Discrete Morse Theory

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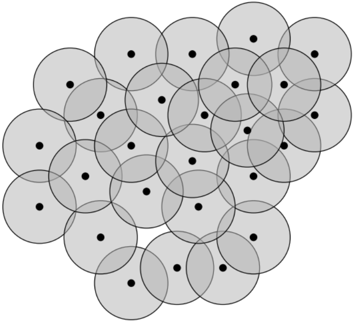
Motivation: Wireless Sensor networks

Cell phone towers have sensors to pick up cell phone signals

- Cell phone system covers the largest area possible

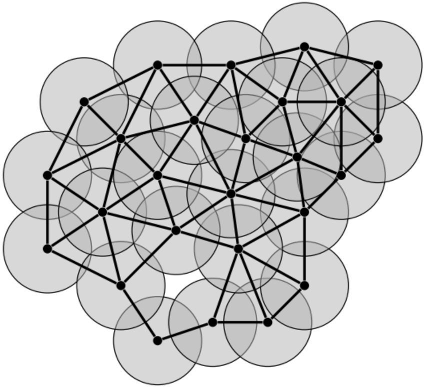


Point = cell tower, circle = tower's sensing area

Can we learn whether a particular region is covered so that no matter where you are, you will have cell phone service?

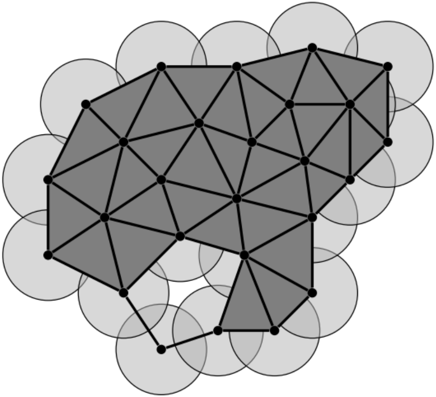
Motivation: Wireless Sensor networks

Only info we know: what cell towers are in the radius of other cell towers
Connect any two towers where their ranges overlap

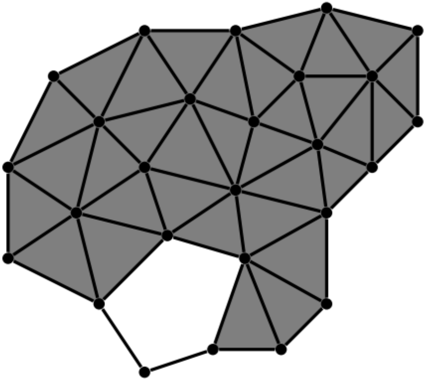


Motivation: Wireless Sensor networks

Fill in regions where 3 tower ranges overlap

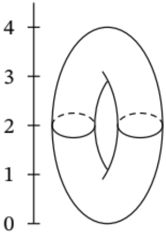


filled in communication graph



simplicial complex

Classical Morse Theory



To study this torus, we break it down into fundamental piece

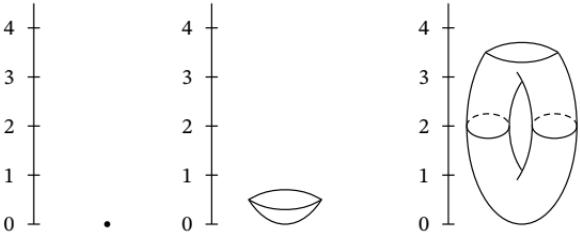
- point
- curve
- cup

Stretching and bending these pieces can get back the torus

Height Function

What's an easier way to get those fundamental pieces?
We define a "height function" : for any height z, we slice through the torus at height z and keep everything below

- $M_0 =$ point
- $M_{0.5} =$ cup
- $M_{3.5} =$ almost a torus
- Notice $M_4 = M_{3.5} +$ a cap

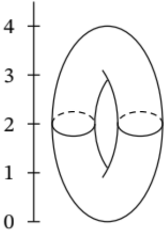


Critical Points

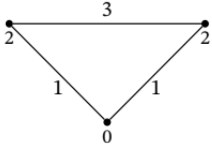
Gluing occurs at critical points of our height function

- 0, 1, 3, 4 are critical points

This recovers the topology of our object and gives us info about our object (eg. its holes)

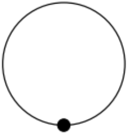


Discrete Morse Theory



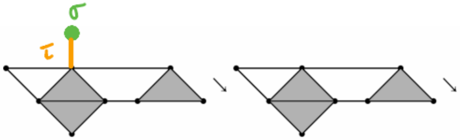
Discrete Morse Theory:

