Covering Spaces

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Outline

Section 1

Basics

Preamble

Motivation

To understand the structures of objects better, we often look for "extensions" of them that may be easier to deal with.

Convention

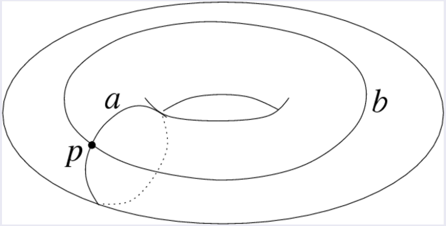
Since we are in the category of topological spaces, a "map" will refer to a continuous function, and a "space" will refer to a set endowed with a topology. The topologies we consider will all be the standard ones; expect nothing exotic.

The Fundamental Group

Idea

Informally, it is the set of all "classes" of loops at a given base point, which forms a group under concatenation. We denote the fundamental group of a space B at b_0 as $\pi_1(B, b_0)$, or when the base point is not relevant, simply $\pi_1(B)$.

Visual

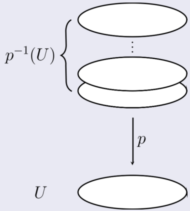


Diving In

Definition

Let $p : E \to B$ be a surjective map. We say that p is a **covering** if for every $b \in B$, there exists a neighborhood U_b in B containing b such that $P^{-1}(U_b) = \bigcup_\alpha V_\alpha$ where each V_α is open, mapped homeomorphically onto U_b by p, and for any $\alpha_1 \neq \alpha_2$, $V_{\alpha_1} \cap V_{\alpha_2} = \emptyset$. We say that E is a **covering space** of B.

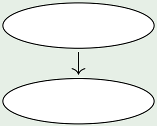
Visual



Examples

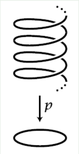
Example

Let X be a space. $id_X : X \to X$ is a covering.



Example

$p : \mathbb{R} \to S^1$ defined by $x \mapsto (cos(2\pi x), sin(2\pi x))$ is a covering.



Basic Properties

Liftings

$$\begin{array}{ccc} & & E \\ & \overset{\tilde{f}}{\nearrow} & \downarrow p \\ [0,1] & \xrightarrow{f} & B \end{array}$$

Embeddings

$$\pi_1(E, e_0) \overset{p_*}{\hookrightarrow} \pi_1(B, b_0)$$

Lifting Correspondence

$$\pi_1(B, b_0) \ni [f] \longmapsto \tilde{f}(1) \in p^{-1}(b_0)$$

Section 2

Classification

Existence

Idea

Every subgroup of the fundamental group of a "nice" space corresponds to a covering space.

Theorem

Let $b_0 \in B$. If B is path connected, locally path connected, and semilocally simply connected, then for every $H \leq \pi_1(B, b_0)$, there exists a covering $p : E \to B$ such that $p_*(\pi_1(E, e_0)) = H$.

Convention

Hereafter, we assume that B is path connected and locally path connected, and that E is path connected.

Equivalence

Idea

When are two coverings essentially the same?

Definition

Let $p : E \to B$ and $p' : E' \to B$ be coverings. An **equivalence** is a homeomorphism $h : E \to E'$ such that $p' \circ h = p$.

Theorem

Let $p : E \to B$ and $p' : E' \to B$ be coverings. They are equivalent if and only if $\pi_1(E)$ and $\pi_1(E')$ are conjugate viewed as subgroups of $\pi_1(B)$.

Transformations

Idea

We want to consider "symmetries" of a covering space.

Definition

An equivalence of a covering space with itself is called a **covering transformation**. The set of all of these forms a group under composition, and is called the **deck transformation group**, denoted $\mathcal{C}(E, p, B)$, where E, p, and B are the covering space, covering, and base space respectively.

Theorem

If E is simply connected, then $\mathcal{C}(E, p, B) \cong \pi_1(B)$.

Section 3

Applications

Fundamental Group of S^1

Theorem

$\pi_1(S^1, b_0) \cong \mathbb{Z}.$

Proof.

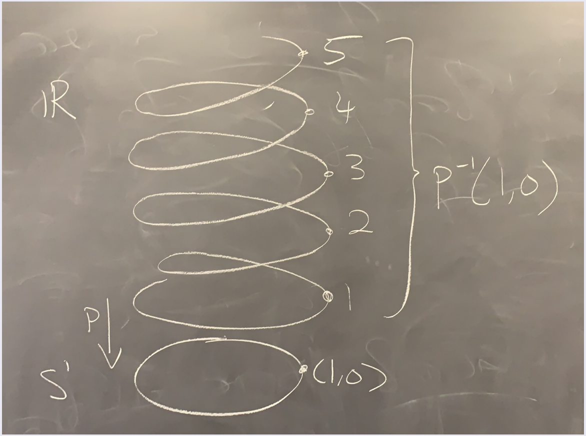
Basic idea.

- $p : \mathbb{R} \to S^1$ as defined before. Let $b_0 = (1, 0)$
- Lifting correspondence $[f] \mapsto \tilde{f}(1)$ is well defined
- Since $\mathbb{R}$ is simply connected, it is a bijection
- Moreover, it is a homomorphism
- $\pi_1(S^1, b_0) \cong p^{-1}(b_0) = \mathbb{Z}$

□

Fundamental Group of S^1 Cont.

Visual



Maps $S^n \to S^1$

Definition

A map is **nullhomotopic** if it is homotopic to a constant map.

Theorem

Every map $f : S^n \to S^1$ is nullhomotopic where $n > 1$.

Proof.

Basic idea.

- f can be lifted (by a more general lifting lemma) to a map $\tilde{f}$ in $\mathbb{R}$ since $\pi_1(S^n)$ is trivial for $n > 1$
- Any map in $\mathbb{R}$ is nullhomotopic
- Homotopy in $\mathbb{R}$ is projected down to S^1

□

Free Groups

Theorem

Every subgroup of a free group is free.

Proof.

Basic idea.

- $\{\text{Free groups } F\} \leftrightarrow \{\text{graphs } B\}$
- $\{\text{Subgroups } H \leq F\} \leftrightarrow \{\text{Coverings } E\}$
- Every covering of a graph is a graph, hence E is a graph
- $\pi_1(E) \cong H$ is a free group

$\square$

References

- [Topology, 2003] James R. Munkres
- [Algebra, 1974] Thomas W. Hungerford
- [Algebraic Topology, 2000] Allen E. Hatcher