# COVERAGE AND HOLE-DETECTION IN SENSOR NETWORKS VIA HOMOLOGY

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## ABSTRACT

We consider coverage problems in sensor networks of stationary nodes with minimal geometric data. In particular, there are no coordinates and no localization of nodes. We introduce a new technique for detecting holes in coverage by means of homology, an algebraic topological invariant. The impetus for these techniques is a completion of network communication graphs to two types of simplicial complexes: the **nerve** complex and the **Rips** complex. The former gives information about coverage intersection of individual sensor nodes, and is very difficult to compute. The latter captures connectivity in terms of inter-node communication: it is easy to compute but does not in itself yield coverage data. We obtain coverage data by using persistence of homology classes for Rips complexes. These homological invariants are computable: we provide simulation results.

## I. INTRODUCTION

Coverage problems in sensor networks are extremely important in a wide variety of contexts, from cell-phone communications to beacon navigation to security and defense. We restrict attention to systems with stationary nodes in $\mathbb{R}^d$ having radially symmetric coverage domains. Depending on the application, these coverage balls may denote broadcast domains, sensing regions, or visibility domains. Knowing the topology of the union of the coverage domains — in particular the location and morphology of holes — is of significant relevance to the coverage problem.

### I-A. Related Work

To our knowledge, there are two prominent approaches to coverage problems. The first is perhaps best described as the 'computational geometry' approach, in which one uses the coordinates of the nodes and standard geometric tools (such as the Delaunay or Voronoi diagrams) to determine coverage. For examples of such an approach, see [1], [2], [3], [4]. One feature of this approach is that the precise geometry of the domain and the exact locations of the nodes must be known.

To circumvent some of these difficulties, many researchers turn to probabilistic methods for coverage. For example, in [5], the author assumes a randomly and uniformly distributed collection of nodes in a domain with a fixed geometry and proves expected area coverage. Other approaches [6], [7], [8] give percolation-type results about coverage and network integrity for randomly distributed nodes. The drawback of these methods is the need for strong assumptions about the exact shape of the domain, as well as the need for a uniform distribution of nodes.

### I-B. Features and Bugs

We initiate a significant departure from both of these techniques: our tools come from algebraic topology, homology theory in particular. This paper does not solve the coverage problem, although the techniques we introduce here can be used to give effective coverage criteria [17]. In order to introduce the homological approach, we restrict attention to a simplified version of the coverage problem: given a collection of nodes $\mathcal{X} \subset \mathbb{R}^d$ covered by balls of a fixed radius, what is the topology of the union of the covering discs? Specifically, are there any holes in the cover? If so, where are the holes located?

We give a sufficient criterion for detecting holes in a coverage network. Our techniques have the following features:

1) Everything is coordinate-free. Nodes have no localization capabilities.
2) The criterion is computable: we present simulations.
3) The methods we present can be generalized to apply to a wide class of coordinate-free sensor networks problems.

Our methods have the following drawbacks:

1) Our criterion is centralized. We believe that a decentralized computation is possible, but it is a significant algorithmic and topological challenge.
2) Our criterion is sufficient to detect a hole, but not necessary.
3) We have not determined the computational complexity of the computation of the homological criterion.

## II. PROBLEM FORMULATION

We assume as little as possible about the nodes and their geometry. Consider a collection of stationary nodes $\mathcal{X} \subset \mathbb{R}^d$. In most practical settings, $d = 2$ or $d = 3$. Our tools are not adapted to any particular $d$; therefore, we leave $d$ as an open variable throughout the rest of the paper. We adopt the following assumptions on our system.

**A1** Nodes have radially symmetric cover domains of radius $r_c$.

**A2** Nodes broadcast their unique ID numbers. Each robot can detect the identity of anyone within radius $r_s$ via a **strong** signal, and via a **weak** signal within a larger radius $r_w$.

**A3** The radii of communication $r_s, r_w$ and the coverage radius $r_c$ satisfy

$$2r_c = r_w \geq r_s\sqrt{d} \tag{1}$$

where $d$ is the dimension of the domain in which the nodes lie.

It is important to note what we do not assume. The coordinates of the nodes are unknown. Neither are there localization or orientation capabilities. Nodes are completely devoid of any information apart from the identities of 'very close' and 'somewhat close' neighbors. This ability to differentiate between strong and weak signals provides a coarse form of distance measurement. For example, if a particular node scans its communications, it can determine whether a certain node is within distance $r_s$ or $r_w$ or neither.

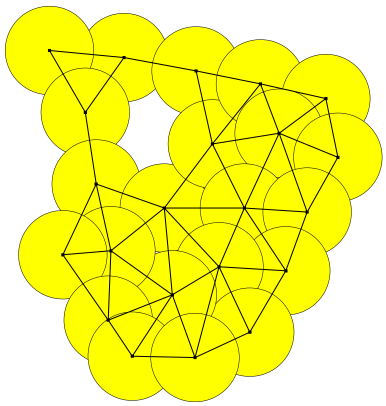


**Fig. 1**. Sensor coverage discs and their union.

## III. TOOLS FROM ALGEBRAIC TOPOLOGY

The mathematical tools we use are by no means novel: with the exception of the simulations, this paper could have been written in the 1930s. However, as these tools are not in the repertoire of researchers in sensor networks, we give a brief (and necessarily incomplete) treatment here. For further reading of various degrees of depth, see [10], [13], [12]; for an introduction in the context of applications and computations see the recent text by Kaczynski et al. [11].

