FATNESS REVISITED

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In 1968, Alan Weinstein invented the concept of a *fat bundle* (named unflat in [Wei68] and renamed fat in [Wei80]), in order to understand when the total space of a fiber bundle with totally geodesic fibers admits a metric with positive sectional curvature. Just the positivity of the vertizontal curvatures (curvature of a 2–plane spanned by a vertical and a horizontal vector) is already a strong condition, which Weinstein named fatness of the bundle.

The present notes grew out of several lectures I gave at the University of Pennsylvania in the spring of 2000 in order to summarize known results about fat bundles, and to understand its significance for attempting to construct metrics with non–negative or positive sectional curvature on principal bundles and their associated bundles.

After [Wei68], the subject was first picked up again by L. Berard Bergery in [BB75], where he classifies homogeneous fat bundles of the form $K/H \to G/H \to G/K$ and found many beautiful examples, which include in particular most of the homogeneous metrics with positive sectional curvature. Surprisingly, since then, no new examples of fat bundles were discovered (except in the special case where the fiber is one dimensional). The fact that fat bundles are rare is also illustrated by one of the main theorems in the subject [DR81], where they proved that every fat S^3 bundle over S^4 must be a Hopf bundle. The theory of fat bundles over 4–manifolds is well developed, due to [DR81], and we summarize the results in these notes. Little is known about fat bundles over a higher dimensional base.

We then turn our attention to using information about fat bundles to constructing a metric on the total space which has sectional curvatures positive, or simply non–negative. The case of S^1 bundles over a base B deserves special attention. In this case, fatness just means that the curvature form of the principal connection is a symplectic 2–form on the base B and the positivity of the sectional curvatures on the total space turns into a simple first order differential equation on the symplectic 2–form. The last subject we discuss is the fact that fatness can also be useful for constructing metrics of nonnegative curvature not only on sphere bundles, but also on the associated vector bundle. This summarizes work by Strake–Walschap, [SW90], Yang [Yan95], and Tapp [Tap00].

Although there are no new results in these notes, our main purpose is to throw some new light on old results by putting everything into a common framework and to encourage others to work on this beautiful subject by posing a list of 10 open problems which seem to be important to us.

We would like to thank Luis Guijarro and Sukhendu Mehrotra for their help in putting these notes into their present form. It has not been completed yet in some parts to the authors satisfaction and needs to be updated to account for more recent developments. Nevertheless it may be useful to post them even in this somewhat unpolished form.

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1. Riemannian Submersions

In this section, we collect basic results about Riemannian submersions, which will be used throughout these notes. We start with some terminology and notation.

Let M and B be (connected) complete Riemannian manifolds and $\pi : M \to B$ a smooth submersion. For $x \in M$, with $b = \pi(x)$, denote by $\mathcal{V}_x$ the kernel of the linear map $\pi_*\colon T_xM \to T_bB$ and $\mathcal{H}_x$ be the orthogonal complement to $\mathcal{V}_x$. The submersion π is said to be *Riemannian* if $\pi_*\colon \mathcal{H}_x \to T_bB$ is an isometry.

For $b \in B$, the submanifolds $\pi^{-1}(b) = F_b$ of M are called *fibers* (which do not have to be connected), M is called the *total space* and B the *base.* A vector field on M is said to be *vertical* if it is always tangent to the fibers, *horizontal* if it is always orthogonal to the fibers. Thus, a submersion is Riemannian if and only if π_* preserves lengths of horizontal vectors.

The vertical subspace $\mathcal{V}_x$ of T_xM above is identified with the tangent space to F_b at x. By $\mathcal{V}$ and $\mathcal{H}$ we denote, respectively, the vertical and horizontal distributions and also the orthogonal projections onto them. Throughout, the letters U, V, W denote vertical vector fields and X, Y, Z horizontal vector fields.

A vector field X on M is said to be *basic* if it horizontal and there exists a vector field $\check{X}$ on B such that $\pi_*(X_x) = \check{X}_{\pi(x)}$. For every vector field $\check{X}$ on B, there exists a unique basic vector field X, *the horizontal lift of* $\check{X}$, on M which is π-related to $\check{X}$. Its value at $x \in M$ is by definition the vector in $\mathcal{H}_x \subset T_xM$ such that $\pi_*(X_x) = \check{X}_{\pi(x)}$.

A smooth path γ in M is said to be *horizontal* if $\dot\gamma$ is horizontal. A *horizontal lift* of a path $\check\beta$ in B is a horizontal path β in M such that $\check\beta = \pi \circ \beta$. For any initial value $\beta(0) \in F_{\check\beta(0)}$ these horizontal lifts exist since β can be viewed as the integral curve of the horizontal lift X of the velocity vector field of $\check\beta$ on the submanifold $\pi^{-1}(\check\beta)$, if $\check\beta$ is regular. The integral curves exists for all t since M is complete and $|X|$ is bounded. If $\check\beta$ is not regular, we can view the lift as a composition of lifts of the regular parts.

Each fiber F_b, being a closed submanifold of M, admits a second fundamental form; all these forms can together be organized into a tensor field T on M given by

$$T_EF = \mathcal{H}\nabla_{\mathcal{V}E}(\mathcal{V}F) + \mathcal{V}\nabla_{\mathcal{V}E}(\mathcal{H}F),$$

where E, F are arbitrary vector fields and ∇ is the Levi-Cevita connection on M. Note that T_UV is the second fundamental form of each fiber. Further, as

$$\langle T_UV, X\rangle = -\langle T_UX, V\rangle,$$

T vanishes identically if and only if the fibers are totally geodesic.

Another tensorial invariant of a Riemannnian submersion is the O'Niell tensor A on M, defined by:

$$A_EF = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F.$$

A basic property of this tensor field is that it is skew-symmetric on $\mathcal{H}$: For any vertical U, and basic X,

$$\langle U, A_XX\rangle = \langle U, \nabla_XX\rangle = -\langle \nabla_XU, X\rangle.$$

As $[X, U]$ is π-related to 0, it is vertical; so $\langle \nabla_XU, X\rangle = \langle \nabla_UX, X\rangle$, and hence, $\langle U, A_XX\rangle = \frac{1}{2}U(\langle X, X\rangle) = 0$ since $\langle X, X\rangle$ is constant along the fibers.

Together with the observation

$$\mathcal{V}[X, Y] = \mathcal{V}\nabla_XY - \mathcal{V}\nabla_YX = A_XY - A_YX,$$

this yields:

$$A_X Y = \frac{1}{2}\mathcal{V}[X, Y].$$

Furthermore, since

$$\langle A_X Y, U\rangle = -\langle A_X U, Y\rangle,$$

the tensor A vanishes iff $A_X Y = 0$. The skew symmetry of A implies that the horizontal lift of a geodesic in B is a geodesic in M since the horizontal lift of $\nabla_{\check{X}}\check{Y}$ is $\mathcal{H}\nabla_X Y$ and $\mathcal{V}\nabla_X Y = A_X Y$.

For any path $\check{\gamma}\colon [0, l] \to B$, and any $x \in F_{\check{\gamma}(0)}$, let $\tau_{\check{\gamma}}(x)$ be the endpoint of the horizontal lift γ of $\check{\gamma}$ starting from x. Since $\tau_{\check{\gamma}}^{-1} = \tau_{\check{\gamma}^{-1}}$, we see that $\tau_{\check{\gamma}}$ is a diffeomorphism from $F_{\check{\gamma}(0)}$ to $F_{\check{\gamma}(l)}$, and hence we can speak of a typical fiber F of a Riemannian submersion.

The *holonomy group* H_b of a Riemannian submersion $M \to B$, $b \in B$, is the group of all diffeomorphisms τ_α of F_b corresponding to closed paths α in B starting at b. Observe that for any path γ, we have $H_{\gamma(1)} = \tau_\gamma \circ H_{\gamma(0)} \circ \tau_\gamma^{-1}$, and hence the holonomy group is well defined up to isomorphism. We will therefore often simply use $(F, H) = (F_{b_0}, H_{b_0})$ for some fixed b_0. We also have the *holonomy bundle* $P = \{\tau_\gamma \mid \gamma(0) = b_0\}$ with projection $\sigma\colon P \to B$, $\sigma(\tau_\gamma) = \gamma(1)$. As we will see shortly, σ is an H principle bundle, where H acts freely on P via the right action $\tau_\alpha \cdot \tau_\gamma = \tau_\gamma \circ \tau_\alpha$. Also notice that the holonomy group is an invariance group of the horizontal distribution and does not depend on the metrics involved. But in order to define it, one needs to assume that horizontal lifts exist for all time (so called completeness of the horizontal distribution), which in our case follows from the completeness of M. $A_X Y$ also depends only on the horizontal distribution and measures the obstruction to integrability.

An important special case for us is when the fibers are totally geodesic:

PROPOSITION 1.1. *Let $\pi\colon M \to B$ be a Riemannian submersion. Its fibers are totally geodesic if and only if τ_γ is an isometry from $F_{\gamma(0)}$ to $F_{\gamma(1)}$ for each path γ in B. In this case one has:*

(a) *The holonomy group H_b is a subgroup of the isometry group of F_b and is itself a Lie group.*
(b) *$\pi\colon M \to B$ is a locally trivial fiber bundle with fiber F and structure group H.*
(c) *$\sigma\colon P \to B$ is a principle H bundle and M is the associated fiber bundle $M = P \times_H F$.*

Proof: Denote by g the metric on M, and assume that the fibers of π are totally geodesic, i.e. $T \equiv 0$. Then, if X is basic, we have

$$(\mathcal{L}_X g)(U, V) = (\nabla_U X, V) + (\nabla_V X, U) = -(X, T_U V) - (X, T_V U) = 0.$$

Thus, g is parallel for basic vector fields. Since the τ_γ are diffeomorphisms associated to the flows of basic vector fields, they preserve g, that is, are isometries.

Conversely, if $(\mathcal{L}_X g)(U, V) = 0$ for every basic field X, we must have $T_U V + T_V U = 2T_U V = 0$, or in other words, the fibers must be totally geodesic.

This also shows that $H = H_b$ a subgroup of the isometry group Iso of F_b. To see that H is a Lie group, we use the fact that Iso is a Lie group and that a connected subgroup of a Lie group is a Lie group as well. First define H^0 to be the subgroup of H corresponding to isometries τ_α with α null homotopic. Since null homotopies can be approximated by differentiable null homotopies, H^0 is a connected subgroup of Iso and hence a Lie group. H^0

is also a normal subgroup of H and the natural homomorphism $\pi_1(B) \to H/H^0$ is onto by definition. Since the fundamental group of a manifold is countable, H/H^0 is countable as well which shows that H is a Lie group with H^0 as its identity component.

To prove (b), let $F = F_{b_0}$ for some fixed b_0 with $H = H_{b_0}$ acting on F. For every normal neighborhood $B_r(b)$ in B, we obtain a trivialization $B_r(b) \times F \to \pi^{-1}(B_r(b))$ by sending $c \times F$ to $\pi^{-1}(c) = F_c$ via $\tau_{\check\gamma}$, where $\check\gamma$ is a fixed path from b_0 to b composed with the unique minimal geodesic from b to c. The coordinate interchanges are then of the form $(c,p) \to (c, f(c)p)$ for some $f\colon B_r(b_1) \cap B_r(b_2) \to H$. Thus π is a locally trivial fiber bundle with fiber F and structure group H.

To prove (c), we observe that one similarly has trivializations $B_r(b) \times H \to \sigma^{-1}(B_r(b))$ by sending $c \times H$ to $\sigma^{-1}(c)$ via $(c, \tau_{\check\alpha}) \to \tau_{\check\gamma} \circ \tau_{\check\alpha}$, $\tau_{\check\gamma}$ as aove. H acts on $B \times H$ on the right in the second coordinate and on P as above, and the trivialization is clearly H equivariant. Hence σ is an H principle bundle. The associated bundle $P \times_H F$ is defined as the quotient of $P \times F$ under the H action $h \cdot (p, f) = (ph^{-1}, hf)$. One easily sees that the map $[\tau_\gamma, f] \to \tau_\gamma(f)$ is a diffeomorphism from $P \times_H F$ to M. □

The same proof shows that a general Riemannian submersion (with M complete) is a locally trivial fiber bundle with structure group H, but H may be an infinite dimensional Lie group.

The following result about Riemannian submersions with totally geodesic fibers will also be useful:

PROPOSITION 1.2. *Let $\pi\colon M \to B$ be a Riemannian submersion. If the fibers of π are totally geodesic, then for any basic X, Y, the vertical fields $A_X Y$ are Killing vector fields along each fiber, and in fact lie in the Lie algebra of the holonomy group H.*

Proof: Fix a fiber F_b and $X, Y \in T_b B$. Since A is a tensor, we may extend X, Y to be horizontal lifts of coordinate vector fields $\check X$ and $\check Y$ on a coordinate neighborhood of B. Then, as $[X,Y]$ is π-related to $[\check X, \check Y] = 0$, we have $A_X Y = \frac{1}{2}\mathcal{V}[X,Y] = \frac{1}{2}[X,Y]$, and, so,

$$(\mathcal{L}_{2A_X Y} g)(U,V) = (\mathcal{L}_{[X,Y]} g)(U,V) = ([\mathcal{L}_X, \mathcal{L}_Y] g)(U,V) = 0$$

which implies that $A_X Y$ is Killing. Furthermore, $[X,Y]_p = \frac{d}{dt}_{|t=0} \phi_t^* \psi_t^* \phi_{-t}^* \psi_{-t}^*(p)$, where ϕ_t, ψ_t are the flows of X and Y, which consists of diffeomorphism τ_γ for some $\gamma \subset B$. Since $\check X, \check Y$ are coordinate flows, this is a derivative of a closed curve in B and hence $A_X Y$ lies in the Lie algebra of the holonomy group H. □

In order to construct metrics on the total space so that the projection is a Riemannian submersion with totally geodesic fibers, we will treat the case of principal bundles first.

Recall that for a principal G–bundle $\sigma\colon P \to B$, a *principal connection* is a differentiable map $\theta\colon TP \to \mathfrak{g}$ with $\theta(g_*(X)) = Ad(g^{-1})\theta(X)$ for all $g \in G$, or simply $g^*\theta = Ad(g^{-1})\theta$. Furthermore, $\theta(V^*) = V$ for all $V \in \mathfrak{g}$ where V^* are the action fields on P: $V^*(x) = \frac{d}{dt}_{|t=0}(x \cdot \exp(tV))$. The principle connection defines a horizontal distribution $\mathcal{H} = \ker\theta$, which is right invariant, i.e., $\mathcal{H}_{xg} = g_*(\mathcal{H}_x)$. Equivalently, one can define a principal connection in terms of a right invariant distribution $\mathcal{H}$, which is complementary to the tangent space of the fibers, by setting $\theta(V^*) = V$ and $\theta(\mathcal{H}) = 0$. The equation $\theta(g_* X) = Ad(g^{-1})\theta(X)$ then follows from the fact that $g_*(V^*) = (\mathrm{Ad}(g^{-1})V)^*$.

For a principal G-bundle the map $\phi_x(g) = xg$ defines a natural identification of G with the fiber F_b for each choice of $x \in F_b$. Since $\phi_{xg} = \phi_x \circ L_g$, these identifications are well defined up to left translations.

Given a complete metric g_B on B, a left invariant metric g_G on G, and a connection form θ, we can define a *connection metric* g_P on P by requiring that:

(a) The tangent spaces of the fibers are orthogonal to the horizontal space $\mathcal{H} = \ker\theta$.
(b) $\sigma_*\colon (\mathcal{H}_x, g_P) \to (T_{\sigma(x)}B, g_B)$ is an isometry for every $x \in P$.
(c) $\phi_x\colon (G, g_G) \to (F_b, g_B)$ is an isometry for every $x \in F_b$.

The metric on F_b is well defined since g_G is left invariant. The metric can also alternatively be described as:

$$g_P(X,Y) = g_G(\theta(X), \theta(Y)) + g_B(\sigma_*(X), \sigma_*(Y)). \tag{1.3}$$

To see that ϕ_x is an isometry in this metric, one observes that $(\phi_x)_*(V) = V^*$ since the flow of a left invariant vector field V on G is given by right translations.

