Category Theory in Geometry

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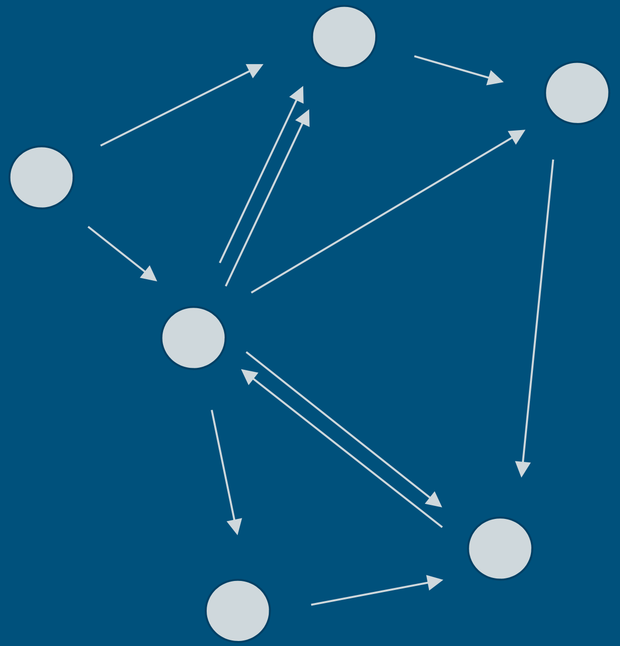
Categories

Category: a collection of objects and morphisms between objects

- Every object c has an identity morphism I_c
- For morphisms f : c → d and g : d → e, there is a composite morphism gf : c → e

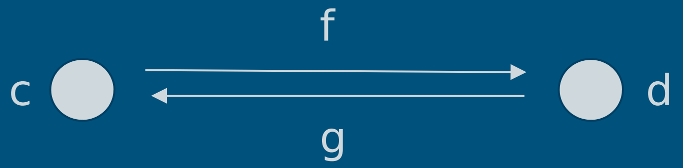
Examples:

- Sets & functions
- Groups & group homomorphisms
- Topological spaces & continuous functions



Categories

An isomorphism is a morphism $f : c \to d$ with $g : d \to c$ so that $fg = I_d$ and $gf = I_c$



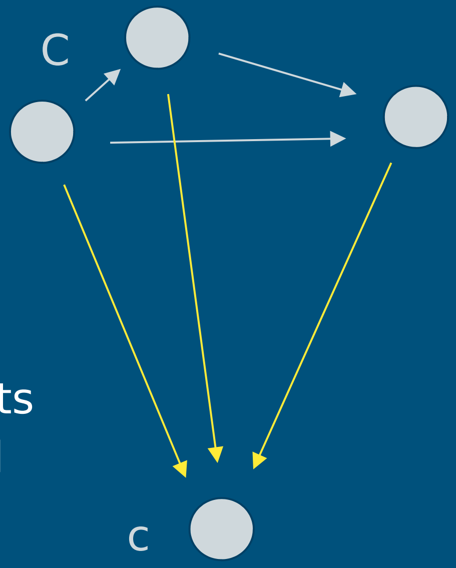
Examples

- ➢ Set: bijections
- ➢ Group: group isomorphisms
- ➢ Top: homeomorphisms

Functors

Functor: a map F : C → D between categories taking objects to objects and morphisms to morphisms

- Preserves identity morphisms
- Preserves function composition



Examples:

- Forgetful: Group → Set sends groups to sets of elements
- C(c, -): C → Set sends x to set of morphisms c → x and morphisms x→y to C(c,x)→C(c,y) by postcomposition
- Constant: C → c sends every object in C to c, every morphism to the identity on c

Diagrams

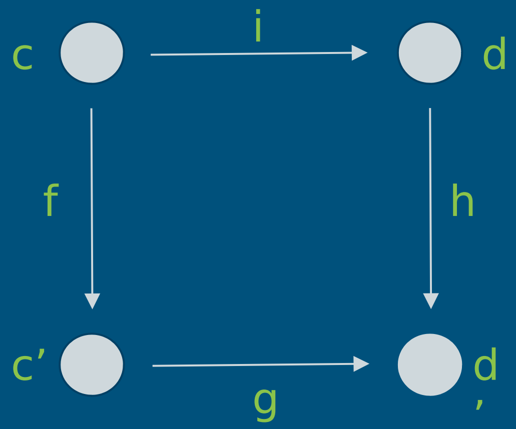


Diagram F : J → C:

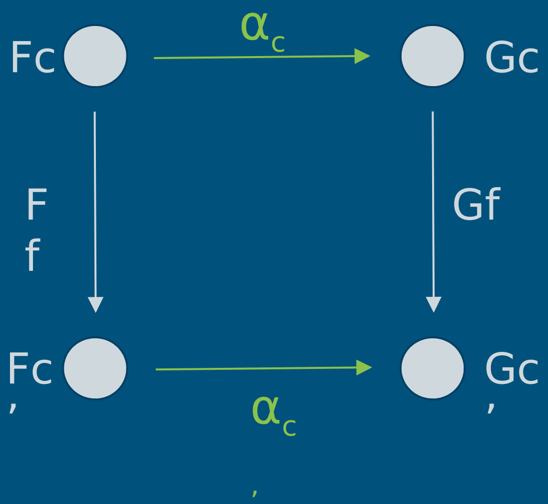
- ➢ An indexing category J of a certain shape
- ➢ A functor F assigning objects and morphism in C to that shape

