Topological T-duality With Monodromy

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Idea

Consider the following manifestations of T-duality:

- **Topological T-duality**
  - Principal torus bundles with flux
  - T-D: exchange Chern class and flux

- **Topological Mirror Symmetry**
  - Torus bundles with singularities
  - Monodromy around singular fibers
  - T-D: dualize monodromy, fill in singular fibers

We would like to bridge the gap between these constructions. 
Topological T-duality can incorporate monodromy, singularities not yet considered.
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We would like to bridge the gap between these constructions. Top T-duality can incorporate monodromy, singularities not yet considered.
Affine Torus Bundles

Let \( V \cong \mathbb{R}^n, \quad \Lambda \cong \mathbb{Z}^n, \quad T^n = V/\Lambda \)

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**Definition**

An **affine torus bundle** on $M$ is a torus bundle

$$\pi : E \to M$$

with transition functions valued in $\text{Aff}(T^n)$.
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$\text{Aff}(T^n) \to \text{Diff}(T^n)$ is a homotopy equivalence for $n \leq 3$. 
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- $\rho : \pi_1(M) \rightarrow \text{GL}(n, \mathbb{Z})$ \textit{monodromy}
- $c \in H^2(M, \Lambda_\rho)$ \textit{twisted Chern class}

where $\Lambda_\rho$ is the local system given by action of $\pi_1(M)$ on $\Lambda \cong \mathbb{Z}^n$
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where $\Lambda_\rho$ is the local system given by action of $\pi_1(M)$ on $\Lambda \simeq \mathbb{Z}^n$

$(\rho, c) \simeq (\rho', c')$ if they are related by an element of $\text{GL}(n, \mathbb{Z})$
Topological T-duality (rank 1 case)

\( \pi : E \rightarrow M, \ \hat{\pi} : \hat{E} \rightarrow M \) circle bundles on \( M \)

\( h \in H^3(E, \mathbb{Z}), \ \hat{h} \in H^3(\hat{E}, \mathbb{Z}) \) flux on \( E, \hat{E} \)
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Diagram:

\[ \begin{array}{c}
E \\
\downarrow \pi \\
\hat{E} \\
\downarrow \hat{\pi} \\
M
\end{array} \]

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\hat{E} \\
\downarrow \hat{\pi} \\
E \\
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M
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\[ \begin{array}{c}
P \\
\downarrow p \\
E \times_M \hat{E}
\end{array} \]

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Topological T-duality (rank 1 case)

**Definition**

$(E, h), (\hat{E}, \hat{h})$ are **T-dual** if

- $\rho = \hat{\rho}$ (dual monodromy)
- $\pi_*(h) = c_1(\hat{E}) \in H^2(M, \mathbb{Z}_{\hat{\rho}})$ (swap Chern class and flux)
- $\hat{\pi}_*(\hat{h}) = c_1(E) \in H^2(M, \mathbb{Z}_{\rho})$
- $p^*h = \hat{p}^*\hat{h}$ (flux coincides on correspondence space)
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**Theorem**

*For any \((E, h)\) there exists a T-dual \((\hat{E}, \hat{h})\) unique up to isomorphism (fibre bundle isomorphisms)*
Topological T-duality (general case)

\[ \pi : E \to M, \quad \hat{\pi} : \hat{E} \to M \] affine \( T^n \)-bundles on \( M \)

\[ h \in H^3(E, \mathbb{Z}), \quad \hat{h} \in H^3(\hat{E}, \mathbb{Z}) \] flux on \( E, \hat{E} \)

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We require a constraint on the flux \( h \) (sim for \( \hat{h} \))
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Leray-Serre spec seq for \( \pi : E \to M \) yields filtration

\[ 0 \subseteq F^{3,3}(\pi) \subseteq F^{2,3}(\pi) \subseteq F^{1,3}(\pi) \subseteq F^{0,3}(\pi) = H^3(E, \mathbb{Z}) \]
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we demand \( h \in F^{2,3}(\pi) \) (\( h \) has “one leg on the fiber”)

\[ F^{2,3}(\pi) \ni h \mapsto [h] \in F^{2,3}(\pi)/F^{3,3}(\pi) = E^{2,1}_{\infty}(\pi) \]

\( E^{2,1}_{\infty}(\pi) \) is a subquotient of \( E^{2,1}_2(\pi) = H^2(M, \Lambda^*_\rho) \)
Topological T-duality (general case)

Definition

\((E, h), (\hat{E}, \hat{h})\) are T-dual if

1. \(\rho\) and \(\hat{\rho}\) are dual representations
2. \(c_1(\hat{E})\) represents \([h]\) in \(E_2^{2,1}(\hat{\pi}) = H^2(M, \Lambda^*_{\hat{\rho}})\) (swap Chern class and flux)
3. \(c_1(E)\) represents \([\hat{h}]\) in \(E_2^{2,1}(\pi) = H^2(M, \Lambda_{\rho})\)
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Theorem

For any \((E, h)\) there exists a T-dual \((\hat{E}, \hat{h})\). The Chern class of \(\hat{E}\) is determined up to a map \(H^0(M, \Lambda^2_{\rho}) \rightarrow H^2(M, \Lambda^*_{\rho})\) given by contraction with \(c_1(E)\).
Main Step in Proof

Can find \((\hat{E}, h')\) where \(c_1(\hat{E})\) represents \([h]\) and \(c_1(E)\) represents \([h']\)
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Leray-Serre spec seq for $E \times_M \hat{E} \to M$

\[ E_2^{p,q} = H^p(M, \wedge^q(\Lambda \rho \oplus \Lambda \hat{\rho})) \]
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Duality pairing of $\Lambda \rho$ and $\Lambda \hat{\rho}$ determines a symplectic form

\[ \omega \in H^0(M, \wedge^2(\Lambda \rho \oplus \Lambda \hat{\rho})) \]
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\(p^*h - \hat{p}^*h'\) is represented by

