Bézout's Theorem

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Section 1

Projective Plane

The Projective Plane

Definition

The **complex projective plane** is the set of equivalence classes of $\mathbb{C}^3 \backslash \{0\}$, such that two vectors are equivalent if one is a scalar multiple of another. We denote $[x : y : z]$ as the equivalence class of the vector (x, y, z).

In practice, we can think of the 2D real projective plane as the real plane adjoined to all directions on the plane.

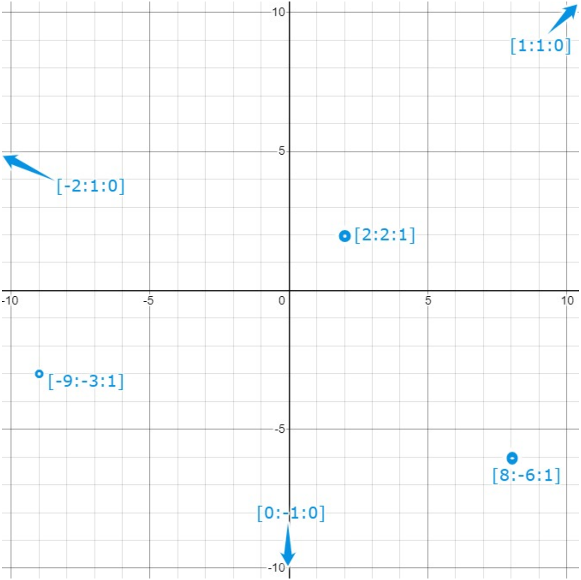

Figure: Points on the Projective Plane

Projective Plane Curves

Definition

Given a polynomial $F \in k[X, Y, Z]$ such that all terms in F are of the same degree d, the set of points in the projective plane for which $F = 0$, called the vanishing set of F, forms a **projective plane curve**.

Any affine planar curve can be extended to a projective plane curve by using the third variable Z to homogenize the equation. For example, the affine curve corresponding to the vanishing of $Y^2 - X^2 - 1$ can be extended to a projective curve corresponding to the vanishing of $Y^2 - X^2 - Z^2$.

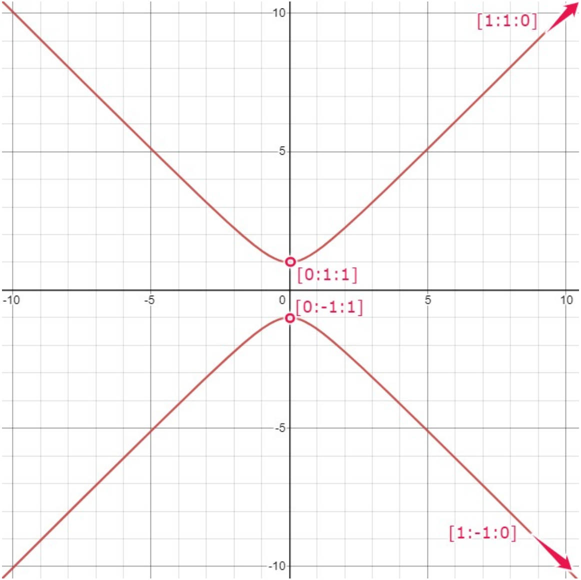

Figure: The Vanishing Set of $Y^2 - X^2 - Z^2$ as a Projective Curve

Section 2

Statement of Bézout's Theorem

Intersection Numbers

We say a point P on a curve F is **simple** if either $F_X(P) \neq 0$ or $F_Y(P) \neq 0$. If a point P is simple on curves F and G, and the curves meet transversally (the tangents to F and G are distinct), we wish to define the intersection number $I(P, F \cap G) = 1$.

Definition

The **intersection number** of two projective plane curves F and G at a point P equals $\dim_k(\mathcal{O}_P(\mathbb{P}^2)/(F, G))$, where $\mathcal{O}_P(\mathbb{P}^2)$ is the field of rational functions on $\mathbb{P}^2$.

Intersection Numbers: Visual Example

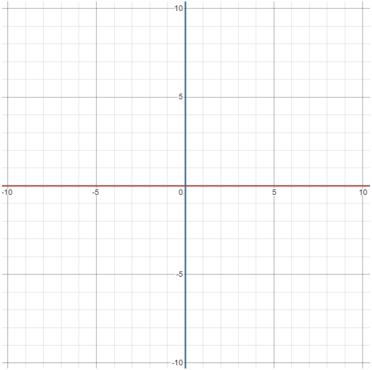

Figure: The forms Y and X have an intersection number of 1 at (0,0).

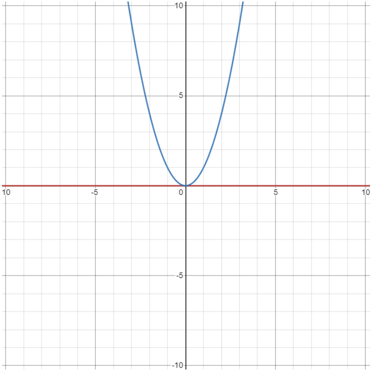

Figure: The forms $YZ - X^2$ and Y have an intersection number of 2 at (0,0).

Bézout's Theorem

Given two projective plane curves F and G, of degrees m and n respectively, such that F and G have no common component, we have

$$\sum_P I(P, F \cap G) = mn$$

Section 3

Applications of Bézout's Theorem

Pascal's Theorem and Pappus's Hexagon Theorem

Pascal's Theorem: Given any conic and six points P_1, P_2, P_3, P_4, P_5, P_6 forming a hexagon on that conic, the intersections of opposite sides of the hexagon are collinear.

Pappus's Theorem: Given any two lines and points P_1, P_2, P_3 on one line and points Q_1, Q_2, Q_3 on the other, the intersections of the three pairs of crossing lines are collinear.

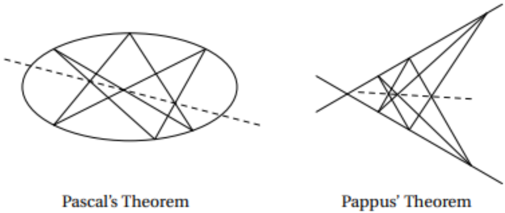

Figure: Pascal's and Pappus's Theorem

Max Noether's Fundamental Theorem

If F, G, and H are projective plane curves, F and G have no common component, and Noether's conditions are satisfied at each point P where F and G intersect, then there exist curves A and B to satify the equation $H = AF + BG$.

Corollary: If F and G are projective curves such that every intersection of F and G is simple, and H is a projective curve such that H passes through every intersection of F and G, there exists a curve B such that B intersects F exactly in the points where H intersects F but G does not.

References

Fulton, William (2008). *Algebraic Curves*.