Directed Reading Program Fall 2020

30 April 2020

Degree and Intersection Theory

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Mentor: Artur B. Saturnino

Text: John Milnor, "Topology from the Differentiable Viewpoint" / Victor Guillemin and Alan Pollack, "Differential Topology."

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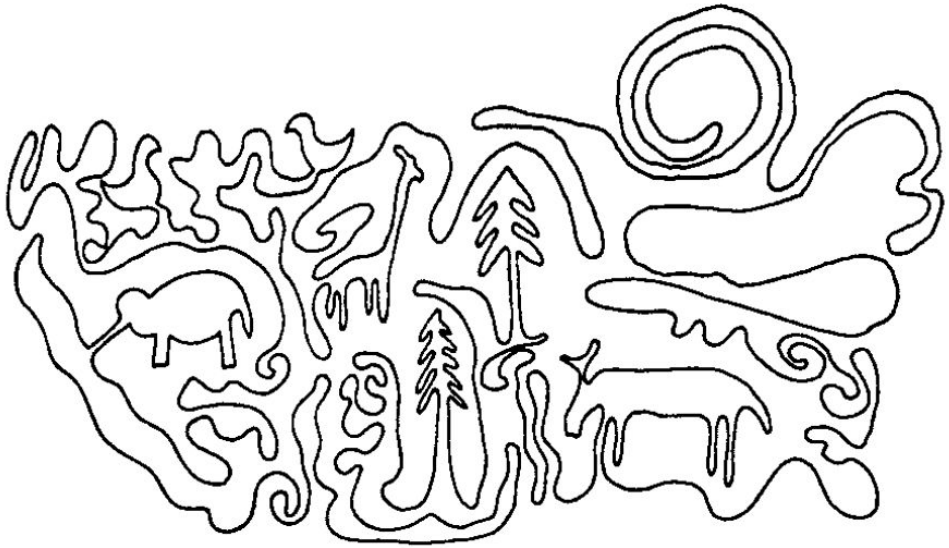
OVERVIEW

The aim of this presentation is to prove the Jordan Brouwer Separation Theorem.



DEFINITIONS

DEGREE OF A MAP

$$deg_2 f = \# f^{-1}(y) \mod 2$$

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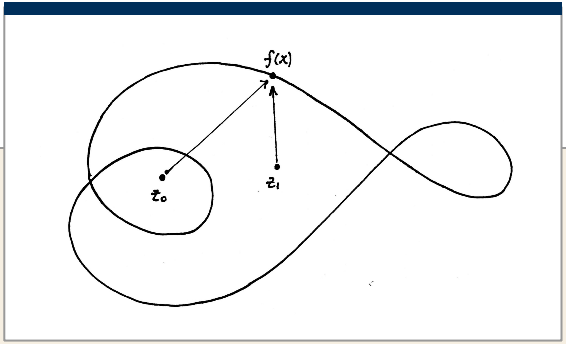
DIRECTIONAL MAP

Given a compact, connected manifold *X* and a smooth map $f : X \longrightarrow \mathbb{R}^n$, the **directional map** of any $z \in \mathbb{R}^n$ not in the image of *f(x)* is defined as:

$$u : X \rightarrow S^{n-1}$$

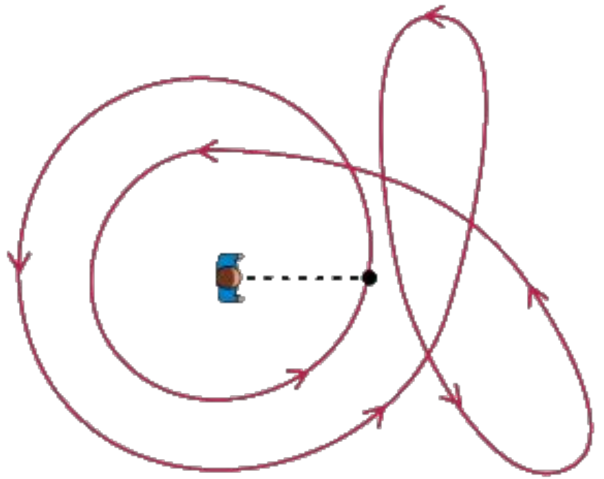
$$u(x) = \frac{f(x)-z}{|f(x)-z|}$$

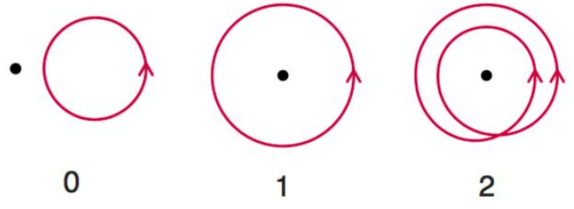


DEFINITIONS

WINDING NUMBER

$$W_2(f, z) = deg_2(u)$$





JORDAN BROUWER SEPARATION THEOREM

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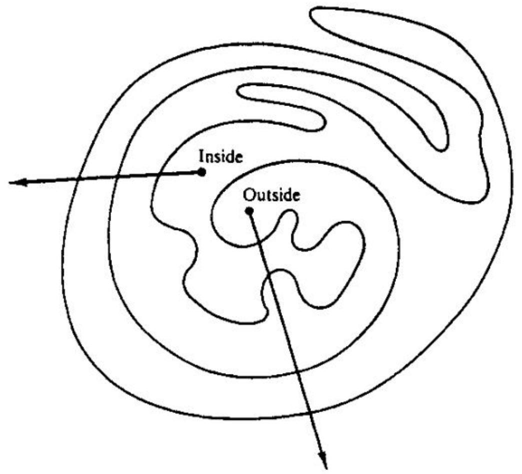
The complement of the compact, connected manifold *X* consists of two connected open sets: the outside D_0 and an inside D_1 .

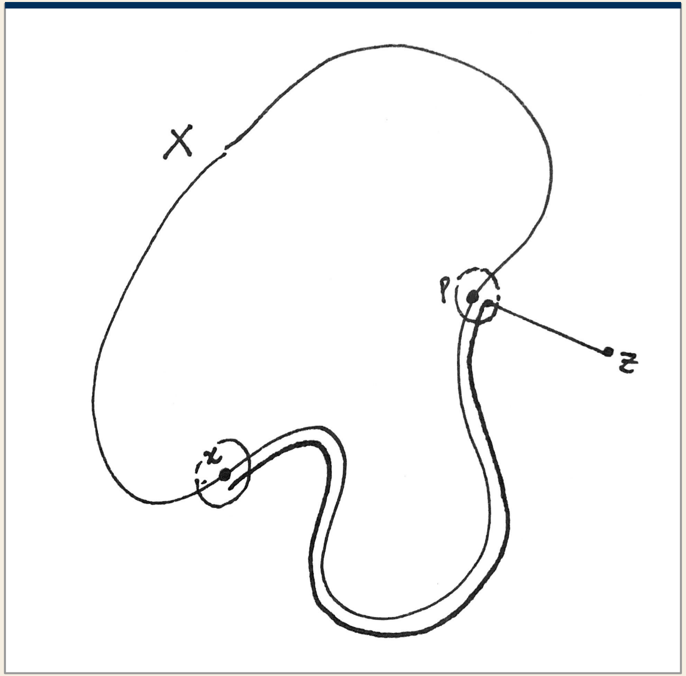


JORDAN BROUWER SEPARATION THEOREM

$\mathbb{R}^n \backslash X = D_0 \cup D_1$ with D_0, D_1 disjoint.

STEP ONE

Any fixed point in R^n - X can be joined to a point in a neighborhood of some $x \in X$ without intersecting X.



