An Elementary Introduction to Hyperbolic Geometry

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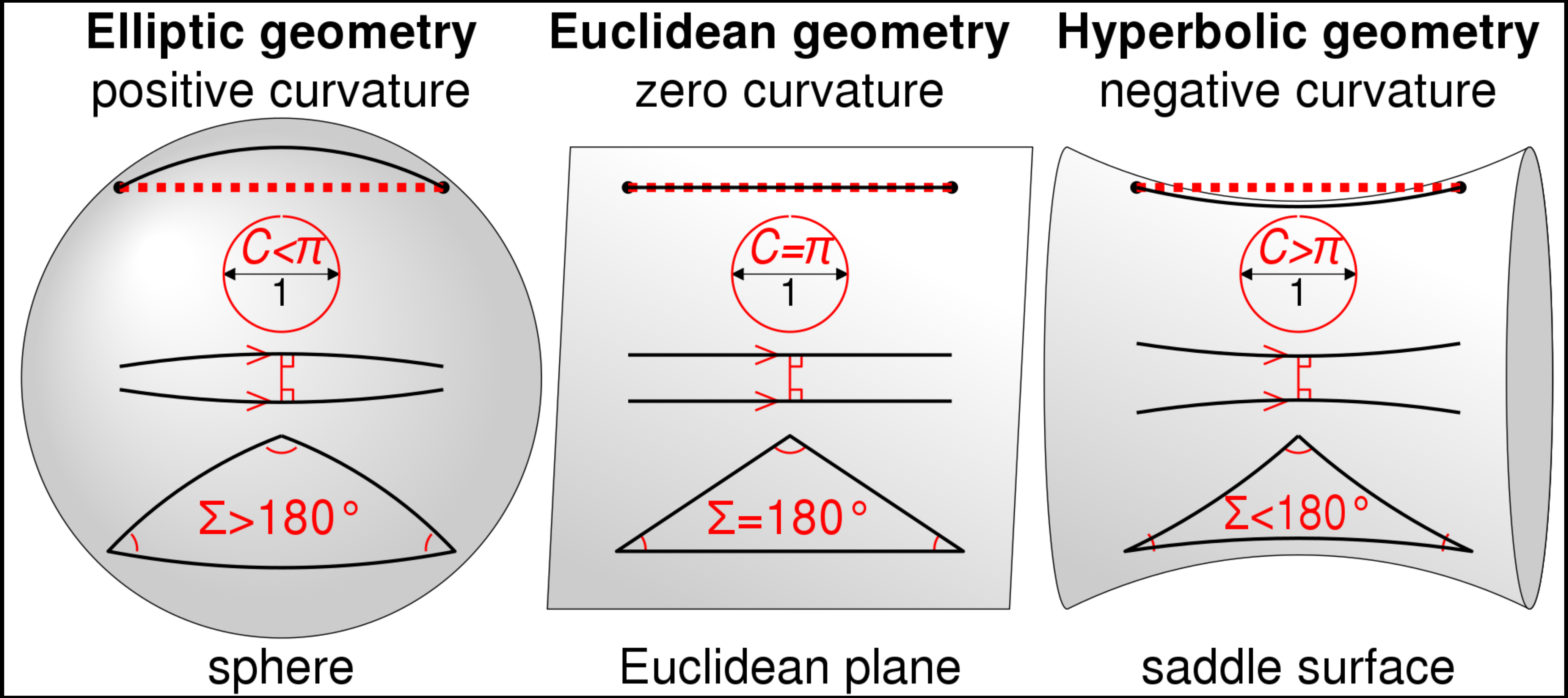
Classification of The Types of Geometries



Definition: The Upper Half-plane Model

The **upper half—plane** $\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}$

Definition: Hyperbolic Length

Let $\sigma\colon [a, b] \to \mathbb{H}$ be a continuously differentiable path in $\mathbb{H}$. Then the hyperbolic length of σ is obtained by integrating the function $\mathrm{f(z)} = \frac{1}{Im(z)}$ along σ, i.e.

$$length_{\mathbb{H}}(\sigma) = \int_{\sigma} \frac{1}{Im(z)} = \int_{a}^{b} \frac{|\sigma'(t)|}{Im(\sigma(t))} dt\,.$$

Definition: Hyperbolic Distance

Let z, z' ϵ $\mathbb{H}$. We define the **hyperbolic distance** $d_{\mathbb{H}}$(z, z') to be

$$d_{\mathbb{H}}(z, z') = \inf \{length_{\mathbb{H}}(\sigma) \mid \sigma \text{ has endpoints } z, z'\}$$

Informally: a geodesic σ between z, z' is the length-minimizing path with z, z' as endpoints.