PROPOSITION 1.4. *The metric g_P on P defined by* (1.3) *is complete and defines a Riemannian submersion $\sigma\colon P{\to}B$ with totally geodesic fibers and with holonomy group H a subgroup of G acting on G from the left.*

Proof: σ is a Riemannian submersion by definition. Since we do not yet know that the metric on P is complete, we first need to show that the horizontal distribution $\mathcal{H}$ has horizontal lifts that exist for all t. To see this, one first takes any lift $\gamma(t)$ of a curve on the base, which exist since the bundle is locally trivial, and then uses the right action in each fiber to choose $a(t) \in G$ such that $\gamma(t)a(t)$ becomes horizontal. Since $\theta((\gamma a)') = \theta(\gamma' a + \gamma a') = \mathrm{Ad}(a^{-1})\theta(\gamma') + a^{-1}a'$, this amounts to solving $a'a^{-1} = -\theta(\gamma')$, $a(0) = e$. If we denote $X = a'a^{-1} \subset \mathfrak{g}$, then a solution is an integral curve of the time dependent vector field $T(g,t) = (X(t)g, d/dt)$ on $G \times \mathbb{R}$, which has solutions for all t since it is a vector field of bounded length in a (complete) biinvariant metric on $G \times \mathbb{R}$

Now we observe that for every horizontal lift γ and every $g \in G$, γg defines another horizontal lift of $\sigma(\gamma)$ since $\mathcal{H}$ is right invariant. Hence, if we use the isometries $\phi_{\gamma(0)}$ and $\phi_{\gamma(1)}$ to identify the respective fibers with G, the diffeomorphism $\tau_{\sigma(\gamma)}$ is the identity, and hence an isometry as well. Thus Proposition 1.1 implies that the fibers are totally geodesic. Similarly, if γ is the horizontal lift of a closed path starting at b_0 and $x_0 \in \sigma^{-1}(b_0)$ fixed, then $\tau_{\sigma(\gamma)}(x) = xg_0$ for $g_0 \in G$ implies that $\tau_{\sigma(\gamma)}(xg) = xg_0g$ for all g. Thus, under the identification ϕ_{x_0}, the isometry $\tau_{\sigma(\gamma)}$ becomes left multiplication by g_0. Hence the holonomy group is a subgroup of G acting on G from the left.

In order to see that g_P is complete, choose a Cauchy sequence $x_i \in P$. Since σ is distance non-increasing, $\sigma(x_i)$ is a Cauchy sequence in B and we choose a subsequence such that $\sigma(x_i) \to y$. Let $\check{\gamma}_i$ be a minimal geodesic from x_i to y (for i sufficiently large) with horizontal lift γ_i. Since $\tau_{\check{\gamma}_i}$ are isometries, one easily sees that $\tau_{\check{\gamma}_i}(x_i)$ is a Cauchy sequence in F_y which has a convergent subsequence since g_G is complete. Finally, if z is its limit, one easily shows that $x_i \to z$. □

Notice that in this construction, the right action of G on P is by isometries iff the metric g_G is biinvariant. Also observe that the holonomy principle bundle P' can be considered as a reduction of the structure group of the principle G-bundle to its holonomy group H. Indeed, if we fix $x_0 \in \sigma^{-1}(b_0)$, the embedding $P' \to P$ given by $\tau_\gamma \to \tau_\gamma(x_0)$ is H-equivariant.

We now define a connection metric on an arbitrary fiber bundle. Assume that in addition to a principal G-bundle $\sigma\colon P\to B$ we have a (complete) Riemannian manifold F with an isometric left action of G. The associated bundle $\pi\colon M=P\times_G F\to B$ is then a fiber bundle with fiber F and structure group G. Notice that an identification of F with a fiber $F_b=\pi^{-1}(b)=\sigma^{-1}(b)\times_G F$, is defined for every fixed $x\in\sigma^{-1}(b)$ by $\psi_x(p)=([(x,p)])$ since every point in $\sigma^{-1}(b)$ is equivalent to x by a unique element of G. Since $\psi_{xg}=\psi_x\circ L_g$, these identifications are well defined up to isometric left translations $L_g\colon F\to F$.

A principal connection θ on P defines a horizontal distribution for π by $\tilde{\mathcal{H}}:=\nu_*(\mathcal{H},0)$, where $\mathcal{H}=\ker\theta$ and $\nu\colon P\times F\to P\times_G F$ is the natural projection. Then for every complete metric g_B on B and G-invariant complete metric g_F on F we define a metric g_M on M such that:

(a) The tangent spaces of the fibers are orthogonal to the horizontal space $\tilde{\mathcal{H}}$.
(b) $\pi_*\colon(\tilde{\mathcal{H}}_x,g_P)\to(T_{\pi(x)}B,g_B)$ is an isometry for every $x\in M$.
(c) $\psi_x\colon(F,g_F)\to(F_b,g_P)$ is an isometry for every $x\in\sigma^{-1}(b)$.

The metric on F_b is again well defined since G acts on F by isometries. Notice that in order to define g_M, we do not choose a metric on G. We only use a principle connection in the principal bundle to define a horizontal distribution $\tilde{\mathcal{H}}$ on M.

Proposition 1.5. *The connection metric g_M on M defined by g_B, g_F and θ is complete and defines a Riemannian submersion $\pi\colon M\to B$ with totally geodesic fibers F and with holonomy group a subgroup of G. Conversely, every Riemannian submersion $\pi\colon M\to B$ with totally geodesic fibers arises in this fashion.*

Proof: π is again a Riemannian submersion by definition. If γ is a horizontal lift of $\check{\gamma}$ under $\sigma\colon P\to B$, then $[(\gamma,x)]\subset P\times_G F=M$ is a horizontal lift of $\check{\gamma}$ under π with initial value $\psi_{\gamma(0)}(x)$. If we use the identifications $\psi_{\gamma(0)}$ and $\psi_{\gamma(1)}$ at the endpoints, the diffeomorphism $\tau_{\check{\gamma}}$ is the identity and hence an isometry. Thus the fibers are totally geodesic and the holonomy group is a subgroup of G. The proof that g_M is complete, can be carried out as in Proposition 1.4.

Conversely, let $\pi\colon M\to B$ is a Riemannian submersion with totally geodesic fibers F and horizonal distribution $\tilde{\mathcal{H}}$. By Proposition 1.1 it is a locally trivial fibre bundle with structure group the holonomy group H acting on F. Recall that we can then write $M=P\times_H F$ where $\sigma\colon P\to B$ is the associated holonomy H-principle bundle. To define a principal connection $\mathcal{H}$ on P, we define $\mathcal{H}_x$, $x\in\sigma^{-1}(b)$ as follows. $x=\tau_\gamma$ for some path γ with $\gamma(0)=b_0$ and $\gamma(1)=b$. For each geodesic δ starting at b we obtain a path $x(t)=\tau_{\delta_{|[0,t]}}\circ\tau_\gamma$ in P with $x(0)=x$. Define $\mathcal{H}_x$ to be the set of all tangent vector $x'(0)$. Since clearly $\mathcal{H}_{xg}=g_*(\mathcal{H}_x)$, this defines a principle connection on P. Furthermore, the induced connection $\nu_*(\mathcal{H}_x,0)$ agrees with $\tilde{\mathcal{H}}$ under the diffeomorphism $P\times_H F\to M\colon[\tau_\gamma,f]\to\tau_\gamma(f)$. □

It is natural to also consider metrics on M defined as follows. If we choose a biinvariant metric Q on G, then Proposition 1.4 defines a metric g_P on P where the right action of G is by isometries. We can then consider the product metric g_P+g_F on $P\times F$ on which G now acts by isometries and the metric $\tilde{g}_M$ on M which is the submersion metric under $\nu\colon P\times F\to P\times_G F=M$. We now describe the relationship between these two metrics. If X^* are the action fields on F for the action of G, then $T_p=\{X^*(p)\mid X\in\mathfrak{g}\}$ is the tangent space to the orbit and we denote by $T_p^\perp$ its orthogonal complement.

PROPOSITION 1.6. *On $M \to B$ there are two submersion metrics with totally geodesic fibers. The metric g_M defined in* (1.3) *and the metric $\tilde{g}_M$ described above. Both have the same horizontal distribution, $\tilde{g}_M = g_M$ on $\mathcal{H}_p$ and $T_p^\perp$, and $\tilde{g}_M(X^*, Y^*) = Q(L(I+L)^{-1}X, Y)$ on T_p, where $g_F(X^*, Y^*) = Q(LX, Y)$ for all $X, Y \in \mathfrak{g}$.*

Proof: The first two claims are clear from the definitions. To see how the metric on F changes under the submersion ν, let $\{(-X^*, X^*) \mid X \in \mathfrak{g}\}$ be its vertical space. The orthogonal complement with respect to $Q + g_F$ is the sum of $\{(0, Y) \mid g_F(Y, \mathfrak{g}^*) = 0\}$ on which $\tilde{g}_M = g_F$ and $\{(Y^*, Z^*) \mid Q(-X, Y) + g_F(X^*, Z^*) = Q(X, -Y + LZ) = 0 \text{ for all } X \in \mathfrak{g}\} = \{((LZ)^*, Z^*) \mid Z \in \mathfrak{g}\}$. Since $\nu_*(X^*, Y^*) = X^* + Y^*$, the vertical lift of X^* under ν is $([L(I+L)^{-1}X]^*, [(I+L)^{-1}X]^*)$ and hence $\tilde{g}_M(X^*, Y^*) = Q(L(I+L)^{-1}X, (I+L)^{-1}Y) + Q(L(I+L)^{-1}X, L(I+L)^{-1}Y) = Q(L(I+L)^{-1}X, Y)$ which proves the claim. □

This change of metric on $F = G \times_G F$ was first considered by Cheeger. Proposition 1.6 shows that it shortens the metric in the direction of the G orbits. Since Riemannian submersions are curvature non-decreasing, this is often used to create more positive curvature on non-negatively curved manifolds. Notice that for the new metric on F the action by G is still by isometries, but other isometries may have been lost. If we replace Q by any right invariant metric on G the formula in Proposition 1.6 still holds, but G will not act by isometries any more.

2. Fat Bundles

For a Riemannian submersion $\pi\colon M \to B$, the most useful curvature identity is

$$\sec_M(x, y) = \sec_B(\pi_*(x), \pi_*(y)) - 3|A_X Y|^2$$

where $x, y \in \mathcal{H}_p$ is orthonormal and X, Y are horizontal extensions of x, y. It implies in particular that the projection is curvature non-decreasing. For us, the following formula, which only holds if the fibers are totally geodesic, will be important. For the convenience of the reader, we give a proof from scratch.

PROPOSITION 2.1. *Assume that $\pi\colon M \to B$ is a Riemannian submersion with totally geodesic fibers. Then, the "vertizontal" sectional curvatures* $\sec(X, U)$ *of M are given by:*

$$\sec(X, U) = \|A_X U\|^2,$$

where $\|X\| = \|U\| = 1$.

Proof: By definition,

$$\begin{aligned} \sec(X, U) &= \langle R(X, U)X, U\rangle \\ &= \langle \nabla_U \nabla_X X - \nabla_X \nabla_U X + \nabla_{[X,U]} X, U\rangle \qquad (\dagger) \end{aligned}$$

for $\|X\| = \|U\| = 1$.

Since R is a tensor, we may assume that X is basic. Then, $[X, U]$ is vertical as $\pi_*[X, U] = [\pi_* X, \pi_* U] = 0$. Thus,

$$\langle \nabla_{[X,U]} X, U\rangle = \langle T_{[X,U]} X, U\rangle = 0$$

by assumption. Now,

$$\nabla_X X = A_X Y + \mathcal{H}\nabla_X X = \frac{1}{2}\mathcal{V}[X, X] + \mathcal{H}\nabla_X X = \mathcal{H}\nabla_X X,$$

and so,

$$\langle \nabla_U \nabla_X X, U\rangle = \langle T_U \nabla_X X, U\rangle = 0.$$

The equation (†), thus, reduces to $K(X,U) = -\langle \nabla_X \nabla_U X, U\rangle$. Finally,

$$\nabla_U X = \mathcal{H}\nabla_U X + T_U X = \mathcal{H}\nabla_U X,$$

so that

$$\nabla_X \nabla_U X = \mathcal{H}\nabla_X \nabla_U X + A_X \nabla_U X$$

and

$$\sec(X,U) = -\langle A_X \nabla_U X, U\rangle = \langle A_X U, \nabla_U X\rangle = \|A_X U\|^2$$

□

In particular, $\sec(X,U) \geq 0$ and $\sec(X,U) > 0$ iff $A_X U \neq 0$. This motivates the following definition:

Definition 2.2. A Riemannian submersion $\pi\colon M \to B$ with totally geodesic fibers is called *fat* if $A_X U \neq 0$ for all $X \wedge U \neq 0$, or equivalently all its vertizontal curvatures are positive.

Remark 2.3. Notice that this condition actually only depends on a choice of horizontal distribution since $A_X Y = \frac{1}{2}\mathcal{V}[X,Y]$ is defined without any choice of metrics. But then it does not easily translate into a curvature condition since the general formula for the vertizontal curvatures (see e.g [Bes87], p. 241) is:

$$\sec(X,U) = \langle (\nabla_X T)_U U, X\rangle - |T_U X|^2 + |A_X U|^2$$

Proposition 2.4. *The following conditions are equivalent to fatness:*

(a) $A_X\colon \mathcal{V} \to \mathcal{H}$ *is injective for all* $X \neq 0$*, or* $A_X\colon \mathcal{H} \to \mathcal{V}$ *is onto for all* $X \neq 0$.
(b) *For each* $U \in \mathcal{V}_p$, $(X,Y) \to < A_X Y, U >$ *is a non–degenerate 2–form on* $\mathcal{H}_p$.
(c) $\dim \mathcal{V} \leq \dim \mathcal{H} - 1$*, and if* $\dim \mathcal{V} = \dim \mathcal{H} - 1$*, then* $A_X\colon \mathcal{H} \cap X^\perp \to \mathcal{V}$ *is an isomorphism.*

Proof: (a) and (b) follow from the definition and $\langle A_X U, Y\rangle = -\langle A_X Y, U\rangle$. For (c), note that A_X is onto, and $A_X X = 0$ by skew-symmetry. □

In the following two propositions, we collect some consequences of the fatness condition.

Proposition 2.5. *Let* $\pi\colon M \to B$ *be a fat submersion. Then, one has the following dimensional restrictions:*

(a) $\dim B = \dim \mathcal{H}$ *is even.*
(b) $\dim \mathcal{V} = \dim \mathcal{H} - 1$ *implies that* $\dim B = 2, 4$ *or 8.*
(c) *If* $\dim \mathcal{V} \geq 2$*, then* $\dim B = 4k$*, while if* $\dim \mathcal{V} \geq 4$*, then* $\dim B = 8k$.

Proof: (a) follows from the fact that $\mathcal{H}$ admits a non-degenerate two-form. To prove (b), observe that for a fixed $u \in \mathcal{V}_p$, as $A_x x = 0$, we can regard $\langle A_x \cdot, u\rangle$ as a non-vanishing vector field on the unit sphere $S^{n-1} \subset \mathcal{H}_p$. In fact, if $u_1, \ldots, u_r$ are linearly independent, the corresponding vector fields on S^{n-1} are also linearly independent. Hence, one gets $\dim \mathcal{V}$ linearly independent vector fields on $S^{(\dim B - 1)}$. This implies severe restrictions (see e.g. [H75], Theorem 8.2). If $\dim \mathcal{V} = \dim \mathcal{H} - 1$, then $S^{(\dim B-1)}$ is parallelizable and hence $\dim \mathcal{V} = 1, 3, 7$, which means that $\dim B = 2, 4$ or 8. (c) follows since if $\mathbb{S}^{n-1}$ admits 2 linearly independent vector fields, then $n = 4k$ (and then it admits 3, given by multiplication with imaginary unit quaternions) and if $\mathbb{S}^{n-1}$ admits 4 linearly independent vector fields, $n = 8k$ (and then it admits 7, given by multiplication with imaginary unit Caley numbers) □

Proposition 2.6. *The fiber F of a fat submersion $\pi\colon M \to B$ is a homogeneous manifold, in fact the holonomy group H acts transitively on F, i.e., $F = H/K$. Consequently, the total space M is diffeomorphic to $P \times_H H/K \cong P/K$, where P is the associated holonomy principal bundle.*

Proof: It is enough to prove that every point $x \in F$ has a homogeneous neighborhood. Define $\phi\colon H \to F$ via $\phi(h) = hx$. For a fat submersion, $A_X\colon \mathcal{H} \to \mathcal{V}$ is onto and by Proposition 1.2, $A_X Y$ are Killing vector fields on F. This implies that ϕ is a submersion at the identity element, and hence the image of ϕ contains a neighborhood of x. □

In light of the previous proposition, we shall concentrate on two special classes of examples: principal bundles, and associated bundles with homogeneous fiber. We will also assume in the remainder that G is a compact Lie group, although some of the following results will hold in general.

2.1. **Principal bundles.** We shall fix a bi-invariant metric Q on $\mathfrak{g}$, which exists by compactness of G, although it will play only an auxiliary role in the discussion below.

