Hyperelliptic Curves with Points over Cyclotomic Extensions

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Galois Theory

Definition (Field Extension/Adjoinment)

A field K is a **field extension** of $\mathbb{Q}$ if $\mathbb{Q}$. A **field extension** K of $\mathbb{Q}$ is denoted $K/\mathbb{Q}$. If $\alpha \notin \mathbb{Q}$, the smallest field extension of $\mathbb{Q}$ containing α, denoted $\mathbb{Q}(\alpha)$, is a **field adjoinment**.

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Let $h(x)$ be a polynomial with coefficients in the field F. A field K is the **splitting field** of $h(x)$ if $K/\mathbb{Q}$ is the smallest field extension over which $h(x)$ can be factored into linear factors.

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> ***Example:*** $\mathbb{Q}(i)$ *is the splitting field of* $h(x) = x^2 - 1 = (x + i)(x - i)$.

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Example: *Let* $K := \mathbb{Q}(i)$*. Define* $F := \mathbb{Q}$*.*
Then $\mathbb{Q}(i)/\mathbb{Q}$ *is a Galois Extension because* $\mathbb{Q}(i)$ *is the splitting field of the polynomials* $f(x) = x^2 + 1$ *and* f *has distinct roots* $i, -i$*.*

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Definition (Automorphism)

Let $K/\mathbb{Q}$ be a field extension of $\mathbb{Q}$. A homomorphism $f : K \to K$ is an **automorphism** over $\mathbb{Q}$ if f is an isomorphism and f fixes $\mathbb{Q}$.

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$$Gal(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}.$$

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Proof: There are two automorphisms that map $\mathbb{Q}(i) \to \mathbb{Q}(i)$ and fix $\mathbb{Q}$: $f(a + bi) = a + bi$ and $f(a + bi) = a - bi$. The automorphisms of $\mathbb{Q}(i)$ form the group $\mathbb{Z}/2\mathbb{Z}$ because $g^2 = f = id$.

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There is a 1-1 correspondence between subfield extensions of a Galois extension and subgroups of the Galois group.

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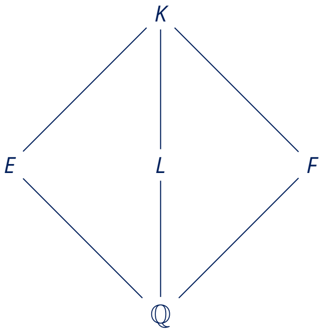
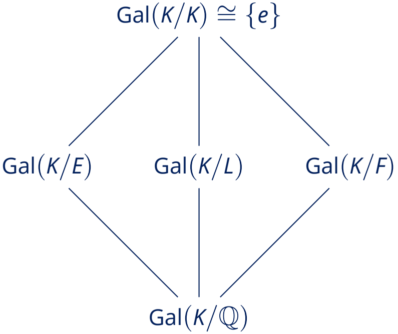
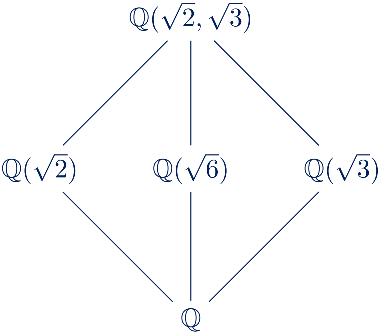
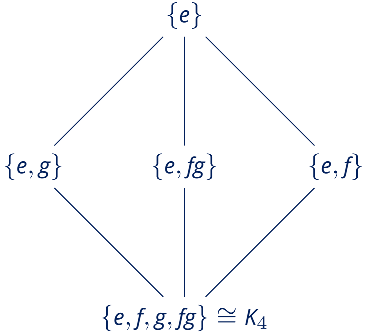




Galois Correspondence

- $e = id$
- $f : \sqrt{2} \to -\sqrt{2}$
- $g : \sqrt{3} \to -\sqrt{3}$





Algebraic Curves

Definition (Plane Curve)

A **plane curve** is the set of points $(\alpha, \beta) \in \mathbb{C}^2$ such that $F(\alpha, \beta) = 0$ for some polynomial $F(x, y)$ with coefficients in $\mathbb{Q}$.

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Example (Hyperelliptic Curves)

A plane curve E is **hyperelliptic** if it is of the form $y^2 = f(x)$ where $f(x) \in \mathbb{Q}[x]$. E is **elliptic** if $\deg(f(x)) = 3$.

