Bézout's Theorem in Algebraic Geometry

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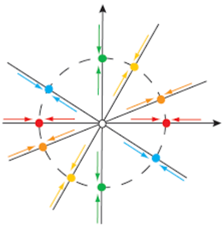
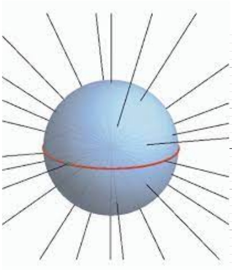
Motivation

Definition

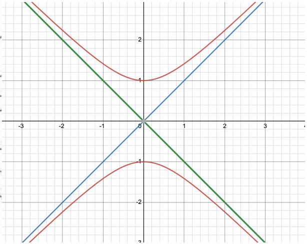
Projective $n-$Space $\mathbb{P}^n(k)$ is defined as $\left(k^{n+1}\backslash\{0\}\right) / \sim$, where $x \sim y \Leftrightarrow x = \lambda y$ for some $\lambda \in k^{\times}$. We use $[x_1, \cdots, x_n]$ to denote the equivalence class of $(x_1, \cdots, x_n)$.



(a) Visualization of $\mathbb{RP}^1$



(b) Visualization of $\mathbb{RP}^2$



(c) Asymptotes

Forms and Projective Plane Curves

Definition (Scale Invariance)

$F \in k[x_1, x_2, \cdots, x_n]$ is homogeneous (form) if $F(\lambda x_1, \lambda x_2, \cdots, \lambda x_n) = \lambda^n F(x_1, x_2, \cdots, x_n)$ at any point $(x_1, x_2, \cdots, x_n)$.

Equivalent Definition

$F \in k[x_1, \cdots, x_n]$ is a form if it is a sum of terms of the same degree.

Examples

$x_1 x_2 + x_1^2$ is a form while $x_1^2 + x_1 x_2^2 + 1$ is not.

Definition

$\gamma \subseteq \mathbb{P}^2(k)$ is a projective plane curve if there is a form F such that $\forall [(x_1, x_2, x_3)] \in \gamma, F(x_1, x_2, x_3) = 0$. We write $\gamma = V(F)$.

Irreducible Polynomials and Components

Definition

A polynomial in $k[x_1, \cdots, x_n]$ is irreducible if it cannot be written as the product of two nonzero polynomials of lower degrees.

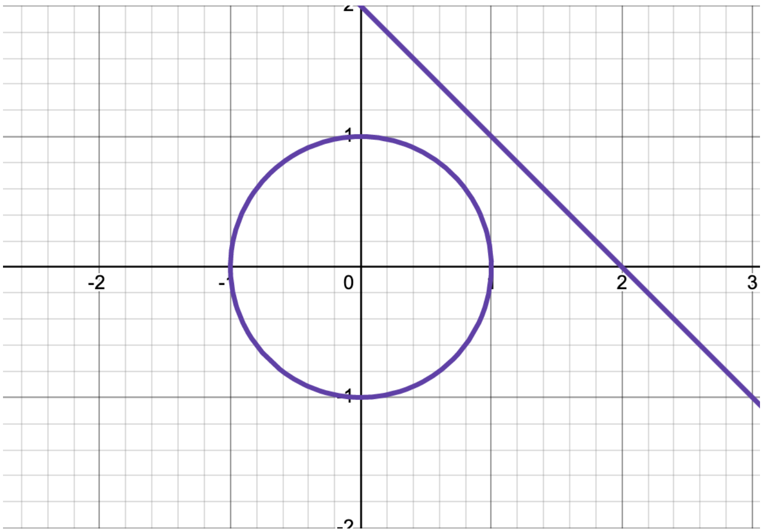
Examples

$x_1^2 + x_2 \in \mathbb{C}[x_1, x_2]$ is irreducible. However $x_1^2 + x_2^2 = (x_1 + ix_2)(x_1 - ix_2)$ is not irreducible in $\mathbb{C}[x_1, x_2]$.

Theorem

Every polynomial $F \in k[x_1, \cdots, x_n]$ can be written uniquely as the product of irreducible polynomials $\prod_{i=1}^{n} F_i^{e_i}$. Here, $F_1, \cdots, F_n$ are called the components of F.

Example and Visualization of Components



(d) Components of $(x + y - 2)(x^2 + y^2 - 1)$

Homogeneizing and Dehomogenizing

Homogenization

$f = f_0 + f_1 + \cdots + f_d \in k[x_1, \cdots, x_n]$, then the homogeneization of f is $f^* = \sum_{i=0}^{d} x_{n+1}^{d-i} f_i \in k[x_1, \cdots, x_n, x_{n+1}]$.