Let $\sigma\colon P \to B$ be a G-principal bundle, θ a principal connection, and $\mathcal{H} = \ker\theta$. The curvature of θ is the 2-form $\Omega : TP \times TP \to \mathfrak{g}$ defined by $\Omega(A, B) = d\theta(\mathcal{H}A, \mathcal{H}B)$. It satisfies $g^*\Omega = Ad(g^{-1})\Omega$ and the following structure equation:

$$d\theta = \Omega + \frac{1}{2}[\theta, \theta]$$

Proposition 2.7. *If a metric g_P on P is given as in* (1.3) *and $X, Y \in \mathcal{H}$ then the O'Neill tensor A satisfies $\theta(A_X Y) = -\Omega(X, Y)$*

Proof: From the general formula

$$2d\theta(A, B) = A\theta(B) - B\theta(A) - \theta([A, B])$$

it follows that $2d\theta(X, Y) = -\theta([X, Y]) = -\theta(\mathcal{V}[X, Y])$ for $X, Y \in \mathcal{H}$, which proves our claim. □

This gives rise to the following definition, which does not depend on any choice of metrics:

Definition 2.8. Given a principal connection θ on a G-principal bundle $P \to B$, we say that:

(a) a vector $\mathsf{u} \in \mathfrak{g}$ is called *fat* if $(X, Y)_{\mathfrak{p}} \to Q(\Omega_p(X, Y), \mathsf{u})$ is nondegenerate on $\mathcal{H}_p = \ker\theta$ for all $p \in P$.
(b) θ is called *fat* if all vectors $\mathsf{u} \in \mathfrak{g}$ are fat.

Notice that if u is fat then so is $\mathrm{Ad}(g)\mathsf{u}$ since

$$Q(\Omega(X, Y), \mathrm{Ad}(g)\mathsf{u}) = Q(\Omega(gX, gY), \mathsf{u}).$$

Hence, fat vectors always occur in adjoint orbits.

Theorem 2.9. *If there exists some fat vector* $\mathsf{u} \in \mathfrak{g}$*, then some characteristic number of the principal bundle is not zero.*

Proof: By assumption we know that $Q(\Omega(X,Y), \mathrm{Ad}(g)\mathsf{u})$ is non-degenerate for every $g \in G$. Let

$$p(w) = \int_G Q(w, \mathrm{Ad}(g)\mathsf{u})^n \, d\,\mathrm{vol}_g$$

Then p is a polynomial of degree at most n on $\mathfrak{g}$, which is Ad invariant. Hence by Chern–Weil theory

$$p(\Omega) = \int_G Q(\Omega, \mathrm{Ad}(g)\mathsf{u})^n d\,\mathrm{vol}_g$$

(where powers now represent wedge products), is the pullback of a closed $2n$–form on B, whose DeRham cohomology class represents a characteristic class. Now if $\dim B = 2n$, fatness implies that $\langle \Omega, \mathrm{Ad}(g)\mathsf{u}\rangle^n$ is a volume form and hence its integral is not zero. □

Corollary 2.10. *A fat principal bundle cannot have a flat connection*

We separate the remaining discussion of principle bundles into two cases, $\dim G = 1$ and $\dim G = 3$.

Circle bundles

The case of $G = S^1$ is special since G is abelian with $\mathfrak{g} = \mathbb{R}$. Hence θ and Ω are ordinary forms on P with $d\theta = \Omega$. Since $\Omega(g_*(A), g_*(B)) = \Omega(A,B)$, there exists a 2 form ω on B with $\Omega = \sigma^*(\omega)$. Chern Weil theory then implies that the 2 form ω is closed with cohomology class $[\omega] = 2\pi e(P)$, where $e(P) \in H^2(B, \mathbb{Z})$ is the Euler class of the S^1 bundle. Hence we obtain:

Corollary 2.11. *For* $G = S^1$*,* θ *is fat iff* $d\theta = \sigma^*(\omega)$ *and* ω *is a symplectic 2–form on* B*. Hence* $\omega^n \neq 0$*, where* $\dim B = 2n$*.*

Conversely, given a symplectic manifold (B^{2n}, ω), such that $\frac{1}{2\pi}[\omega]$ is an integral class, one can consider the S^1 bundle P over B with Euler class $e(P) = \frac{1}{2\pi}[\omega]$. Then there exists a connection form θ on P with $d\theta = \sigma^*(\omega)$. Indeed let θ' be any connection form. Then $d\theta' = \sigma^*(\omega')$ with $[\omega] = [\omega']$. Hence $\omega - \omega' = d\eta$ and we can set $\theta = \theta' + \sigma^*(\eta)$. The connection form θ is then a fat connection by definition. It is determined, up to a change $\theta_2 = \theta_1 + d(\sigma \circ f)$, where $f\colon B \to \mathbb{R}$. In fact since $\theta_2 - \theta_1$ is closed, $\theta_2 - \theta_1 = dg$ for some $g\colon P \to \mathbb{R}$ which we can average over the G action on P to obtain $g = \sigma \circ f$.

Hence fat S^1 principal bundles are essentially in one to one correspondence to symplectic manifolds, and thus are plentiful. Notice also that the one form θ on P is by definition a contact form, since $\theta \wedge d\theta^n \neq 0$.

To come back to metrics associated to fat circle bundles, let us first make a digression about Killing vector fields of constant length. This is relevant in our context since for a connection metric (1.3) an action field V^*, for any $V \in \mathfrak{g}$, has constant length. Since S^1 is also abelian, the right action is by isometries and hence V^* Killing as well.

Let g be a metric on M and ξ a Killing vector field. ξ has constant length iff all integral curves of ξ are geodesics since $X(g(\xi,\xi)) = g(\nabla_X \xi, \xi) = -g(\nabla_\xi\, \xi, X)$. Notice also that if

ξ is a non-vanishing Killing vector field, then in the metric $|\xi|^{-2}g$, the vector field ξ is still Killing, and now has unit length. We will use the following general properties of Killing vector fields:

LEMMA 2.12. *If ξ is a Killing vector field and $J(X) = \nabla_X \xi$, then*
(a) $||J(X)||^2 = g(R(X,\xi)\xi, X)$
(b) $g((\nabla_X J)(Y), Z) = g(R(\xi, X)Y, Z)$

Proof. For (a) we observe that $g(\nabla_X \xi, X) = 0$ and hence

$$\begin{aligned} g(R(X,\xi)\xi, X) &= g(\nabla_X \nabla_\xi \, \xi - \nabla_\xi \nabla_X \xi - \nabla_{[X,\xi]} \, \xi, X) \\ &= -g(\nabla_\xi \nabla_X \xi, X) - g(\nabla_{[X,\xi]} \, \xi, X) \\ &= g(\nabla_X \xi, \nabla_\xi X) + g(\nabla_X \xi, [X,\xi]) = g(\nabla_X \xi, \nabla_X \xi) = |J(X)|^2 \end{aligned}$$

Part b) follows from the general fact that any Jacobi field ξ satisfies:

$$\nabla_A(\nabla \xi)(B) + R(\xi, A)B = 0 \text{ for all } A, B \in T_pM$$

where $\nabla_A(\nabla\xi)(B) = \nabla_A \nabla_B \xi - \nabla_{\nabla_A B}\xi$. Indeed, if $L(A,B)$ is the left hand sinde, then $L(A,A) = 0$ since ξ is a Jacobi field. Furthermore,

$$L(A,B)\xi - L(B,A)\xi = R(A,B)\xi + R(\xi,A)B + R(xi,B)A = 0.$$

□

Let us recall a few definitions. A metric on M is called *K-contact* if it admits a unit Killing field ξ with $\sec(\xi, X) = 1$ for all $\xi \wedge X \neq 0$. It is called regular if the integral curves define a free circle action and quasi regular if they define an almost free circle action. By a Theoreom of Wadsley, the latter condition is equivalent to requiring that all integral curves of ξ are closed. If it is regular, $M \to M/S^1 = B$ is a Riemannian submersion with totally geodesic fibers, i.e. a connection metric which is clearly fat. Although the condition $\sec(\xi, X) = 1$ is seemingly much stronger than the fatness condition $\sec(\xi, X) > 0$, we will see that a fat bundle always admits a K-contact structure.

A metric is called *Sasakian* if it admits a unit Killing vectorfield ξ with $\hat{R}(\xi \wedge X) = \xi \wedge X$ for all $\xi \wedge X \neq 0$, where $\hat{R}$ is the curvature operator. This is equivalent to requiring $R(\xi, X)Y = g(\xi, Y)X - g(X,Y)\xi$, for all $X, Y \perp \xi$. In particular $\sec(\xi, X) = 1$, i.e., the metric is K-contact. As before, the Sasakian structure is called regular if the integral curves of ξ define a free circle action and quasi regular if they define an almost free circle action.

We are interested in the case where the K-contact or Sasakian structure is quasi-regular since we can then consider the orbifold fibration $M \to M/S^1 = B$ which is a Riemannian submersion with totally geodesic fibers. For simplicity we will call such structures simply K-contact resp. Sasakian.

Recall also that a Riemannian manifold M^{2n} is called *almost hermitian*, if there exists an orthogonal almost complex structure, i.e., an endomorphism $J\colon T_pM \to T_pM$ for each $p \in M$, with $J^2 = -\operatorname{Id}$ and $g(JX, JY) = g(X,Y)$. It is called *hermitian* if the complex structure is integrable. An almost hermitan structure is called *almost Kähler* if in addition the two form $\omega(X,Y) = g(JX,Y)$ is closed. This implies that $\omega^n \neq 0$, i.e., the manifold is symplectic. It is called Kähler if $\nabla\omega = 0$ or equivalently $\nabla J = 0$. This implies in particular that the almost complex structure is integrable. Finally, the metric is called *Kähler Einstein* if it is Kähler and the Ricci curvature is constant.

We now have the following beautiful relationship between these concepts:

Proposition 2.13. *Let $\sigma\colon (P, g_P) \to (B, g_B)$ be a Riemannian submersion with totally geodesic one dimensional fibers of length 2π.*

(a) *g_P is K-contact iff g_B is almost Kähler.*
(b) *g_P is Sasakian iff g_B is Kähler.*
(c) *g_P is Sasakian Einstein iff g_B is Kähler Einstein.*

Proof: Since S^1 is abelian, it acts by isometries on P. Hence the action field ξ is a Killing field and we normalize it to have length 1. The principle connection is then given by $\theta(X) = g(\xi, X)$ with horizontal distribution $\mathcal{H} = \ker\theta$. As in Lemma 2.12 we let $J(X) = \nabla_X \xi$. Clearly $g(JX, \xi) = X(g(\xi,\xi)) = 0$ and thus $J\colon \mathcal{H} \to \mathcal{H}$. Since ξ is Killing, J is skew symmetric and since g is K-contact, Lemma 2.12 implies that J ia an isometry and thus $J^2 = -\mathrm{Id}$, i.e. J is a complex structure on $\mathcal{H}$. We can now define a complex structure $\tilde J$ on B in terms of J. Let X be tangent to B and $\bar X$ a horizontal lift and set $\tilde J X = J\bar X$. This is independent of the lift $\bar X$, since $g^*(J) = J$, which follows from the fact that $g^*(\xi) = \xi$ and that $g \in \mathrm{S}^1$ acts by isometries. Thus $\tilde J$ is an orthogonal complex structure and hence g_B is almost hermitian. To see that it is almost Kähler, observe that $2d\theta(X,Y) = -\theta([X,Y]) = g(-\nabla_X Y + \nabla_Y X, \xi) = g(Y, \nabla_X\xi) - g(X, \nabla_Y \xi) = 2g(\nabla_X\xi, Y)$ for all $X, Y \in \mathcal{H}$ and thus $d\theta(X,Y) = g(JX,Y)$. Since $\Omega = d\theta = \sigma^*(\omega)$ for some closed 2-form ω on B, it follows that $\omega(X,Y) = g_B(\tilde J X, Y)$. If in addition, g_P is Sasakian, Lemma 2.12 implies that $\nabla_X J = 0$ for $X \in \mathcal{H}$. Since $g_p(\nabla_{\bar X}\bar Y, \bar Z) = g_B(\nabla_X Y, Z)$, this implies that $\nabla_X \tilde J = 0$ as well and thus g_B is Kähler .

Finally, we compare the Ricci tensor of g_P and g_B. If g_P is K-contact, $Ric_P(\xi) = 2n$, where $\dim B = 2n$. To compute $\mathrm{Ric}_P(X)$ for $X \in \mathcal{H}$, observe that the O'Neill tensor A of the submersion σ satisfies $g_P(A_X Y, \xi) = g_P(\nabla_X Y, \xi) = g_p(JX, Y)$ and thus $|A_X Y|^2 = g_P(JX,Y)^2$. For a given unit vector $X \in \mathcal{H}$, choose an orthonormal basis $e_i, i = 1, \dots 2n$ of $\mathcal{H}$ with $e_1 = X$ and $e_2 = JX$. Using O'Neill's formula, we obtain $Ric_P(X) = 1 + \sum_{i=2}^{i=2n} \sec_P(X, e_i) = 1 + \sum_{i=3}^{i=2n} \sec_B(\sigma_*(X), e_i) - 3 = Ric_B(\sigma_*(X)) - 2$. If g_P is Einstein, we have $Ric_P = 2n g_P$, and thus $Ric_B = (2n+2) g_B$.

Conversely, if g_B is almost Kähler with almost complex structure $\tilde J$ and closed two form $\omega(X,Y) = g_B(\tilde J X, Y)$, we choose the connection form θ such that $d\theta = \sigma^*(\omega)$ and define $J\bar X = \tilde J X$. Thus $d\theta(\bar X, \bar Y) = g_P(J\bar X, \bar Y)$. Since, as above, we also have $d\theta(X,Y) = g_P(\nabla_X\xi, Y)$, it follows that$JX = \nabla_X \xi$. Since J and hence $\tilde J$ is an isometry, Lemma 2.12 implies that $\sec(\xi, X) = 1$. Similarly, if g_B is Kähler , g_P is Sasakian and if g_B is Einstein, with Einstein constant normalized to be $2n+2$, g_P is Einstein as well. □

The following result shows that a fat bundle, i.e., metrics with $\sec(\xi, X) > 0$ implies the existence of another metric with $\sec(\xi, X) = 1$.

Proposition 2.14. *For a circle bundle $\sigma\colon P \to B$ with B symplectic, there exist metrics g_P and g_B such that σ is a Riemannian submersion with g_P K-contact and g_B almost Kähler .*

Proof. Let ω be the symplectic form on B. By Proposition 2.13, it is sufficient to find a metric g_B which is almost Kähler , i.e., a metric g_B and an almost complex structure J with $\omega(X,Y) = g_B(JX, Y)$. Let g_B be any metric on B and define an endomorphism L by setting

$\omega(X,Y) = g_B(LX,Y)$. Since ω is symplectic, L is skew symmetric and non-singular. If we set $L = JS = SJ$ with J orthogonal and S symmetric and positive definite, the uniqueness of this decomposition implies that J is skew symmetric as well, and thus $J^2 = -\operatorname{Id}$. We can now define a new metric by setting $\tilde{g}_B(X,Y) = g_B(SX,Y)$ and thus $\omega(X,Y) = \tilde{g}_B(JX,Y)$ □

On the other hand, there are obstructions to the existence of Sasakian metrics. E.g., Sasakian implies that the Betti numbers satisfy b_{2i+1} odd for $i \leq ?$. Similarly, there are obstructions to the existence of Kähler metrics since one has $b_{2i} \leq b_{2i+2}$ for $i \leq ?$.

There are of course many examples of Kähler metrics and Kähler Einstein metrics. Notice also that (2.13) still holds if the structure on P is only quasi-regular, i.e. the action of S^1 is only almost free, as long as one allows an orbifold structure on the quotient. The proof is the same since the computations were local. This gives rise to many further examples, e.g. Sasakian Einstein metrics on every exotic 7-sphere.