JORDAN BROUWER SEPARATION THEOREM

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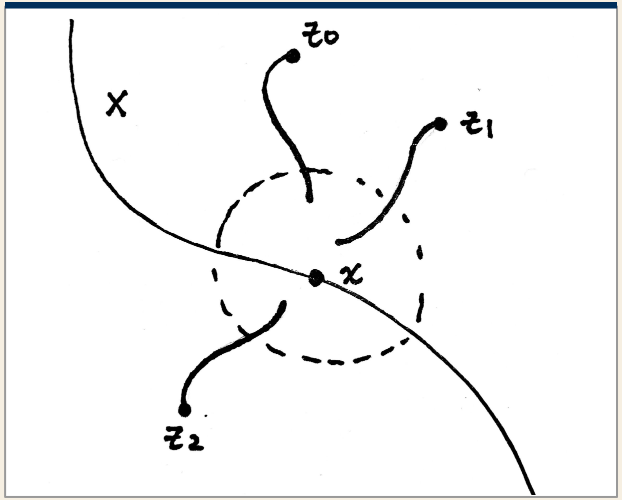
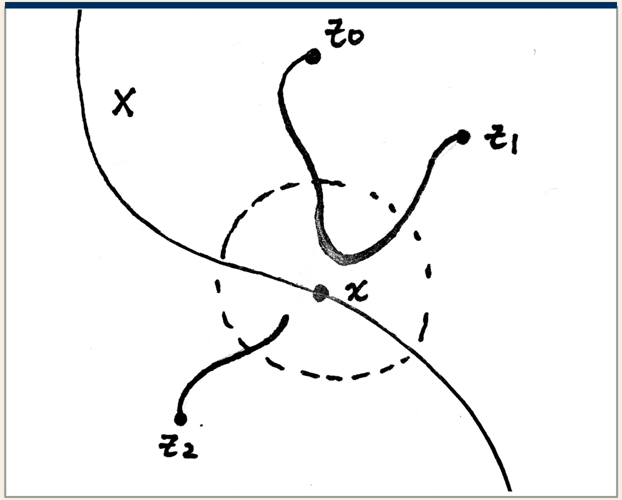
STEP TWO

R^n - X has, at most, 2 connected components.





JORDAN BROUWER SEPARATION THEOREM

$\mathbb{R}^n \backslash X = D_0 \cup D_1$ with D_0, D_1 disjoint.

STEP TWO

R^n - X has, at most, 2 connected components. Points in the same connected component have the same winding number.

Homotopy between z_0, z_1 directional maps.

$$u_t(x) = \frac{x - z_t}{|x - z_t|}$$

Degree is invariant under homotopy.

$$deg_2(u_0) = deg_2(u_1)$$

$$W_2(X, z_0) = W_2(X, z_1)$$

JORDAN BROUWER SEPARATION THEOREM

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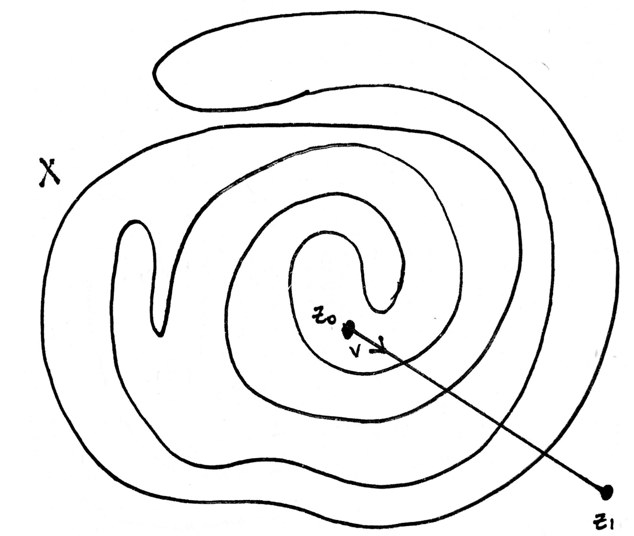
STEP THREE

