Chapter 3

GAMES WITH PERFECT INFORMATION

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Contents

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1. Introduction

The most seriously played games of perfect information (which we will call PI-games) are Chess and Go. But there are numerous other interesting PI-games: Checkers, Chinese Checkers, Halma, Nim, Hex, their misére variants, etc. Perfect information means that at each time only one of the players moves, that the game depends only on their choices, they remember the past, and in principle they know all possible futures of the game (a full definition is given in Section 2). For example War is not a PI-game since the generals move simultaneously, and Bridge and Backgammon are not PI-games because chance plays a role in them. However, as we shall see in Section 7, some cases of Pursuit and Evasion can be studied by means of PI-games, in spite of the simultaneity and continuity of the movements of the players. There exists a marvelous book, *Winning Ways* (Vols. 1 and 2), by Berlekamp, Conway and Guy (1982) which gives many old and new examples of PI-games and a deep (and light) development of their theories. Thus my first duty as a surveyor of this subject is to refer the reader to this book and to the literature quoted in its 24 sections of references. But, to the less assiduous reader, I will suggest Martin Gardner's several chapters on games [Gardner (1983, 1986)], and the *Boardgame Book* by Bell [Bell (1979) and the references therein], and other relevant chapters of this Handbook.

This survey will not overlap much with the above literature since I will focus here on *infinite* PI-games. If one wanted to play such a game one would have to play infinitely many moves! So those games are not intended to be played in reality, and their theory has (as yet) no relevance for practical play of the finite games. The main chapters of game theory, which stem from von Neumann's Minimax Theorem, are much closer to real applications. But the theory of infinite PI-games is motivated by its beauty and manifold connections with other parts of mathematics. For example, it gave new insights or new points of view in descriptive set theory [see Kechris et al. (1977, 1979, 1981, 1985) and Moschovakis (1980)], general topology, some chapters of analysis developed by G. Choquet and others, and number theory [see the surveys of Piotrowski (1985) and Telgarsky (1987)]. As mentioned above they give also a natural mathematical theory for some games of pursuit and evasion (see Section 7 of this chapter). Finite and infinite PI-games are also used in model theory [see, for example, Ehrenfeucht (1961), Lynch (1985, 1992) and Hodges (1985), and references therein] and in recursion theory [see, for example, Yates (1974, 1976)].

Before dropping the subject of finite PI-games (to which we return only briefly in Sections 5 and 6), let me emphasize the question: *What additions to the general theory would be needed to make it relevant for Chess or Go*? We can only say that in 1990 the state of the art is still confusing. On the one hand there exist machines playing Chess and Checkers well above the amateurish level, see Michie (1989) and references therein. Those machines rely essentially upon the high speed of special digital processors which allow them to examine a large number of possible future developments of the play and to choose a good move on account of this analysis. On the other hand we feel, and are told by masters, that the best human players do not think in this way! Moreover, those machines do not benefit from playing, they do not learn. So, in particular, we hesitate to call them intelligent since the ability to learn appears to us to be the single most important feature of intelligence. We think that a theory of long-range strategies or plans of attack and methods for the construction of a book or a classification of good moves will have to be developed for a more general and practical theory. We know only one general concept, called the temperature of a position, studied in Berlekamp et al. (1982), which appears to be practical; a similar concept is used in Chess-playing programs in order to decide if a given position should be analyzed further or not. In spite of the present shortcomings of the theory of finite PI-games its prospects are bright: a relevant general theory should yield a computer program such that anybody could code his favorite game (for me it would be Hex, see Section 5), and after playing a number of games with the computer, the machine should get better and better, and eventually display an overwhelming superiority. But this goal still appears to be far ahead in the future, especially for the game Go (see Chapter 1 of this volume).

This chapter on PI-games is self-contained in the sense that in principle the proofs given below do not require any specialized knowledge. The necessary background is given, for example, in the beautiful short text of Oxtoby (1971).

2. Basic concepts

A triple $\langle A, B, \varphi \rangle$, where A and B are abstract sets and $\varphi: A \times B \to \bar{\mathbb{R}}$, where $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $\mathbb{R}$ is the set of real numbers, is called *a game*. A is called the set of *strategies* of player I and B the set of *strategies* of player II. This game is played as follows: player I chooses $a \in A$ and player II chooses $b \in B$. Both choices are made independently and without any knowledge about the choice of the other player. Then II pays to I the value $\varphi(a, b)$. [Of course $\varphi(a, b) < 0$ means that II gets from I the value $|\varphi(a, b)|$.] Occasionally it will be convenient to use also a dual definition in which I pays to II the value $\varphi(a, b)$.