SU(2) *and* SO(3) *principle bundles*

The case of G principal bundles with $\dim G = 3$ is also special. Proposition 2.5 implies that $\dim B = 4n$ and we have the following analogue of Corollary 2.11:

PROPOSITION 2.15. *If an $G = \mathrm{SU}(2)$ or $SO(3)$ principle bundle admits a fat principle connection, then there exists a closed 4 form α on B, such that $\frac{1}{8\pi^2}[\alpha]$ represents the first Pontryagin class, and $\alpha^n \neq 0$, where $\dim B = 4n$.*

Proof. If $e_i, i = 1,2,3$ is an orthonormal basis of $\mathbb{R}^3$, then $\Omega = \sum \Omega_i e_i$ where Ω_i are ordinary 3 forms on P, which must be symplectic forms if θ is fat. The expression $\sum \Omega_i \wedge \Omega_i$ is then independent of the choice of basis and is $Ad(G)$ invariant. Hence it is of the form $\sigma^*(\alpha)$ for some 4 form α on B. Chern Weil theory then implies that α is closed with $[\alpha] = 8\pi^2 p_1(P)$ where $p_1(P)$ is the first Pontrjagin class of the bundle. Since Ω_i are symplectic 2 forms, one easily shows that $\alpha^n \neq 0$, where $\dim B = 4n$. □

The only known examples of such fat principle bundles are when (P, g_P) is 3-Sasakian. Recall that a manifold P is called *3-Sasakian* if it admits an almost free isometric action by SU(2) or SO(3) whose orbits are totally geodesic of curvature 1, and such that $\hat{R}(\xi^* \wedge X) = \xi^* \wedge X$ for all $\xi \in \mathfrak{g}$ and any X. This is equivalent to requiring $R(\xi^*, X)Y = g(\xi^*, Y)X - g(X,Y)\xi^*$ and hence all vertizontal curvatures are equal to 1. The projection $P \to P/G = B$ is then by definition a fat G orbifold principal bundle. Furthermore, recall that a manifold (B, g_B) is called *quaternionic Kähler* if there exists a 3 dimensional subbundle E of the bundle $O(TB)$ of isometric linear maps of TB, which is invariant under parallel translation and for each $p \in B$ is spanned by orthogonal almost complex structures J_i whith $J_1 \circ J_2 = -J_2 \circ J_1 = J_3$. We then have:

PROPOSITION 2.16. *If the metric g_P is 3-Sasakian, then the induced metric g_B is quaternionic Kähler with positive scalar curvature.*

Proof: In addition to Lemma 2.12, we also need the following:

Lemma 2.17. *If ξ_i be two Killing vector fields of constant length, $J_{\xi_i}(X) = \nabla_X \xi_i$ and X, Y orthogonal to ξ_i, then*

$$g(J_{\xi_1} \circ J_{\xi_2}(X), Y) = g(J_{[\xi_1, \xi_2]}(X), Y) - g(R(\xi_1, Y)X, \xi_2)$$

Proof. Proof to be added. □

Let ξ_i, $i = 1, 2, 3$, be the Killing vector fields which are action fields corresponding to an orthonormal basis e_i of $\mathfrak{g}$, and such that $[\xi_1, \xi_2] = \xi_3$. As in Lemma 2.12, set $J_i(X) = \nabla_X \xi_i$. If g_P is 3-Sasakian, Lemma 2.12 and Lemma 2.17 imply that J_i are orthogonal almost complex structures on $\mathcal{H} = \ker \theta$ with $J_1 \circ J_2 = -J_2 \circ J_1 = J_3$. These do not descend to almost complex structures on B since $g^*(\xi) = \xi \circ \mathrm{Ad}(g^{-1})$ for $\xi \in \mathfrak{g}$ and $g \in G$. On the other hand, the subspace of orthogonal linear maps on $\mathcal{H}$ spanned by J_i descends to a well defined subspace of such linear maps on B. Since also $\nabla J_i = 0$ on $\mathcal{H}$, this subspace on B is invariant under parallel translation, and thus g_B is quaternionic Kähler . □

The converse is somewhat more complicated than in the case of circle bundles.

Proposition 2.18. *Let (B, g_B) be a quaternionic Kähler orbifold with positive scalar curvature and P the* $\mathrm{SO}(3)$*-principal bundle of the 3 dimensional parallel vector bundle E. Then P admits a 3-Sasakian orbifold metric with respect to the almost free action by* $\mathrm{SO}(3)$.

Proof. Proof to be added..... □

The simplest example is the Hopf fibration $S^3 \to S^{4n+3} \to \mathbb{H}P^n$ where the constant curvature one metric is 3-Sasakian and $\mathbb{H}P^n$ quaternionic Kähler. As is well known, in contrast to (2.13), both a quaternionic Kähler and a 3-Sasakian metric is automatically Einstein.

It is somewhat surprising that in all known examples of S^3 fat principal bundles the vertizontal curvatures are not only positive, but in fact equal to 1. This motivates the question whether there is an analogue to Proposition 2.14:

Problem 1. *Given a fat G-principle bundle with* $\dim G = 3$, *can one change the metric on B such that all vertizontal curvatures are equal to 1?*

It is conjectured that the only quaternionic Kähler manifolds with positive scalar curvature are quaternionic symmetric spaces. They are classified and will be described in the next section.

It is also important to notice that the same proof works if one only requires that the G action is almost free. The quotient is then a quaternionic Kähler orbifold. This seemingly small change allows many more examples, e.g. a 3-Sasakian structure on many of the positively curved Eschenburg spaces.

Surprisingly, no other fat principal bundles are known, which suggests the following problem:

Problem 2. *Are there any fat G-principle bundles with* $\dim G > 3$*? The answer would be no, if one could for example show that not only the vertizontal sectional curvatures are positive, but the vertical one's also.*

2.2. **Associated bundles.** Let $\sigma\colon P \to M$ be a G principal bundle and $\pi\colon M = P\times_G F \to B$ the associated fiber bundle with fiber F. Lemma (2.6) says that if π is fat, we can assume that $F = G/H$ and that G is the holonomy group of π.

PROPOSITION 2.19. *The Riemannian submersion $\pi\colon P\times_G F \to B$ with totally geodesic fibers G/H is fat iff $Q(\Omega(X,Y),u)$ is non-degenerate in X,Y for all u with $Q(u,\mathfrak{h}) = 0$.*

Proof: This see this we claim that

$$A_XY = -\frac{1}{2}\,\mathrm{pr}_{\mathfrak{h}^\perp}(\Omega(X,Y))$$

where we identify a fiber with G/H and its tangent space with $\mathfrak{h}^\perp \subset \mathfrak{g}$. Recall that by (2.7) we have $A_XY = \frac{1}{2}\mathcal{V}[X,Y] = -\frac{1}{2}\Omega(X,Y)$ in the principal bundle $P \to B$, where we identify the tangent space of the fiber with $\mathfrak{g}$ via θ. In the fiber bundle π we also have $A_XY = \frac{1}{2}\mathcal{V}[X,Y]$ where now the vertical space is identified with $\mathfrak{h}^\perp \subset \mathfrak{g}$. This proves the claim. □

We will therefore make the following definition for a principal connection:

Definition 2.20. A connection θ in a G-principal bundle $P \to B$ is called H-fat for some $H \subset G$ if all vectors $u \in \mathfrak{h}^\perp$ are fat, i.e. $Q(\Omega(X,Y),u)$ is non-degenerate for all $u \in \mathfrak{h}^\perp$.

Although $M = P\times_G G/H = P/H$ as manifolds, this equality means that we can consider two metrics on M. We can start with a metric $g_{G/H}$ on the fiber G/H where we need to know that the left translations by G are isometries. Such metrics can be viewed as a submersed metric $g_{G/H}$ under $G \to G/H$ for a metric g_G on G which is left invariant under G and right invariant under H. If $\mathcal{H}$ is a horizontal distribution induced by a principal connection on $P \to B$, then (1.5) describes a metric g_M on M where the fibers are isometric to $(G/H, g_{G/H})$. This metric can also be described as follows: Choose a metric on P as in (1.4) where the metric on the fiber is g_G. Then the right action of G on P is not by isometries, but the right action of H is, and the submersed metric on P/H under $P \to P/H$ is the same as g_M. A second metric can be defined by choosing a biinvariant metric Q on G and use it to define a metric g_P on P as in (1.4). This metric is now right invariant under G and defines a submersed metric $\tilde{g}_M$ under $(P\times G, g_P + g_{G/H}) \to P\times_G G/H = M$. The relationship between g_M and $\tilde{g}_M$ was described in (1.6). In particular the horizontal distribution is the same in both cases, as is g_B, but the metric on the fibers is $g_{G/H}$ for g_M and $\tilde{g}_M(X^*,Y^*) = Q(L(I+L)^{-1}X,Y)$ for $X,Y \in \mathfrak{h}^\perp$, where $g_G(X,Y) = Q(LX,Y)$. One often has $L = \lambda I$ which implies that $\tilde{g}_M = \frac{\lambda}{\lambda+1} g_M$ on the fibers. Since $\frac{\lambda}{\lambda+1} < 1$ it shortens the metric on the fibers, but as $\lambda \to \infty$ the metric $\tilde{g}_M$ converges to g_M.

We will discuss in some detail the case of a sphere bundle, which we will come back to in section ?.

Consider a sphere bundle $S^k \to M \to B$ with structure group $O(k+1)$, principal bundle $O(k+1) \to P \to B$, and associated vector bundle $\mathbb{R}^{k+1} \to E = P\times_{O(k+1)} \mathbb{R}^{k+1} \to B$. A principal connection θ in P is in one–to–one correspondence with a metric connection ∇ in E by declaring a horizontal lift of a curve γ in B to be a parallel frame along γ. This defines a horizontal distribution on $S^k \to M \to B$ which is fat iff θ is $O(k)$-fat for $O(k) \subset O(k+1)$.

PROPOSITION 2.21. *A sphere bundle $S^k \to M \to B$ is fat iff the curvature R of the associated metric connection has the property that $< R(X,Y)v, w >$ is non degenerate in X, Y for all $v \wedge w \neq 0$.*

Proof: The curvature Ω of θ has values in $\mathfrak{so}(k+1)$. For each $p \in P$, which is an orthonormal frame in $E_{\pi(p)}$, we can identify $\Lambda^2 E_{\pi(p)}$ with $\Lambda^2 \mathbb{R}^{k+1}$ and hence with $\mathfrak{so}(k+1)$. Hence $\Omega_p(X,Y)$ lies in $\Lambda^2 E_{\pi(p)}$. We claim that under this identification $\Omega_p(X,Y) = -\frac{1}{2}R(X,Y)$. We already know that $\Omega(X,Y) = -\frac{1}{2}A_X Y$ and in the proof of Lemma (1.2) we saw that $A_X Y = [X,Y]_p = \frac{d}{dt}_{t=0}\phi_t^*\psi_t^*\phi_{-t}^*\psi_{-t}^*$, where ϕ_t, ψ_t are the flows of X, Y, which are horizontal lifts of coordinate vector fields in B. Furthermore, ϕ_t, ψ_t are diffeomorphisms τ_γ for some γ in B, which in our case consist of parallel translation along γ. Hence $\Omega_p(X,Y) = -\frac{1}{2}R(X,Y)$ follows from the well known interpretation of $R(X,Y)$ in terms of parallel translation.

Now consider $E_{12} \in \mathfrak{so}(k)^\perp \subset \mathfrak{so}(k+1)$, where $E_{12} = e_1 \wedge e_2$ is decomposable. Then the adjoint orbit of E_{12} is the set of all decomposable vectors in $\mathbb{R}^{k+1}$. Since all vectors in $\mathfrak{so}(k)^\perp$ are linear combinations of $e_1 \wedge e_i, i = 2 \cdots k+1$, they are all decomposable. Furthermore, we know that the adjoint orbit of a fat vector is also fat, and hence all decomposable vectors are fat. Thus θ is $O(k)$–fat if and only if $(X,Y) \to < \Omega(X,Y), v \wedge w > = < R(X,Y)v, w >$ is non degenerate for all $v \wedge w \neq 0$. ☐

In the case of $k = 2$ we have the following special property:

PROPOSITION 2.22. *A sphere bundle $S^2 \to M \to B$ is fat iff the corresponding principal bundle $SO(3) \to P \to B$ is fat.*

Proof: This follows from the fact that the adjoint orbit of any vector $u \in so(2)^\perp \subset \mathfrak{so}(3)$ is the whole sphere of radius $|u|$ since the Adjoint action of $SO(3)$ is its tautological representation on $\mathbb{R}^3$. ☐

3. EXAMPLES OF FAT BUNDLES

As we explained in the last section, fat circle bundles are defined in terms of symplectic manifolds and are hence plentiful. Surprisingly all other known fat bundles are homogeneous bundles. This means one has inclusions $H \subset K \subset G$, which defines the homogeneous fibration $K/H \to G/H \to G/K$ which can be viewed as the fibration associated to the principal bundle $K \to G \to G/K$ with fiber K/H, since $G \times_K K/H = G/H$.

An Ad(K) invariant splitting $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ induces a principal connection θ on $G \to G/K$ with G-invariant horizontal space given by $\mathcal{H}_e = \mathfrak{m}$, and an induced horizontal homogeneous distribution $\tilde{\mathcal{H}}$ for the fibration $G/H \to G/K$ with $\tilde{\mathcal{H}}_{(H)} = \mathfrak{m}$. An Ad(H)–invariant splitting $\mathfrak{K} = \mathfrak{h} \oplus \mathfrak{p}$ with a choice of an Ad(H) invariant metric on $\mathfrak{p}$ and an Ad(K) invariant metric on $\mathfrak{m}$ induce homogeneous metrics on K/H and G/K via the identifications $T_{(H)}K/H \simeq \mathfrak{p}$ and $T_{(K)}G/K \simeq \mathfrak{m}$. We can now define a metric on G/H, such that on $T_{[H]}G/H \simeq \mathfrak{p} \oplus \mathfrak{m}$, $\mathfrak{p}$ and $\mathfrak{m}$ are orthogonal and the metric on $\mathfrak{p}$ and $\mathfrak{m}$ are as given before.

PROPOSITION 3.1. *The fibration $K/H \to G/H \to G/K$ with the homogeneous metrics described above, is a Riemannian submersion with totally geodesic fibers and with $A_X U = [X,U]$ for $X \in \mathfrak{m}$ horizontal, and $U \in \mathfrak{p}$ vertical. Hence the fibration is fat iff $[X,U] \neq 0$ for all $0 \neq X \in \mathfrak{m}$, $0 \neq U \in \mathfrak{p}$.*

Proof: The map $K/H \to G/H \to G/K$ is clearly a Riemannian submersion. By construction, the holonomy group of $\mathcal{H}$ and $\tilde{\mathcal{H}}$ are the left translations by K on G and on K/H respectively. Hence the fibers K/H are totally geodesic. Since $A_XY = \frac{1}{2}\mathcal{V}[X,Y] = \frac{1}{2}[X,Y]_{\mathfrak{p}}$ we have $Q(A_XU,Y) = -Q(A_XY,U) = -Q([X,Y],U) = Q([X,U],Y)$ and since $[X,U] \in \mathfrak{m}$ by $Ad(K)$ invariance, it follows that $A_XU = [X,U]$. □

Berard–Bergery classified all such homogeneous H–fat connections, and first proves the following general theorem:

Proposition 3.2. *If $K/H \to G/H \to G/K$ is fat bundle with* $\dim(K/H) > 1$, *then G/K is a symmetric space,* $\mathrm{rk}\ K = \mathrm{rk}\ G$, *and a normal metric on K/H has* $\sec > 0$.

Proof: To be added. □

The case where $\dim(K/H) = 1$ are the well known Boothby–Wang fibrations:

Proposition 3.3. *Let $S^1 \subset G$, $Z(S^1)$ the centralizer of S^1 in G, and $K = \exp(\mathfrak{k})$ with $\mathfrak{k}$ the orthogonal complement of of the Lie algebra of S^1 inside the Lie algebra of $Z(S^1)$. Then $Z(S^1) = S^1 \cdot K$ and $S^1 \to G/K \to G/Z(S^1)$ is a principal circle fibration which is fat. Furthermore, $G/Z(S^1)$ is a symplectic manifold (in fact a Kähler manifold) and G/K a contact manifold.*

Proof: The claim that these bundles are always fat is due to the fact that in the isotropy representation of $G/Z(S^1)$ on $\mathfrak{m}$, the circle S^1 has no fixed vector, since otherwise the centralizer of S^1 would be larger. □

The base $G/Z(S^1)$ can also be described as the adjoint orbits of G on $\mathfrak{g}$ with their canonical Kähler-Einstein metric. This includes for example, many circle bundles over G/T with T a maximal torus (although not all of them work!) as one extreme case, and the circle bundle over the hermitian symmetric spaces $G/K \cdot S^1$, $S^1 \to G/K \to G/K \cdot S^1$ as the other extreme (corresponding to the case where $Z(S^1)$ is a maximal subgroup of G).

A homogeneous fat principal bundle corresponds to the case where H is normal in K, and if $K/H \neq S^1$, the only possibilities are $K/H = S^3$ or $SO(3)$, since K/H must have $\sec > 0$.
The later case can be described uniformly in terms of homogeneous quaternionic symmetric space $G/(K \cdot Sp(1))$ (so called Wolf spaces). correspond precisely to those symmetric spaces G/H with H locally of the form $K \cdot Sp(1)$ and such that the restriction of the isotropy representation of $K \cdot Sp(1)$ on $G/K \cdot Sp(1)$ to the subgroup $Sp(1)$ is equivalent to the usual Hopf action of $Sp(1)$ on $\mathbb{R}^{4n} = \mathrm{H}^n$.

Proposition 3.4. *If $G/K \cdot Sp(1)$ is a quaternionic symmetric space, then the principal bundle fibration $K \cdot Sp(1)/K \to G/K \to G/K \cdot Sp(1)$ is fat. Furthermore, $K \cdot Sp(1)/K = S^3$ or $SO(3)$ and $Sp(1)/K$ is 3-Sasakian.*

Proof: This principal bundle is fat (and they all arise in this fashion) since the action of $Sp(1)$ on $\mathfrak{m} = T_{[K\cdot Sp(1)]}G/(K \cdot Sp(1)) \simeq \mathrm{H}^n$ is given by the Hopf action, which is free on $\mathfrak{m} - \{0\}$. □

One gets the following examples, and in most cases, the condition $[X,U] \neq 0$ is trivial to verify:

(E.1) The simplest examples are, of course, the Hopf fibrations:

a) $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ for $SU(n) \subset U(n) \subset SU(n+1)$,
b) $S^3 \to S^{4n+3} \to \mathbb{HP}^n$ for $Sp(n) \subset Sp(n)Sp(1) \subset Sp(n+1)$,
c) $S^2 \to \mathbb{CP}^{2n+1} \to \mathbb{HP}^n$ for $Sp(n)U(1) \subset Sp(n)Sp(1) \subset Sp(n+1)$,
d) $S^7 \to S^{15} \to S^8$ for $Spin(7) \subset Spin(8) \subset Spin(9)$,

which must be fat because the fibers are totally geodesic and the total space has positive curvature.