### III-A. Simplicial complexes

All of the topological objects we work with in this paper belong to a certain class of spaces called simplicial complexes. Given a set of points $V$, a $k$-simplex is an unordered subset $\{v_0, v_1, \ldots, v_k\}$ where $v_i \in V$ and $v_i \neq v_j$ for all $i \neq j$, see Fig. 2. The **faces** of this $k$-simplex consist of all $(k-1)$-simplices

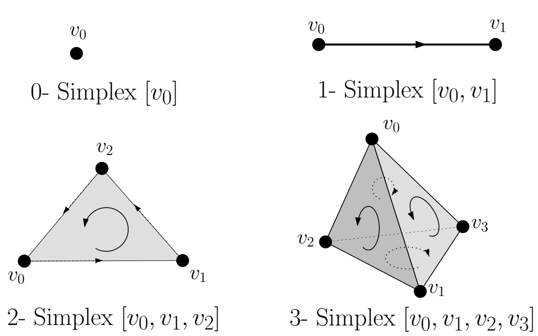



**Fig. 2**. Oriented simplices of dimension zero through three.

of the form $\{v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k\}$ for $0 \leq i \leq k$. A **simplicial complex** is a collection of simplices which is closed with respect inclusion of faces: see Figure 3 for non-examples. Triangulated surfaces form a concrete example, where the vertices of the triangulation correspond to $V$, edges correspond to 1-simplices, and faces correspond to 2-simplices. The orderings of the vertices correspond to an orientation. Any abstract simplicial complex on a (finite) set of points $V$ has a geometric realization in some $\mathbb{R}^n$. See Figures 8-10 for examples of geometric realizations. See [12], [13] for an elementary introduction.

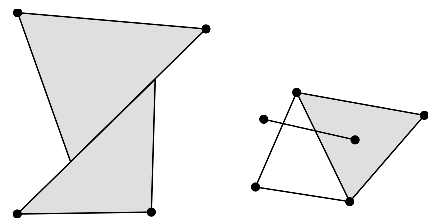


**Fig. 3**. Non-examples of simplicial complexes.

### III-B. Simplicial homology

Homology is an algebraic procedure for counting 'holes' of various types. There are numerous variants of homology: we describe **simplicial homology with real coefficients**. Let $X$ denote a simplicial complex. The homology of $X$, denoted $H_*(X)$, is a sequence of vector spaces $\{H_k(X) : k = 0, 1, 2, 3 \ldots\}$, where $H_k(X)$ is called the **$k$-dimensional homology of** $X$. The dimension of $H_k(X)$, called the $k^{th}$ **Betti number of** $X$, is a coarse measurement of the number of different holes in the space $X$ that can be sensed by using subcomplexes of dimension $k$.

For example, the dimension of $H_0(X)$ is equal to the number of path-connected components of $X$. These are the types of 'holes' in $X$ that points can detect — are two points connected by a sequence of edges or not? The simplest basis for $H_0(X)$ consists of a choice of vertices in $X$, one in each path-component of $X$. Likewise, the simplest basis for $H_1(X)$ consists of loops in $X$, each of which surrounds a different 'hole' in $X$. For example, if $X$ is a graph, then $H_1(X)$ is a measure of the number and types of cycles in the graph.

Let $X$ denote a simplicial complex. Define for each $k \geq 0$, the vector space $C_k(X)$ to be the vector space whose basis is the set of **oriented** $k$-simplices of $X$; that is, a $k$-simplex $\{v_0, \ldots, v_k\}$ together with an order type denoted $[v_0, \ldots, v_k]$ where a change in orientation corresponds to a change in the sign of the coefficient:

$$[v_0, \ldots, v_j, \ldots, v_i, \ldots, v_k] - [v_0, \ldots, v_i, \ldots, v_j, \ldots, v_k].$$

For $k$ larger than the dimension of $X$, $C_k(X) = 0$. The **boundary map** is defined to be the linear transformations $\partial : C_k \to C_{k-1}$ which acts on basis elements $[v_0, \ldots, v_k]$ via

$$\partial[v_0, \ldots, v_k] := \sum_{i=0}^{k} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_k], \quad (2)$$

as illustrated in Fig. 4.