Examples

$x_1 x_2 + 1 \to x_1 x_2 + x_3^2$ $\qquad$ $x_1^2 + x_2 \to x_1^2 + x_2 x_3$

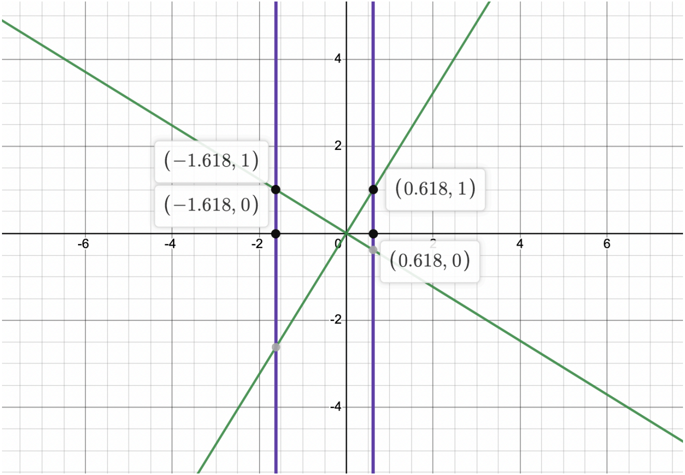
Dehomogenization

$F \in k[x_1, \cdots, x_n, x_{n+1}]$ is a form, then the dehomogeneization of F is $F_* = F(x_1, \cdots, x_n, 1) \in k[x_1, \cdots, x_n]$.

Examples

$x_1^2 + x_2 x_3 \to x_1^2 + x_2$ $\qquad$ $x_1 x_2 x_3 + x_2^2 x_1 + x_3^3 \to x_1 x_2 + x_2^2 x_1 + 1$

An Example



(e) Dehomogeneization of $x^2 - y^2 + xy$
Homogeneization of $x^2 + x - 1$

Properties of Homogeneizing/Dehomogeneizing

Properties

- $(fg)^* = f^* g^*$
- $x_{n+1}^a (f+g)^* = x_{n+1}^b f^* + x_{n+1}^c g^*$.
- $(FG)_* = F_* G_*$,
- $(F+G)_* = F_* + G_*$

Remark

Here, $a = deg(f) + deg(g) - deg(f+g), b = deg(g), c = deg(f)$. These properties can be summarized as "weakly additive" and "multiplicative."

Intersection Number

Now we introduce the concept of intersection number. Firstly, we need the definition of local ring.

Let $p = [(x_1, \cdots, x_n, 1)] = [(p', 1)]$, we have

Definition

Let $\mathcal{O}_p(\mathbb{P}^2) = \{\frac{f}{g} | g(p) \neq 0, f, g \text{ are forms}\}$. This is the local ring at point p. Similarly, let $\mathcal{O}_{p'}(k^2) = \{\frac{f}{g} | f, g \in k[x_1, x_2], g(p) \neq 0\}$.

Remark

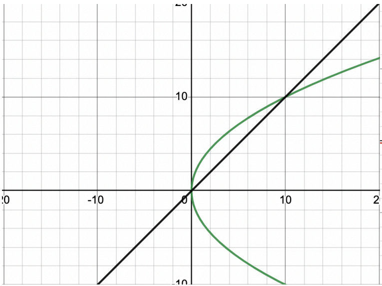
There is a canonical isomorphism $\mathcal{O}_p(\mathbb{P}^2) \cong \mathcal{O}_{p'}(k^2)$.

Intersection Number Continued

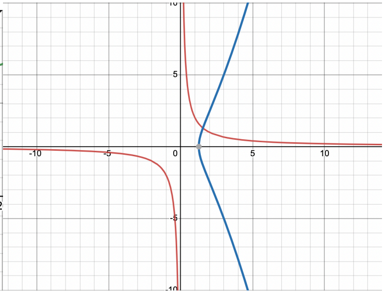
Now we can define Intersection Number of projective plane curves.