(E.2) We already mentined the fact circle bundles with their Boothby-Wang fibrations.

(E.3) The only other fat S^3 or $SO(3)$ principal bundles correspond to quaternioc symmetric spaces. There is precisely one such space for every simple Lie group. In the case where G is a classical simple Lie group, one gets the following examples:

(a) $SO(3) \to T_1\mathbb{CP}^n \to Gr_2(\mathbb{C}^{n+1})$, $n \geq 2$ coming from the groups $S(U(n-1) \times Z(U(2))) \subset S(U(n-1) \times U(2)) \subset SU(n+1)$. The fiber is $SO(3)$ since $U(2)/Z(U(2)) = SO(3)$ and the total space $SU(n+1)/S(U(n-1) \times Z(U(2))) = T_1\mathbb{CP}^n$ since $\mathbb{CP}^n = SU(n+1)/S(U(n)U(1))$ with $U(n) \simeq S(U(n)U(1))$, and embedding

$$A \in U(n) \to \begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} \subset SU(n+1)$$

and isotropy representation

$$v = \begin{pmatrix} 0 & v \\ -\bar{v}^t & 0 \end{pmatrix} \to (\det A) \cdot A(v)$$

and hence the isotropy group of $v = (0, \ldots, 0, 1) \in T_1\mathbb{CP}^n$ is equal to $\begin{pmatrix} B & & \\ & z & \\ & & z \end{pmatrix}$ with $(\det B)z^2 = 1$, i.e, $S(U(n-1)Z(U(2)))$

(b) $SO(3) \to SO(n)/S^3 \times SO(n-4) \to \Lambda^o_{4,n}$, $n \geq 5$, (the oriented 4 planes in $\mathbb{R}^n$), coming from the groups $S^3 \times SO(n-4) \subset SO(4) \times SO(n-4) \subset SO(n)$, where $S^3 \subset SO(4)$ is one of the normal subgroups, and $SO(4)/S^3 = SO(3)$.

(c) $S^3 \to Sp(n+1)/Sp(n) \to \mathbb{HP}^n$, i.e, E.1(b).

For the exceptional simple Lie groups (there exists exactly one quaternionic symmetric space $G/(K \cdot Sp(1))$ for each simple Lie group G!), the following examples are particularly interesting:

$$SO(3) \to G_2/S^3_- \to G_2/SO(4) \qquad SO(3) \to G_2/S^3_+ \to G_2/SO(4)$$

where $S^3_\pm \subset SO(4)$ are the two simple factors. Also notice that $G_2/S^3_- = V_2\mathbb{R}^7$, the 2− frames in $\mathbb{R}^7$.

(E.4) The fat homogeneous S^2 bundles, i.e. $K/H = S^2$, all arise as the associated S^2 bundles to the principal $SO(3)$ or S^3 bundles in (E.2). This follows from one of our earlier observations that an S^2 bundle is fat iff the corresponding $SO(3)$ principal bundle is fat.

(E.5) The fat bundles, where K/H is a higher dimensional symmetric space of sec > 0 are all of the following type:

(a) $\mathbb{RP}^5 \to SU(n+1)/S(U(n-3)Sp(2)) \to G_4(\mathbb{C}^{n+1})$, $n \geq 4$, coming from the inclusions $S(U(n-3)Sp(2)) \subset S(U(n-3)U(4)) \subset SU(n+1)$ with fiber $U(4)/(Sp(2) \cdot Z(U(4))) = SU(4)/(Sp(2) \bigcup iSp(2)) = SO(6)/O(5) = \mathbb{RP}^5$ The case $n = 4$ corresponds to the Berger space $B^{13} \to \mathbb{CP}^4$ (see later).

(b) $\mathbb{RP}^7 \to SO(n)/SO(n-8)Spin(7) \to G_8^0(\mathbb{R}^n)$, $n \geq 9$, coming from the inclusions $SO(n-8)Spin(7) \subset SO(n-8)SO(8) \subset SO(n)$, with fiber $SO(8)/Spin(7) = \mathbb{RP}^7$. The case $n = 9$ is the $\mathbb{Z}_2$ quotient of the Hopf fibration (E.1(d))

(c) $S^4 \to Sp(n)/Sp(n-2)Sp(1)Sp(1) \to G_2(\mathrm{H}^n)$, $n \geq 3$, coming from the inclusions $Sp(n-2) \times Sp(1) \times Sp(1) \subset Sp(n-2)Sp(2) \subset Sp(n)$. The case $n = 3$ is the flag manifold $F^{12} = Sp(3)/Sp(1)^3$ (see later).

(d) $S^8 \to F_4/Spin(9) = F^{24} \to \mathrm{Ca}\mathbb{P}^2$, coming from the inclusion $Spin(8) \subset Spin(9) \subset F_4$

(e) If we start with two principal fibrations $L_i \to G_i/K_i \to G_i/K_i \cdot Sp(1)$, $i = 1, 2$ as in E.3, and $L_i = S^3$ or $SO(3)$, one gets the associated fat fibration

$$L = L \times L/\triangle L \to (G_1/K_1) \times (G_2/K_2)/\triangle L \to (G_1/K_1 \cdot Sp(1)) \times (G_2/K_2 \cdot Sp(1))$$

(E.6) A fat lens space fibration:

$$S^3/\mathbb{Z}_{p,q} \to U(2)/S^1_{p,q} \to SU(n+1)/S(U(n-1) \cdot S^1_{p,q}) \to G_2(\mathbb{C}^{n+1}), \quad n > 2$$

with $S^1_{p,q} = \mathrm{diag}(z^p, z^q)$ coming from the inclusion

$$S(U(n-1) \cdot S^1_{p,q}) \subset S(U(n-1)U(2)) \subset SU(n+1)$$

with fiber $U(2)/S^1_{p,q} = SU(2)/\mathrm{diag}(z^p, z^q) = S^3/\mathbb{Z}_{p+q}$, $z^{p+q} = 1$, since $SU(2) \subset U(2)$ clearly acts transitively. This bundle is fat iff $p \cdot q > 0$ (see ?? for a discussion of the case $n = 2$ where the total space is the Allof–Wallach space $W_{p,q}$).

(E.7) A fat lens space fibration

$$S^3/\mathbb{Z}_q \to S^3 \times S^1/S^1_{p,q} \to G \times S^1/(K \cdot S^1_{p,q}) \to G/(K \cdot Sp(1))$$

where $G/(K \cdot Sp(1))$ is a quaternionic symmetric space as in E.3, and $K \cdot S^1_{p,q} \subset K \cdot Sp(1) \cdot S^1 \subset G \times S^1$ with $S^1_{p,q} \subset Sp(1) \times S^1$ of the form $\mathrm{diag}(z^p, z^q)$. This bundle is fat iff $p \neq 0$ and $q \neq 0$.

Notice that as a manifold, we can regard this as a $\mathbb{Z}_p$ quotient of the examples in E.3, since $G \times S^1/K \cdot S^1_{p,q} = (G/K)/\mathbb{Z}_q$, where $\mathbb{Z}_q \subset L = S^3$ or $SO(3)$ of the form $z^p \subset L$, with $z^q = 1$ (and L acts freely on G/K).

This finishes the complete description of all possible homogeneous fat bundles. Surprisingly, so far one does not know any examples of fat bundles which are not homogeneous!

Of particular importance are of course the homogeneous fat bundles where the total space also admits a connection metric with sec > 0. The homogeneous spaces with sec > 0 were classified by Berger–Wallach–Berard Bergery, and surprisingly they all fit into this framework, except for the Berger example $B^7 = SO(5)/SO(3)$, with $SO(3)$ maximal in $SO(5)$. One has the following beautiful theorem:

THEOREM 3.5 (Wallach). *Let $H \subset K \subset G$ and let Q be a biinvariant metric on $\mathfrak{g}$. Assume that the following three conditions are satisfied:*

a) *G/K is a rank 1 symmetric space, i.e. $[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{k}$, where $\mathfrak{m} = \mathfrak{k}^\perp$*

b) *The normal metric induced by* Q *on* K/H *has* $\sec > 0$, *i.e. for* $\mathfrak{p} = \mathfrak{h}^\perp \bigcap \mathfrak{k}$, $[X,Y] \neq 0$ *for* $X, Y \in \mathfrak{p}$, $X \wedge Y \neq 0$.

c) *Fatness, i.e.* $[X,U] = 0$ *for* $X \in \mathfrak{m}$, $U \in \mathfrak{p}$ *not 0.*

Then the homogeneous metric on G/H *given by* $Q_t = tQ|_{\mathfrak{p}} + Q|_{\mathfrak{m}}$ *has* $\sec > 0$ *for any* $0 < t < 1$.

Wallach's original proof was complicated, but Eschenburg [**?**] gave a beautiful simple proof which requires almost no computations: one starts by regarding the metric Q_t as the base space of the Riemannian submersion $G \times K \to G$, $(g,k) \to gk^{-1}$ where the metric on $G \times K$ is the biinvariant metric $Q|_{\mathfrak{g}\times 0} + \frac{t}{1-t} Q|_{0\times\mathfrak{k}}$. This immediately implies that the metric Q_t on G (and hence also the induced metric on G/H) has $\sec \geq 0$, and one easily shows using O'Neill's formula, that the only 2–planes $(X,Y) \in \mathfrak{g}$ with zero curvature in Q_t are the ones where $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ and $[X_{\mathfrak{m}} + tX_{\mathfrak{k}}, Y_{\mathfrak{m}} + tY_{\mathfrak{k}}] = 0$. Since $[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{k}$ by assumption (a), and $[\mathfrak{k},\mathfrak{m}] \subset \mathfrak{m}$, this reduces to $[X_{\mathfrak{k}}, Y_{\mathfrak{k}}] = 0$ (we have reduced the set of 2 planes with zero curvature in Q, which only consists of those with $[X,Y] = 0$). Now if $(X,Y) \in \mathfrak{p} \oplus \mathfrak{m} \simeq T_{[H]}G/H$ is a 2-plane with zero curvature, we get $[X_{\mathfrak{p}}, Y_{\mathfrak{p}}] = 0$ and $[X_{\mathfrak{m}}, Y_{\mathfrak{m}}] = 0$, and (a) and (b) together imply that $X_{\mathfrak{p}}, Y_{\mathfrak{p}}$ and $X_{\mathfrak{m}}, Y_{\mathfrak{m}}$ must be linearly dependent. Hence we can assume that $X \in \mathfrak{m}$ and $Y \in \mathfrak{p}$, and (c) guarantees that (X,Y) has $\sec > 0$.

Remark 3.6. If one assumes in addition that K/H is a symmetric space, one also gets that Q_t with $1 < t < \frac{4}{3}$ has $\sec > 0$, but this is less obvious.

¿From Berard-Bergery's classification of fat bundles, one now easily gets the following list of examples:

P.1: $T^2 \subset U(2) \subset SU(3)$ gives rise to the S^2 bundle

$$S^2 = U(2)/T^2 \to F^6 = SU(3)/T^2 \to \mathbb{CP}^2$$

P.2: $Sp(1)^3 \subset Sp(2)Sp(1) \subset Sp(3)$ gives rise to the S^4 bundle

$$S^4 = Sp(2)Sp(1)/Sp(1)^3 \to F^{12} = Sp(3)/Sp(1)^3 \to \mathbb{HP}^2$$

P.3: $Spin(8) \subset Spin(9) \subset F_4$ gives rise to the S^8 bundle

$$S^8 \to F^{24} = F_4/Spin(8) \to \mathrm{Ca}\mathbb{P}^2$$

We can regard these as flag manifolds over $\mathbb{C}$, H and $\mathbb{C}$Ә.

P.4: $Sp(2) \cdot U(1) \subset S(U(4) \cdot U(1)) \subset SU(5)$ gives rise to the fat $\mathbb{RP}^5$ bundle (see ??)

$$\mathbb{RP}^5 = S(U(4)U(1))/Sp(2)U(1) \to B^{13} = SU(5)/Sp(2)U(1) \to SU(5)/S(U(4)U(1)) = \mathbb{CP}^4$$

P1-P4 are precisely the examples where K/H is also a rank one symmetric space.

P.5: $S^1_{p,q} \subset U(2) \subset SU(3)$, where $S^1_{p,q} = \mathrm{diag}(z^p, z^q, \bar{z}^{p+q})$ with $|z| = 1$ and $(p,q) = 1$, gives rise to the fat bundle (see ??)

$$U(2)/S^1_{p,q} \to W^7_{p,q} = SU(3)/S^1_{p,q} \to SU(3)/U(2) = \mathbb{CP}^2$$

with fiber a lens space $U(2)/S^1_{p,q} = SU(2)/\,\mathrm{diag}(z^p, z^q) = S^3/\mathbb{Z}_{p,q}$, with $z^{p+q} = 1$ if $p + q \neq 0$, and fiber $U(2)/S^1_{1,-1} = S^2 \times S^1/(x,y) \sim (-x,-y)$ if $p + q = 0$. But in this case, the connection is fat iff $pq > 0$; in fact, $\mathfrak{p} = (S^1)^\perp$ are the

$$X = \begin{pmatrix} i(2q+p)x_1 & x_2 & 0 \\ -\overline{x_2} & -i(q+2p)x_1 & 0 \\ 0 & 0 & i(q-p)x_1 \end{pmatrix} \qquad \text{where } x_1 \in \mathbb{R}, x_2 \in \mathbb{C}$$

and $\mathfrak{m}$ are the

$$Y = \begin{pmatrix} 0 & y \\ -\overline{y}^\perp & 0 \end{pmatrix}$$

and finally

$$[X,Y] = \begin{pmatrix} i(2q+p)x_1 & -\overline{x_2} \\ -\overline{x_2} & -i(q+2p)x_1 \end{pmatrix} \cdot Y - i(p-q)x_1 \cdot Y = 0$$

iff $i(p-q)y_1$ is an eigenvalue of that matrix A, and since $\operatorname{tr}(A) = i(q-p)y_1$, the other eigenvalue would be $2i(q-p)y_1$ and hence $\det A = (2q+p)(q+2p)y_1^2 + |y_2|^2$ would be $2(p-q)^2y_1^2$. The difference is $pqy_1^2 + |y_2|^2$, which is positive if $p, q > 0$. This shows that this bundle is fat iff $p \cdot q > 0$. But by choosing a different embedding of $U(2)$ in $SU(3)$ we can achieve $p \cdot q > 0$ iff $p \cdot q(p+q) \neq 0$.

Notice that we can also fiber $W^7_{p,q} \to \mathbb{CP}^2$ with fiber $SU(2)/\mathbb{Z}_p$ or $SU(2)/\mathbb{Z}_q$ by choosing these different embeddings of $U(2)$, but those bundles will not be fat! We can normalize $S^1_{p,q}$ by interchanging coordinates, and changing z to $\overline{z}$ so that $p \geq q \geq 0$ to get a unique representative parametrized by $0 \leq \frac{q}{p} \leq 1$ with the extreme cases $W_{1,0}$ and $W_{1,1}$. $W_{1,0}$ fibers over $\mathbb{CP}^2$ with fiber S^3, but as such is not a fat bundle.

In fact, it has no homogeneous metric of positive curvature since the fixed point set of $\operatorname{diag}(-1,1,-1) \in S^1_{1,0}$ is equal to $S^2 \times S^1/\mathbb{Z}^2$ which is totally geodesic and admits no metric of positive curvature. $S^2 \times S^1/\mathbb{Z}^2$ is also the fiber of one of the fibrations $W_{1,0} \to \mathbb{CP}^2$. In this normalization $W_{p,q}$, $p \geq q > 0$, becomes a fat bundle over $\mathbb{CP}^2$ with fiber $S^3/\mathbb{Z}_{p+q}$.

$W_{1,1}$ is special in that it is a principal $SO(3)$ bundle over $\mathbb{CP}^2$, as was first observed by Chaves and rediscovered later by Shankar. In fact, all bundles over $\mathbb{CP}^2$ can be viewed as bundles associated to the principal bundle $U(2) \to SU(3) \to SU(3)/U(2) = \mathbb{CP}^2$ with fiber $U(2)/\operatorname{diag}(z^p, z^q)$:

$$SU(3)/S^1_{p,q} = SU(3) \times_{U(2)} U(2)/\operatorname{diag}(z^p, z^q)$$

and

$$W_{1,1} = SU(3) \times_{U(2)} U(2)/Z(U(2)) = SU(3) \times_{U(2)} SO(3) = SU(3)/Z(U(2)) \times_{SO(3)} SO(3)$$

becomes a principal $SO(3)$ bundle over $\mathbb{CP}^2$.

