"A=B" by Marko Petkovšek, Herbert S. Wilf and Doron Zeilberger

Reviewed by Ian Wanless, Department of Computer Science, Australian National University, Canberra ACT 0200, Australia.

This remarkable book tells of a revolution akin to the one in symbolic integration nearly three decades ago. Until recently, combinatorial identities had to be proved by some clever argument, say by finding an appropriate bijection. Now computers have taken over. Three of the experts who enabled this automation have combined to produce a very readable account of their work.

A hypergeometric series $\sum_n a(n)$ is one in which the ratio $a(n+1)/a(n)$ of consecutive terms is a rational function of the index n. This is quite a general concept, including as it does the series for exponential, trigonometric and Bessel functions, to name a few. In combinatorics, hypergeometric series often crop up in perplexing identities.

Recently, in the context of walks in a regular tree, I encountered the hypergeometric sum

$$f_1(n) = \sum_{k=0}^{n} F(n,k) \quad \text{where} \quad F(n,k) = \binom{2n}{k} \frac{2n-2k+1}{2n-k+1}(x-1)^k. \tag{1}$$

Naturally, I would like to know if this sum has a closed form and find it if it does. With "A=B" I don't need to leaf through a catalogue of combinatorial identities and transformations to see if some trick will make my sum neater. There is now an algorithm to answer this question! All the software is available on the web at

http://www.cis.upenn.edu/˜wilf/AeqB.html (2)

and "A=B" describes how to use it and why it works.

Let's do my example (1). The first thing the software does is to find a recurrence involving the *summand*. In this case

$$F(n+1,k) - x^2 F(n,k) = G(n,k+1) - G(n,k), \tag{3}$$

where $G(n,k)$ is a (not particularly pretty) hypergeometric function. When we sum (3) over k, the right hand side telescopes and we get a recurrence for the desired sum;

$$f_1(n+1) - x^2 f_1(n) = -\frac{1}{n+1}\binom{2n}{n}(x-1)^{n+1}x. \tag{4}$$

Now we can apply the routine for solving recurrences, and find that (4) has no closed form solution for $f_1(n)$. Note that this is not a case of the computer failing to find the solution. We actually know, thanks to the theory in "A=B", *that there isn't one.*

Had I been interested in summing $F(n,k)$ over a different range, the result may have been quite different. From (3) we get that $f_2(n) = \sum_{k=0}^{2n} F(n,k)$ satisfies the recurrence

$$f_2(n+1) - x^2 f_2(n) = (2x-1)(x-1)^{2n+1}. \tag{5}$$

This time there is a closed form solution, namely $f_2(n) = (2-x)x^{2n} + (x-1)^{2n+1}$. Note that the recurrence (3) is easy enough to check (though not so easy to discover!). The computer does a lot of churning, but the proof of the identity it produces is human checkable.

The authors are quite honest about the limitations of their methods. They define a 'closed form' as a "linear combination of a fixed number of hypergeometric terms". They then prove that there is no 'closed form' for the number of derangements (fixed point free permutations) of n,

$$d(n) = n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}.$$

Thus, the familiar formula $d(n) = \lfloor n!/e + 1/2 \rfloor$ cannot be found (or proved) using their methods. The definition of 'closed form' is thus not as general as we might like.

However the methods of “A=B” have wider application than just proving combinatorial identities. They are of interest wherever recurrence relations arise, for example in number theory or in the study of classical polynomials. They can also be used for asymptotic analysis, as in the example (pg.134),

$$\sum_{k=0}^{n} \binom{3n}{k} \sim 2\binom{3n}{n}$$

as $n \to \infty$. There is also the interesting possibility of shifting from the discrete case (sum) to the continuous (integral). This potential is mentioned, but not fully explored in “A=B”.

While clearly explaining the historical context and proving the relevant theorems, much of “A=B” is directed toward practical problem solving using a computer algebra package. It would be very hard to read the book without wanting to try out the ideas it contains. To assist you, a wealth of examples are worked through in both Maple and Mathematica. Some of the impact of the book will be lost on a reader familiar with neither package (anyone for an existence proof of such a mathematician?). Conscious of this issue, the authors use margin icons to delimit the practical and theoretical sections.

The cheeky title of “A=B” is not initially as revealing as say “Hypergeometric identities” would have been. However it certainly serves to identify the book, is easy to remember and does encapsulate the substance of the book, viz identities. The authors of more cumbersome titles would do well to remember that you cannot fit the text of a work into the title. We have come to expect a sense of humour from Wilf (the author of “Generatingfunctionology”), and he and his co-authors have not disappointed.

A testimony to the importance of “A=B” is that the American Maths Society awarded two of the authors (W+Z) the 1998 Steele Prize for seminal contribution to mathematics, for one of the papers on which “A=B” is based. Throw out your catalogue of identities, visit the web site (2) and buy “A=B”.