Singularity structure and massless dyons of pure $\mathcal{N}=2, d=4$ theories with A_r and C_r

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Disclaimer: all figures are mouse-drawn, not to scale.

Based on works w/ Keshav Dasgupta

Seiberg-Witten and Argyres-Douglas theories

- Seiberg-Witten curves, massless dyons (low ranks), and some aspects of singularity in moduli space were studied for
  - SU(2) w/o and w/ matter [Seiberg Witten 94]
  - SU(n) [Klemm Lerche Yankielowicz Theisen 94, Argyres Faraggi 94], SU(n) w/o and w/ matter [Hanany Oz 95]
  - SO(2r) [Brandhuber-Landsteiner 95] SO(2r+1) [Danielsson Sundborg 95], SO with matter [Hanany 95]
- The hyper-elliptic curve $y^2 = f(x)$ degenerates into a cusp form

$$y^2 = (x-a)^m \times \cdots$$

  when $m \geq 3$ branch points collide on x-plane. For SU(n) SW curves, this give exotic theories, discovered and studied [Argyres Douglas 95], where we have *mutually non-local* massless dyons.
  (Under symplectic transformation, you cannot bring everything purely electronic.)

Main idea

Want to build Gaiotto's $\mathcal{N} = 2$ theory (see Yuji's talk) in F theory

- $\mathcal{N} = 2$ $Sp(2r)$ SW theories via r D3-branes
- compute massless dyon charges for pure $SU(r+1), Sp(2r)$
- study wall crossing
- provide plethora of Argyres-Douglas (AD) theories

physics	$\leftrightarrow$	geometry
massless d.o.f.	$\leftrightarrow$	singularity $\Delta_x f = 0$
exotic theories (i. e. AD)	$\leftrightarrow$	$\Delta_u \Delta_x f = 0$

Double discriminant captures higher singularity of hyper-elliptic curves.

Discriminant

Δ_x: discriminant *with respect to* x

$$f_n(x) = \sum_{i=1}^{n} a_i x^i = a_n \prod_{i=1}^{n} (x - e_i)$$

$$\Delta_x \left(f_n(x)\right) = a_n^{2n-2} \prod_{i<j} (e_i - e_j)^2$$

Order **o**f **v**anishing (multiplicity of roots) $\geq 2 \leftrightarrow \Delta_x = 0$

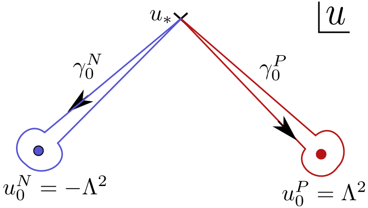
Recipe for locating Argyres-Douglas loci

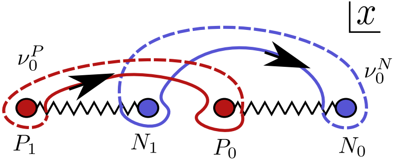
- Start with hyper-elliptic Seiberg-Witten curve $y^2 = f(x; u, v, \cdots)$
- Demanding $\Delta_x f = 0$ and $\Delta_u \Delta_x f = 0$ gives two massless BPS dyons. [Argyres Plesser Seiberg Witten '95]
- Check **o**rder **o**f **v**anishing (o.o.v.) of each solution to $\Delta_u \Delta_x f = 0$
- If o.o.v. $\geq$ 3, Argyres-Douglas loci: The hyperelliptic curve degenerates into a cusp-like singularity $y^2 = (x-a)^3 \times \cdots$ and two mutually non-local dyons become massless. (checked up to rank 5)

Review of monodromy of $SU(2) = Sp(2)$ SW curve

How did they get massless monopole and dyon?

$$y^2 = (x^2 - u)^2 - \Lambda^4$$

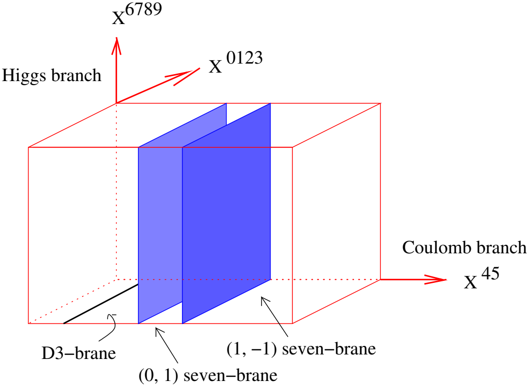




Start at a generic point in moduli space, then branch points on x-plane are separated. As you move around a singular locus in u-plane, a pair of branch points approach & change with each other. Their trajectory gives vanishing cycle.