Another way of looking at this fact is that in general, $N(H)/H$ acts freely on G/H on the right, and $N(S^1_{p,q})/S_{p,q} = S^1$ if $(p,q) \neq (1,1)$, but $N(Sq_{1,1})/S^1_{1,1} = U(2)/S^1_{1,1} = U(2)/Z(U(2)) = SO(3)$. Since $\mathbb{Z}_2 \times \mathbb{Z}_2 \subset SO(3)$, one gets the counterexamples to the Chern conjecture due to Shankar. We will also use the fact that $H^4(W_{p,q}, \mathbb{Z}) = \mathbb{Z}_{p^2+q^2+pq}$ and hence get infinitely many distinct bundles over $\mathbb{CP}^2$.

The $SO(3)$ principal bundle $SO(3) \to W_{1,1} \to \mathbb{CP}^2$ has $w_2 \neq 0$ (since otherwise, $W_{1,1}$ would have a 2–fold spin cover, although the $W_{p,q}$ are simply connected) and $p_1[P] = -3$ since in general, for $SO(3) \to P \to B^4$ one can show that $H^4(P, \mathbb{Z}) = \mathbb{Z}_{(p_1(P)[B])}$ and $p_1(P)[B] = 1 \bmod 4$ in our case. The Allof–Wallach metric on $W_{1,1}$ is clearly a connection metric for the principal bundle, and hence it is a fat bundle. Also notice that the metric on the fiber is a biinvariant metric, since $Q_t|_{U(2)}$ is biinvariant and $S^1_{1,1} = Z(U(2))$. As observed before, $W^1_{1,1}$ is also $T_1\mathbb{CP}^2$ where the S^3 fibration is given by the other embedding of $U(2)$ and is not fat.

In general, the metric on the fiber $U(2)/\operatorname{diag}(z^p, z^q)$ induced by the Allof-Wallach metric Q_t, $Q(A,B) = -\frac{1}{2}\operatorname{tr} AB$ on $\mathfrak{su}(3)$ is given by the normal homogeneous metric induced by $Q_t|_{U(2)}(A,B) = -\frac{t}{2}\operatorname{tr} AB - \frac{t}{2}(\operatorname{tr} A)(\operatorname{tr} B)$, for $A, B \in \mathfrak{u}(2) \subset \mathfrak{su}(3)$, since the embedding is of the form

$$\begin{pmatrix} A & 0 \\ 0 & -\operatorname{tr} A \end{pmatrix} \in \mathfrak{su}(3)$$

The induced metric on $U(2)/\operatorname{diag}(z^p, z^q) = S^3/\mathbb{Z}_{p+q}$ can be described as a Berger type metric in general. A computation shows that it is given by

$$\frac{p^2+q^2-pq}{(p^2+q^2)^2} \cdot t \cdot g_0|_{\mathcal{V}} + t g_0|_{\mathcal{H}}$$

where $S^1 \to S^3 \to S^2$ is the Hopf fibration with $g_0 = S^3(1)$. In particular, the metric on $S^3/\mathbb{Z}_{p+q}$ is the round sphere metric iff $p+1$, $q=0$, i.e. for the fibration $S^3 \to W_{1,0} \to \mathbb{CP}^2$ which is not fat. In the case of the $SO(3)$ bundle $W_{1,1}$ the metric $\frac{1}{4}g_0|_{\mathcal{V}} + g_0|_{\mathcal{H}}$ is the biinvariant metric on $SO(3)$. But in all other cases, the metric is a Berger type metric.

Let us present the argument in the case of the S^3 bundles, i.e. if $p+q=1$. In that case, as explained later, the action of $U(2)$ on $S^3 = U(2)/\operatorname{diag}(z^p, z^q)$ extends to the linear action of $U(2)$ on $\mathbb{C}^2 : v \to (\det A)^{-p} A(v)$. If we start with the biinvariant metric $Q(AB) = -\frac{1}{2}\operatorname{tr} AB$ on $\mathfrak{su}(3)$, then it induces the following metric on $U(2) \subset SU(3)$:

$$Q(A,B) = -\frac{1}{2}\operatorname{tr} AB - \frac{1}{2}(\operatorname{tr} A)(\operatorname{tr} B) \quad \text{for } A, B \in \mathfrak{u}(2)$$

The orthogonal complement of $\mathfrak{h}$, where $H = S^1_{p,q}$, with respect to this metric is spanned by $A = \operatorname{diag}(-iq, ip))$ and $W = \begin{pmatrix} 0 & w \\ -\overline{w} & 0 \end{pmatrix}$, where $w \in \mathbb{C}$. We have $Q(W,W) = |w|$, $Q(W,A) = 0$, $Q(A,A) = p^2+q^2-pq$. The corresponding action fields on S^3 have value at $1 \in S^3$ equal to $W^*(1) = (0,w) \in T_1 S^3 = \{(ia,w) : a \in \mathbb{R}, w \in \mathbb{C}\}$ and $A^*(1) = (-i(p^2-pq+q), 0)$. Hence for the induced action on S^3, the W direction has the usual length, but $(i,0)$ has length squared equal to $\frac{p^2+q^2-pq}{(p^2-pq+q)^2} = \frac{3p(p-1)+1}{(2p(p-1)+1)^2}$ since $p+q=1$. Notice that this length is 1 iff $p=0,1$ (the fiber is the round sphere), and is <1 for $p>1$ (a Berger type metric).

4. Fat Bundles over 4–manifolds

Fat bundles over 2–manifolds are not very interesting, since they consist only of circle bundles over B^2 and a circle bundle is fat iff the curvature is a symplectic 2–form on the base, i.e. a volume form in this case. Hence every nontrivial circle bundle over B^2 is fat.

But flat bundles over 4–manifolds B^4 are already quite interesting. For simplicity, we assume that B is simply connected. Due to the dimension restriction in the first section, we only need to look at principal S^1 and S^3 or $SO(3)$ bundles over B^4, or at principal $SO(4)$ bundles which are $SO(3)$ fat, or at principal $U(2)$ bundles which are $U(1)$ fat. We first review the classification of principal $SO(4)$ bundles over B^4 which will be essential for later on. The basic result is

Theorem 4.1. ***(Dold–Whitney)***

(a) *Principal $SO(4)$ bundles $SO(4) \to P \to B^4$ are classified by $w_2(P) \in H^2(B, \mathbb{Z}_2) = H^2(B, \mathbb{Z}) \bigotimes \mathbb{Z}_2$, $p_1(P) \in H^4(B, \mathbb{Z})$, and $e(P) \in H^4(B, \mathbb{Z})$, and the latter two we often identify with integers by evaluating on an orientation class $[B]$ (not all values are allowed).*

(b) *$SO(3) \to P \to B$ are classified by $w_2(P) \in H^2(B, \mathbb{Z}_2)$ and $p_1(P) \in H^4(B, \mathbb{Z})$, where the allowed values for w_2 are arbitrary and $p_1(P)[B] = e^2[B] \bmod 4$, where $e = w_2 \bmod 2$.*

If we start with a principal bundle $SO(4) \to P \to B$, we get two associated $SO(3)$ principal bundles: Let $S^3_\pm \subset SO(4)$ be the two simple subgroups of $SO(4)$ coming from left multiplication (S^3_-) and right multiplication (S^3_+) with $S^3 = Sp(1)$ on $\mathrm{H} = \mathbb{R}^4$. Then $SO(4)/S^3_\pm \simeq SO(3)$ and hence we get that $SO(3) \to P_\pm = P/S^3_\pm \to B$ are principal $SO(3)$ bundles. Now one easily shows that:

a) $p_1(P_\pm) = p_1(P) \pm 2e(P)$, and $w_2(P) = w_2(P_\pm)$.

b) P is uniquely determined by P_+ and P_-, and conversely, given two principal $SO(3)$ bundles P_+ and P_- with $w_2(P_+) = w_2(P_-)$, there exists a unique $SO(4)$ principal bundle P associated to P_- and P_+ as above. This also determines the allowed values for w_2, p_1 and e for $SO(4)$ principal bundles.

c) The structure group of the $SO(4)$ bundle $P \to B$ reduces to $U(2) \subset SO(4)$ iff the structure group of one of the bundles P_+ or P_- reduces to $SO(2) \subset SO(3)$. Indeed, if P reduces to $U(2) \to P^* \to B$, then $P = P^* \times_{U(2)} SO(4)$, and hence if $S^3_- \subset U(2) \subset SO(4)$ (or S^3_+ for the other embedding), then $P_- = P^* \times_{U(2)} (SO(4)/S^3_-) = P^* \times_{U(2)} SO(3)$ with $U(2) \subset SO(3)$ given by $U(2) \to U(2)/SU(2) = SO(2) \subset SO(3)$ and hence $P_- = P^* \times_{U(2)} SO(3) = P^*/SU(2) \times_{SO(2)} SO(3)$, or P_- is the $SO(3)$ extension of $S^1 \to P^*/SU(2) \to B$.

d) Similarly, the structure group of $SO(4) \to P \to B$ reduces to $SU(2) = S^3_\pm \subset SO(4)$ iff one of P_+ or P_- is trivial as an $SO(3)$ principal bundle.

Another way of looking at $P_\pm$ is as follows: let $E = P \times_{SO(4)} \mathbb{R}^4 \to B$ be the associated vector bundle, which comes with an inner product. Using the $*$ operator on 2–forms (where $*^2 = 1$), we get the descomposition $\Lambda^2 E = \Lambda^2_+ E \oplus \Lambda^2_- E$ into ± 1 eigenspaces of $*$. $\Lambda^2_\pm E$ are now 2–three dimensional vector bundles over B and it is not hard to see that the principal frame bundle of $\Lambda^2_\pm E$ is precisely $P_\pm$. We can apply these construction in particular to the tangent bundle τ of B and get two $SO(3)$ bundles $P_\pm(\tau) = P(\tau)_\pm$. In this case the Hirzebruch signature theorem implies that the signature $s(B) = \frac{1}{3} p_1(P(\tau))[B]$, and also $e(P(\tau))[B] = \chi(B)$. Hence $p_1(P_\pm(\tau))[B] = 3s(B) \pm 2\chi(B)$. The main theorem of the subject is

Theorem 4.2 (Derdzinski–Rigas). *If $SO(3) \to P \to B^4$ with principal connection θ is $SO(3)$ fat, then there exists a conformal structure on B^4 and an orientation on B^4 such that the curvature Ω is self–dual, i.e. $Q(\Omega, \mathsf{u}) \in \Lambda^2_+ \tau$ for any $\mathsf{u} \in \mathfrak{so}(3)$*

Corollary 4.3. *A fat $SO(3)$ principal bundle $P \to B$ is isomorphic to $P_+(\tau)$ or $P_-(\tau)$*

Proof: Fix a basis u_i of $\mathfrak{so}(3)$. Then $Q(\Omega_p, \mathsf{u}_i)$ is a basis of $\Lambda^2_+\tau$ and hence an element of $P_+(\tau)$. This gives an $SO(3)$ equivariant isomorphism $P \to P_+(\tau)$.

COROLLARY 4.4. *If $S^3 \to P \to B^4$ is a fat bundle, then B^4 is spin, and P is isomorphic to the 2–fold spin cover of $P_+(\tau)$ or $P_-(\tau)$.*

THEOREM 4.5. *If $SO(4) \to P \to B$ is a principal bundle with connection θ that is $SO(3)$ fat ($SO(3) \subset SO(4)$) then*

(a) [Wei80]*? There exists an orientation on B^4 and an orientation on $E = P \times_{SO(4)} \mathbb{R}^4$ such that P_+ is isomorphic to $P_+(\tau)$.*
(b) [DR81] *With these orientations,*

$$|p_1(P_-)[B]| < p_1(P_+)[B] = p_1(P_+(\tau)[B] = 3s(B) + 2\chi(B)$$

Proof of (a): Corresponding to $\Lambda^2\mathbb{R}^4 = \Lambda^2_+\mathbb{R}^4 \oplus \Lambda^2_-\mathbb{R}^4$ one has (under the natural $SO(4)$ equivariant isomorphism $\Lambda^2\mathbb{R}^4 \simeq \mathfrak{so}(4)$) that $\mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$, a direct sum of simple ideals, and the natural embedding $SO(3) \subset SO(4)$ corresponds to $\triangle\mathfrak{so}(3) \subset \mathfrak{so}(3) \oplus \mathfrak{so}(3)$. $\Omega : TP \to \mathfrak{so}(4) = \mathfrak{so}(3) \oplus \mathfrak{so}(3)$ splits up into $\Omega = (\Omega_-, \Omega_+)$, $\Omega_\pm : TP \to \mathfrak{so}(3)$. Also $\theta = (\theta_-, \theta_+)$ and $\theta_\pm$ can be regarded as connections on the principal bundles $P_\pm$ with curvature $\Omega_\pm$. θ is $SO(3)$ fat iff $Q(\Omega(X,Y), \mathsf{u})$ is nondegenerate for all $\mathsf{u} \in \mathfrak{so}(3)^\perp$ or equivalently $\mathrm{pr}_{\mathfrak{so}(3)^\perp}(\Omega(X,Y)) \neq 0$ for all $X \wedge Y \neq 0$, or $\Omega_-(X,Y) \neq \Omega_+(X,Y)$. Under the $\mathrm{Ad}(SO(4))$ orbit, this corresponds to $A\Omega_-(X,Y) \neq B\Omega_+(X,Y)$ for $A, B \in SO(3)$, or equivalently $|\Omega_-(X,Y)| \neq |\Omega_+(X,Y)|$. A change of orientation in $E = P \times_{SO(4)} \mathbb{R}^4$ interchanges P_+ and P_- and hence we can assume $|\Omega_-(X,Y)|^2 < |\Omega_+(X,Y)|^2$. In particular $\Omega_+(X,Y) \neq 0$, which means that P_+ is $SO(3)$ fat and by Theorem 4.1 isomorphic to $P_+(\tau)$ for appropriate orientation on B^4.
Proof of (b): For any $SO(3)$ principal bundle P with connection θ and curvature Ω we have by Chern–Weyl theory

$$p_1(P)[B] = \frac{1}{4\pi^2}\int_B \Omega \wedge \Omega = \frac{1}{4\pi^2}\int_B < \Omega, *\Omega >$$

. If Ω is self–dual, $p_1(P) = \frac{1}{4\pi^2}\int_B |\Omega|^2$ and in general $|p_1(P)[B]| \leq \frac{1}{4\pi^2}\int_B |\Omega|^2$ by Cauchy–Schwartz. In ?? can ?? know that Ω_+ on P_+ is self dual and hence

$$|p_1(P_-)[B]| \leq \frac{1}{4\pi^2}\int_B |\Omega_-|^2 < \frac{1}{4\pi^2}\int_B |\Omega_+|^2 = p_1(P_+)[B] = p_1(P_+(\tau))[B] = 3s(B) + 2\chi(B)$$

□

COROLLARY 4.6. [DR81] *The only $SO(4)$ principal bundle over S^4 that is $SO(3)$ fat is the $SO(4)$ bundle associated to the Hopf bundle $S^3 \to S^7 \to S^4$.*

Proof: The frame bundle of $\tau = \tau(S^4)$ is $SO(4) \to SO(5) \to SO(5)/SO(4) = S^4$ and hence $P_\pm = SO(5)/S^3_\pm$, with $S^3_\pm \subset SO(4)$.

