Fun with the Fundamental Group Functor

Directed Reading Program

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Homotopy

- A **homotopy** between two continuous functions f and g from a topological space X to a topological space Y is defined to be a continuous function $H : X \times [0, 1] \to Y$ such that for all $x \in X$
    - $H(x, 0) = f(x)$ and
    - $H(x, 1) = g(x)$.
- Intuition : deforming one function into another
- For Spaces X and Y, having a homotopy from X to Y is an **equivalence relation** on the set of continuous function from X to Y.

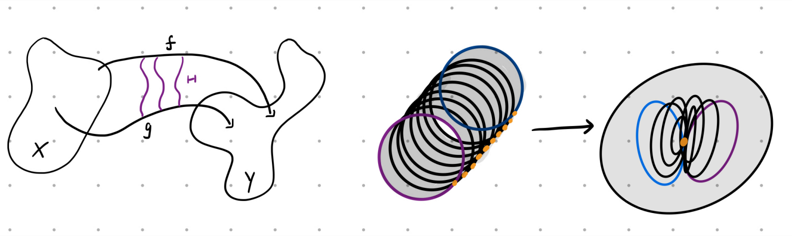


Figure 1: Homotopy

Homotopy Equivalence

- Given two topological spaces X and Y, a **homotopy equivalence** between X and Y is a pair of continuous maps $f : X \to Y$ and $g : Y \to X$, such that $g \circ f$ is **homotopic** to the identity map Id_X and $f \circ g$ is **homotopic** to Id_Y.
- Intuition: homotopy equivalent spaces are spaces that can be deformed continuously into one another.

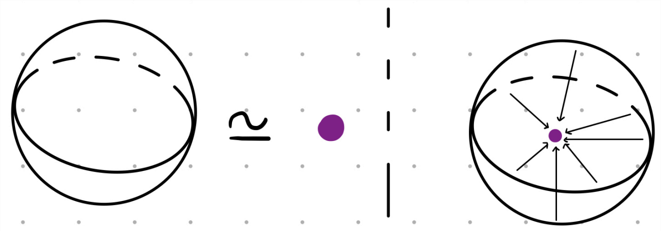


Figure 2: Homotopy Equivalence

Path Homotopy

- Path: a **path** in a topological space X is a continuous function $f : [0,1] \to X$ with initial point $f(0)$ and terminal point $f(1)$.
- A **homotopy of paths** from f to g is a family $H : [0,1] \times [0,1] \to X$ such that
    - The endpoints $H(0) = x_0$ and $H(1) = x_1$ are independent of t
    - $H(s,0) = f(s)$, $H(s,1) = g(s)$, $H(0,t) = x_0$, and $H(1,t) = x_1$
- Intuition: continuously deforming a path when keeping its endpoints fixed.

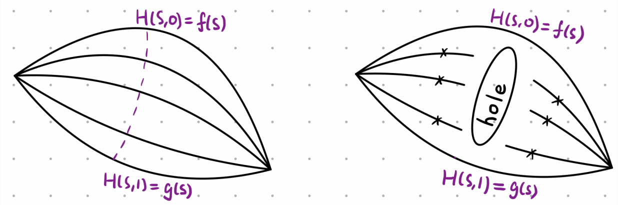


Figure 3: Path Homotopy

Product Path

- Given two paths $f, g : [0,1] \to X$ such that $f(1) = g(0)$, there is a concatenation of path $f \cdot g$ that traverses first f then g

$$(f \cdot g)(s) = \begin{cases} f(2s) & 0 \leq s \leq \frac{1}{2} \\ g(2s-1) & \frac{1}{2} \leq s \leq 1. \end{cases}$$

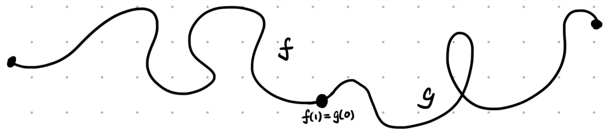
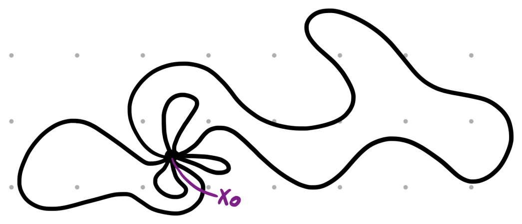


Figure 4: $(f \cdot g)(s)$

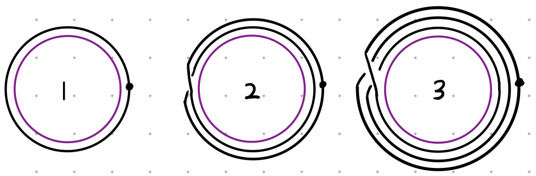
Fundamental Group

- **Loops** are paths $f : [0,1] \to X$ with the same starting and ending point $f(0) = f(1) = x_0$, and their common starting and ending point x_0 is called the **basepoint**.
- The **fundamental group** is the sets of path homotopy classes of the set of all loops, denoted $\pi_1(X, x_0)$.
    - Basepointed topological spaces $\to$ Groups
    - Multiplication is concatenation of paths
    - Basepoint preserving continuous functions $\to$ Group homomorphisms
    - if f and g are homotopic they give the same group homomorphism