- analogue to the "height function": put numbers on each part of the simplicial complex (generally increase numbers as dimension of simplex increases)
- 0 is the local min, 3 is the local max
- we can build a circle out of these two pieces



Collapse

- Free pair: a pair of simplicies $\sigma^{(p-1)}, \tau^{(p)}$ such that σ is a face of τ and τ has no other coface
- Elementary collapse: removing a free pair from a simplicial complex



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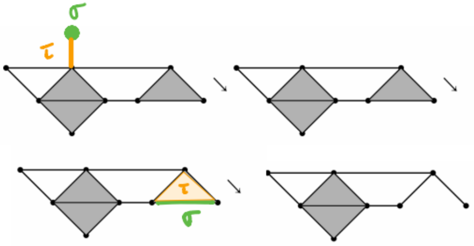

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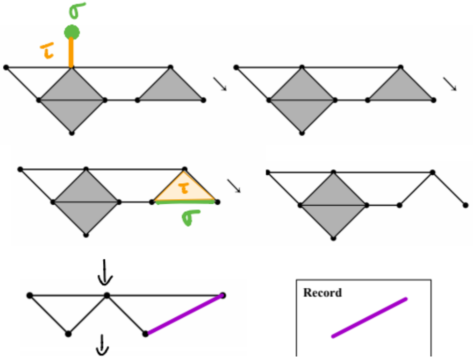

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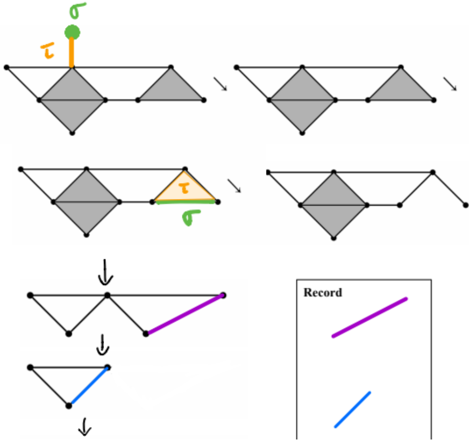

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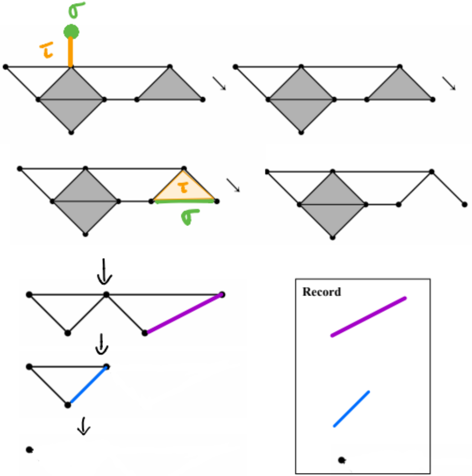


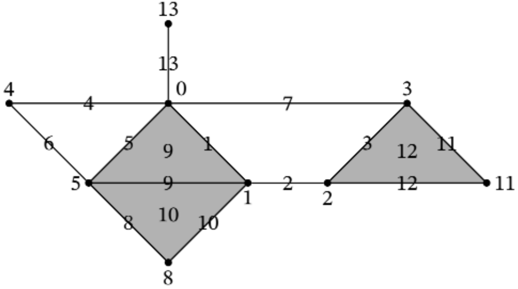
Discrete Morse Theory

Let K be a simplicial complex.
A function $f : K \to \mathbb{R}$ is **weakly increasing** if $f(\sigma) \leq f(\tau)$ whenever $\sigma \subset \tau$.

- ie. lower dimensional simplex has a lower corresponding number

A **basic discrete Morse Function** $f : K \to \mathbb{R}$ is a weakly increasing function which satisfies two properties:

- 2-1
- if $f(\sigma) = f(\tau)$, then either $\sigma \subset \tau$ or $\tau \subset \sigma$

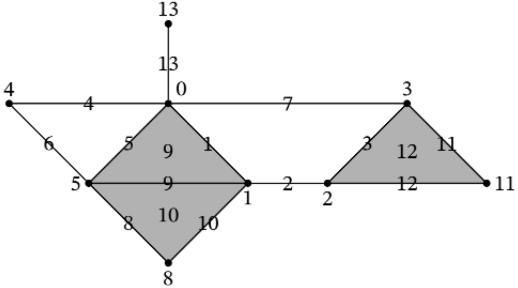


Critical Values

Let $f : K \to \mathbb{R}$ be a basic discrete Morse function.