Under the 2–fold cover $Sp(2) \to SO(5)$ one has $Sp(1) \times Sp(1) \to SO(4)$ (since $S^4 = \mathbb{HP}^1 = Sp(2)/Sp(1) \times Sp(1)$) and hence the spin cover is $\widetilde{P_\pm(\tau)} = Sp(2)/Sp(1) \times \{1\}$ or $Sp(2)/\{1\} \times Sp(1) = S^7$ with induced $Sp(1)$ action given by left or right multiplication on $S^7 \subset \mathrm{H}^2$. Hence $P_\pm(\tau)$ are the two induced Hopf bundles $SO(3) \to \mathbb{RP}^7 \to S^4$ of the left and right Hopf bundles. Notice that $w_2(P_\pm(\tau)) = w_2(P(\tau)) = 0$ and $p_1(P_\pm(\tau))[B^4] = p_1(P(\tau)) \pm 2e(P(\tau))[B^4] = \pm 4$. Now if P is an $SO(4)$ bundle over S^4 that is $SO(3)$ fat, we have that for appropriate orientations that $P_+ \simeq P_+(\tau)$ is the Hopf bundle, $|p_+(P_-)[S^4] < p_1(P_+(\tau)) = 4$ and since $w_2(P_-) = 0$ we need $p_1(P_-)[S^4] = 0 \mod 4$, and hence P_- is trivial, which implies that P is the $SO(4)$ bundle associated to the Hopf bundle. □

We now apply Theorems 4 and 5 to the case of $B^4 = \mathbb{CP}^2$, but before we do this, let us shortly disgress and look at the bundles

$$S^3/\mathbb{Z}_{p+q} = U(2)/S^1_{p,q} \to W_{p,q} = SU(3)/S^1_{p,q} \to \mathbb{CP}^2$$

discussed in Example (P.5). If $p+q = 1$, there are S^3 bundles over $\mathbb{CP}^2$ and hence correspond to a certain $SO(4)$ principal bundle $SO(4) \to P_{p,q} \to \mathbb{CP}^2$ (but are not fat, unless $p = q = 1$. Let us determine what bundle this is by computing $P_\pm$ for this bundle. The transitive action of $U(2)$ on $S^3 = U(2)/\operatorname{diag}(z^p, z^q)$ extends to the linear action of $U(2)$ on $\mathbb{C}^2$ given by $v \xrightarrow{\phi} (\det A)^{-p} \cdot A(v)$ and hence $E = SU(3) \times_{U(2)} \mathbb{C}^2$ and $P = SU(3) \times_{U(2)} SO(4)$ with homomorphism $\phi : U(2) \to SO(4)$. Hence, if $\phi(SU(2)) = S^3_- \subset SO(4)$, we get $P_- = SU(3) \times_{U(2)} (SO(4)/S^3_-) = SU(3) \times_{U(2)} SO(3)$ with homomorphism $U(2) \to SO(3)$ given by the composition $U(2) \xrightarrow{\det} U(2)/SU(2) = U(1) \to SO(3)$, $e^{i\theta} \to R((-2p+1)\theta)$. Indeed, $\det \operatorname{diag}(e^{i\theta}, 1) = e^{i\theta}$ and $\phi(\operatorname{diag}(e^{i\theta}, 1)) = \operatorname{diag}(e^{i\theta(-p+1)}, e^{-ip\theta}) \subset SO(4)$, and $S^3_- \times S^3_+ \to SO(4)$ is sending $(e^{i\theta}, e^{i\psi}) \to (e^{i(\theta-\psi)}, e^{i(\theta+\psi)})$. Hence P_- is the $SO(3)$ extension of the $SO(2)$ bundle with Euler class $e = -2p+1$ (and hence $w_2(P_\pm) = w_2(P) = 1$). Indeed $P_- = SU(3)/SU(2) \times_{U(1)} SO(3) = {S^5}_{U(1)} SO(3)$ where $U(1)$ acts via the Hopf action on S^5 and $U(1) \to SO(3)$ sends $e^{i\theta} \to R((-2p+1)\theta)$.

Similarly, $P_+ = SO(3)_{U(2)} SO(4)/S^3_+ = SU(3)_{U(2)} SO(3)$ with homomorphsim $U(2) \to U(2)/Z(U(2)) = SO(3)$. Hence $P_+ = SU(3)/Z(U(2))_{SO(3)} SO(3) = W_{1,1}$ is our fat $SO(3)$ principal bundle over $\mathbb{CP}^2$.

Notice that the tangent bundle τ of $\mathbb{CP}^2$ corresponds to $p = -1$ and hence $P_+(\tau) = W_{1,1}$, $P_-(\tau) = {S^5}_{U(1)} SO(3)$ which is the $SO(3)$ extension of the $SO(2)$ bundle with Euler class 3. It follows that $p_1(P_+(\tau))[\mathbb{CP}^2] = -3$ or better $p_1(P_+(\tau))[\overline{\mathbb{CP}^2}] = 3$ and $p_1(P_-(\tau))[\mathbb{CP}^2] = e^2 = 9$, which agrees with $p_1(P_\pm(\tau)) = 3s(B) \pm 2\chi(B)$.

Theorem 4.2 now implies that $P_+(\tau)$ and $P_-(\tau)$ can be the only possible $SO(3)$ fat bundles over $\mathbb{CP}^2$. Hence, if we apply Theorem 4.5, we get the following two possibilities for principal $SO(4)$ bundles P over $\mathbb{CP}^2$ which are $SO(3)$ fat: $P_+ \simeq P_+(\tau) = W_{1,1}$ and $|p_1(P_-)| < p_1(P_+(\tau))[\overline{\mathbb{CP}^2}] = 3$ and hence $p_1(P_-)[\mathbb{CP}^2] = 1$ which means that P_- is the $SO(3)$ extension of the Hopf bundle $S^1 \to S^5 \to \mathbb{CP}^2$. This means that P is the $SO(4)$ extension of the $U(2)$ bundle $U(2) \to SU(3) \to \mathbb{CP}^2$ under the homomorphism $U(2) \hookrightarrow SO(4)$, $v \to A(v)$. This bundle probably does not have a fat $SO(3)$ extension, since the associated sphere bundle is $SU(3)_{U(2)} S^3 = SU(3) U(2)_{U(2)/\operatorname{diag}(z,1)} = SU(3)/\operatorname{diag}(z, 1, \bar{z}) = W_{1,0}$ which does not have a homogeneous connection metric with positive curvature. The other possibility is that $P_+ \simeq P_-(\tau)$ which is the $SO(3)$ extension of the S^1 bundle $S^3/\mathbb{Z}_3 \to \mathbb{CP}^2$ with Euler class 3α and since $p_1(P_-(\tau)) = 9$, we get $|p_1(P_-)| < 9$ or $p_1(P_-)[\mathbb{CP}^2] = 1, -3, 5, -7$. At last we get

Corollary 4.7 (Chaves). *There are at most 5 principal $SO(4)$ bundles over $\mathbb{CP}^2$ which are $SO(3)$ fat.*

Actually, we believe that the following is true:

PROBLEM 3. ***(a)*** *Show that for a fat* $SO(3)$ *principal bundle* $SO(3) \to P \to B^4$ *one cannot reduce the structure group to* $SO(2)$ *(or more generally if* $G \to P \to B$ *is a fat principal bundle, the structure group does not reduce!)*
(b) *Show that the principal* $SO(4)$ *bundle* $\mathrm{SU}(3)_{\mathrm{U}(2)}\,\mathrm{SO}(4)$ *with usual embedding* $U(2) \subset SO(4)$ *does not have an* $SO(3)$ *fat connection (since its structure group reduces, this is also related to (a))*

If both are true, it would follow that $\mathbb{CP}^2$ has no $SO(4)$ principal bundles which are $SO(3)$ fat, and that the only $SO(3)$ principal bundle which is fat is $W_{1,1} \to \mathbb{CP}^2$. It would also follows that $S^2 \times S^2$ and $\mathbb{CP}^2 \# \ldots \# \mathbb{CP}^2$ have no $SO(4)$ bundles that are $SO(3)$ fat, and no $SO(3)$ fat bundles since one easily shows that for these two manifolds, $P_\pm(\tau)$ both reduce to $SO(2)$ bundles.

But notice that nevertheless (as observed by Chaves), one has infinitely many $S^3/\mathbb{Z}_{p+q}$ bundles over $\mathbb{CP}^2$ which are fat, given by $S^3 = U(2)/S^1_{p,q} \to SU(3)/S^1_{p,q} \to \mathbb{CP}^2$ if $p \cdot q > 0$. The metrics are only $U(2)$ invariant and are Berger type metrics.

One should therefore look at the more general question of whether there are $U(2)$ principal bundles $U(2) \to P \to B^4$ which are $S^1_{p,q}$ fat for some p, q. If $p+q = 1$ this will correspond to fat S^3 bundles over B^4 where the metric on the fiber is only $U(2)$ invariant and not necessarily a round sphere. As in the case of $SO(4)$ bundles, we can assocaite an $SO(2)$ bundle $P_- = P/SU(2)$, $SU(2) \subset U(2)$, $U(2)/SU(2) = S^1$ and an $SO(3)$ bundle $P_+ = P/Z(U(2))$, $U(2)/Z(U(2)) = SO(3)$.

THEOREM 4.8. *If* $U(2) \to P \to B^4$ *is* $S^1_{p,q} = \mathrm{diag}(z^p, z^q)$ *fat, then* $(p,q) \neq (1,-1)$ *and if* $p+q \neq 0$*, then for some choice of orientation on* B^4 *we have that* $P_+ \simeq P_+(\tau)$ *is a fat* $SO(3)$ *principal bundle. Furthermore, if* $p+q \neq 0$*,* $p-q \neq 0$*, then*

$$|p_1(P_-)| < \left(\frac{p+q}{p-q}\right)^2 p_1(P_+(\tau))[B] = \left(\frac{p+q}{p-q}\right)^2 (3s(B) + 2\chi(B))$$

Proof: If θ is fat with curvature Ω, we have $\Omega : TP \to \mathfrak{u}(2) = \mathfrak{so}(3) \oplus \mathbb{R}$, $\theta = (\theta_+, \theta_-)$, $\Omega = (\Omega_+, \Omega_-)$. $\mathfrak{h}^\perp$, where $H = S^1_{p,q}$ always contains a vector $\mathsf{u} \in \mathfrak{so}(3)$, and hence $Q(\Omega, \mathsf{u}) = Q(\Omega_+, \mathsf{u})$ must be non–degenerate, which means that θ_+ in P_+ is fat. Since $\mathfrak{h}$ is generated by $(p-q)\,\mathrm{diag}(i,-i) + (p+q)\,\mathrm{diag}(i,i) \in \mathfrak{so}(3) + \mathbb{R}$, the $\mathrm{Ad}(U(2))$ orbit of (Ω_-, Ω_+) is non–zero in $\mathfrak{h}^\perp$ iff $|p+q||\Omega_+(X,Y)| - |p-q||\Omega_-(X,Y)| \neq 0$ and it must be > 0 since there always exist X, Y with $\Omega(X,Y) \neq 0$.

Hence $|p+q||\Omega_+(X,Y)| > |p-q||\Omega_-(X,Y)|$ which implies that $p+q \neq 0$. If $p+q \neq 0$ and $p-q \neq 0$, then the same argument in the proof of Theorem 5 implies the inequality on $P_1(P)$. ☐

If $B = S^4$, then P_+ is the Hopf bundle and P_- is trivial, and hence P is the $U(2)$ extension of the Hopf bundle. But notice that example E.7 indeed shows that this bundle is $S^1_{p,q}$ fat for most p, q.

For $B^4 = \mathbb{CP}^2$, we at least get e.g. in the case $q\; p+q = 1$, i.e. S^3 bundles, a contradiction, unless $(p,q) = (1,0)$, and if $P_+ \simeq W_{1,1}$ (it must be if problem 1 is correct) we get $p_1(P_-) = 1$, which means $P = SU(3) \to \mathbb{CP}^2$, which might be $\mathrm{diag}(z, 1)$ fat. Of course, again, we have $P/\mathrm{diag}(z, 1) = W_{1,0}$!

We end this section with the following natural problems:

Problem 4. *Develop an analogue of Theorems 3 and 4 for principal G–bundles over B^8 using the recently developed analogues of self duality in dimension 8* [Joy96] [FK98] [**?**]. *Study its application to S^8: are there any $SO(k)$ principal bundles which are $SO(k-1)$ fat for $k = 6, 7, 8$ (classified by p_2 for $k = 6, 7$ and p_1 and e for $k = 8$). In particular, is the Hopf bundle the only such bundle for $k = 8$? Also study $B = \mathbb{HP}^2$, $\mathbb{CP}^4$, $G_2/SO(4)$ and $G_2(\mathbb{R}^6)$, where one has natural examples of fat bundles (Berard–Bergery). How about $SO(6)$ principal bundles over S^6 (classified by p_2) which are $SO(5)$ fat?*

5. Nonnegative curvature vector bundles and sphere bundles

We now look at the question which motivated the study of fat bundles, namely when does the metric on the total space have positive or nonnegative sectional curvature? We will concentrate on $SO(k)$ principal bundles and their associated S^{k-1} sphere bundles, but also study the question of whether the associated k–dimensional vector bundle has nonnegative curvature, which turns out to be closely related.

For vector bundles, we remark that a theorem of Walschap [GW00] implies that if a vector bundle has nonnegative curvature (with the soul being the zero section), then the boundary of a small tubular neighborhood, which is the sphere bundle of the vector bundle, also has nonnegative curvature (the converse is unknown, see problem 9). Also, if the vector bundle has nonnegative curvature, then one can change the metric outside a small tubular neighborhood of the soul such that $\sec \geq 0$, and the normal exponential map is a diffeomorphism (Guijarro, [Gui98]). This reduces the problem to one only near the soul, and is the motivation why problem 9 might be correct.

We start with the simplest case, that of a principal S^1-bundle or a 2-dimensional vector bundle (oriented for simplicity). By the above remarks, the vector bundle has $non-negativecurvature$ invariant under the S^1 action.

Let θ be a connection on the principal S^1 bundle $S^1 \to P \xrightarrow{\pi} B$ which induces a connection metric $g_t(X, Y) = t\theta(X)\theta(Y) + g_B(\pi_*(X), \pi_*(Y))$ for some $t > 0$. This situation was examined by D.G.Yang in [Yan95] and earlier by Chaves-Derdzinsky-Rigas in [CR92].

Let Ω be the curvature of θ, $\Omega = \theta$, and hence $\Omega = \pi^*(\alpha)$ where α is a closed 2-form on B with $[\alpha]$ the Euler class $e(P)$. Conversely, for every closed 2-form α with $[\alpha] = e(P) \in H^2(B, \mathbb{Z})$ there exists a connection θ with $d\theta = \pi^*(\alpha)$, unique up to gauge transformations. In the following, we will use the notation $k(X, Y) =< R(X, Y)Y, X >$ for the unnormalized sectional curvature.

Theorem 5.1. *(Yang?)*
(a) *g_t on P has $non-negativecurvature$ iff*

$$(\nabla_X \alpha)(X, Y)^2 \leq |i_X \alpha|^2 \left(k_B(X, Y) - \frac{3}{4} t^2 \alpha(X, Y)^2 \right)$$

(b) *g_t has* sec > 0 *(for sufficiently small t) iff*

$$(\nabla_X\alpha)(X,Y)^2 < |i_X\alpha|^2 k_B(X,Y)$$

for all $X \wedge Y \neq 0$

Notice that $K(X,U) = t^2|i_X\alpha|^2$ by our previous discussion and $K(X,Y) = K_B(X,Y) - \frac{3}{4}t^2\alpha(X,Y)^2$ by ONeill formula. The term $\nabla_X\alpha$ enters since $\langle R(Y,X)X,U\rangle = \frac{t}{2}(\nabla_X\alpha)(X,Y)$. Notice also that every 2-plane in P has the form $(X, \cos\theta Y + \sin\theta U)$ for X, Y horizontal and one shows that the sectional curvature of this 2-plane is equal to

$$\frac{1}{4}t^2\cos^2\theta\|i_X\alpha\|^2 + \sin^2\theta(K_B(X,Y) - \frac{3}{4}t^2\alpha(X,Y)^2) - t\sin\theta\cos\theta(\nabla_X\alpha)(X,Y)$$

from which (a) follows immediately, and for (b) we remark that $K_B > 0$ is required and hence $K_B \geq \delta \geq 0$ for some δ which means that $<$ in (a) is equivalent to (b) for t small. Also, notice that if (a) or (b) holds for one t_0, then it holds for all $t \leq t_0$.

Da Gang Yang [Yan95] used (a) to prove:

Theorem 5.2 (D.G.Yang). *Every circle bundle over $\mathbb{CP}^n\#-\mathbb{CP}^n$ has a connection metric with $non-negative curvature$.*

Question. *What about circle bundles over $\mathbb{CP}^n\#\mathbb{CP}^n$?*

Corollary 5.3 (Berard-Bergery). $\mathrm{Ric}(g_t) \geq 0$ *iff for $X \in TB$,*

$$|\delta\alpha(x)|^2 \leq |\alpha|^2(2\,\mathrm{Ric}_B(X) - t^2|i_X\alpha|^2)$$

Hence, if $\mathrm{Ric}_B > 0$ and if we choose α harmonic and t small, then $\mathrm{Ric}(g_t) \geq 0$ and is positive at a point if $|\alpha| > 0$. One can then make $\mathrm{Ric} > 0$, unless $\alpha = 0$ identically .

Let us know look at the case $\sec_P > 0$ for a connection metric on P. Theorem 5.1(b) of course implies $\sec_B > 0$ and α symplectic, i.e. α fat.

Problem 5. Given a manifold B with $\sec_B > 0$ where B is also a symplectic manifold, choose an $e \in H^2(B,\mathbb{Z})$ which contains a symplectic 2–form and let $S^1 \to P \to B$ be the S^1 bundle with Euler class e. Can one always find a symplectic form α with $[\alpha] = e$ such that 5.1(b) is satisfied (i.e, a connection metric with sec > 0 on P)? We can normalize the metric on B such that $\sec_B \geq 1$ and hence we need $(\nabla_X\alpha)(X,Y)^2 < |i_X\alpha|^2 \cdot |X \wedge Y|^2$. Notice that this could be implied by the stronger inequality (*) $|\nabla_X\alpha|^2 < |i_x\alpha|^2$ which now becomes a problem in symplectic geometry: when can you find α in its cohomology class such that * is satisfied? E.g, use some kind of heat flow method to change α ?