Geodesics in $\mathbb{H}$

Example: The imaginary axis is a geodesic.

Proof. Let $\sigma(\mathrm{t}) = \mathrm{it}$, $\mathrm{a} \le \mathrm{t} \le \mathrm{b}$. Then σ is a path from ia to ib.

$$length_{\mathbb{H}}(\sigma) = \int_b^a \frac{|\sigma'(t)|}{Im(\sigma(t))} dt = \int_a^b \frac{1}{t} dt = \ln(b) - \ln(a).$$

Now let $\xi(\mathrm{t}) = \mathrm{x}(\mathrm{t}) + \mathrm{iy}(\mathrm{t})\colon [a, b] \to \mathbb{H}$ be any path from ia to ib. Then

$$length_{\mathbb{H}}(\xi) = \int_a^b \frac{\sqrt{x'(t)^2 + y'(t)^2}}{y(t)} dt \ge \int_a^b \frac{|y'(t)|}{y(t)} dt \ge \int_a^b \frac{y'(t)}{y(t)} dt = ln\big(y(t)\big)\Big|_a^b$$

$$= \ln b - \ln a$$

$$= length_{\mathbb{H}}(\sigma).$$

Definition: Möbius Transformations

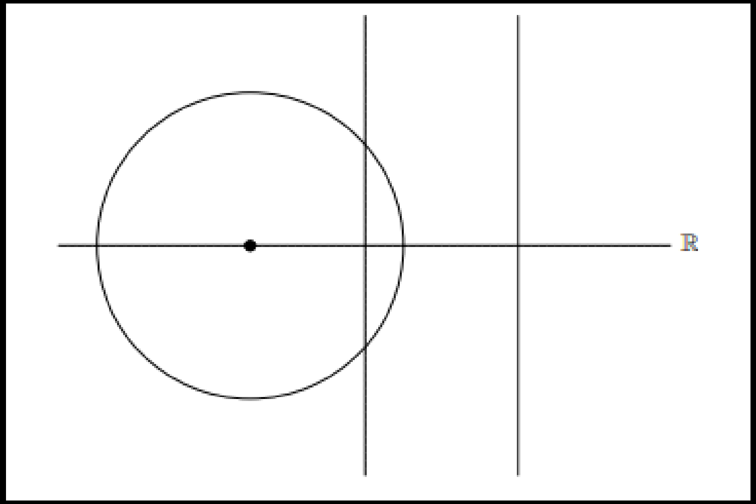
- Let a,b,c,d ϵ $\mathbb{R}$ be such that ad $-$ bc > 0. Define the map $\gamma\colon \mathbb{H} \to \mathbb{H}$ by

$$\gamma(z) = \frac{az + b}{cz + d}.$$

- Transformations of $\mathbb{H}$ of this form are called Möbius transformations of $\mathbb{H}$.

Application of Möbius Transformations (Pt. 1)

Idea: In our example, we saw that the imaginary axis is a geodesic. We now assert that any vertical line and any circle meeting the real axis orthogonally are geodesics as well. To do this we show that these curves can be mapped to the imaginary axis via Möbius transformations.



Application of Möbius Transformations (Pt. 2)

- Let L be the vertical line $\text{Re}(z) = r$, where $r \epsilon \mathbb{R}$. The translation $z \mapsto z - r$ is a Möbius transformation of $\mathbb{H}$ that maps L to the imaginary axis $\text{Re}(z) = 0$.
- Let K be a semi-circle with the endpoints $t, v \epsilon \mathbb{R}$, where $t < v$. Consider the following map:

$$\gamma(z) = \frac{z - v}{z - t}.$$

- Since $-t + v > 0$, this is a Möbius transformation of $\mathbb{H}$. $\gamma(v) = 0$ and $\gamma(t) = \infty$, and we have thus mapped K to the imaginary axis $\text{Re}(z) = 0$.

References

- J. Anderson, Hyperbolic Geometry, 1st ed., Springer Undergraduate Mathematics Series, Springer-Verlag, Berlin, New York, 1999.
- C. Walkden, MATH32051 Hyperbolic Geometry, lecture notes, The University of Manchester, delivered 18 September 2019.