F-theoretic (quantum) 3/7 brane picture of $Sp(2)$

[Sen '96] [Banks-Douglas-Seiberg '96] & [Vafa 96]



Two (p,q) 7-branes are monopole and dyon in the original SW theory.

- Put r D3-branes as probes to get $Sp(2r)$ gauge theory.

  [Douglas Lowe Schwarz 96]

SW curve for pure $SU(r+1) = A_r$

From [Klemm Lerche Yankielowicz Theisen 94]

$$y^2 = f_{SU(r+1)} \equiv \left(x^{r+1} + u_1 x^{r-1} + u_2 x^{r-2} + \cdots + u_r\right)^2 - \Lambda^{2r+2} = f_+ f_-$$

$$f_\pm \equiv x^{r+1} + u_1 x^{r-1} + u_2 x^{r-2} + \cdots + u_r \pm \Lambda^{r+1}$$

$$f_+ \equiv \prod_{i=0}^{r} (x - P_i) \qquad f_- \equiv \prod_{i=0}^{r} (x - N_i)$$

Note that $f_\pm$ do not share roots ($f_+ - f_- = 2\Lambda^{r+1} \neq 0$).
On the x-plane, only P_i's (or N_i's) can collide among themselves.
$\rightarrow$ Discriminant factorizes $\Delta_x f_{SU(r+1)} = \# \left(\Delta_x f_+\right) \left(\Delta_x f_-\right)$

SW curve for pure $Sp(2r) = C_r$

By taking no-flavor limit of [Argyres Shapere 95], obtain

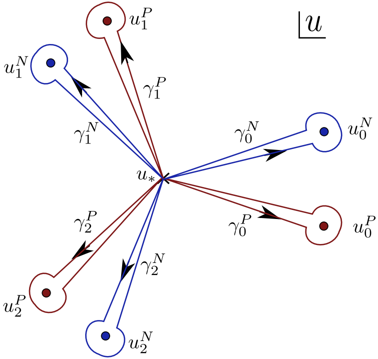
$$
\begin{aligned}
y^2 = f_{Sp(2r)} &\equiv \left(\prod_{a=1}^{r}\left(x-\phi_a^2\right)\right)\left(x\prod_{a=1}^{r}\left(x-\phi_a^2\right)+\Lambda^{2r+2}\right) \\
&= f_C f_Q
\end{aligned}
$$

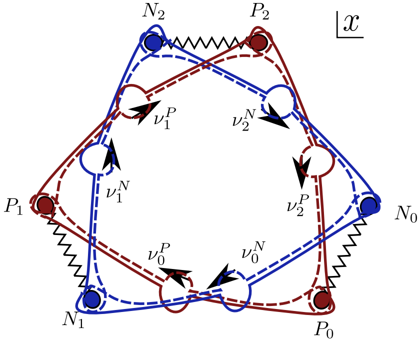
$$
\begin{aligned}
f_C &\equiv \prod_{a=1}^{r}\left(x-\phi_a^2\right) = \prod_{i=1}^{r}(x-C_i) \\
&= x^r + u_1 x^{r-1} + u_2 x^{r-2} + \cdots + u_r \\
f_Q &\equiv f_C x + \Lambda^{2r+2} = \prod_{i=0}^{r}(x-Q_i)
\end{aligned}
$$

Similarly as in SU(r+1) curve, $\Delta_x f_{Sp(2r)} = \#\left(\Delta_x f_C\right)\left(\Delta_x f_Q\right)$

Vanishing cycles of SW curve for pure SU(r+1)

SU(3) [Klemm Lerche Yankielowicz Theisen 94]