Example 5.4. *The inequality is of course satisfied if $\nabla\alpha = 0$, i.e. on a Kahler manifold. But the only known Kahler manifold with* $\sec_B > 0$ *is biholomorphic to $\mathbb{CP}^n$ and hence $P = S^{2n+1}$. Of course, the weaker condition 5.1(a) is satisfied for all of the circle bundles in example E2.*

Example 5.5. *Let $B = SU(3)/T^2 = W^6$ be the Allof-Wallach flag manifold, and $P = SU(3)/S^1_{p,q}$. Since $S^1_{p,q} \subset T^2$, we get the S^1 bundle $S^1 = T^2/S^1_{p,q} \to W^7_{p,q} \to SU(3)/T^2$ and the Allof-Wallach metric is as decribed earlier, and hence is a connection metric with* sec > 0 *if $p \geq q > 0$ (excluding only $W_{1,0}$). Although $SU(3)/T^2$ is also a Kahler manifold, none of* *the Allof-Wallach metrics can be Kahler metrics (although the two sets of metrics intersect in their closure!). The cohomology ring of* $SU(3)/T^2$ *(using Borel Theory) is given by*

$$H^*(SU(3)/T^2, \mathbb{Z}) = \mathbb{Z}[u, v]/ < u^3 = 0,\ u^2 + uv + v^2 = 0 >$$

where $u, v \in H^2(B, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$ *are the image under transgression of the description* $T^2 = \mathrm{diag}(z, w, \overline{zw}) \subset SU(3)$, *i.e. the* z *and the* w *are circles. Hence one gets for the Euler class* $e(W_{p,q}) = -qu + pv$. *Hence every circle bundle over* $SU(3)/T^2$ *which is simply connected (corresponding to indivisible elements in* $H^2(B, \mathbb{Z})$ *i.e.* $(p, q) = 1$*) are of the form* $W_{p,q}$ *for some* p, q.

Notice that $H^4(B, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$ *with generators* u^2, v^2 *(and relation* $uv = -u^2 - v^2$*), and* $H^6(B, \mathbb{Z}) = \mathbb{Z}$ *with generator* u^2v *(and* $uv^2 = -u^2v$, $v^3 = 0$*). Hence* $e^3 = (-qu + pv)^3 = qp(q + p)uv^2$.

Thus only the circle bundle with $pq(p + q) = 0$ *(corresponding to* $W_{1,0}$*) does not contain a symplectic form in its cohomology class (and hence cannot have any connection metric with* $\sec > 0$*). Of course in all other cases the Alof-Wallach metric induces a symplectic form* α *that satisfies 5.1b.*

Example 5.6. *The only other known symplectic manifold with* $\sec > 0$ *is the inhomogeneous Eschenburg flag manifold* $E^6 = SU(3)//T^2$ *where* T^2 *acts freely on* $SU(3)$ *as*

$$\begin{pmatrix} z & & \\ & w & \\ & & z^2w^2 \end{pmatrix} SU(3) \begin{pmatrix} 1 & & \\ & 1 & \\ & & z^2w^2 \end{pmatrix}^{-1}$$

and the metric g_t *on* $SU(3)$ *(shrank in the direction of* $U(2)$*) induces* $\sec > 0$ *on* E^6. *We can again choose* $S^1_{p,q} \subset T^2$, $(p, q) = 1$, *and consider the Eschenburg manifolds*

$$\begin{pmatrix} z^p & & \\ & z^q & \\ & & z^{p+q} \end{pmatrix} SU(3) \begin{pmatrix} 1 & & \\ & 1 & \\ & & z^{2p+2q} \end{pmatrix}^{-1}$$

Now the metric g_t *from* $SU(3)$ *induces* $\sec > 0$ *on* $E_{p,q}$ *iff* $p \cdot q > 0$ *(Eschenburg). This is only half of all possibles* S^1 *bundles over* E^6*! Notice also that* $E_{1,-1} = W_{1,0}$. *Eschenburg showed that*

$$H^*(E^6, \mathbb{Z}) = \mathbb{Z}[u, v]/ < u^2 + 3uv + v^2 = 0,\ u^3 = 0 >$$

(since $u^2 + 3uv + v^2$ *is indefinite, and* $u^2 + uv + v^2$ *is positive definite,* W^6 *and* E^6 *do have non-isomorphic cohomology rings!)*

Again $H^4(E^6, \mathbb{Z}) = \mathbb{Z} + \mathbb{Z}$ *with generators* u^2, uv *(and* $v^2 = -u^2 - 3uv$*) and* $H^6(E^6, \mathbb{Z}) = \mathbb{Z}$ *with generator* u^2v *(and* $uv^2 = -3u^2v$, $v^3 = 8u^2v$*). One can also show that* $e(W_{p,q}) = (-p + 2q)u + (p + q)v$ *and hence* $e^3 = (p + q)(2p^2 + 2q^2 + pq^2?)u^2v$. *Then* $e^3 = 0$ *iff* $p + q = 0$ *and hence* $E_{1,-1}$ *is the only circle bundle whose Euler class does not contain any symplectic form! The challenge is now to solve problem 5 in this particular case, hopefully for all* $p + q \neq 0$. *This would be very interesting for the following reason:*

Eschenburg showed that $H^4(E_{p,q}, \mathbb{Z}) = \mathbb{Z}_{p^2+q^2+3pq}$ *but now* $p^2 + q^2 + 3pq$ *is indefinite and for each* $n \in \mathbb{Z}$ *there are infinitely many solutions of* $p^2 + q^2 + 3pq = n$ *(in the Allof-Wallach case* $p^2 + q^2 + pq = n$ *there are only finitely many!). In the case of* $n = \pm 1$, *one gets infinitely many Eschenburg spaces* $E_{p,q}$ *with* $H^*(E_{p,q}, \mathbb{Z}) = H^*(S^2 \times S^5, \mathbb{Z})$*!*

But even in the case $p \cdot q > 0$, the Eschenburg metrics on $E_{p,q}$ are not connection metrics since one easily shows that the length of the fibers in $S^1 \to E_{p,q} \to SU(3)//T^2$ is not constant. Hence the first question is

Problem 6. *For what values of p, q $(p + q \neq 0)$ does $(p + 2q)u + (p + q)v \in H^2(E^6, \mathbb{Z})$ contain a symplectic 2-form? If yes, does it satisfy 5.1b or does a heat flow change it to one that satisfies 5.1b?*

But the Eschenburg metrics belong to the following more general class of metrics on circle bundles, described by a connection θ and a function $f : B \to \mathbb{R}$

$$g_t = f \cdot \theta(x)\theta(y) + g_B(\pi_*(x), \pi_*(y))$$

i.e. the horizontal span is still given by an S^1 invariant distribution, but the fiber over $p \in B$ has now length $2\pi f(b)$ and hence are not anymore totally geodesic (the metrics agree with Riemannian submersions $P \to B$ where the S^1 action still acts by isometries).

One now shows (see e.g. Tapp) that $\sec_P > 0$ iff

$$(\nabla_x \alpha)(x, y)^2 < (|i_x\alpha|^2 + \mathrm{Hess}(f)(x, x))k_B(x, y)$$

for any $x, y \in TB$.

Now it does not follow any more that α must be symplectic. A good exercise would be to compute α and f for the Eschenburg metrics and see if one can change them so that this inequality is satisfied for all p, q (most p, q?) (another "heat flow?)

The spaces $E^7_{p,q}$ and E^6 can also be viewed as bundles over $\mathbb{CP}^2$ since the free T^2 action on $SU(3)$ extends to a free $U(2)$ action:

$$\begin{pmatrix} A & 0 \\ 0 & \det A \end{pmatrix} SU(3) \begin{pmatrix} 1 & 0 \\ 0 & (\det A)^2 \end{pmatrix}^{-1} \qquad A \in U(2)$$

and $SU(3)//U(2)$ is again $\mathbb{CP}^2$ since if we first divide by $SU(2) \subset U(2)$ we get $SU(2)\backslash SU(3) = S^5$ and the ?? action of $Z(U(2)) = \mathrm{diag}(z, z)$, $|z| = 1$, becomes the action on $S^5 \subset \mathbb{C}^3$ given by multiplication with $\mathrm{diag}(z, z, \bar{z})$ which is equivalent to the Hopf action. As before we conclude that $E_{p,q} \to \mathbb{CP}^2$ is a bundle with fiber $U(2)/\mathrm{diag}(z^p, z^q) = S^3/\mathbb{Z}_{p+q}$ and hence a sphere bundle if $p + q = 1$ (not possible if $p \cdot q > 0$). Also, as was observed by Shankar, $E_{1,1} = SU(3)//Z(U(2)) = SU(3)_{U(2)}SO(3)$ becomes an $SO(3)$ principal bundle and since $H^4(E_{1,1}) = \mathbb{Z}_5$ we have $w_2 = 1$, $p_1 = 5$ for this principal bundle. Notice that Theorem 4.1 implies that this principal bundle has no connection metric with $\sec > 0$ *(in the Eschenburg metric the fibers are not totally geodesic). Also, the S^3 bundles $E_{p,q} \to \mathbb{CP}^2$ with $p + q = 1$ have $P_+ = E_{1,1}$, $P_- = SO(3)$-extension of the $SO(2)$ bundle over $\mathbb{CP}^2$ with $e = -2p + 1$ (where P is the $SO(4)$ principal bundle corresponding to the S^3 bundle).*

If we look at the principal G bundle with $G \neq S^1$, then we can find a connection metric on the total space with $\sec > 0$ only if G admits a left invariant metric with $\sec > 0$, i.e. only if $G = SO(3)$ or S^3 (Wallach). This case was examined by Chaves-Derdzinski-Rigas:

Theorem 5.7. *A connection metric on $G \to P \to B$ with $G = SO(3)$ or S^3 and with biinvariant metric Q on G has* $\sec > 0$ *(for t sufficiently small, t the scale in the fibers) iff*

$$(\nabla_x \Omega_u)(x,y)^2 < |i_x \Omega_u|^2 k_B(x,y)$$

for all x, y and $u \in \mathfrak{g}$, where $\Omega_u = Q(\Omega, u)$,

Notice that if this inequality holds for one $u \in \mathfrak{g}$, then it does for all u since the adjoint orbit of u is the sphere in $\mathfrak{g}$ of radius $|u|$.

The inequality in Theorem 5.7 comes again from looking at planes of the form $(x, \cos\theta y + \sin\theta u)$ and is equivalent to these planes having positive sectional curvature. Now it is somewhat more surprising that this condition is also sufficient since there are many other planes on P.

We now look at the case of higher dimensional vector bundles and sphere bundles.

For a vector bundle $\mathbb{R}^k \to E \to B$ the situation was examined in Strake-Walschap [SW90] and Tapp [Tap00]. We consider connection type metrics of the following type:

Let $<,>$ be a metric in the vector bundle and ∇ a compatible connection with curvature R^∇. ∇ defines the horizontal distribution on E which we use to define the metric on E and the metric on the fibers $\mathbb{R}^k$ we choose to be rotationally symmetric: $dt^2 + f^2(t)d\theta^2$ for some function f. For this metric on $\mathbb{R}^k$ to have $\sec \geq 0$ we need $-f''/f \geq 0$ (curvature of $\partial/\partial t, \partial/\partial\theta$) and $(1 - f'^2)/f^2 \geq 0$ (curvature of $\partial/\partial\theta_1, \partial/\partial\theta_2$). But it turns out that this function is not relevant, except that it helps if we make $-f''/f$ very large at $t = 0$ to ensure large curvature for these 2-planes near the 0-section, e.g. if we choose $f^2 = \varepsilon^2 t^2/\varepsilon^2 + t^2$, $-f''/f = 3\varepsilon^2/(\varepsilon^2 + t^2)^2$ which becomes large at $t = 0$ if ε small.

Notice that $|f| \leq \varepsilon$, i.e. the fibers becomes very narrow (this is precisely the metric one gets on $TS^n = SO(n+1)_{SO(n)}\mathbb{R}^n$ when choosing $Q = -\frac{1}{2}\operatorname{tr} AB$ on $SO(n+1)$ and the flat metric on $\mathbb{R}^n$, where $\varepsilon = 1$). This can be explained by our earlier remark that $non-negativecurvature$ becomes essentially a condition near the 0-section. Here is a necessary condition for this metric to have $non-negativecurvature$:

Theorem 5.8 (Strake-Walschap). *If the metric on E has $non-negativecurvature$, then*

$$< (\nabla_x R^\nabla)(x,y)u, v >^2 \leq |R^\nabla(u,v)x|^2 (k_B(x,y) - \frac{3}{4}\varepsilon^2 |R^\nabla(x,y)u|^2)$$

for all x, y, u, v and for some fixed $\varepsilon > 0$.

Here we set $< R(u,v)x, y > = < R(x,y)u, v >$.

The choice of an arbitrary small $\varepsilon > 0$ corresponds to the choice of the metric on $\mathbb{R}^k$ above. The condition comes from looking at 2-planes at $u \in E$ spanned by $(\overline{x}, \overline{y} + tv)$, where $\overline{x}$, $\overline{y}$ are the horizontal lifts of $x, y \in TB$.

This inequality implies that $< \nabla_x R^\nabla(x,y)u, v >^2 \leq |R^\nabla(u,v)x|^2 k_B(x,y)$ but not necessarily conversely. In [SW90] it was also shown that if $k_B > 0$ and ∇ was radially symmetric (i.e. $\nabla_x R^\nabla(x,y)u = 0$) then there exists a connection metric on E with $non-negativecurvature$. But in [GSW00] it was shown that e.g. a vector bundle over an irreducible symmetric space B which admits a connection with $\nabla R = 0$ must be a homogeneous vector bundle.

Tapp examined when this kind of connection metric in the sphere bundle has $\sec > 0$, where one has the choice of making the radius of the (round) sphere arbitrarily small. He obtained the following result, which improves some previous results in [SW90].

THEOREM 5.9 (Tapp). *A connection metric on the sphere bundle has* sec > 0 *for sufficiently small radius if and only if*

$$< (\nabla_x R^\nabla)(x,y)u, v >^2 < |R^\nabla(u,v)x|^2 k_B(x,y)$$

for all $x \wedge y \neq 0$, $u \wedge v \neq 0$. *In addition, one can then also find a metric on* E *with* sec > 0 *such that the metric on the sphere bundle (with sufficiently small radius) has* sec > 0.

In particular, one has $k_B > 0$ (clear from O'Neill) and $R^\nabla(u,v)x \neq 0$ for all $u \wedge v \neq 0$, $x \neq 0$, which is equivalent to $(x,y) \rightarrow < R^\nabla(x,y)u, v >$ being nondegenerate for all $u \wedge v \neq 0$, i.e. fatness, which is our old necessary condition for sec > 0 on the sphere bundle.

The condition on Theorem 5.9 again comes from looking at 2-planes of the form $(\overline{x}, \overline{y} + tv)$ at $u \in SE$, $v \perp u$, and the extrinsic and intrinsic curvature of such 2-planes (in the vector and sphere bundles) is the same!. But now it is even more surprising (and more difficult to prove) that this condition is also sufficient.

Notice also that Theorem 5.7 and 5.9 say that an S^2 bundle has a connection metric with sec > 0 if and only if its $SO(3)$ principal bundle has one with sec > 0!

We end these notes with a list of problems:

PROBLEM 7. *Is the condition in Theorem 5.8 also sufficient for a connection metric to have* $non-negativecurvature$ *on a vector bundle? (the answer is yes if the dimension of the fiber is 2, by Theorem 5.1)*

PROBLEM 8. *Is the condition in Theorem 5.8 also necessary and sufficient for a connection metric on the sphere bundle to have* $non-negativecurvature$*? Again the answer is yes if the dimension of the fiber is 2.*

PROBLEM 9. *If the sphere bundle of the vector bundle has a metric with* $non-negativecurvature$, *does the vector bundle has a metric with* $non-negativecurvature$ *(possibly inducing the given metric on the sphere bundle)? Even for connection metrics this is unknown (but see Theorem 5.9 for a partial result). All known examples come from metrics on the principal bundle, and hence, this is true in all examples.*

PROBLEM 10. *Develop an analogue of Theorem 5.8 and Theorem 5.9 for connection metrics on vector bundles and sphere bundles, where the metrics on the fiber* S^k *is only homogeneous, but not* $SO(k+1)$ *invariant (e.g.* $U(n)$ *invariant on* S^{2n-1}, *or* $Sp(n)$ *invariant on* S^{4n-1}*). It would be interesting to produce some exampels of connection metricsof this more general type with nonnegative or positive curvature which do not admit* $SO(k+1)$ *invariant metrics with nonnegative or positive curvature.*

PROBLEM 11. *Are there connection metrics on vector bundles (and/or sphere bundles) over* S^4 *with* $non-negativecurvature$*? It is difficult to see any obstruction! The examples in* [GZ] *are not of that type. If yes, the same question can be asked for vector bundles and sphere bundles over* S^6 *and* S^8 *(for* S^7 *bundles over* S^8 *4095 of them are exotic 15-spheres!) The methods in* [GZ] *do not seem to work well for vector bundles over* S^6 *or* S^8*!*

PROBLEM 12. *Are there metrics on vector bundles and/or sphere bundles over* S^4 *(maybe connection metrics?) where the induced metric on* S^4 *has* sec > 0 *or* sec $\equiv 1$*? (for the*

Gromoll-Meyer sphere, there is sec > 0 *on* S^4*!). In the* [GZ] *examples, every point on* S^4 *has zero curvatures.*

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