As usual we denote the set $\{0, 1, 2, \ldots\}$ by ω and $\alpha = \{\xi: \xi < \alpha\}$ for all ordinal numbers α. In particular $\{0, 1\} = 2$. For any sets X and Y, Y^X denotes the set of all functions $f: X \to Y$.

The intuitive idea of an *infinite game of perfect information* is the following.

There is a set P called the set of *choices*. Player I chooses $p_0 \in P$, next player II chooses $p_1 \in P$, than I chooses $p_2 \in P$, etc. There is a function $f\colon P^\omega \to \bar{\mathbb{R}}$ such that "at the end" player II pays to I the value $f(p_0, p_1, \ldots)$. More precisely, and consistently with the previous general definition of a game, $\langle A, B, \varphi \rangle$ is said to be a *game of perfect information* (a PI-game) if there exists a set P such that A is the set of all functions

$$a\colon \bigcup_{n<\omega} P^n \to P\,, \quad \text{where } P^0 = \{\emptyset\}\,,$$

B is the set of all functions

$$b\colon \bigcup_{0<n<\omega} P^n \to P$$

and there exists a function $f\colon P^\omega \to \bar{\mathbb{R}}$ such that

$$\varphi(a, b) = f(p_0, p_1, p_2, \ldots)\,,$$

where $p_0 = a(\emptyset)$, $p_1 = b(p_0)$, $p_2 = a(p_1)$, $p_3 = b(p_0, p_2)$, $p_4 = a(p_1, p_3), \ldots$ (see Figure 1).

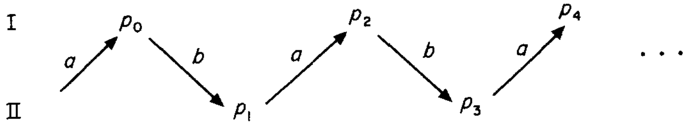

Figure 1.

From now on a game $\langle A, B, \varphi \rangle$ defined in this way will also be denoted by $\langle P^\omega, f \rangle$.

The sequence $p = (p_0, p_1, \ldots)$ defined as above will also be called *a game* and any finite sequence $(p_0, \ldots, p_{n-1}) \in P^n$ is called a *position*.

In the case when f is the *characteristic function of a set* $X \subseteq P^\omega$, i.e., $f(p) = 1$ if $p \in X$ and $f(p) = 0$ if $p \not\in X$, the game $\langle P^\omega, f \rangle$ will also be denoted by $\langle P^\omega, X \rangle$. In this case we will say that player I *wins the game* p if $f(p) = 1$ and that II *wins the game* p if $f(p) = 0$.

A game $\langle A, B, \varphi \rangle$ is called *determined* if

$$\inf_{b\in B} \sup_{a\in A} \varphi(a, b) = \sup_{a\in A} \inf_{b\in B} \varphi(a, b)\,, \tag{$*$}$$

and the common value of both sides of this equation is called *the value of the game* $\langle A, B, \varphi \rangle$. The assertion that a game is determined is also expressed by the phrase *the game has a value*. *Note: if the game is not determined, then the left-hand side of* $(*)$ *is larger than the right-hand side of* $(*)$.

If the game has the value V and there exists an a_0 such that $\varphi(a_0, b) \geqslant V$ for all b, then a_0 is called an *optimal strategy* for I. If $\varphi(a, b_0) \leqslant V$ for all a, then b_0 is called an *optimal strategy* for II.

A game can be determined but no optimal strategies need to exist. For example, this is so if $A = B =$ the open interval $(0, 1)$, and $\varphi(a, b) = a + b$. However, if the game is determined, then for every $\varepsilon > 0$ there exists a strategy a_0 which secures $\varphi(a_0, b) > V - \varepsilon$ for all $b \in B$ [or $\varphi(a_0, b) > \varepsilon$ if $V = +\infty$] and a strategy b_0 which secures $\varphi(a, b_0) < V + \varepsilon$ for all $a \in A$ [or $\varphi(a, b_0) < -\varepsilon$ if $V = -\infty$]. It follows that, *if the set of values of* φ *is finite and the game is determined, then both players have optimal strategies*. We will say that $\langle P^\omega, X\rangle$ *is a win for* I or *a win for* II if $\langle P^\omega, X\rangle$ has the value 1 or 0, respectively.

