Electrical Networks and Pólya's Random Walk Theorem

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Outline of Project

Project Goals

1. Examine applications of electrical network theory to random walks
2. Classify the behavior of random walks on graphs in different dimensions (≤ 2 vs. ≥ 3)

Project References

- Peter G. Doyle and J. Laurie Snell, *Random Walks and Electric Networks*. The Mathematical Association of America, 1984.
- Padraic Bartlett, *Electrical Networks and Random Graphs*. Lectures 5 & 7 from Math 7H (2014) at University of California, Santa Barbara. Accessed last Dec 9, 2020 from `http://web.math.ucsb.edu/~padraic/ucsb_2014_15/math_honors_f2014/math_honors_f2014_lecture5.pdf`

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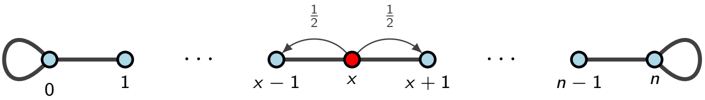
Motivation: 1-D Random Walk

- A random walker starts at node x and has a $\frac{1}{2}$ probability of moving to the left/right



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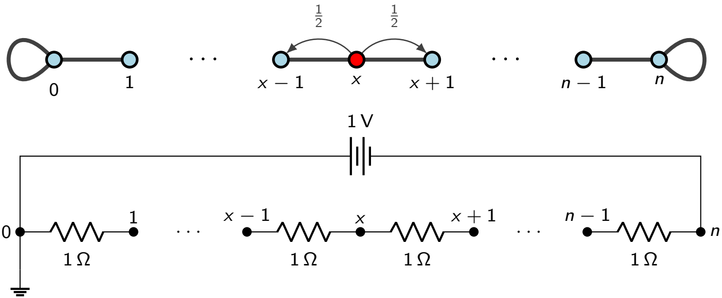


Probability of reaching n before 0	$p(x) = \frac{1}{2}p(x-1) + \frac{1}{2}p(x+1)$	$p(0) = 0$ $p(n) = 1$
Voltage at node x	$v(x) = \frac{1}{2}v(x-1) + \frac{1}{2}v(x+1)$	$v(0) = 0$ $v(n) = 1$

- From this, $p(x) = x/n$. As $n \to \infty$, $p(x) \to 0$, i.e. the random walker must return to the origin.

Pólya's Random Walk Theorem

- A walk is **recurrent** if it is certain that the random walker will return to the origin
- A walk is **transient** if the **escape probability** $p_{esc} > 0$, i.e. there is a positive probability that the random walker will *never* return to the origin
- (Definitions as in Doyle and Snell, modified from Pólya's original definitions)

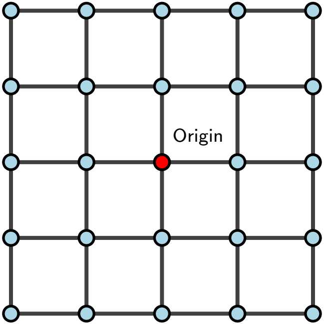
Theorem

Simple random walks on a d-dimensional lattice $\mathbb{Z}^d$ *are:*

- *Recurrent for* $d = 1, 2$
- *Transient for* $d \geq 3$

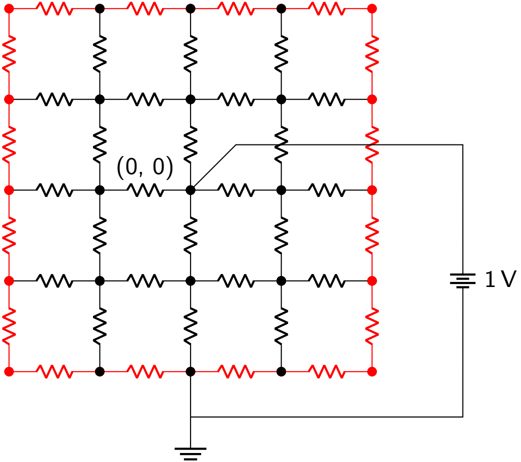
Random Walks on $\mathbb{Z}^2$

- Is it certain that the random walker will return to the origin? *(Recurrent)*
- Or, is there a non-zero probability that the walker will never return to the origin? *(Transient)*



Electrical network on $\mathbb{Z}^2$

- It can be shown that the **escape probability** $p_{esc} \propto 1/R_{eff}$, where R_{eff} is the **effective resistance** from the origin to infinity
- To determine p_{esc} electrically, compute R_{eff} between the origin and far-away grounded points



Proof of Pólya's Theorem for $\mathbb{Z}^2$: Shorting Nodes

- **Shorting**: Treat certain subsets of nodes as one node (electrically: connect nodes with perfectly conducting wires, i.e. set the resistance of certain edges to 0)
- **Rayleigh's Monotonicity Law**: Shorting nodes only decreases the effective resistance

Proof of Pólya's Theorem for $\mathbb{Z}^2$: Shorting Nodes

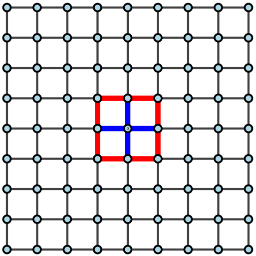
- **Shorting**: Treat certain subsets of nodes as one node (electrically: connect nodes with perfectly conducting wires, i.e. set the resistance of certain edges to 0)
- **Rayleigh's Monotonicity Law**: Shorting nodes only decreases the effective resistance
- **Goal**: To prove that random walks on $\mathbb{Z}^2$ are recurrent, i.e.