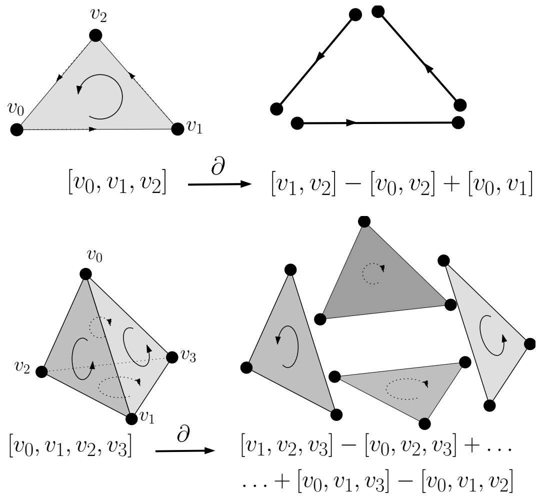



**Fig. 4**. The boundary operator on a 2-simplex [top] and a 3-simplex [bottom].

This gives rise to a **chain complex**: a sequence of vector spaces and linear transformations

$$\cdots \xrightarrow{\partial} C_{k+1} \xrightarrow{\partial} C_k \xrightarrow{\partial} C_{k-1} \cdots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

Consider the following two subspaces of $C_k$: the **cycles** (those subcomplexes without boundary) and the **boundaries** (those subcomplexes which are themselves boundaries).

$$\begin{array}{lcll} k\text{-cycles} & : Z_k(X) & = & \ker(\partial : C_k \to C_{k-1}) \\ k\text{-boundaries} & : B_k(X) & = & \operatorname{im}(\partial : C_{k+1} \to C_k) \end{array} \tag{3}$$

A simple lemma demonstrates that $\partial \circ \partial = 0$; that is, the boundary of a chain has empty boundary. It follows that $B_k$ is a subspace of $Z_k$. The $k$-cycles in $X$ are the basic objects which count the presence of a 'hole of dimension $k$' in $X$. But, certainly, many of the $k$-cycles in $X$ are measuring the same hole; still other cycles do not really detect a hole at all — they bound a subcomplex of dimension $k+1$ in $X$.

We say that two cycles $\xi$ and $\eta$ in $Z_k(X)$ are **homologous** if their difference is a boundary:

$$[\xi] = [\eta] \quad \leftrightarrow \quad \xi - \eta \in B_k(X).$$

The $k$-dimensional **homology** of $X$, denoted $H_k(X)$ is the quotient vector space,

$$H_k(X) = \frac{Z_k(X)}{B_k(X)}. \tag{4}$$

Specifically, an element of $H_k(X)$ is an equivalence class of homologous $k$-cycles. This inherits the structure of a vector space in the natural way: $[\xi] + [\eta] = [\xi + \eta]$ and $c[\xi] = [c\xi]$ for $c \in \mathbb{R}$.

One may show that the homology $H_*(X)$ is a topological invariant of $X$: it is indeed an invariant of **homotopy type**. A **homotopy** between two continuous functions $f_0 : X \to Y$ and $f_1 : X \to Y$ is continuous a 1-parameter family of continuous functions $f_t : X \to Y$ connecting $f_0$ to $f_1$. Two spaces $X$ and $Y$ are said to be of the same homotopy type if there exist functions $f : X \to Y$ and $g : Y \to X$ with $g \circ f$ homotopic to the identity map on $X$ and $f \circ g$ homotopic to the identity map on $Y$. More specifically, homeomorphic spaces are homotopy equivalent, as are spaces obtained from a larger space by 'deforming' or 'collapsing' in a continuous manner. Homology groups are invariant under such operations.

Readers familiar with the Euler characteristic of a triangulated surface will not find it odd that intelligent counting of simplicies yields a topological invariant. Indeed, for any simplicial complex, the Euler characteristic is the alternating sum of the Betti numbers.

### III-C. Induced homomorphisms

Consider two simplicial complexes $X$ and $X'$. Let $f : X \to X'$ be a continuous simplicial map: $f$ takes each $k$-simplex of $X$ to a $k'$-simplex of $X'$, where $k' \leq k$. Then, the map $f$ induces a linear transformation $f_\# : C_k(X) \to C_k(X')$. It is a simple lemma to show that $f_\#$ takes cycles to cycles and boundaries to boundaries; hence there is a well-defined linear transformation on the quotient spaces

$$f_* : H_k(X) \to H_k(X') \quad : \quad f_* : [\xi] \mapsto [f_\#(\xi)].$$

This is called the **induced homomorphism** of $f$ on $H_*$. This operation satisfies two elementary properties: (1) the identity map $Id : X \to X$ induced the identity map on homology; and (2) the composition of two maps $g \circ f$ induces the composition of the induced homomorphisms: $(g \circ f)_* = g_* \circ f_*$.

## IV. SIMPLICIAL COMPLEXES IN SENSOR NETWORKS

### IV-A. Nerves and Rips

The problem of computing the topological type of a union of sets is classical, and easily handled using the concept of a nerve.

*Definition V:* Given a collection of sets $\mathcal{U} = \{U_\alpha\}$, the **nerve complex** of $\mathcal{U}$, $\mathcal{N}(\mathcal{U})$, is the abstract simplicial complex whose $k$-simplices correspond to nonempty intersections of $k+1$ distinct elements of $\mathcal{U}$.

