Proof and Models

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Logic as viewed in the past 2000 years

Classical Logic is "computationally trivial"

Logic is self-evident

Foundations to carry out math

Mathematical Logic

Logic as an Object of Mathematical Study

- We can study logic within math
- How do we characterize a proof?
- Can we exhaust all proofs?
- What does something unprovable look like?
- Is this computationally trivial?

Characterize Proofs

- Proofs can be seen as lists of sentences that refer to one another based on a set of rules
- Starting statements will be axioms that we take for granted

Models

- A theory is a set of sentences built on primitive formulas
- A model of that theory is a set of sentences that can be assigned a truth assignment

Examples of Models

- Natural Numbers
- Graphs
- Groups, like Integers mod 2

Semantic Truth

$M \models \phi$ if and only if ϕ holds for every instance of M

- In other words, ϕ is true for every truth assignment where M is satisfied

Completeness Theorem

Completeness is broken up into two parts:

- Soundness: $M \vdash \phi \to M \vDash \phi$
- Adequacy: $M \vDash \phi \to M \vdash \phi$

Change of Quantifiers

- $M \vdash \phi$ means that there exists a proof ϕ of from M
- $M \vDash \phi$ means that all truth assignments of M satisfy ϕ

It is generally easier to prove existence than universal properties

Consistency

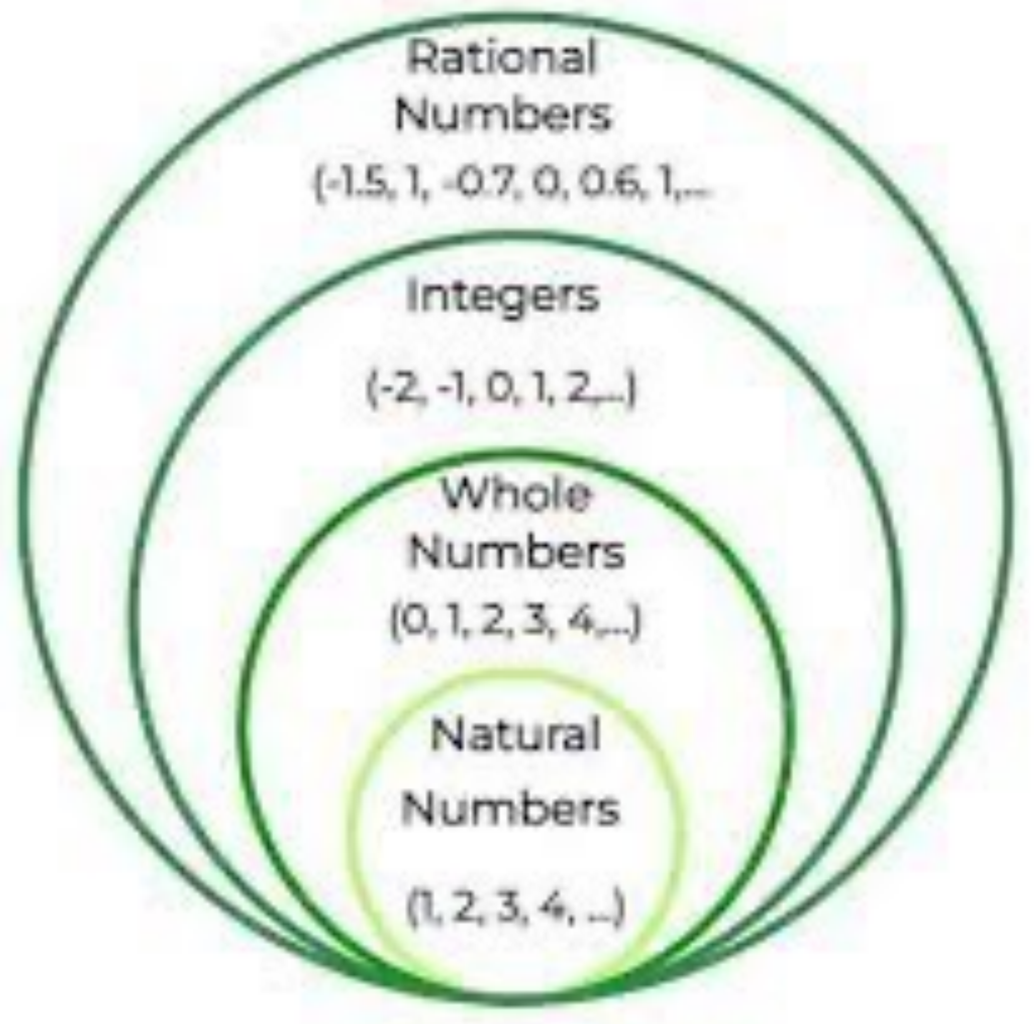
- Something that is extremely hard to prove using proof theory can be intuitively expressed within a model

$$\mathrm{PA} \nvdash \bot \iff \mathrm{PA} \nvDash \bot$$

Consistency of Peano Arithmetic

- $\mathrm{PA} \vDash \bot$ if and only if every model M of Peano Arithmetic is unsatisfiable
- $\mathrm{PA} \nvDash \bot$ if and only if there exists some satisfiable model of Peano Arithmetic

The Natural Numbers



Set of Natural Numbers

Unprovability ≅ Countermodel

$$\mathrm{ZF} \nvdash \mathrm{C} \iff \mathrm{ZF} \nvDash \mathrm{C}$$

To prove unprovability of choice, we only need to find a model that satisfies $\{\mathrm{ZF}\} \cup \neg\{\mathrm{C}\}$

Similarly prove unprovability of negation of choice, we need to find a model that satisfies ZFC

Philosophically

- Philosophically, this means that everything that is unprovable is unprovable for a reason
- It’s because there exists a model that satisfies the hypotheses but not the statement

Corollary of Compactness

- Completeness helps us utilize the finite property of proofs

We state that a set of well formed formulas S tautologically implies a statement φ (or $S \vDash \varphi$) if and only if every truth assignment that satisfies S also satisfies φ. Prove that if Σ is a set of well formed formulas such that $\Sigma \vDash \varphi$, then there exists some finite subset Σ_0 of Σ such that $\Sigma_0 \vDash \varphi$.