Definition

$I(p, F \cap G) = dim_k \left(\mathcal{O}_p(\mathbb{P}^2)/(F_*, G_*)\right).$

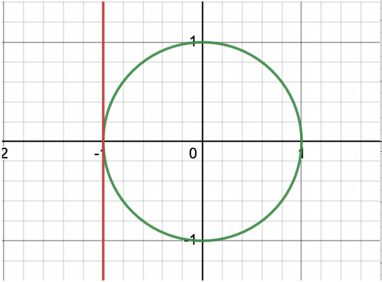


(f) $I(0, F \cap G) = 1$
$y^2 - x$ and $x - y$

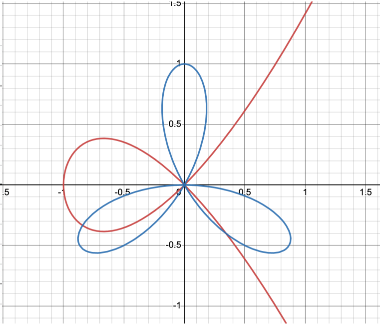


(g) $I(p, F \cap G) = \infty$
$x^3 - y^2 - 2$ and
$(x^3 - y^2 - 2)(xy - 2)$

More Examples



(h) $I((-1,0), F \cap G) = 2$
$y^2 + x^2 - 1$ and $x + 1$



(i) $I(0, F \cap G) = m_f m_g = 2 \times 3 = 6$
$y^3 - x^3 - x^2$ and
$(x^2 + y^2)^2 - y^3 + 3x^2y$

Bézout's Theorem

Main Theorem

If F, G have no common components,
$\sum_{p \in F \cap G} I(p, F \cap G) = deg(F)deg(G)$.

Lemma

If $\gamma_1 = V(F), \gamma_2 = V(G)$ such that F and G have no common components, then $|\gamma_1 \cap \gamma_2| < \infty$.

Remark

- Two projective plane curves always intersect because k is algebraically closed.
- The lemma guarantees that the sum is well-defined.

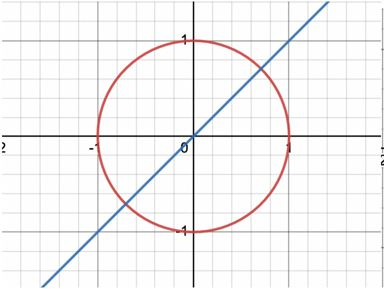
Examples

Remark

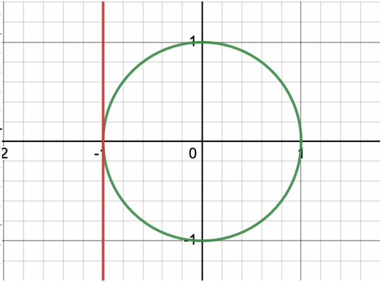
Note that $I([x_1, x_2, x_3], F \cap G) = I((\frac{x_1}{x_3}, \frac{x_2}{x_3}), F_* \cap G_*)$.

Remark

With this result, Bézout's Lemma also applies to intersecting plane curves.



(j) $y^2 + x^2 - 1$ and $x - y$



(k) $x^2 + y^2 - 1$ and $x + 1$

Idea of Proof

Idea of proof: Here we write $k[x_1, x_2, x_3] = A$, and the ideal of degree d forms $B_d, A/(F, G) = B$.

- Step 1: Reducing.

(1): $\sum_p I(p, F \cap G) = \sum_p I(p, F_* \cap G_*)$.

(2): $V(I) = \{P_1, \cdots, P_N\}$

$k[x_1, \cdots, x_n]/I \cong \prod_{i=1}^{N} \mathcal{O}_{P_i}(k^2)/I\mathcal{O}_{P_i}(k^2)$

(3): $\Rightarrow \sum_p I(p, F_* \cap G_*) = dim_k(k[x, y]/(F_*, G_*))$.

Proof Continued

- Step 2: When $d \geq deg(F) + deg(G)$, we use an exact sequence to show that $dim_k(B_d) = deg(F)deg(G)$.
  $0 \longrightarrow A \xrightarrow{\phi} A^2 \xrightarrow{\psi} A \xrightarrow{\pi} B \longrightarrow 0$ where $\phi : H \mapsto (GH, -FH), \psi : (A, B) \mapsto AF + BG, \pi$ is the canonical projection. Then we restrict the sequence from A to forms of various degrees, so that the second to last term becomes B_d.
- Step 3: $B \to B, \overline{H} \mapsto \overline{x_3 H}$ is injective.
- Step 4: Use homogeneization and dehomogeneization to show that a basis of B_d is also a basis of $k[x, y, 1]/(F_*, G_*)$.

Bibliography

Algebraic Curves–An Introduction to Algebraic Geometry, William Fulton.
http://www.math.uchicago.edu/ may/VIGRE/VIGRE2011/REUPapers/Menon.pdf