Hence, the vertices of $\mathcal{N}$ correspond to the elements of $\mathcal{U}$ themselves. An edge in $\mathcal{N}$ exists between two vertices if and only if the corresponding elements of $\mathcal{U}$ intersect. Higher dimensional simplices are regulated by mutual intersections of collections of elements of $\mathcal{U}$. Among the many uses of nerves in topology, the following classical result would appear to be of great importance in applications:

*Theorem 1 (The Čech Theorem):* The nerve complex of a collection of convex sets has the homotopy type of the union of the sets.

For a proof, see, e.g., [14]. Unfortunately, nerves are very difficult to compute without precise locations of the nodes and a global coordinate system. We therefore turn to a different method for obtaining a simplicial complex from a sensor network, using only pairwise communication data.

The following type of complex goes back to the 1927 paper of Vietoris on the foundations of homology theory [15]: similar objects were reinvented by Rips in the 1980's in the context of geometric group theory and have been used extensively since.

*Definition VI:* Given a set of points $\mathcal{X} = \{x_\alpha\} \subset \mathbb{R}^n$ in Euclidean $n$-space and a fixed radius $\epsilon$, the **Vietoris-Rips complex** of $\mathcal{X}$, $\mathcal{R}(\mathcal{X})$, is the abstract simplicial complex whose

$k$-simplices correspond to unordered $(k+1)$-tuples of points in $\mathcal{X}$ which are pairwise within Euclidean distance $\epsilon$ of each other.

We will abbreviate this to the term 'Rips' complex. Since Rips complexes are determined by pairwise distances, they are completely determined by the communication graph of the system: any time you see a triangle in the communication graph, you fill in an abstract 2-simplex. Any time you see a complete subgraph on $k+1$ vertices, you fill it in with an abstract $k$-simplex.

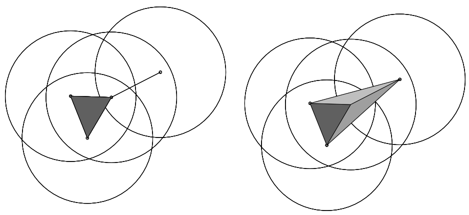


**Fig. 5**. [left] An example of a Rips complex for a set of points: edges are determined by whether each boundary nodes lies within a radial disc centered about the other. [right] The nerve complex associated to this collection of discs.

It would seem reasonable to conjecture that the radius $R$ Rips complex of a set of nodes would be topologically equivalent to the nerve complex of the balls of a particular radius at the nodes. Unfortunately, this is not true. Figure 6 gives an example for which the Rips complex fails to capture the nerve: the nerve is topologically a circle with a single hole in the middle, whereas the Rips complex is an octahedron, which is simply connected. These two complexes have different homology groups and are thus not of the same homotopy type. Similar examples can be constructed for arbitrary choice of coverage radius.

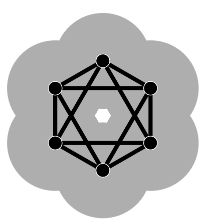


**Fig. 6**. A Rips complex not homotopy equivalent to the union of cover discs.

There is a way to modify the Rips complex to resolve these issues. We define the $\ell^\infty$ communication graph to be the graph whose vertices are $\mathcal{X}$ and whose edges are those nodes whose $\ell^\infty$ distance is less than or equal to $r_s$. Denote by $\mathcal{R}_s^\infty$ the Rips complex of this graph. The following result is very simple.

*Lemma VII:* $\mathcal{R}_s^\infty$ is precisely the nerve complex of the cover of $\mathcal{X}$ by closed $d$-dimensional cubes of side-length $r_s$.

*Proof:* In the case where $d=1$, the result is immediate. For the general case, since distances are measured in the $\ell^\infty$ norm (each coordinate is measured) and cover elements are cubes, the entire problem decomposes into cross-products of the $d=1$ case. $\diamond$

If our sensors were outfitted with a natural $\ell^\infty$ geometry, Lemma VII would immediately imply that the topological properties of the cover (by cubical sensor domains) are completely determined by the topology of $\mathcal{R}_s^\infty$. Unfortunately, it is physically unrealistic to assume that communication can be carried out with precise $\ell^\infty$ geometry — sensors are assumed to be free of all coordinates *and orientations*.

We therefore consider what can be done with radially symmetric communication and sensing. The core idea is that we can 'squeeze' the $\ell^\infty$ Rips complex (hence a nerve complex) between two standard Rips complexes of different radii.

## VIII. COORDINATE-FREE DETECTION OF HOLES

The following theorem is our principal criterion for hole detection:

*Theorem 2:* A sensor network satisfying assumptions **A1**-**A3** has a coverage hole if there is a nonzero homology class in $\mathcal{R}_s$ which is also nonzero as a homology class in $\mathcal{R}_w$. That is, the homomorphism

$$\iota_* : H_k(\mathcal{R}_s) \to H_k(\mathcal{R}_w) \qquad (5)$$

induced by the inclusion $\iota : \mathcal{R}_s \hookrightarrow \mathcal{R}_w$ is nonzero for some $k > 0$.