\((c_1(E), -c_1(\hat{E}))\)

so \(p^*h - \hat{p}^*h' = \hat{p}^*\hat{\pi}^*(a)\)

we then set \(\hat{h} = h' + \hat{\pi}^*a\)
Twisted Cohomology

Let \((E, h), (\hat{E}, \hat{h})\) be T-dual

Assume \(E, \hat{E}\) smooth, \(H, \hat{H}\) closed 3-forms representing \(h, \hat{h}\).
Twisted Cohomology

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\(H^* (E, H)\) defined as the \(\mathbb{Z}_2\)-graded cohomology of \((\Omega^* (E), d_H)\) where

\[
d_H \alpha = d \alpha + H \wedge \alpha
\]

Theorem

If \(\rho\) is \(\text{SL}(n, \mathbb{Z})\)-valued then we have an isomorphism

\[ T: H^k (E, H) \cong H^{k-n} (\hat{E}, \hat{H}) \]

\[ T \alpha = \int_{\hat{T} n} e^B \pi^* (\alpha) \]

where \(B\) is a certain 2-form on \(E \times \hat{E}\) which restricted to the fibers \(T n \times \hat{T} n\) is the natural symplectic form \(\omega\).
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Non-oriented case

What if $\rho$ is not $\text{SL}(n, \mathbb{Z})$-valued?
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$H^*(E, (w_1, H))$ defined as cohomology of $(\Omega^*(E, \mathbb{R}w_1), d_{\nabla,H})$

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Twisted $K$-theory

$\rho$ determines a flat vector bundle $V_\rho = \Lambda_\rho \otimes \mathbb{R}$

set $w_1 = w_1(V_\rho), \ W_3 = W_3(V_\rho)$
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We use \( K \)-theory with isomorphism classes of twists \( H^1(\_ , \mathbb{Z}_2) \times H^3(\_ , \mathbb{Z}) \) (e.g. using graded bundle gerbes)
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Theorem

We have isomorphisms

\[
K^k(E, h) \cong K^{k-n}(\hat{E}, (w_1, \hat{h} + W_3)) \\
K^k(E, (w_1, h + W_3)) \cong K^{k-n}(\hat{E}, \hat{h})
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$$K^k(E, (w_1, h + W_3)) \cong K^{k-n}(\hat{E}, \hat{h})$$

In the special case $(w_1, W_3) = (0, 0)$ this reduces to

$$K^k(E, h) \cong K^{k-n}(\hat{E}, \hat{h})$$
Proof (following Bunke Rumpf Schick)

Represent $h, \hat{h}$ as bundle gerbes
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Exists an isomorphism $u : p^* h \rightarrow \hat{p}^* \hat{h}$ s.t. on the fibers $T_m \times \hat{T}_m u$ looks like the Poincaré line bundle
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Choose trivializations

$$\tau : 0 \to h|_{T_m}$$
$$\hat{\tau} : 0 \to \hat{h}|_{\hat{T}_m}$$
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$$
p^*(h)|_{T_m \times \hat{T}_m} \overset{p^*\tau^{-1}}{\longrightarrow} 0 \overset{\hat{p}^*\hat{\tau}}{\longrightarrow} \hat{p}^*(\hat{h})|_{T_m \times \hat{T}_m}
$$

differs from $u|_{T_m \times \hat{T}_m}$ by an element of $H^2(T_m \times \hat{T}_m, \mathbb{Z})$
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differs from \( u|_{T_m \times \hat{T}_m} \) by an element of \( H^2(T_m \times \hat{T}_m, \mathbb{Z}) \)

Modulo \( p^* H^2(T_m, \mathbb{Z}) + \hat{p}^* H^2(\hat{T}_m, \mathbb{Z}) \) required to be the Poincaré line bundle
Proof (following Bunke Rumpf Schick)

Fourier-Mukai type transformation

\[ T : K^i(E, h) \to K^{i-n}(\hat{E}, (w_1, \hat{h} + W_3)) \]
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\[ K^i(E \times_M \hat{E}, p^* h) \xrightarrow{u} K^i(E \times_M \hat{E}, \hat{p}^* \hat{h}) \]

\[ \begin{array}{ccc}
K^i(E, h) & \xrightarrow{p^*} & K^i(E \times_M \hat{E}, \hat{p}^* \hat{h}) \\
\downarrow \hat{p}_* & & \downarrow \hat{p}_* \\
K^{i-n}(\hat{E}, (w_1, \hat{h} + W_3)) & & K^{i-n}(\hat{E}, (w_1, \hat{h} + W_3))
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Fourier-Mukai type transformation

\[ T : K^i(E, h) \to K^{-n}(\hat{E}, (w_1, \hat{h} + W_3)) \]

\[ K^i(E \times_M \hat{E}, p^* h) \xrightarrow{u} K^i(E \times_M \hat{E}, \hat{p}^* \hat{h}) \]

Locally \( T \) looks like K-theoretic Fourier-Mukai

use Mayer-Vietoris
Monodromy can be incorporated into topological T-duality using local coefficients
Conclusions

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- Higher rank T-duality presents the same existence and uniqueness challenges (although monodromy tends to make the T-dual less ambiguous).
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- One should consider de Rham cohomology twisted by $H^1(\_ , \mathbb{Z}_2) \times H^3(\_ , \mathbb{R})$ and $K$-theory by $H^1(\_ , \mathbb{Z}_2) \times H^3(\_ , \mathbb{Z})$. 
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THANK YOU