Matrix factorizations via group actions on categories, etc.

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June 8th, 2011

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$\mathbf{V}$ vector space (or chain complex)

Pairs of ways of giving $\mathbf{V}$ *extra structure*.

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(Note: The two $\mathbb{G}_\mathbf{a}$ in the bottom row are *dual* lines.)

Starting point: **MF** as categorified Fourier transform

"Non-commutative *singularity theory*"

"Commutative"		"Non-Commutative"
$\mathbf{M}$ a *smooth variety*/$\mathbf{k}$	$\Rightarrow$	$\mathcal{C} = \mathbf{Perf}\,\mathbf{M}$ a $\mathbf{k}$-linear *dg-category*
$\mathbf{f}\colon \mathbf{M} \to \mathbb{G}_\mathbf{m}$ or $\mathbf{w}\colon \mathbf{M} \to \mathbb{A}^1$	$\Rightarrow$	*some* extra structure on $\mathcal{C}$

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Observation of C. Teleman:

$$\left\{\begin{matrix}\mathbf{k}[\mathbf{x},\mathbf{x}^{-1}]\text{-linear} \\ \text{structure on } \mathcal{C}\end{matrix}\right\} \Leftrightarrow \Gamma(\mathbf{M}, \mathcal{O}_\mathbf{M})^\times \Leftrightarrow \left\{\begin{matrix}\text{(homotopy) } \mathbf{S}^1 = \mathbf{B}\mathbb{Z} \\ \text{action on } \mathcal{C}\end{matrix}\right\}$$

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$$\mathbf{M} \text{ smooth variety}/k \qquad \mathcal{C} = \mathbf{Perf\,M} \in \mathbf{dgcat_k}$$

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One can show:

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3. Invert β to get $\mathbf{k}(\!(\beta)\!)$-linear $\Leftrightarrow \mathbb{Z}/\mathbf{2}$-graded dg-cat

$$\mathcal{C}^{\mathrm{Tate}} \stackrel{\text{def}}{=} \mathcal{C}^{\mathbf{S}^1} \otimes_{\mathbf{k}[\![\beta]\!]} \mathbf{k}(\!(\beta)\!) \simeq \mathbf{DSing\,M_1}$$

which has an explicit model in terms of *matrix factorizations*:

$$\mathbf{d_{ev}}: \mathbf{V_{ev}} \rightleftarrows \mathbf{V_{odd}}: \mathbf{d_{odd}} \qquad \mathbf{d_{ev}}\mathbf{d_{odd}} = \mathbf{f\,id}_{\mathbf{V_{odd}}}, \mathbf{d_{odd}}\mathbf{d_{ev}} = \mathbf{f\,id}_{\mathbf{V_{ev}}}$$

Closed string sector/Hochschild invariants

Relate "non-commutative singularity theory" to less categorical things.

Functorial construction:

$$\underbrace{\left\{\begin{matrix}\mathbf{R}\text{-linear} \\ \text{dg-cat } \mathcal{C}\end{matrix}\right\}}_{\text{Categorical}} \Rightarrow \underbrace{\left\{\begin{matrix}\mathbf{R}\text{-linear cplxs} \\ \mathbf{HH}^{\bullet}_{/\mathbf{R}}(\mathcal{C}), \mathbf{HH}^{/\mathbf{R}}_{\bullet}(\mathcal{C})\end{matrix}\right\}}_{\text{Linear algebraic}}$$

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Group action description good for *functoriality*:

1. Get $\mathbf{S^1}$-action on $\mathbf{HH}^{\bullet}(\mathcal{C})$ and $\mathbf{HH}_{\bullet}(\mathcal{C})$.
2. Get natural maps.

$$\mathbf{HH}^{\bullet}_{/\mathbf{k}(\!(\beta)\!)}(\mathcal{C}^{\mathrm{Tate}}) \longrightarrow \mathbf{HH}^{\bullet}_{\mathbf{k}}(\mathcal{C})^{\mathrm{Tate}}$$

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3. (Least formal part of this talk:)

Theorem (Lin-Pomerleano, P.)

Suppose $\mathbf{1}$ *is the only critical value of* $\mathbf{f}$*. Then, these are equivalences of complexes.*

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Recall that Hochschild invariants have *rich extra structure* functorially attached! E.g., genus $\mathbf{0}$ piece of this is:

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Group action description good for *functoriality*:

1. Get $\mathbf{S^1}$-action on $\mathbf{HH}^\bullet(\mathcal{C})$ and $\mathbf{HH}_\bullet(\mathcal{C})$ *as $\mathbf{E_2}$-algebra/$\mathbf{fE_2}$-module*.
2. Get natural maps *as $\mathbf{E_2}$-algebra/$\mathbf{fE_2}$-module*.

$$\mathbf{HH}^\bullet_{/\mathbf{k}(\!(\beta)\!)}(\mathcal{C}^{\mathrm{Tate}}) \longrightarrow \mathbf{HH}^\bullet_{\mathbf{k}}(\mathcal{C})^{\mathrm{Tate}}$$

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Theorem (Lin-Pomerleano, P.)