A simplex σ is **critical** if they have a unique value

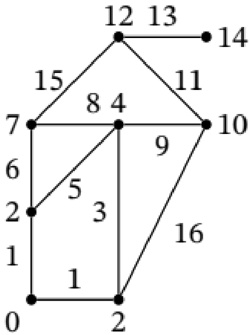
The value $f(\sigma)$ is the **critical value**



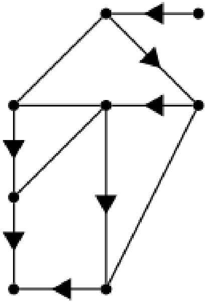
critical simplicies: 0, 6, 7

Gradient Vector Fields

We can graphically show what the discrete Morse function is doing to the complex K.



complex K



Gradient Vector Field

We draw an arrow if a lower dimensional simplex has a greater value than its coface

Gradient Vector Field

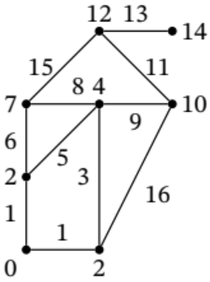
Let f be a discrete Morse function on K. The **induced gradient vector field** V_f

$$V_f := \{(\sigma^{(p)}, \tau^{(p+1)} : \sigma < \tau, f(\sigma) \geq f(\tau)\}$$

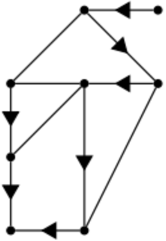
(σ, τ) is an arrow. σ is a tail, τ is a head

Easier to see critical points in this example:

- an edge is critical iff it is not the head of an arrow
- a vertex is critical iff it is not the tail of an arrow



complex K



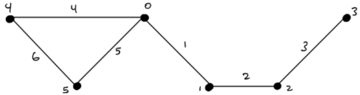
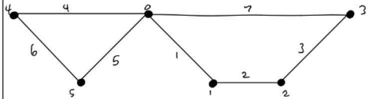
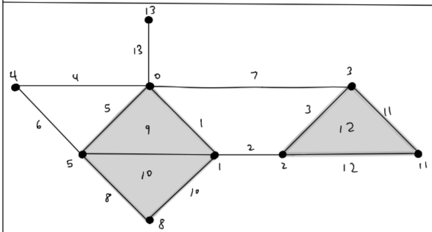
Gradient Vector Field

Persistence with discrete Morse functions

For a discrete Morse function, we can write its Betti numbers $\rightarrow$ this forms a **homological sequence**

- 0th dimension: number of connected components
- 1st dimension: number of holes

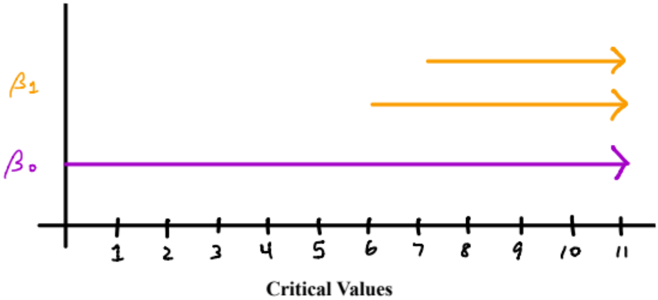
Betti Numbers

Simplex	β_0	β_1
	1	0
	1	1
	1	2
	1	2

Constructing Barcodes

Critical points occurred at points 0, 6, 7

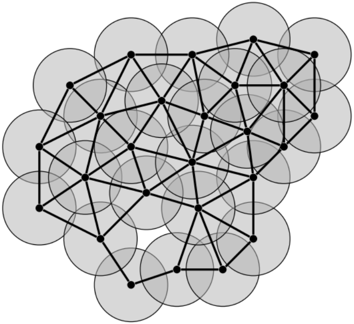
β_0	1	1	1	1
β_1	0	1	2	2



Back to Wireless Sensor networks

Goal: detect holes in the region (areas with no service)

- nontrivial 1st homology $\rightarrow$ there's a hole



- computing number of holes runs on the order of kn^3 where n is the number of lines VS discrete Morse Theory reduces the computational effort

Special Thank You to my amazing mentor, Jacob Van Hook!!

References

Nanda, V. (2021). Computational Algebraic Topology: Lectures on Algebraic Topology.
Scoville, N. A. (2019). Discrete Morse theory (Vol. 90). American mathematical society.

Thank you!
Questions?