For $SU(r+1)$, in a region of moduli space given by

$$u_1 = \cdots = u_{r-2} = 0$$

$$u_r/\Lambda^{r+1} \in \mathbb{I},$$

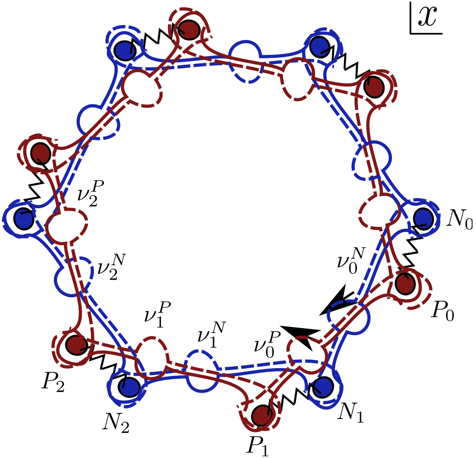
monodromy satisfies

$$\nu_i^P \cap \nu_{i+1}^P = \nu_i^N \cap \nu_{i+1}^N = 1$$

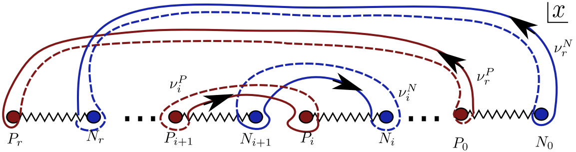
$$\nu_i^P \cap \nu_i^N = -2$$

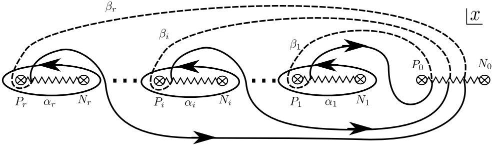
$$\nu_i^P \cap \nu_{i+1}^N = 2$$

all other intersection numbers vanish.



Identify massless dyons of A_r and C_r curves at $\Delta_x f = 0$





$$\nu_0^P = \beta_1$$

$$\nu_r^P = -\beta_r + \sum_{i=1}^{r-1} \alpha_i$$

$$\nu_0^N = \beta_1 + \sum_{i=1}^{r} \alpha_i + \alpha_1$$

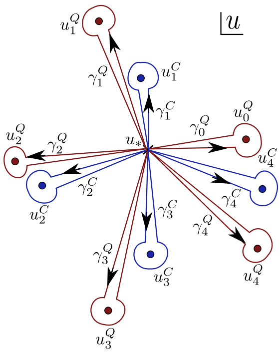
$$\nu_r^N = -\beta_r - 2\alpha_r$$

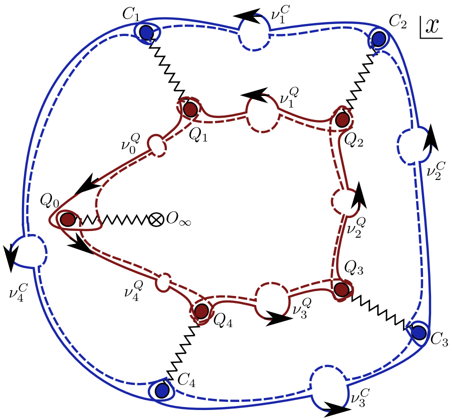
$$\nu_i^P = \beta_{i+1} - \beta_i - \alpha_i$$

$$\nu_i^N = \beta_{i+1} - \beta_i + \alpha_{i+1} - 2\alpha_i$$

$$i = 1, \cdots, r-1$$

$Sp(2r)$ monodromy





In a moduli region by $u_2 = \cdots = u_{r-1} = 0$ and $u_r = \text{const}$: choose small enough u_r to keep $\nu_Q \cap \nu_C = 0$ (checked up to rank 5).

With the same choice of symplectic basis as $SU(r+1)$ before, the vanishing cycles are written as

$$\nu_0^Q = \beta_1$$

$$\nu_r^Q = -\beta_r - \sum_{i=1}^{r} \alpha_i - \alpha_r$$

$$\nu_r^C = \beta_1 - \beta_r - \sum_{i=2}^{r} \alpha_i$$

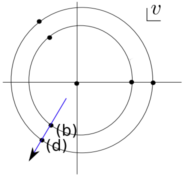
$$\begin{aligned} \nu_i^Q &= \beta_{i+1} - \beta_i - \alpha_i \\ \nu_i^C &= \beta_{i+1} - \beta_i + \alpha_{i+1} \\ i &= 1, \cdots, r-1 \end{aligned}$$

whose non-vanishing intersection numbers come from only

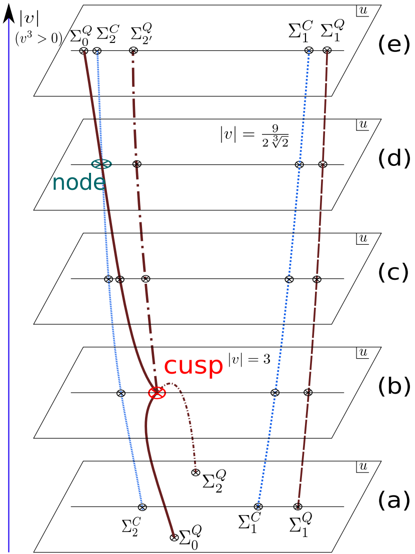
$$\nu_i^Q \cap \nu_{i+1}^Q = 1, \qquad \nu_r^Q \cap \nu_0^Q = \nu_i^C \cap \nu_{i+1}^C = -1$$