Suppose $\mathbf{1}$ is the only critical value of $\mathbf{f}$. Then, these are equivalences of $\mathbf{E_2}$-algebra/$\mathbf{fE_2}$-module.

Closed string sector/Hochschild invariants, contd.

Great! Now want to *compute* e.g., the $\mathbf{S^1}$-equivariant $\mathbf{E_2}$-algebra $\mathbf{HH^\bullet(Perf\ M)}$ (with circle action corresponding to $\mathbf{f} \in \mathbf{\Gamma(M, \mathcal{O}_M)^\times}$).

(Complication: $\mathbf{E_2}$ structures not combinatorial + formality doesn't make sense a priori. Can pass to $\mathbf{hoGerst_2}$, after making an auxillary universal choice of $\mathbf{DQ_\Phi : hoGerst_2 \simeq E_2}$.)

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Goal for the rest of the talk, is to make sense of

Theorem (Imprecise formulation)

The equivariant $\mathbf{E_2}$*-algebra* $\mathcal{A} = \mathbf{HH^\bullet(Perf\ M)}$ *...*

1. *...depends only on the* $\mathbf{E_2}$*-algebra* $\mathcal{A}$ *and a dg Lie map* $\mathbf{f : k[+1] \to \mathcal{A}[+1]}$*, not the dg-cat* $\mathcal{C}$ *itself!*
2. *...has a description in terms of "*$\mathbf{E_2}$ *adjoint action."*
3. *One can leverage the* $\mathbf{E_2}$*- (really,* $\mathbf{hoGerst_2}$*-) formality of* $\mathcal{A}$ *(really,* $\mathbf{DQ_\Phi}\,\mathcal{A}$*) to get explicit description. (Looks as expected.)*

MF as Fourier transform (*infinitesimal* version)

$$\left\{\mathbf{w}\colon \mathbf{M} \to \mathbb{A}^{\mathbf{1}}\right\} \Leftrightarrow \{\mathbf{k[x]}\text{-linear structure}\} \Leftrightarrow \underbrace{\left\{\mathbf{B}\widehat{\mathbb{G}_{\mathbf{a}}}\text{-action}\right\}}_{\text{der. formal gp!}}$$

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Theorem (/Definition)

Suppose $\mathcal{C} \in$ **dgcat**. *Then, TFA(naturally)E as* ∞*-groupoids=spaces:*

$$\left\{\begin{matrix} \textit{dg Lie alg} \\ \mathbf{k}[+1]\textit{-action on } \mathcal{C} \end{matrix}\right\} \Leftrightarrow \mathbf{Map}_{\mathbf{Lie}}(\mathbf{k}[+1], \mathbf{HH}^\bullet(\mathcal{C})[+1])_\bullet$$

$$\left\{\begin{matrix} \textit{(der.) formal gp} \\ \mathbf{B}\widehat{\mathbb{G}}_{\mathbf{a}}\textit{-action on } \mathcal{C} \end{matrix}\right\} \Leftrightarrow \mathbf{Map}_{\mathbf{Fun(DArt,sSet)}}(\mathbf{B}^2\widehat{\mathbb{G}}_{\mathbf{a}}, \mathbf{dgcat}_{\widehat{\mathcal{C}}})_\bullet$$

$$\left\{\begin{matrix} \textit{curved } \mathbf{k}[\![\beta]\!]\textit{-linear} \\ \textit{deformations of } \mathcal{C} \end{matrix}\right\} \Leftrightarrow \mathbf{MC}_\bullet\left(\mathfrak{m}_{\mathbf{k}[\![\beta]\!]} \otimes \mathbf{HH}^\bullet(\mathcal{C})[+1]\right)$$

$$\{\mathbf{k}[\mathbf{x}]\textit{-linear structure on } \mathcal{C}\} \Leftrightarrow \mathbf{Map}_{\mathbf{E_2}}(\mathbf{k}[\mathbf{x}], \mathbf{HH}^\bullet(\mathcal{C}))_\bullet$$

$\mathbf{E}_2$-adjoint actions

1. Start with

$$\overbrace{\mathbf{B}^2\widehat{\mathbb{G}_\mathbf{a}} \xrightarrow{\mathbf{e}^{\beta\mathbf{w}}} (\mathbf{dgcat})_{\widehat{\mathcal{C}}}}^{\text{Only bit depending on } \mathbf{w}} \xrightarrow{\mathbf{HH}^\bullet(-)} (\mathbf{E_2}\text{-alg})_{\widehat{\mathbf{HH}^\bullet\mathcal{C}}} \xrightarrow{-[+1]} (\mathbf{Lie}\text{-alg})_{\widehat{\mathbf{HH}^\bullet(\mathcal{C})[+1]}}$$