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Example: *The curve* $E : y^2 = x^3 - x + 1$ *is an elliptic curve.*

Bringing it all Together

Definition (Field Adjoining a Point on a Curve)

Let $P = (\alpha, \beta), \alpha, \beta \in \overline{\mathbb{Q}}$ be a point on $E : F(x, y) = 0$ over $\mathbb{Q}$.

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Definition (Parameterization)

Consider a plane curve $E : F(x, y) = 0$. Let $x(t), y(t)$ be polynomials with coefficients in $\mathbb{Q}$. Then the polynomial $F(x(t), y(t)) = 0$ is a **parameterization** of $F(x, y)$. The roots of $F(x(t), y(t))$ give us points on E.

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Example: *Consider* $F(x, y) = x^2 + y^2 - 1 = 0$*. Let* $x(t) = 2t$ *and* $y(t) = t - 1$*. Then* $F(x(t), y(t)) = 5t^2 - 2t = 0$ *is a parameterization of* $F(x, y)$.

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Which finite groups can be realized as Galois groups over $\mathbb{Q}$?

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Our Question

Fix a plane curve $C : F(x, y) = 0$. What parameterizations $x(t), y(t) \in \mathbb{Q}[t]$ give us polynomials with Galois group $G \ncong S_n$?

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- Particularly, we are searching for curves and parameterizations that give us cyclotomic polynomials, whose Galois groups are always abelian (i.e. not S_n).
- Can we get a Galois group of the form $(\mathbb{Z}/n\mathbb{Z})^\times$ from an elliptic curve?

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Our Method

Claim [Keyes]

The parameterization $x(t) = t$, $y(t) = \frac{g(t)}{h(t)}$ on the hyperelliptic curve $F : y^2 = f(x)$ gives the following:

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$$\Theta(t) = g(t)^2 - h(t)^2 f(t) = 0$$

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$$\Theta(t) = g(t)^2 - h(t)^2 f(t) = 0$$

For each root α such that $\Theta(\alpha) = 0$, we adjoin $\mathbb{Q}(\alpha, \frac{g(\alpha)}{h(\alpha)})$ to get a Galois extension.

The field $\mathbb{Q}(\alpha, \frac{g(\alpha)}{h(\alpha)})$ is equal to $\mathbb{Q}(\alpha)$. [1]

Cyclotomic Polynomials

Definition (Cyclotomic Polynomial)

The nth **cyclotomic polynomial** denoted $\Phi_n(x)$ is the monic polynomial of minimal degree with Galois group $(\mathbb{Z}/n\mathbb{Z})^\times$.

Example: *The 4th cyclotomic polynomial is* $\Phi_4(x) = x^2 + 1$

- Alternatively: $\Phi_n(x) = \prod_{a \in (\mathbb{Z}/n\mathbb{Z})^\times} (x - \zeta_n^a)$
- Let p be prime. Then $\Phi_{p^n}(x) = \Phi_p(x^{p^{n-1}})$.
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Remark

Set $\Theta(t) := \Phi_n(t)$, the nth cyclotomic polynomial. If $\Theta(\alpha) = 0$ for some $\alpha \in \bar{\mathbb{Q}}$, then $\mathbb{Q}(x(\alpha), y(\alpha))/\mathbb{Q}$ gives a cyclotomic field extension of $\mathbb{Q}$.

Results

We computed examples using SageMath and generated a conjecture on how Cyclotomics factor. This conjecture implies results about our factoring method. We proved the following:

- **Theorem:** Let $n = p^m$ where $p \equiv 1 \mod 4$. Then $R(x) = x^{\frac{d}{2}} - \Phi_n(x)$ is reducible with a square factor.
- **Theorem:** Let $n = 3^m \cdot 2^\ell$. Then $R(x) = x^{\frac{d}{2}} - \Phi_n(x)$ is a perfect square.
- Recall our parametrization: $\Theta(t) = g(t)^2 - h(t)^2 f(t) = 0$. Then $\Phi_n(x) = \Theta(x)$, $g(x)^2 = x^{\frac{d}{2}}$, and $h(x)^2 f(x) = R(x)$.
- Thus $y^2 = f(x)$ is a hyperelliptic curve with a point over the *nth* cyclotomic field.

References

C. D. Keyes, *Growth of points on hyperelliptic curves over number fields*, (2019).