$$p_{esc} \propto \frac{1}{R_{eff}} = 0 \iff R_{eff} = \infty$$

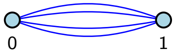
- **Technique**: Short nodes on $\mathbb{Z}^2$ such that:

$$R_{eff} \geq R_{shorted} = \infty$$

Proof of Pólya's Theorem for $\mathbb{Z}^2$



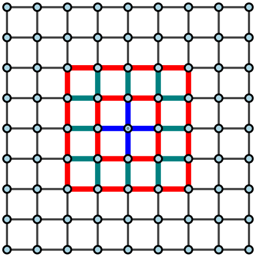
(Shorted nodes in red)



4 edges

$\frac{1}{4}\ \Omega$

Proof of Pólya's Theorem for $\mathbb{Z}^2$



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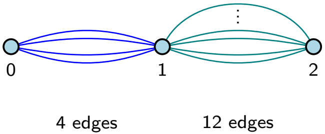

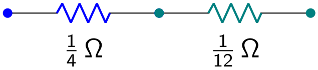

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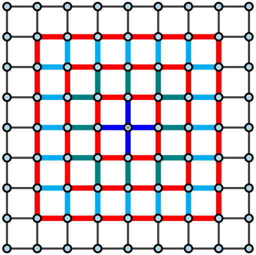

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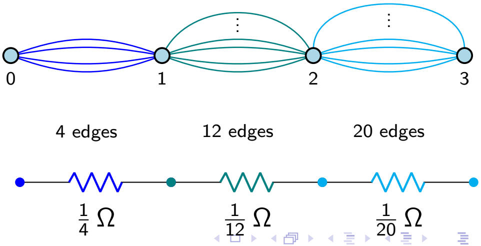

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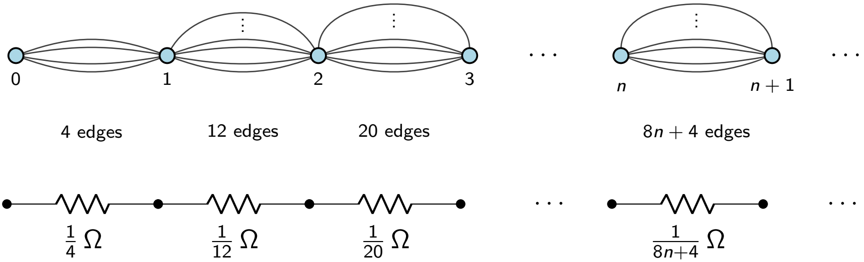

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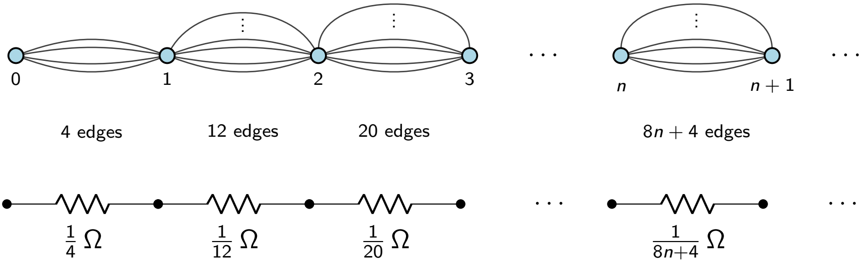
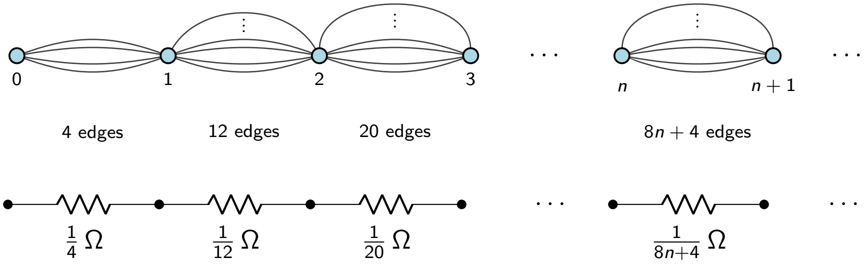


- Recalling Rayleigh's Monotonicity Law,

$$R_{eff} \geq R_{shorted} = \sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$$

Proof of Pólya's Theorem for $\mathbb{Z}^2$

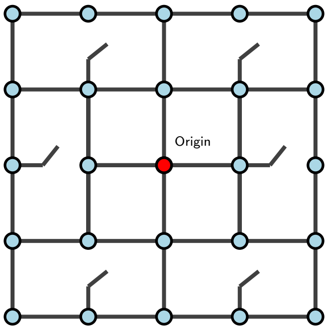


- Recalling Rayleigh's Monotonicity Law,

$$R_{eff} \geq R_{shorted} = \sum_{n=0}^{\infty} \frac{1}{8n+4} = \infty$$

- Thus, random walks on $\mathbb{Z}^2$ are recurrent! □

Proof Idea for Higher Dimensions



- **Cutting**: Removing an edge from the network (increases resistance of edge)
- **Rayleigh's Monotonicity Law**: Cutting edges only increases the effective resistance
- **Goal**: To prove that random walks on $\mathbb{Z}^3$ are **transient**, i.e.

$$p_{esc} \propto \frac{1}{R_{eff}} > 0 \iff R_{eff} < \infty$$

- **Technique**: Cut edges outside an intricate tree such that:

$$R_{eff} \leq R_{cut} < \infty$$

Acknowledgements

Thank you for listening!

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