Examples of the Fundamental Group

- Example 1 : $\pi_1(S^1, x_0) \cong \mathbb{Z}$
  - Clockwise positive, anti-clockwise negative



- Example 2 : $\pi_1(D^2, x_0) \cong \{0\}$
  - Intuition: D^2 is homotopy equivalent to a point so they have isomorphic fundamental groups. There's only one function from $[0, 1]$ to a point so $\pi_1(D^2, x_0) \cong \{0\}$

Category

- A **Category** is a collection of **objects**$\{X, Y, Z \ldots\}$ and **morphisms**$\{f, g, h \ldots\}$ between objects. For a pair of objects X and Y we have a collection of morphisms $\{f, g, h, \ldots\}$ from X to Y so that
  - Each object has a designated **identity morphism** $Id_X : X \to X$
  - For any pair of morphisms f, g where $f : X \to Y$ and $g : Y \to Z$, we have $g \circ f : X \to Z$.
- A category is also subject to the following two rules:
  - For any $f : X \to Y$, we have $Id_Y \circ f = f = f \circ Id_X$
  - Compositions are **associative**: $(g \circ h) \circ f = g \circ (h \circ f)$

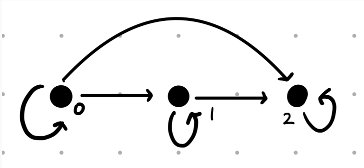
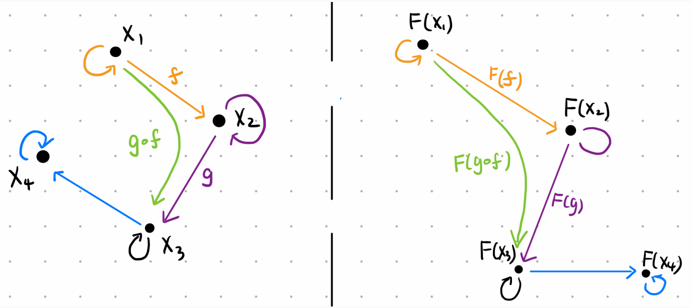


Figure 5: The Category [2]

More Examples of Categories

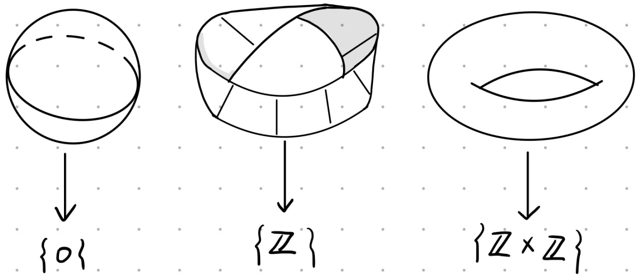
Categories	Objects	Morphisms
Set	Sets	Functions
Vect_R	Vector spaces over R	Linear Functions
Grp	Groups	Group homomorphisms
Top	Topological spaces	continuous functions
Top_*	Base pointed topological space	continuous functions that maps basepoints to each other

- A **Functor** F is a mapping $F : C \to D$ that relates two categories C and D such that it associates
  - Each $x \in Obj(C)$ to a $F(x) \in Obj(D)$
  - Each $f : x_1 \to x_2$ in C to a $F(f) : F(x_1) \to F(x_2)$ in D.
- A Functor also satisfies the following two conditions
  - For each object $x \in Obj(C)$, $F(Id_x) = Id_{F(x)}$
  - For all morphisms $g, f \in C$, $F(g \circ f) = F(g) \circ F(f)$



An Example of functor

- The Fundamental group is a functor.
  - $\text{Top}_* \to \text{Grp}$
  - Basepoint preserving continuous functions $\to$ Group homomorphisms
  - two homotopic basepoint preserving continuous functions give the same group homomorphism



Example of using functor in algebraic topology

Want To Show: No retraction from a disc to a circle

- Intuition: one has a 'hole' in it and the other does not, so they must be different in some way.
- Alternative framing: Is there a continuous function $r : D^2 \to S^1$ that fixes the boundary?

Proof:
Let $i : S^1 \to D^2$ be the inclusion. Suppose for contradiction that r exists, such that $r \circ i = Id_{S^1}$.

$$x \in S^1 \to i(x) = x, r(x) = x.$$

Example Continued

So now i, r are morphisms in Top_*, and we can apply the fundamental group.

$$\pi_1(S^1, x) \cong \mathbb{Z}$$

$$\pi_1(D^2, x) \cong \{0\}.$$

These give us

$$\pi_1(i) : \mathbb{Z} \to \{0\}$$

$$\pi_1(r) : \{0\} \to \mathbb{Z},$$

which is saying

$$id_{\mathbb{Z}} = \pi_1(id_{S^1}) = \pi_1(r \circ i) = \pi_1(r) \circ \pi_1(i) = 0.$$

Contradiction $\implies$ the assumption that r exists is false.

References

References

- Hatcher’s Algebraic Topology
- Riehl’s Category Theory in Context
- Wikipedia - https://en.wikipedia.org/wiki/File:Fundamental groupofthecircle.gif
- https://www.math3ma.com/blog/what-is-a-functor-part-1

Thank You

Thank you so much for listening!