Remarks:

1) If $k \geq d$, then $\iota_* = 0$ always.
2) The type of hole is determined by $k$; e.g., if $d=3$, then a hole of dimension $k=1$ means that the cover has a 'tubular' hole running through it, whereas as $k=2$ hole means that there is a 'hollow' portion of the cover.
3) This criterion is a sufficient criterion, but not necessary. In an extreme case, one can choose $r_s$ to be exceedingly small, in which case $H_k(\mathcal{R}_s)=0$ for all $k>0$ and the criterion automatically fails.

*Proof of Theorem 2:* We claim that there exists a chain of inclusions

$$\mathcal{R}_s \subset \mathcal{N}_c \subset \mathcal{R}_w, \qquad (6)$$

where $\mathcal{N}_c$ is the nerve complex of the cover of $\mathcal{X}$ by balls of radius $r_c$. Assume for the moment that these inclusions hold, and denote them as $j : \mathcal{R}_s \hookrightarrow \mathcal{N}_c$ and $j' : \mathcal{N}_c \hookrightarrow \mathcal{R}_w$. Since $\iota = j' \circ j$, we conclude that the induced homomorphisms on homology satisfy $\iota_* = j'_* \circ j_*$. Hence, if $\iota_*$ is nonzero, then it takes a nonzero element of $H_k(\mathcal{R}_s)$ to a nonzero element of $H_k(\mathcal{R}_w)$, which itself is in the image of $j'_*$. This implies that $H_k(\mathcal{N}_c) \neq 0$. From the Čech Theorem, we conclude that the cover has a hole of dimension $k$.

We now demonstrate that Equation (6) holds. Clearly, the inclusion $j' : \mathcal{N}_c \hookrightarrow \mathcal{R}_w$ holds because (since $r_c = r_w/2$) both complexes have the same 1-dimensional skeleton and the Rips complex has all possible simplices of this 1-skeleton filled in.

The inclusion $j : \mathcal{R}_s \hookrightarrow \mathcal{N}_c$ is not as direct. Clearly, $\mathcal{R}_s \subset \mathcal{R}_s^\infty$. From Lemma VII, $\mathcal{R}_s^\infty$ equals the nerve of the cover of $\mathcal{X}$ by cubes of side length $r_s$. These cubes are contained inside of balls of radius $r_c \geq r_s\sqrt{d}$. (See Fig. 7 for an illustration of this nesting.) Hence, $\mathcal{R}_s \subset \mathcal{N}_c$. Equation (6) holds and the theorem is proved. $\diamond$

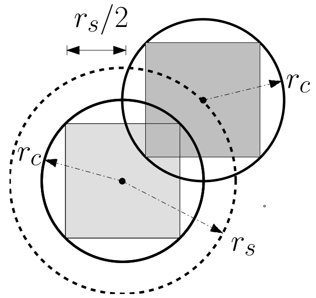



**Fig. 7**. The nesting of sensing and covering discs with the $\ell^\infty$ balls used in $\mathcal{R}_s^\infty$.

The proof that we use involves the $\ell^\infty$-Rips complex. This is not the optimal way to obtain Equation (6). A result of de Silva (see [17]) states that the optimal ratio $r_w/r_s$ for which the equation holds is $\sqrt{2d/(d+1)}$; hence, the results of this note hold with replacing $\sqrt{d}$ in **A3** with the aforementioned constant.

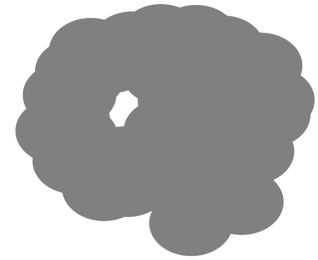
## IX. SIMULATIONS

To a mathematician, homology is 'easy' to compute: a linear algebra computation. However, in practice, the number of simplices in a complex increases exponentially with the dimension of that complex. To make matters worse, the standard algorithm for computing generators in homology is of quintic order in the number of simplices: this would discourage computation of homology in all but very low dimensions. Fortunately, the past few years have witnessed an explosion in algorithms and software for computing homology of simplicial complexes which make nontrivial computations possible (see [11] and references therein).

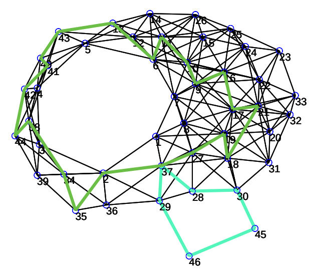
In order to verify the theoretical results developed in the previous section, we have successfully run several simulations using a publicly available computational homology software known as `CHomP` [16]. These simulations have been written using `MATLAB` as the frontend (primarily for generating the simplicial complexes from various point-data sets, data formatting and for visualization.) The `CHomP` routines have been used for simplification of the Rips complexes, computing the homology groups and for localizing the non-trivial generators of the homology groups.

Note that in the figures and examples which follow, we illustrate the cover using coordinates. The frontend keeps track of coordinates for purposes of drawing pictures. However, `CHomP` has no information about coordinates: the homology criterion uses *only* connectivity data as per our assumptions.