Why can games like Chess or Go be viewed as games of the form $\langle P^\omega, f\rangle$? The interpretation is the following: P is the set of all possible configurations of pieces on the board. Any infinite sequence of configurations is accepted as a game but the first player who violates the rules loses, unless the previous position is a win for one of the players or a draw. So f takes on three possible values: 1 (White wins), 0 (a draw) and -1 (Black wins), and $\langle P^\omega, f\rangle$ represents the desired game. We should add that this mathematical abstraction ignores some aspects of the reality. For example, the rules about timing are essential in most Chess tournaments but here they are ignored.

If $f: P^\omega \to \bar{\mathbb{R}}$ has the property that there exists an n such that $f(p_0, p_1, \ldots)$ does not depend on the choices p_i with $i > n$, then $\langle P^\omega, f\rangle$ is called a *finite game*. (Note that this does not imply that P is finite.)

Proposition 2.1. *Every finite game has a value.*

Proof. It is clear that the proposition is true for $n = 0$ and it is easy to see that, if it is true for $n = k$, then it is also true for $n = k + 1$. $\square$ (For another proof see Proposition 3.2.)

Proposition 2.1 was first stated as a mathematical theorem by Zermelo (1913). As we shall see in the next section, it fails for some infinite PI-games. The main goal of this chapter, the theorems in Sections 3, 8 and 9, will be refinements of this proposition relaxing the condition of finiteness in various ways.

3. Open games are determined

The first published paper devoted to general infinite PI-games is due to Gale and Stewart (1953). The material of this section is contained in that paper.

For any set P we introduce the discrete topology in P and the corresponding *product topology* in P^ω. That is, the basic neighborhoods of $p = (p_0, p_1, \ldots) \in P^\omega$ are of the form

$$U(p_0, \ldots, p_{m-1}) = \{q \in P^\omega : q_i = p_i \text{ for } i < m\} .$$

In contrast to Proposition 2.1 we have:

Proposition 3.1. *There exist sets $X \subseteq \{0,1\}^\omega$ such that the game $\langle \{0,1\}^\omega, X \rangle$ is not determined.*

Proof. Note that if we fix one strategy for one of the players, then all games in $\{0,1\}^\omega$ which remain possible constitute a perfect set, i.e., a set which is non-empty, closed and dense in itself. Now, it is an old and well known theorem of Bernstein [see Oxtoby (1971)] that there exists a partition of any Polish space S into two parts such that none of them includes a perfect set. This depends on the fact that every perfect set has no less elements than the set of all perfect subsets of S, and the Axiom of Choice (a well ordering of the space and of the set of its perfect subsets). To conclude the proof it suffices to pick for X any of the parts of a Bernstein partition of $\{0,1\}^\omega$. □

The existence of non-determined infinite PI-games follows also from each of the Theorems 4.1, 4.2, 4.4 and 4.5 of the next section.

However, we can rescue a part of Proposition 2.1 for the case of infinite games. The first step in this direction is the following.

Proposition 3.2. *If the set $X \subseteq P^\omega$ is closed or open, then the game $\langle P^\omega, X \rangle$ is determined.*

Proof. Assume that X is closed. If player II does not have a winning strategy (i.e., a strategy which secures $p \not\in X$), then it is clear that player I can maintain that advantage, i.e., I has a strategy a_0 which secures that for every $n < \omega$ the position $(p_0, \ldots, p_{n-1})$ does not yield a winning strategy for II. In particular, a_0 guarantees that for all n we have

$$U(p_0, \ldots, p_{n-1}) \cap X \neq \emptyset .$$

Since X is closed, it follows that $(p_0, p_1, \ldots) \in X$, and so a_0 is a winning strategy for I.

