Knots and Virtual Knots

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Introduction

Knots are closed non self intersecting curves in 3 dimensions. Namely, a knot is a continuous injective map $K : \mathbb{S} \to \mathbb{R}^3$.

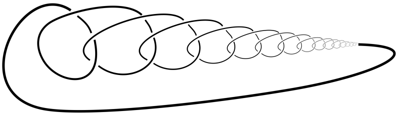


Figure: Wild knot.

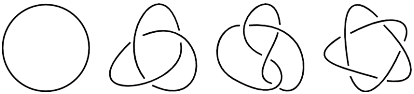


Figure: Tame knots.

Wild knots exhibit pathological behavior so we will deal with tame knots.

Introduction

We represent knots in 2 dimensions by drawing knot diagrams which are simply an augmented projection of the knot.

Figure: Curve in 3D.

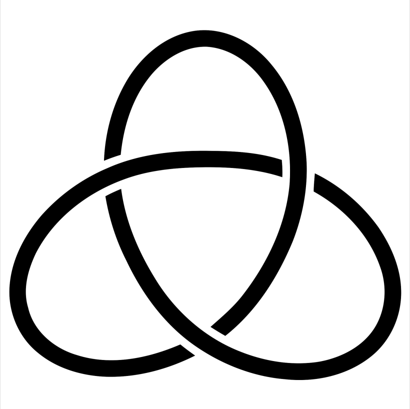


Figure: Diagram in 2D.

Introduction

We say that two knots are equivalent if one can be continuously transformed into the other. These continuous transformations have analogues on the knot diagram, the Reidemeister moves.

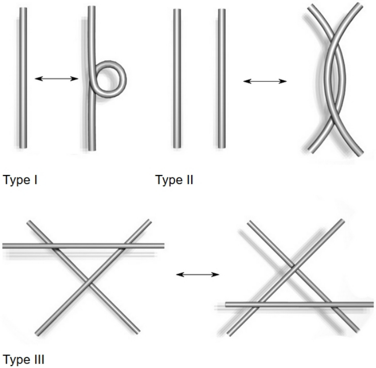


Figure: Reidemeister moves.

Invariants

Distinguishing knots can be very hard. To solve this problem we compute a variety of knot invariants. One easy example is tricolorability. We color arcs with one of three colors under the condition that all crossings have just one or all three colors.

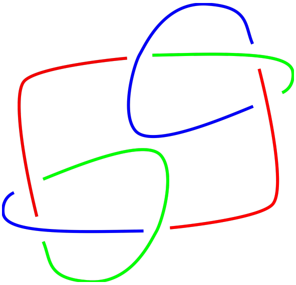


Figure: Tricolorable.

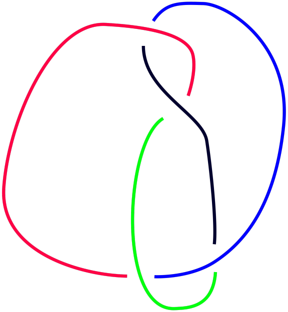


Figure: Not tricolorable.

Fundamental Group

A more complicated invariant of a knot is the fundamental group of its complement, namely $\pi_1(\mathbb{R}^3 \setminus K)$. One way of presenting it is using Wirtinger presentations. We take each arc to be a generator and each crossing gives us a relation.

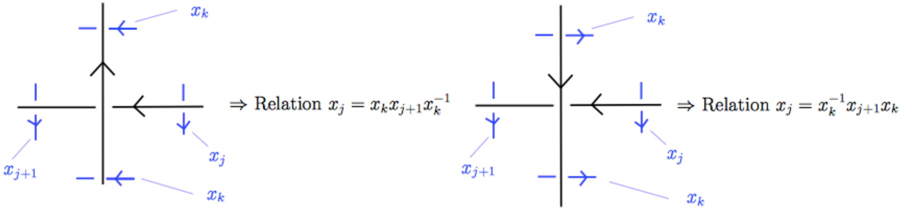
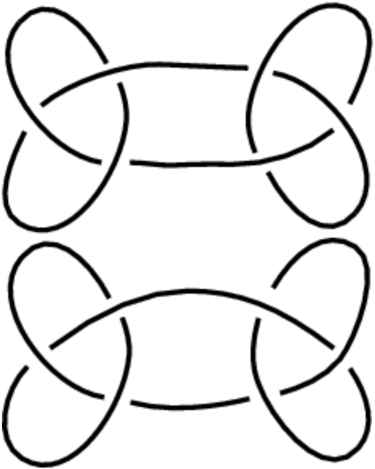


Figure: Relations for the Wirtinger Presentation.

Fundamental Group

Although a very powerful invariant, fundamental groups are not a complete invariant. The following knots are distinct but have the same knot group.



Gauss Diagrams

One way of representing knots is using Gauss Diagrams. We number the crossings and put then in order of appearance in a circle, then we draw arrows from the overcrossings to the undercrossings along with a sign to indicate handedness.

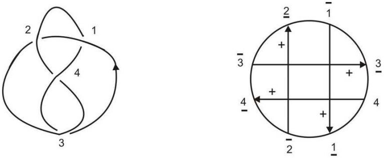
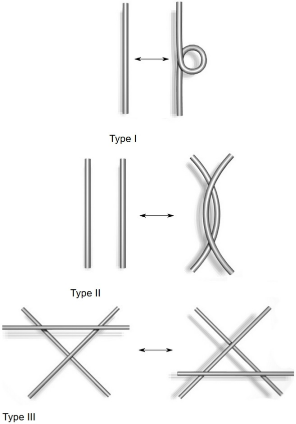
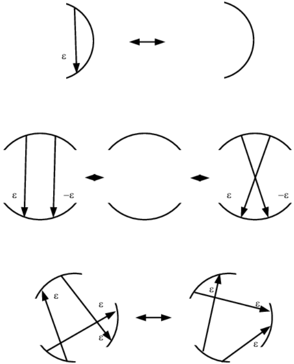


Figure: Example for the figure eight knot.

Gauss Diagrams

The same way we had Reidemeister moves for the knot diagram, we also have the equivalent moves for Gauss diagrams.





Virtual Knots

Notice that not all Gauss diagrams are equivalent to knots. Gauss diagrams define a larger set of objects we call Virtual Knots.

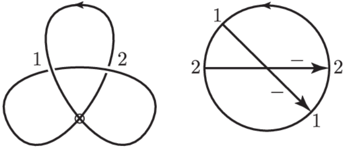
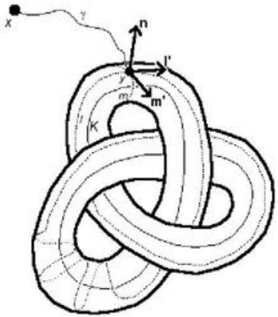


Figure: A non-trivial virtual knot.

Peripheral System

Given the fundamental group of a knot complement, a peripheral subgroup is a subgroup generated by a meridian and a longitude.



The peripheral system, the fundamental group augmented with a peripheral subgroup is a complete invariant of knots.