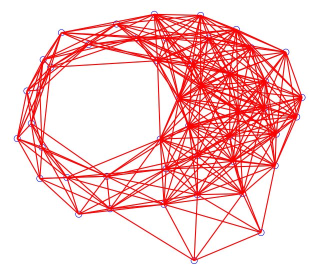
Results of one such simulation appear in Figure 8. Figure 8.(a) illustrates the union of sensor coverage discs in $\mathbb{R}^2$, each of radius $r_c = \frac{1}{\sqrt{2}}$. Clearly, there is a hole inside the union of sensor domains. Figure 8.(b) shows the Rips complex $\mathcal{R}_s$ generated by the detection of a strong signal within radius $r_s = 1$. Using `CHomP`, we detect two generators for the first homology group $H_1(\mathcal{R}_s)$ of this complex. Representative cycles for each equivalence class of generators are depicted in the same figure in *green* and *blue*. The generator depicted in *blue* envelops a non-collapsible cycle of 5 nodes in the lower part of the complex whereas the generator colored as *green* envelops another non-collapsible cycle of nodes in the left part of the complex. The



(a) The union of coverage discs in $\mathbb{R}^2$.



(b) Rips Complex $\mathcal{R}_s$.



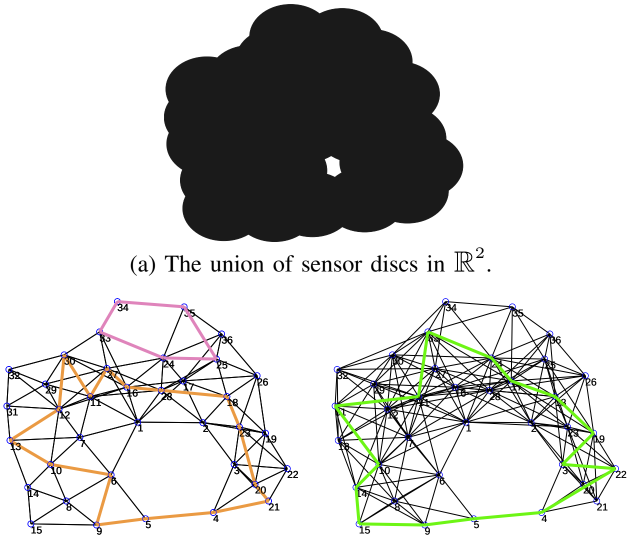
(c) Rips Complex $\mathcal{R}_w$.

**Fig. 8**. Simulation results with $r_c = \frac{1}{\sqrt{2}}, r_s 1$ and $r_w = \sqrt{2}$.

Rips complex $\mathcal{R}_w$ generated by the detection of a weak signal within $r_w = 2r_c = \sqrt{2}$ appears in Figure 8.(c). First, note that $\mathcal{R}_s \subset \mathcal{R}_w$. A greater radius of detection adds several new edges to the complex, which in turn induce many higher dimensional simplices. The result of the induced homomorphism of the inclusion map $\iota_* : H_1(\mathcal{R}_s) \to H_1(\mathcal{R}_w)$ can be understood as follows. The cycle of nodes enveloped by the *blue* generator in $\mathcal{R}_s$ no longer remains non-collapsible in $\mathcal{R}_w$ due to the addition of many 2-simplices. Therefore the same *blue* generator when seen in $\mathcal{R}_w$ becomes trivial and vanishes. However, the *green* generator remains non-collapsible when seen in $\mathcal{R}_w$, despite the shortening of the non-collapsible cycle it envelops. Therefore, in the language of Theorem 2, $\iota_* : H_1(\mathcal{R}_s) \to H_1(\mathcal{R}_w)$ is non-zero and indicates the presence of a 1-dimensional hole, as clearly verified from Figure 8.(a).

Another such simulation appears in Figure 9. Again, there are two generators (colored as *purple* and *orange*,) in $H_1(\mathcal{R}_s)$, each bounding a non-trivial cycle in $\mathcal{R}_s$. However, only one of them survives the inclusion map to $H_1(\mathcal{R}_w)$, showing the presence of a hole. The inclusion map picks the right generator so that the true hole is identified and the false cycle gets killed. We give no illustrations for the criterion in dimension three as these would be altogether unilluminating; the computation involved would not be significantly harder.

The homology computation also indicates the location of holes in the network. As pointed out earlier, a cycle representing a generator of a homology group actually represent its entire equivalence class. Two cycles are equivalent if their difference is a boundary. Therefore one can use this property to collapse the generators till they lock exactly onto the non-collapsible cycles they envelop. We denote such cycles as **minimal**. For simplicial complexes, this kind of collapsing or minimization has a nice geometrical interpretation. If two edges of a cycle in dimension 1



(a) The union of sensor discs in $\mathbb{R}^2$.

(b) Rips Complex $\mathcal{R}_s$.

(c) Rips Complex $\mathcal{R}_w$.