If X is open the theorem follows by symmetry. □

Corollary 3.3. *If $f: P^\omega \to \bar{\mathbb{R}}$ has the property that for every $x \in \bar{\mathbb{R}}$ the set $\{p \in P^\omega : f(p) < x\}$ or the set $\{p \in P^\omega : f(p) \leq x\}$ is open or closed, then the game $\langle P^\omega, f \rangle$ is determined.*

Proof. By Proposition 3.2 there exists $v \in \bar{\mathbb{R}}$ such that for every $x < v$ the game $\langle P^\omega, A(x)\rangle$, where $A(x) = \{p \in P^\omega: f(p) \leq x\}$, is a win for II, and for every $x > v$ the game $\langle P^\omega, A(x)\rangle$ is a win for I. It follows that v is a value of $\langle P^\omega, f\rangle$. □

Of course, if the game $\langle P^\omega, f\rangle$ is finite, then the function f is continuous and Corollary 3.3 yields Proposition 2.1. Much stronger results than Proposition 3.2 and Corollary 3.3 will be presented in Section 8.

4. Four classical infinite PI-games

We discuss here four games which are related to classical concepts of real analysis.

The first interesting infinite PI-game was invented by S. Mazur about 1935 [see Mauldin (1981, pp. 113–117)]. We define a slightly different (but now standard) version of that game which we call Γ_1. A set Q and a set $X \subseteq Q^\omega$ are given. The players choose alternately finite non-empty sequences of elements of Q. (As in Section 2, player I makes the first choice.) Those sequences are juxtaposed to form one sequence q in Q^ω. If $q \in X$, I wins. If $q \not\in X$, II wins. We take Q with the discrete topology and Q^ω with the product topology.

Mazur pointed out that *if X is of the first category, then II has a winning strategy for Γ_1*. Then he asked if the converse is true and offered a bottle of wine for the solution. S. Banach won the prize proving the following theorem.

Theorem 4.1. *If II has a winning strategy for Γ_1 then X is of the first category.*

Proof. If $p_0, \ldots, p_n$ are finite sequences of elements of Q, let $p_0p_1 \cdots p_n$ denote their juxtaposition. Let b_0 be a winning strategy for II. It is clear that in every neighborhood $U(p_0)$ there is a neighborhood of the form $U(p_0p_1)$, where p_0 is the first choice of I and $p_1 = b_0(p_0)$. Hence, proceeding by transfinite recursion we can construct a family F_0 of disjoint neighborhoods of the form $U(p_0p_1)$ such that their union is everywhere dense in Q^ω. Repeating the same construction within each neighborhood belonging to F_0 we obtain a family F_1 of disjoint neighborhoods $U(p_0p_1p_2p_3)$ such that $p_1 = b_0(p_0)$, $p_3 = b_0(p_0, p_2)$, $U(p_0p_1) \in F_0$ and $\bigcup(F_1)$ is everywhere dense in Q^ω. We continue in this way forming a sequence of families $F_0, F_1, \ldots$. It is clear from this construction that if $q \in \bigcap_{i<\omega} \bigcup(F_i)$, then q is the juxtaposition of a game consistent with b_0. Hence, since b_0 is a winning strategy we have $X \cap \bigcap_{i<\omega} \bigcup(F_i) = \emptyset$. And, since all $\bigcup(F_i)$ are dense and open in Q^ω, X is of the first category. □

Our second example Γ_2 was invented by L. Dubins. The rules are similar to those of Γ_1 except that here it is assumed that $Q = \{0, 1\}$ and that the choices of II are sequences of length one, i.e., elements of Q, while the choices of I are still arbitrary finite sequences of elements of Q, but this time he can also choose the empty sequence. It is easy to see that *I has a winning strategy for Γ_2 iff X has a perfect subset* (perfect means non-empty, closed and dense in itself). [*Hint*: use again the fact that the set of all sequences which can occur when I plays any fixed strategy is perfect.] Furthermore, it is easy to see that *II has a winning strategy if X is at most countable.* Davis (1964) proved the converse:

Theorem 4.2. *If II has a winning strategy for Γ_2, then X is at most countable.*

Proof. Let b_0 be a winning strategy for II. We claim that for every $x \in X$ there exists a finite sequence $p_0, p_2, \ldots, p_{2k}$ (possibly empty) of choices of I, such that $x \in U(p_0 p_1 \cdots p_{2k+1})$, where $p_1 = b_0(p_0)$, $p_3 = b_0(p_0, p_2), \ldots,$ $p_{2k+1} = b_0(p_0, \ldots, p_{2k})$, and such that for every choice p_{2k+2} of I we have $x \not\in U(p_0 p_1 \cdots p_{2k+2} p_{2k+3})$, where $p_{2k+3} = b_0(p_0, p_2, \ldots, p_{2k+2})$. Indeed, if no such $p_0, p_2, \ldots, p_{2k}$ existed, then I could play forever in such a way that $x \in U(p_0 p_1 \cdots p_n)$, where $p_0, p_1, \ldots, p_n$ is determined by his choices p_{2k} and by b_0. Hence b_0 would not be a winning strategy.