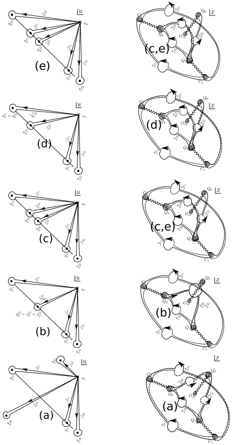
The spectra change as we move in moduli space. (C_2 **example***)*

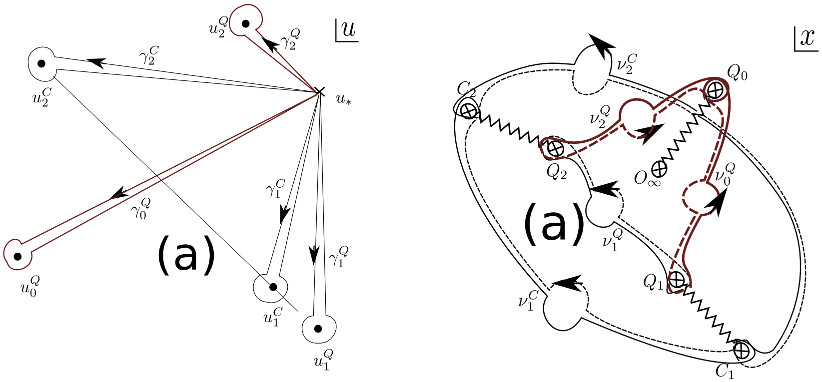
Example:
$Sp(4) = C_2$



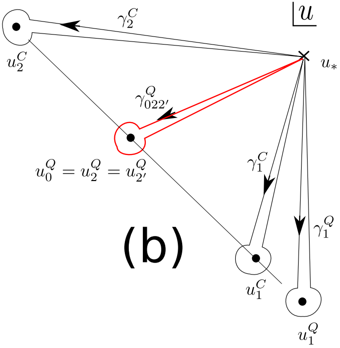
$\Delta_x f = 0$ at Σ which intersect at $\Delta_u \Delta_x f = 0$

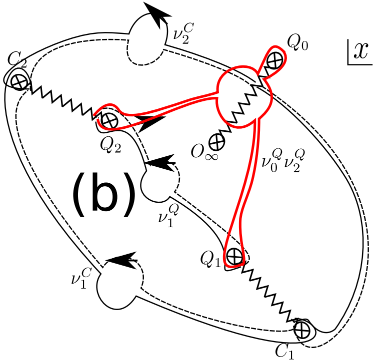






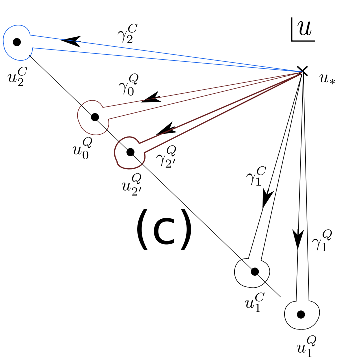
Two cycles ν_0^Q and ν_2^Q vanish at two different moduli loci u_0^Q and u_2^Q respectively.

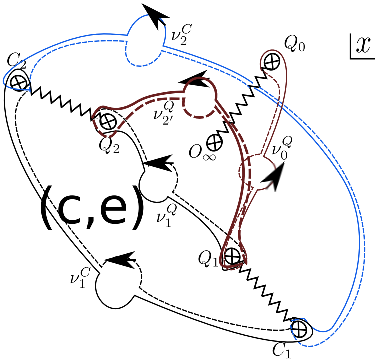




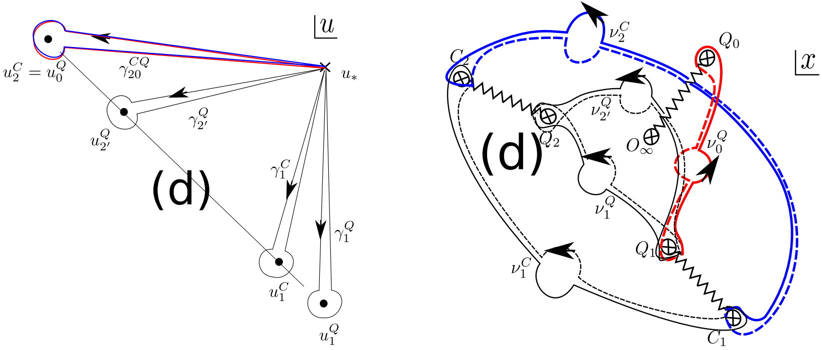
As we vary in a moduli space (where u_0^Q and u_2^Q intersect tangentially), we hit a locus where they vanish simultaneously. The curve degenerates into $y^2 \sim (x-a)^3 \times \cdots$. (Argyres-Douglas)

The red curve on the right does not give a well defined cycle.