Consider a ray, $r = \{z + t\vec{v} : t \geq 0\}$ that intersects X.



$$W_2(X, z_0) = W_2(X, z_1) + l \mod 2$$

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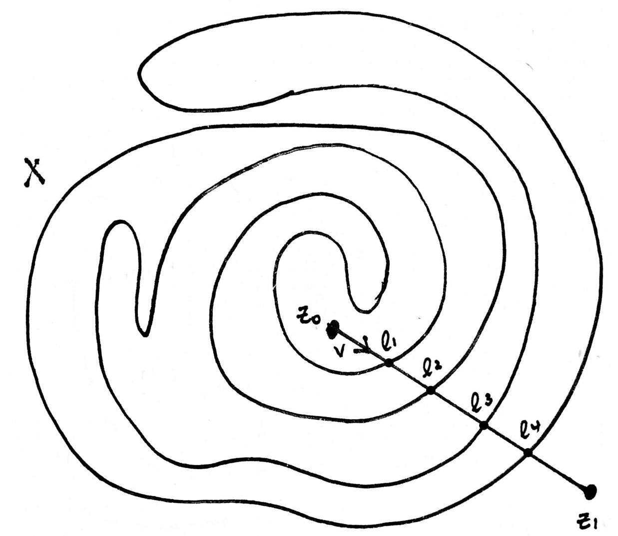


$$u^{-1}(z_0, v) = \{l_1 . l_2, l_3, l_4\}$$

$$W_2(X, z_0) = W_2(X, z_1) + l \mod 2$$

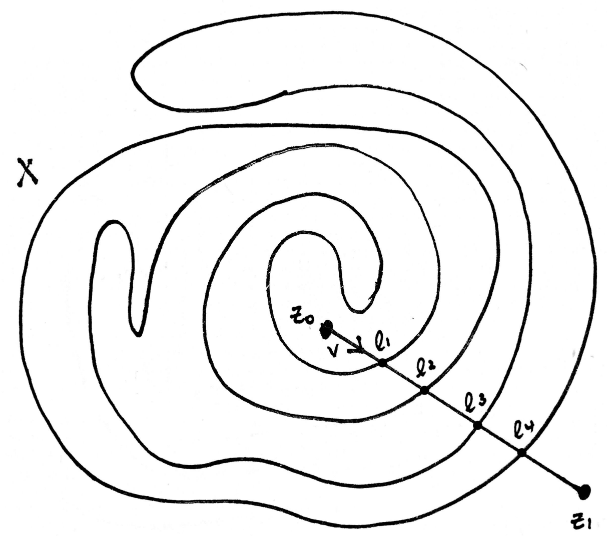
JORDAN BROUWER SEPARATION THEOREM

$$\mathbb{R}^n \backslash X = D_0 \cup D_1 \text{ with } D_0, D_1 \text{ disjoint.}$$

STEP FOUR

R^n - X has precisely two components.

$$D_0 = \{z : W_2(X, z) = 0\}$$
$$D_1 = \{z : W_2(X, z) = 1\}$$



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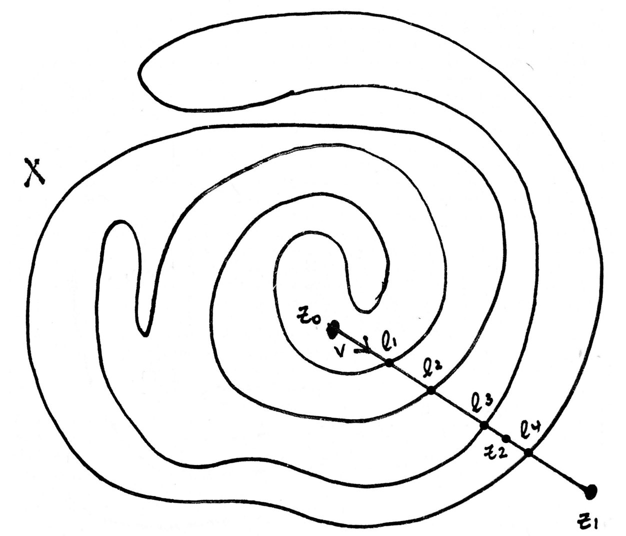


