Noether's Theorem and Symplectic Geometry

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Statement of the theorem

Heuristics

For every continuous symmetry of a system, there corresponds a conserved quantity.

More formally

If a Hamiltonian function H on a symplectic manifold (M^{2n}, ω) admits the one-parameter group of canonical transformations given by a hamiltonian F, then F is a first integral of the system of Hamiltonian function H.

The one-parameter group pf transformations captures the symmetry, and the "first integral" F captures the conserved quantity.

Some physics

Principle of Least Action $\xrightarrow{\text{calculus of variations}}$ Lagrangian Mechanics

Lagrangian Mechanics $\xrightarrow{\text{Legendre transform}}$ Hamiltonian Mechanics

Hamilton's Equations

$H(q, p, t)$ is a real valued function, where $q = (q_i)$ is position, $p = (p_i)$ momentum, t time.

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

is called Hamilton's equations

More physics

Phase space

Phase space keeps track of possible configurations of a mechanical system. It consists of points (q, p), where q is a position vector, p is a momentum vector.

Knowing the position and momentum of a particle tells you its trajectory under the Hamilton's equations.

p can be thought of as a 1-form acting on $\dot{q}$.

Basic definitions

Cotangent bundle

Given a n-dimensional differentiable manifold N, its cotangent bundle is defined to be the $2n$-dimensional manifold M consisting of the union of cotangent spaces of N.

Symplectic manifold

A symplectic manifold M is a $2n$-dimensional differentiable manifold together with a non-degenerate, closed differential 2-form ω on M^{2n}.

A cotangent bundle has a natural symplectic structure (we can define such an ω to make it into a symplectic manifold).

Tying physics with math

Given a Hamiltonian function $H : M \to \mathbb{R}$, we an consider dH, which would be a differential 1-form.

Hamiltonian vector field

The Hamiltonian vector field on our symplectic manifold (M, ω) associated with the Hamiltonian function H is defined to be the vector field X_H such that $\omega(X_H, -) = dH$

It turns out that if we trace out the "integral curves" of this vector field, these are exactly trajectories satisfying the Hamiltonian equations. This is due to both the closedness and non-degeneracy of our form ω.

Flow

A vector field induces a flow: it locally tells you how to map every point of the manifold to some other point.
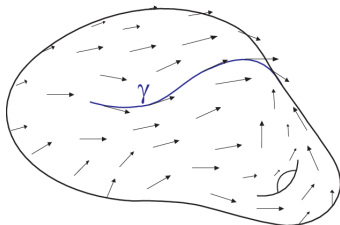
More formally, the vector field induces a family of diffeomorphisms.
In particular, if we look at the Hamiltonian vector field X_H of some H, the group of diffeomorphisms, g_H^t, induced by that vector field preserves the symplectic structure in precomposition.

Poisson bracket

We can in fact define a Lie Algebra structure (a novel way of "multiplying") over functions from $M \to \mathbb{R}$.

Poisson bracket

The Poisson bracket takes as input two functions and outputs one function:

$$(H, F) = \left. \frac{d}{dt} \right|_{t=0} H(g_F^t(x))$$

This bracket can be shown to be skew-symmetric.
If $H(g_F^t(x)) = H(x)$, then we say that H **admits the one-parameter family** of diffeomorphisms g_F^t.

Observation

A function F is a first integral of the flow with Hamiltonian H (F doesn't change along possible trajectories) if and only if $(F, H) \equiv 0$.

Understanding the theorem

Noether's Theorem

If a Hamiltonian function H on a symplectic manifold (M^{2n}, ω) admits the one-parameter group of canonical transformations given by a hamiltonian F, then F is a first integral of the system of Hamiltonian function H.

Proof: Since H admits g_F^t, $H(g_F^t(x)) = H(x)$ and so $\frac{d}{dt}\big|_{t=0} H(g_F^t(x)) = 0$, which means $(H, F) = 0$. By skew symmetry, $(F, H) = 0$. Therefore, F is a first integral for the system with Hamiltonian function H.

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