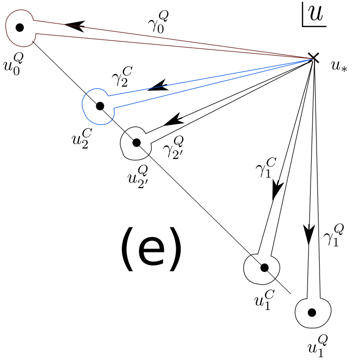


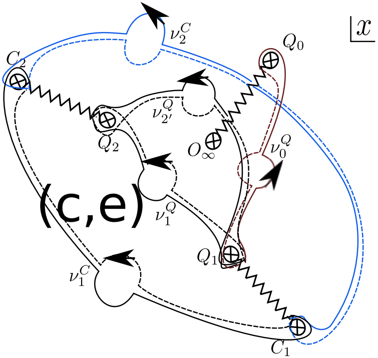


Now u_0^Q and $u_{2'}^Q$ are separated, but u_0^Q runs toward another singularity locus u_2^C.
($\nu_{2'}^Q = \nu_0^Q + \nu_2^Q$; Not related by symplectic transformation)



This time, singularity loci u_0^Q and u_2^C intersect (node-like crossing). Vanishing cycles ν_0^Q and ν_2^C seem mutually non-local, but the curve looks like $y^2 = (x-a)^2(x-b)^2 \times \cdots$ and it is not Argyres-Douglas form. ($\therefore$ generalized AD)





Now u_0^Q and u_2^C are separated, but the vanishing cycles did not change, unlike (b $\rightarrow$ c)

Massless dyons coexist at a codim-$2_{\mathbb{C}}$ loci $\{\Delta_x f = \Delta_u \Delta_x f = 0\}$ [Argyres Plesser Seiberg Witten '95], where the curve looks like either of following two:

- $y^2 = (x-a)^3 \times \cdots$
  - curve has cusp-like singularity (Argyres-Douglas)
  - $\Delta_x f = 0$ locus also intersects at $\Delta_u \Delta_x f = 0$ (o.o.v ≥ 3) with cusp-like singularity (tangential intersection)
  - two massless dyons are mutually non-local.
- $y^2 = (x-a)^2(x-b)^2 \times \cdots$
  - curve has node-like singularity
  - $\Delta_x f = 0$ locus also intersects at $\Delta_u \Delta_x f = 0$ (o.o.v ≤ 2) with node-like singularity
  - two massless dyons are mutually local, with some exception (*generalized AD*: non-Argyres-Douglas & non-local loci) occurring at $SU(5\uparrow), Sp(4\uparrow)$

Generalized Argyres-Douglas loci

curve degen.n	$y^2 = (x-a)^3 \times \cdots$	$y^2 = (x-a)^2(x-b)^2 \cdots$	
o.o.v of $\Delta_u \Delta_x$	3	2	
shape of curve	cusp	node	
shape of $\Delta_x = 0$	cusp	node	
intersection	mutually non local	local	non-local
name	Argyres-Douglas	ML	gen AD

- generalized Argyres-Douglas: non-Argyres-Douglas & non-local loci ($\exists$ conformal limit?)

Conclusion

- Discriminant $\Delta_x f = 0$ of the SW curve $y^2 = f$: identified all $2r+1$ and $2(r+1)$ massless dyons for $Sp(2r)$ and $SU(r+1)$
- At double discriminant $\Delta_u \Delta_x f = 0$: massless dyons coexist. If order of vanishing ≥ 3, then Argyres-Douglas. (checked up to rank 5)

Questions still remain...

- Behaviour in the Argyres-Douglas neighborhood?
- Rank r curve classification as rank 2 of [Argyres Crescimanno Shapere Wittig '05][Argyres Wittig '05]
- Global behaviour in moduli space: wall crossing [ShapereVafa99, GaiottoMooreNeitzke09]