Now, to prove that X is at most countable, it suffices to show that given $p_0, p_2, \ldots, p_{2k}$ there is at most one point $x \in X$ with the above property. So suppose to the contrary that there are two such points $x, x' \in U(p_0 p_1 \cdots p_{2k+1})$, where $p_1 = b_0(p_0), \ldots, p_{2k+1} = b_0(p_0, p_2, \ldots, p_{2k})$. Let q be the longest initial segment of x which equals an initial segment of x'. Then I can choose p_{2k+2} such that $p_0 p_1 \cdots p_{2k+1} p_{2k+2} = q$. Now, since $Q = \{0, 1\}$, either x or x' belongs to $U(p_0 p_1 \cdots p_{2k+3})$, where $p_{2k+3} = b_0(p_0, p_2, \ldots, p_{2k+2})$, contrary to our supposition about those points. This concludes the proof. □

The third example Γ_3 is defined as follows. A set S is given. Player I splits S into two parts. Player II chooses one of them. Again I splits the chosen part into two disjoint parts and II choses one of them, etc. I wins iff the intersection of the chosen parts is not empty and II wins iff it is empty. It is easy to see that *I has a winning strategy iff $|S| \geq 2^{\aleph_0}$*, and that *II has a winning strategy if $|S| \leq \aleph_0$*. R.M. Solovay proved the converse.

Theorem 4.3. *If II has a winning strategy for Γ_3, then $|S| \leq \aleph_0$.*

Proof. The idea is similar to that of the former proof. It suffices to consider the case when $S \subseteq \mathbb{R}$, and to restrict player I to such partitions of S which are induced by the partition of $\mathbb{R}$ into the rays $\{x: x < r\}$ and $\{x: x \geq r\}$, where r is a rational number. Now we argue in the same way as in the proof for Γ_2 that if b_0 is a winning strategy for II, then for every $x \in S$ there exists a sequence of partitioning rationals $r_0, \ldots, r_n$ such that $x \in b_0(r_0, \ldots, r_n)$ but for every r_{n+1}, we have $x \notin b_0(r_0, \ldots, r_n, r_{n+1})$, and also that for every $r_0, \ldots, r_n$ there exists at most one such x. Of course this implies $|S| \leq \aleph_0$. □

Our fourth example Γ_4 is due to L. Harrington and his analysis of Γ_4 given below simplifies former work of Mycielski and Swierczkowski (1964). We consider the Cantor set $\{0, 1\}^\omega$ with its natural Borel probability measure μ, given by $\mu(U(q)) = 1/2^n$ for any $q \in \{0, 1\}^n$. A set $X \subseteq \{0, 1\}^\omega$ is given and Γ_4 is played as follows. Player I chooses a rational number $\varepsilon > 0$ and a number $p_0 \in \{0, 1\}$. Then II chooses a clopen (i.e., closed and open) set $A_1 \subseteq \{0, 1\}^\omega$ with $\mu(A_1) < \varepsilon/4$. At stage n player I chooses $p_{n-1} \in \{0, 1\}$ and then II chooses a clopen set $A_n \subseteq \{0, 1\}^\omega$ with $\mu(A_n) < \varepsilon/4^n$. Player I wins iff $(p_0, p_1, \ldots) \in X \setminus \bigcup_{1 \leq i < \omega} A_i$. Player II wins otherwise.

Harrington has proved two facts about Γ_4.

Theorem 4.4. *If X has inner measure zero, then I has no winning strategy for* Γ_4.