Using Completeness

- If $\Sigma \models \varphi$ then $\Sigma \vdash \varphi$
- Proofs are finite, so take the finite subset of Σ that proves ϕ
- Since this proves ϕ, it also semantically implies ϕ

Proofs using Finitely Many Hypotheses

- If we have finitely many hypotheses, we can deterministically determine if a set proves a statement
- Computationally trivial: Just use truth tables

Gödel’s Incompleteness Theorem

- Peano Arithmetic is Incomplete

Inspiration

“This statement cannot be proven”

- If it’s true, it can’t be proven
- If it’s false, it can be proven, meaning that if it cannot be proven, it is true
- Thus, this statement is true if and only if it cannot be proven

Issue: We can’t write this with symbols in formal logic

Gödel Coding

- An Injective mapping from statements to numbers
- Uniqueness, so why not Fundamental Theorem of Arithmetic

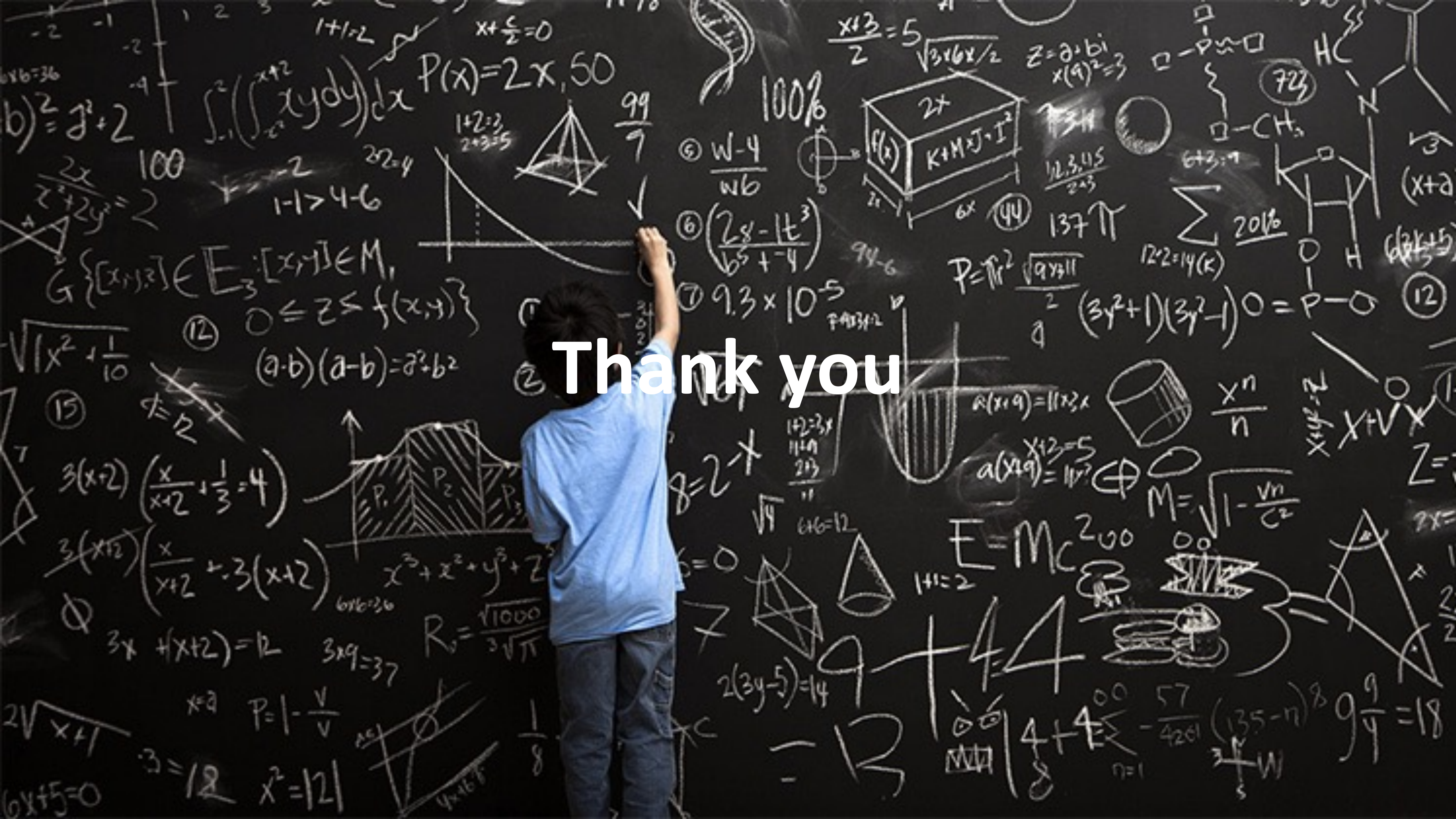
Statement

- “The formula with Gödel Number ___ cannot be proven”
- We can show that ___ exists such that it points to itself.

Gödel’s Second Incompleteness Theorem

- No system can prove its own consistency

Classifying Simple Logic Systems


Thank you