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2. First bit at the level of tangent dgla:

$$\mathbf{w} \colon \mathbf{k}[+1] \longrightarrow \mathbf{HH}^\bullet(\mathcal{C})[+1]$$

3. Second bit at the level of tangent dgla:

$$\overbrace{\underbrace{\mathbf{HH}^\bullet(\mathcal{C})[+1] \longrightarrow \mathbf{Der}_{\mathbf{E_2}}(\mathbf{HH}^\bullet(\mathcal{C}))}_{\mathbf{E_2} \text{ adjoint action}} \longrightarrow \mathbf{Der}_{\mathbf{Lie}}(\mathbf{HH}^\bullet(\mathcal{C})[+1])}^{\mathbf{Lie} \text{ adjoint action}}$$

ends up depending only on $\mathbf{E_2}$-alg $\mathbf{HH}^\bullet(\mathcal{C})$, not $\mathcal{C}$ itself.

$\mathbf{E_2}$-adjoint actions (contd.)

1. Lie *adjoint action* is compatible with Lie bracket. Gives natural lift

$$\mathcal{L} \in \mathbf{Lie}\text{-alg}(\mathcal{C}) \quad \Rightarrow \quad \mathcal{L}^{\mathrm{ad}} \in \mathbf{Lie}\text{-alg}(\mathcal{L}\text{-mod}^{\mathbf{Lie}}(\mathcal{C}))$$

2. Analogous construction for associative alg: $\mathbf{A}$ is not an associative algebra in $\mathbf{A}$-mod, but it *is* in Lie-modules over its underlying Lie algebra $\mathbf{A}_{\mathbf{Lie}}$!

$$\mathbf{A} \in \mathbf{E_1}\text{-alg}(\mathcal{C}) \Rightarrow \begin{cases} \mathbf{A}_{\mathbf{Lie}} \in \mathbf{Lie}\text{-alg}(\mathcal{C}) \\ \mathbf{A}^{\mathrm{ad}} \in \mathbf{E_1}\text{-alg}(\mathbf{A}_{\mathbf{Lie}}\text{-mod}^{\mathbf{Lie}}(\mathcal{C})) \end{cases}$$

3. Analogous construction for $\mathbf{E_k}$-algebras, e.g., $\mathbf{k = 2}$:

$$\mathcal{A} \in \mathbf{E_2}\text{-alg}(\mathcal{C}) \Rightarrow \begin{cases} \mathcal{A}[+1] \in \mathbf{Lie}\text{-alg}(\mathcal{C}) \\ \mathcal{A}^{\mathrm{ad}} \in \mathbf{E_2}\text{-alg}(\mathcal{A}[+1]\text{-mod}^{\mathbf{Lie}}(\mathcal{C})) \end{cases}$$

4. Analogous (explicit) constructions for $\mathbf{Gerst_2}$, $\mathbf{hoGerst_2}$ algebras. "Compatible" with above under $\mathbf{DQ_\Phi : hoGerst_2 \simeq E_2}$ and π_*.

Leveraging formality

Theorem (Dolgushev-Tamarkin-Tsygan)

There exists a $\mathbf{hoGerst_2}$ *(extends to* $\mathbf{hoCalc_2}$*) quasi-isomorphism*

$$\mathbf{DQ_\Phi\, HH^\bullet(Perf\, M) \simeq \pi_* HH^\bullet(Perf\, M) \simeq R\Gamma(M, \wedge^{-\bullet} T_M)}$$

Here, $\wedge^{-\bullet}\mathbf{T_M}$ $(+\ \mathbf{\Omega^\bullet_M})$ equipped with the usual Gerst. (+ BV) structure: $\wedge$ product, Schouten-Nijenhuis bracket, (+ Lie derivative, de Rham diff.), etc.

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Corollary

There exists a $\mathbf{hoGerst_2}$ *(extends to* $\mathbf{hoCalc_2}$*) quasi-isomorphism*

$$\mathbf{DQ_\Phi}\left(\mathbf{HH^\bullet(Perf\ M)}^{\mathrm{Tate}}\right) \simeq \mathbf{R\Gamma}\left(\mathbf{M}, \left(\wedge^{-\bullet}\mathbf{T_M}(\!(\beta)\!), \beta \cdot \mathbf{i_{dw}}\right)\right)$$

Here, $\mathbf{i_{dw}} = [\mathbf{w},]$ occurs as the adjoint action restricted along

$$\mathbf{w}\colon \mathbf{k[+1]} \to \wedge^{-\bullet}\mathbf{T_M[+1]}.$$