Proof. Suppose to the contrary that I has a winning strategy a_0. Let A be the set of all sequences $(p_0, p_1, \ldots)$ which can occur when I uses a_0. Let $\tilde{A}$ be the set of all sequences of clopen sets $A_i \subseteq \{0, 1\}^\omega$ which can occur when I uses a_0. Since $\{0, 1\}^\omega$ has countably many clopen subsets we have a natural identification of $\tilde{A}$ with ω^ω. Providing ω^ω with the natural product topology we see that $a_0 : \tilde{A} \to A$ is a continuous surjection. Hence A is an analytic set (the definition is given at the beginning of Section 9). It follows that A is measurable [see Oxtoby (1971)] and, since $A \subseteq X$, $\mu(A) = 0$. One checks now that there is a sequence of clopen sets $A_1, A_2, \ldots$, with $\mu(A_n) < \varepsilon/4^n$ (ε being given to I by a_0) such that $A \subset A_1 \cup A_2 \cup \cdots$. This contradicts the assumption that a_0 was a winning strategy for I. □

Theorem 4.5. *If II has a winning strategy for* Γ_4, *then* $\mu(X) = 0$.

Proof. Let b_0 be a winning strategy for II. Suppose to the contrary that X has outer measure $\alpha > 0$. Let I play $\varepsilon < \alpha$. Then for each n there are only 2^n plays $(p_0, \ldots, p_{n-1})$ of I, and so at most 2^n answers $A_n = b_0(\varepsilon, p_0, \ldots, p_{n-1})$. Hence $\mu(\bigcup_{(p_0,\ldots,p_{n-1})} A_n) \leq \varepsilon/2^n$. Let A be the union of all the sets A_n which II could play using b_0 given the above ε. So $\mu(A) \leq \varepsilon < \alpha$. Hence I could play $(p_0, p_1, \ldots) \in X \setminus A$, contradicting the assumption that b_0 was a winning strategy. □

Three classical properties of sets are now seen to have a game theoretical role:

Corollary 4.6. *Given a complete separable metric space M and a set $X \subseteq M$ which does not have the property of Baire, or, is uncountable but without any perfect subsets, or is not measurable relative to some Borel measure in M, one can define a game $\langle \{0,1\}^\omega, Y \rangle$ which is not determined.*

Proof. By a well-known construction [see Oxtoby (1971)] we can assume without loss of generality that $M = \{0,1\}^\omega$, with its product topology and with the measure defined above. Let $X \subseteq M$ be a set without the property of Baire, U be the maximal open set in M such that $X \cap U$ is of the first category and V be the maximal open set such that $V \cap (M \setminus X)$ is of the first category. We see that the interior of $M \setminus (U \cup V)$ is not empty (otherwise X would have the property of Baire). Thus there is a basic neighborhood $W \subseteq M \setminus (U \cup V)$. We identify W with $\{0,1\}^\omega$ in the obvious way and define S to be the image of $X \cap W$ under this identification. So we see that S is not of the first category and, moreover, for each p_0 of I, $U(p_0) \cap (\{0,1\}^\omega \setminus S)$ is not of the first category. Thus, by the result of Banach (4.1), neither II nor I has a winning strategy for the game Γ_1.

Now Γ_1 is a game of the form $\langle P^\omega, Z \rangle$, where P is countable. We can turn such a game into one of the form $\langle \{0,1\}^\omega, Y \rangle$ using the fact that the number of consecutive 1's chosen by a player followed by his choice of 0 can code an element of P (the intermediate choices of the other player having no influence on the result of the game).

For the alternative assumptions about X considered in the corollary we apply the results of Davis and Harrington to obtain non-determined games Γ_2 and Γ_4. (For Γ_2 we use the fact that a perfect set in $\{0,1\}^\omega$ has cardinality $2^{\aleph_0}$. For Γ_4 we need a set S with inner measure 0 and outer measure 1. Its construction from X is similar to the above construction of S for Γ_1.) □

All known constructions of M and X satisfying one of the conditions of Corollary 4.6 have used the Axiom of Choice, and, after the work of Paul Cohen, R.M. Solovay and others, it is known that indeed the Axiom of Choice is unavoidable in any such construction. Thus Corollary 4.6 suggested the stronger conjecture of Mycielski and Steinhaus (1962) that *the Axiom of Choice is essential in any proof of the existence of sets $X \subseteq \{0,1\}^\omega$ such that the game $\langle \{0,1\}^\omega, X \rangle$ is not determined*. This has been proved recently by Martin and Steel (1989) (see Section 8 below).