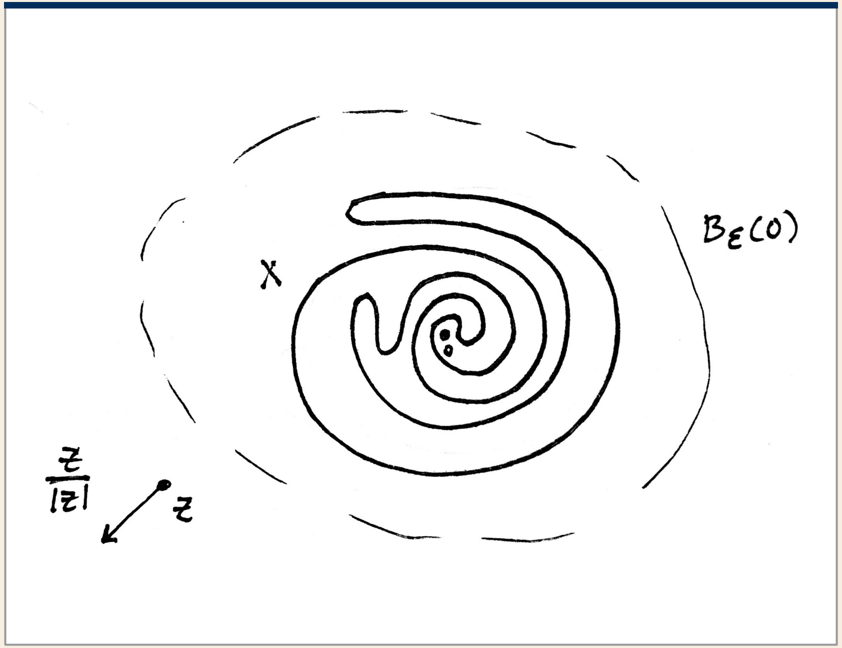
$$u^{-1}(z_0, v) = \{l_1 . l_2, l_3, l_4\} \quad u^{-1}(z_2, v) = \{l_4\}$$
$$W_2(X, z_0) = W_2(X, z_2) + 1$$

JORDAN BROUWER SEPARATION THEOREM

$\mathbb{R}^n \backslash X = D_0 \cup D_1$ with D_0, D_1 disjoint.

STEP FIVE

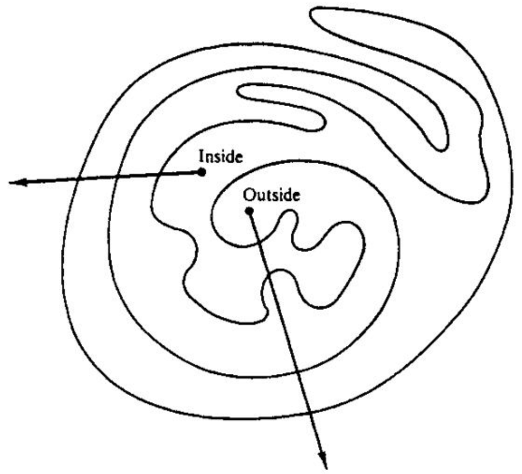
If *z* is very large, then $W_2(X, z) = 0$.
(i.e., D_0 is the "outside" of X).



JORDAN BROUWER SEPARATION THEOREM

THEOREM

We've shown that a simple, closed curve in R^n can be separated into an "inside" and "outside," which can be identified by the mod 2 winding number.



FINAL THOUGHTS

- The idea of a direction map is seen in the proof of other theorems.

  Ex: Poincare-Hopf Theorem
- Really interesting results from counting points!
- Thank you Artur for the past two semesters in the Directed Reading Program!

THANK YOU