \ & V_\omega = igcup_{n\in\omega} V_n \end{aligned}$$

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Fact: Everything in V_{ω} is **hereditarily finite**.

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It doesn't stop here

We have to keep going! $V_{\omega+1} = \mathcal{P}(V_{\omega})$. In fact, $\omega \notin V_{\omega}$, but $\omega \in V_{\omega+1}$.

- $\mathbb{Z} \in V_{\omega+4}$
- $\mathbb{Q} \in V_{\omega+7}$
- $\mathbb{R} \in V_{\omega+9}$
- $\mathbb{C} \in V_{\omega+12}$
- $(\mathbb{Q},+,0)\in V_{\omega+22}$

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- This is the limit of $\omega + n$ for n = 1, 2, ...
- $V_{\omega+\omega}$ is the closure of the power set operation on $\omega+n$ for n=1,2,...
- Fact: $V_{\omega+\omega}$ is a universe for all regular mathematics, and is the smallest such.

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It (still) doesn't stop here

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$$\omega + \omega + 1 = S(\omega + \omega),$$

• $\omega + \omega + \omega = 3\omega,$
• $\omega \cdot \omega = \omega^2,$
• $\omega^2 \cdot \omega^2 = \omega^4,$
• $\omega^{\omega},$
• $\omega^{\omega^{\omega}},$

• $\dots \mathcal{E}_0 \dots$

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Do these large numbers e	exist?		

Idea

Every "image" of a function defined by a formula on a set is still a set.

Axiom (Replacement)

For each formula ϕ without Y free,

 $\forall A \forall x \in A \exists ! y \phi(x, y) \implies \exists Y \forall x \in A \exists y \in Y \phi(x, y)$

Define $\phi: \omega \to \mathbf{V}$ by $n \mapsto \omega + n$. Thus, $\phi(\omega) = \omega + \omega$ is a definable set. So, all these sets *do* exist.

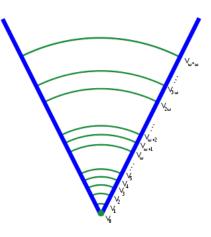
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Axiom (Foundation)

 $\forall x [\exists y (y \in x) \implies \exists y (y \in x \land \neg \exists z (z \in x \land z \in y))]]$

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Set Theory, 1980] Kenneth Kunen