**Fig. 9**. Simulation results for a second data set with $r_c \frac{1}{\sqrt{2}}, r_s = 1$ and $r_w = \sqrt{2}$.

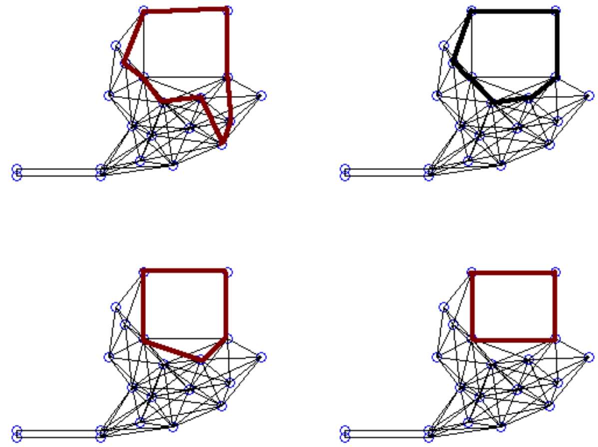


**Fig. 10**. Minimization of generators to detect location of network holes.

| # Nodes | # Edges | $T_w^{\text{exp}}$ | $T_s^{\text{exp}}$ | $T_w^{\text{hom}}$ | $T_s^{\text{hom}}$ |
|---|---|---|---|---|---|
| 20 | 45 | 0.14 | 0.13 | 0.49 | 0.47 |
| 67 | 345 | 1.46 | 1.41 | 0.60 | 0.60 |
| 84 | 285 | 3.15 | 3.11 | 0.73 | 0.58 |
| 154 | 1059 | 26.2 | 26.1 | 1.25 | 0.63 |
| 214 | 1243 | 92.2 | 90.2 | 1.08 | 0.69 |

**Table I**. Run times for five simulations: all times listed in seconds. Here, $T^{\text{exp}}$ is the time to build and export the Rips complex (weak and strong respectively) and $T^{\text{hom}}$ is the time to compute the homology of the complex.

belong to a 2-simplex, then one can collapse the cycle to the third edge. This procedure can be generalized to higher dimensions as well [11]. For the purpose of demonstration, such minimization is depicted in Figure 10 for a certain Rips complex generated in our simulations. The non-trivial generator is minimized till it captures the nodes that surround the hole. Although, this method can be applied to all simplicial complexes, it has one serious drawback. For more than one hole in the network, this kind of minimization depends on the initial choice of representative cycles. Ideally, one may wish for each generator surrounding exactly one hole. However, the homology group can possibly be generated equivalently if one hole is enveloped by more than one generators. In that case, the minimization would at best partly detect the location of the holes in the network.

All computations are performed on a Pentium 4, 1.60 GHz, 512 Mb RAM, Windows 2000 machine, running Matlab 6.1 as the front end for `CHomP`. Table I gives the runtimes for a variety of systems. It is noted that the runtime is dominated by the construction of the Rips complexes: these can have a large number of simplices as a function of the number of nodes. It is a pleasant observation that the homology computation is relatively quick, even though we have not optimized the homology computation code for out (rather particular) type of systems. The signal complexity of the process — the number of communications required between nodes — is a function of the number of edges in the Rips complex, listed in the second column.

## X. Generalizations and Conclusions

The methods presented are novel and of potentially great use in sensor networks. The use of topological methods allows one to dispense with assumptions about coordinates, distances, and orientations: this is a boon. The results presented in this note are by no means the best possible results, the reliance on having $r_c$ equal to exactly $r_w/2$ is something that would not necessarily hold in a physical setting.

We have chosen to present a limited result for ease of exposition. In recent work with Vin de Silva [17], we have developed a homological coverage criterion which is dual to the hole-detection criterion presented here. We briefly outline the ingredients of this more general theory.

In addition to Assumptions **A1**-**A3**, there are further assumptions about the domain $\mathcal{D} \subset \mathbb{R}^d$ in which the nodes lie. First, $\mathcal{D}$ is assumed to be bounded with boundary $\partial\mathcal{D}$ which is not too 'pinched'. Second, nodes can detect if they are within some radius $r_f$ of $\partial\mathcal{D}$. These 'boundary' nodes generate subcomplexes $\mathcal{F}_s \subset \mathcal{R}_s$ and $\mathcal{F}_w \subset \mathcal{R}_w$. The result in [17] states that $\mathcal{U}$ contains all of $\mathcal{D}$ (except for possibly a neighborhood of $\partial\mathcal{D}$) whenever the induced map

$$\iota_* : H_d(\mathcal{R}_s/\mathcal{F}_s) \to H_d(\mathcal{R}_w/\mathcal{F}_w)$$

is nonzero. Here, the quotients $\mathcal{R}_s/\mathcal{F}_s$ and $\mathcal{R}_w/\mathcal{F}_w$ mean the abstract simplicial complexes obtained by collapsing $\mathcal{F}_s$ or $\mathcal{F}_w$ to a single vertex.

