Representations of Matrix Lie Groups

Directed Reading Project

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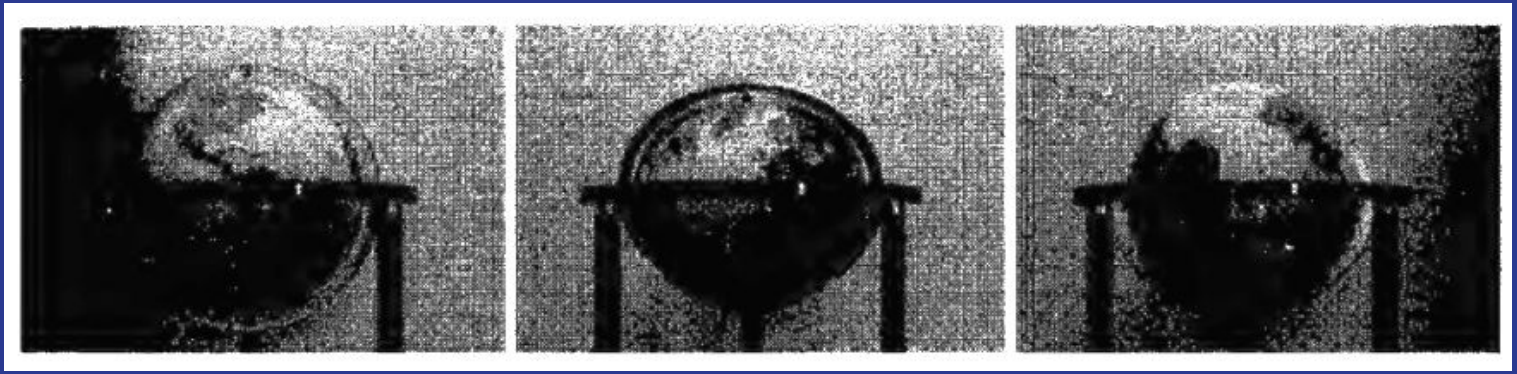
General Linear Group

General Linear Group over the Reals

- denoted $GL_n(\mathbb{R})$
- the group of all nxn invertible matrices with real entries

SO(3)

SO(3): the group of all rotations about the origin of a 3D Euclidean space



Matrix Lie Group

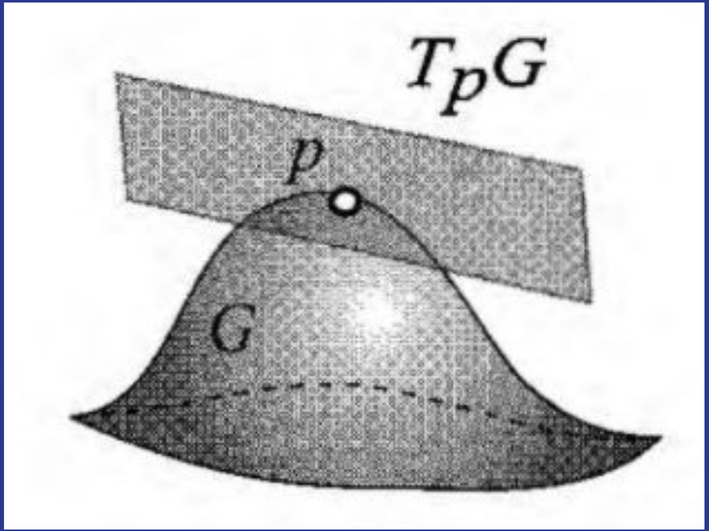
Matrix Lie Group: A subset G of $GL_n(\mathbb{R})$ such that

- it is a subgroup of $GL_n(\mathbb{R})$
- if A_n is any sequence of matrices in G, and A_n converges to some matrix A, then either $A \in G$, or A is not invertible
  - ie. G must be a subgroup that is topologically closed

Examples of Matrix Lie Groups

- $GL_n(\mathbb{R})$
- $GL_n(\mathbb{C})$
- $O(n)$ orthogonal groups
  - reflections in n-dimensional space
- $SO(n)$ special orthogonal groups
  - rotations in n-dimensional space

Lie Algebra



Lie Algebra of a matrix group $G \subset GL_n(K)$:

- the tangent space to G at I

Examples:

- Lie algebra of G in R^3 is a 2D subspace of R^3
- Lie algebra of $GL_n(K)$ is all nxn matrices
- Lie algebra of SO(3) is skew symmetric matrices

Lie Bracket

Lie Bracket of two vectors A and B in $\mathfrak{g}$:

$$[A, B] = \frac{d}{dt}\Big|_{t=0} Ad_{a(t)} B$$

Alternate definition: for all A,B $\in \mathfrak{g}$, [A,B] = AB - BA

Properties:

For all $A, A_1, A_2, B, B_1, B_2, C \in \mathfrak{g}$ *and* $\lambda_1, \lambda_2 \in \mathbb{R}$,

(1) $[\lambda_1 A_1 + \lambda_2 A_2, B] = \lambda_1 [A_1, B] + \lambda_2 [A_2, B]$.

(2) $[A, \lambda_1 B_1 + \lambda_2 B_2] = \lambda_1 [A, B_1] + \lambda_2 [A, B_2]$.

(3) $[A, B] = -[B, A]$.

(4) *(Jacobi identity)* $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$.

Adjoint Action

Action of a matrix group G on $\mathbb{R}^m$:

- a homomorphism from G to $GL_m(\mathbb{R})$
- determines how elements of G act on vectors in $\mathbb{R}^m$

Adjoint Action

- a smooth homomorphism Ad: G -> $GL_d(\mathbb{R})$
  g -> Ad_g where $Ad_g(A)=gAg^{-1}$

Adjoint Actions of SO(3)

$$\left\{ E_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, E_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, E_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$$

Lie bracket structure: $[E_1, E_2] = E_3$, $[E_2, E_3] = E_1$, $[E_3, E_1] = E_2$

This determines a vector space isomorphism $f: \mathbb{R}^3 \to so(3)$

$$(a, b, c) \overset{f}{\mapsto} \begin{pmatrix} 0 & -c & b \\ c & 0 & -a \\ -b & a & 0 \end{pmatrix}$$

Consider the map Ad: $SO(3) \to GL_3(\mathbb{R})$

- it's the inclusion map (sends every matrix to itself)

Adjoint Action of Sp(1)

Sp(1)

- Group of unit quaternions

Adjoint action of Sp(1)

- Ad: Sp(1) -> O(3)

However, note:

- Sp(1) is path connected -> output must be path connected
- So output must be SO(3)

Adjoint action should be Sp(1) -> SO(3)

Double Covers

Local Diffeomorphism

- $\exists$ a neighborhood of any point of the domain, restricted to which the function is a diffeomorphism onto its image

Double Cover

- a surjective 2-to-1 local diffeomorphism between compact manifolds

Ad: Sp(1) -> SO(3) is a double cover

- 2-to-1: Ker(Ad) = {1, -1}
- Ad is a local diffeomorphism at I
- surjective

Double Covers: More Examples

- for every n>2, there is a matrix group which double-covers SO(n)
  - Sp(1) -> SO(3)
  - Sp(1) x Sp(1) -> SO(4)
  - Sp(2) -> SO(5)
  - SU(4) -> SO(6)

References

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Tapp, Kristopher. Matrix Groups for Undergraduates. American Mathematical Society, 2005.

THANK YOU!

Questions?