Natural Transformations

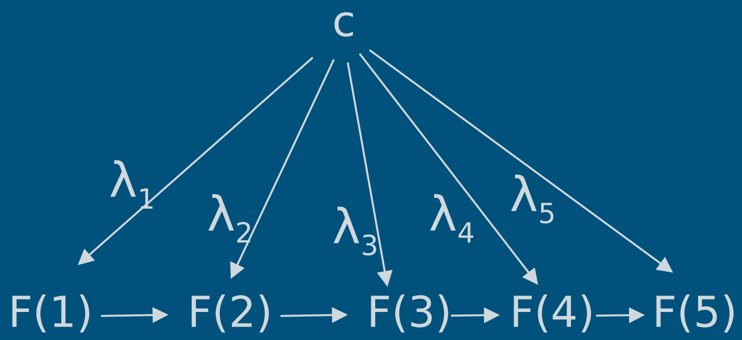
Natural transformation $F \Rightarrow G$ of functors $F, G : C \to D$:

- A collection of morphisms called components $\alpha_c : Fc \to Gc$
- For all $f : c \to c'$, the diagram commutes

If the components are isomorphisms, we have a natural isomorphism $F \cong G$



Cones



Cone over a diagram $F : J \to C$:

- A natural transformation between the constant functor $c : J \to c$ and the diagram $F : J \to C$
- The components λ_j are called legs

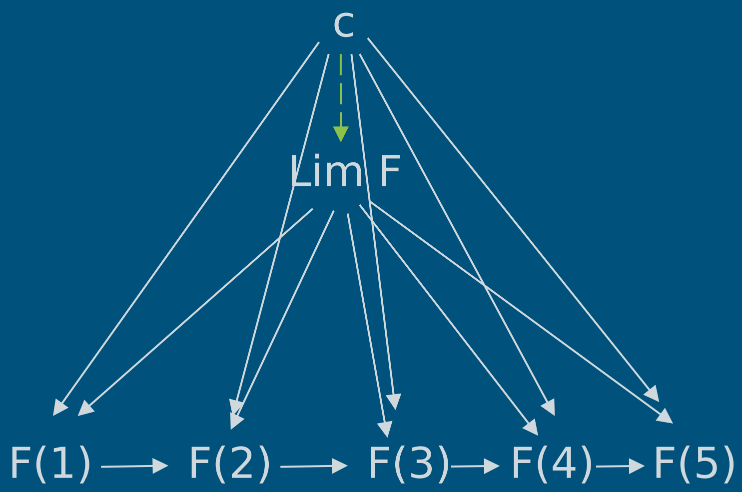
Universal Properties

A functor $F : C \to Set$ is representable if there is an object c in C so that $C(c, -) \cong F$

- Recall $C(c, -)$ takes an object c' to the set of morphisms $c \to c'$

The functor F encodes a universal property of c

Limits



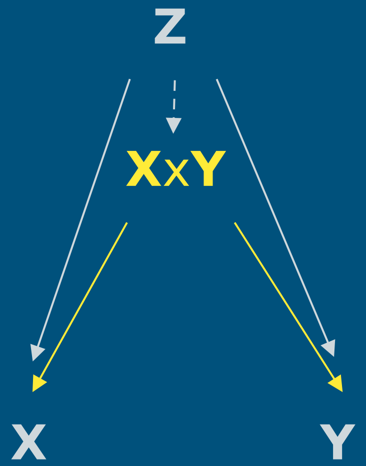
A limit is a universal cone:

- There is a natural isomorphism C(- , lim F) ≅ Cone(- , F)
- Morphisms c → Lim F are in bijection with cones with summit c over F

Limits in Geometry

Product

Diagram shape



Spaces

Product of spaces

Limits in Geometry

Pullback

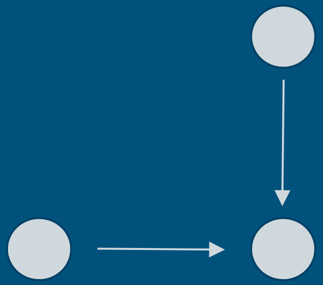
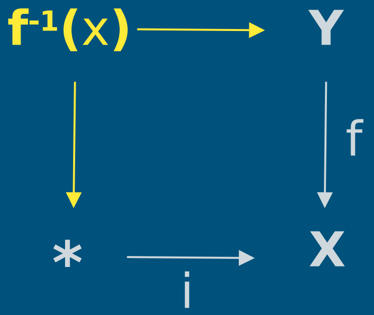


Diagram shape



Spaces

Fiber of x=i(*)

Conclusion

Category Theory is everywhere

- ➢ Mathematical objects and their functions belong to categories
- ➢ Maps between different types of objects/functions are functors
- ➢ Universal properties such as limits describe constructions like products and fibers

Reference

“Category Theory in Context” by Emily Riehl