This coverage criterion has several nice features. There are no assumptions on the geometry or topology of $\mathcal{D}$; for example, it is not assumed that $\partial\mathcal{D}$ is connected. In addition, the radius $r_c$

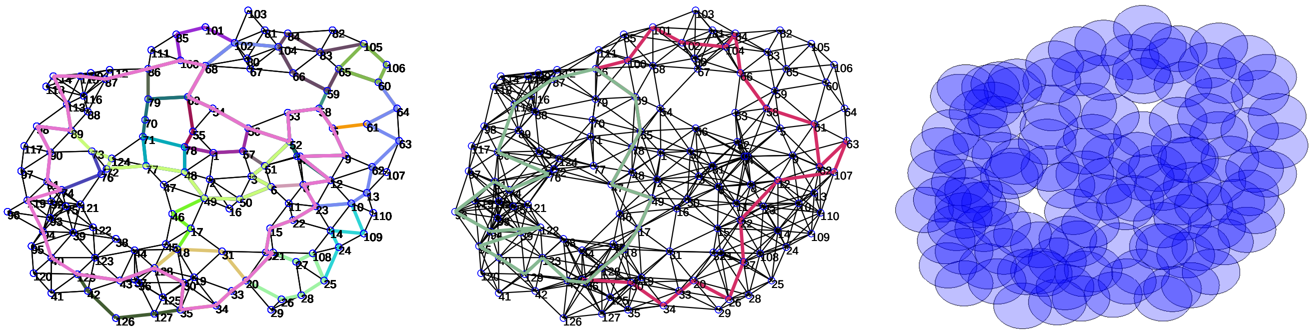


**Fig. 11**. An example of a system of 129 nodes: [left] $\mathcal{R}_s$, [center] $\mathcal{R}_w$, [right] $\mathcal{U}$.

needs to satisfy only an inequality, not an equality as in this note (since, of course, increasing $r_c$ can only improve coverage). The minimal ratio $r_w/r_s$ is different than in this paper.

In a further work [**?**], extensions to systems involving mobile nodes and nodes which go off-line and on-line as a function of time are also possible. In addition, with appropriate control of the boundary nodes, the homological coverage criteria hold for systems with a single communication radius. This reduces the need for the weak-radius Rips complex (which saves expensive computations) and does not require the dual-radius assumption on the sensor nodes.

Perhaps the most crucial direction for this work is to develop a local coverage criterion from which a distributed coverage algorithm could be developed. This seems theoretically possible, since homology is a local operation.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## XI. References


[1] H. Zhang and J. Hou, "Maintaining Coverage and Connectivity in Large Sensor Networks," invited paper in *International Workshop on Theoretical and Algorithmic Aspects of Sensor, Ad hoc Wireless and Peer-to-Peer Networks*, Florida, Feb. 2004

[2] D.Tian, N.D.Georganas, "A coverage-preserving node scheduling scheme for large wireless sensor networks," in Proc. *1st ACM international workshop on Wireless sensor networks and applications*, Atlanta, Georgia, USA, 2002.

[3] S. Meguerdichian, F. Koushanfar, M. Potkonjak, and M. Srivastava, "Coverage problems in wireless ad-hoc sensor network," in *IEEE INFOCOM*, pp. 13801387, 2001.

[4] X.-Y. Li, P.-J. Wan, and O. Frieder, "Coverage in wireless ad-hoc sensor networks" IEEE Transaction on Computers, Vol. 52, No. 6, pp. 753-763, 2003.

[5] H. Koskinen, "On the coverage of a random sensor network in a bounded domain," in *Proceedings of 16th ITC Specialist Seminar*, pp. 11-18, 2004.

[6] B. Liu and D. Towsley, "A study of the coverage of large-scale sensor networks," in *IEEE International Conference on Mobile Ad-hoc and Sensor Systems*, 2004.

[7] F. Xue and P. R. Kumar, "The number of neighbors needed for connectivity of wireless networks." *Wireless Networks*, 169–181, vol. 10, no. 2, March 2004.

[8] C. Hsin and M. Liu, "Network Coverage Using Low Duty-Cycled Sensors: Random and Coordinated Sleep Algorithms," in Proc. International Workshop on Information Processing in Sensor Networks (IPSN), April 2004.

[9] V. de Silva, R. Ghrist and A. Muhammad, "Blind swarms for coverage in 2-d," submitted.

[10] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002.

[11] T. Kaczynski, K. Mischaikow, and M. Mrozek, *Computational Homology,* Applied Mathematical Sciences 157, Springer-Verlag, 2004.

[12] M. A. Armstrong, *Basic Topology*, Springer-Verlag, 1997.

[13] J. Munkres, *Elements of Algebraic Topology*, Addison Wesley, 1993.

[14] R. Bott and L. Tu, *Differential Forms in Algebraic Topology*" Springer-Verlag, Berlin, 1982.

[15] L. Vietoris, "Über den höheren Zusammenhang kompakter Räume und eine Klasse von zusammenhangstreuen Abbildungen," *Math. Ann.* 97 (1927), 454–472.

[16] Computational Homology Program home page, `http://www.math.gatech.edu/~chom/`, 2004.

[17] V. de Silva, R. Ghrist and A. Muhammad, "Coordinate-free coverage in sensor networks with controlled boundaries," in preparation.