In the same order of ideas, Theorem 4.3 shows that *the Continuum Hypothesis is equivalent to the determinacy of a natural class of PI-games.*

For a study of some games similar to Γ_2 but with $|Q| > 2$, see Louveau (1980). Many other games related to Γ_2 and Γ_3 were studied by F. Galvin et al. (unpublished); see also the survey by Telgarsky (1987).

5. The game of Hex and its unsolved problem

Before plunging deeper into the theory of infinite games we discuss in this and the next section a few particularly interesting finite games. We begin with one of the simplest finite games of perfect information called Hex which has not been solved in a practical sense. Hex is played as follows. We use a board with a honeycomb pattern as in Figure 2. The players alternatively put white or black stones on the hexagons. White begins and he wins if the white stones connect the top of the board with the bottom. Black wins if the black stones connect the left edge with the right edge.

Theorem 5.1. (i) *When the board is filled with stones, then one of the players has won and the other has lost.*

(ii) *White has a winning strategy.*

Proof (in outline). (i) If White has not won and the board is full, then consider the black stones adjacent to the set of those white stones which are

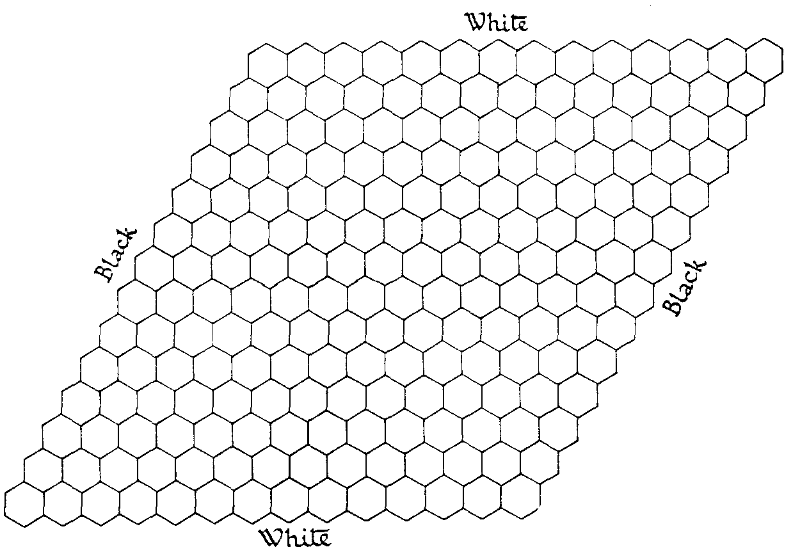

Figure 2. A 14×14 Hex board.

connected by some white path to the upper side of the board. Those stones are all black and together with the remaining black stones of the upper line of the board they contain a black path from the left edge to the right edge. Thus Black is the winner. [For more details, see Gale (1979).]

(ii) By (i) and Proposition 2.1 one of the players has a winning strategy. Suppose to the contrary that it is Black who has such a strategy b_0. Now it is easy to modify b_0 so that it becomes a winning strategy for White. (*Hint*: White forgets his first move and then he uses b_0.) Thus both players would have a winning strategy, which is a contradiction. □

Problem. Find a useful description of a winning strategy for White!

This open problem is a good example of the general problem in the theory of finite PI-games which was discussed at the end of Section 1. In practice Hex on a board of size 14×14 is an interesting game and the advantage of White is hardly noticeable. It is surprising that such a very concrete finitistic existential theorem like Theorem 5.1(ii) can be meaningless from the point of view of applications. [Probably, strict constructivists would not accept our proof of Theorem 5.1(ii).]

Hex has a relative called Bridge-it for which a similar theorem is true. But for Bridge-it a useful description of a winning strategy for player I *has been found* [see Berlekamp et al. (1982, p. 680)]. However, this does not seem to help for the problem on Hex. Dual Hex, in which winning means losing in Hex, is also an interesting unsolved game. Here Black has a winning strategy.

Of course for Chess we do not know whether White or Black has a winning strategy or if (most probably) both have strategies that secure at least a draw.

It is proved in Even and Tarjan (1976) that some games of the same type as Hex are difficult in the sense that the problem of deciding if a position is a win for I or for II is complete in polynomial space (in the terminology of the theory of complexity of algorithms).

It is interesting that Theorem 5.1(i) implies easily the Brouwer fixed point theorem [see Gale (1979)].

6. An interplay between some finite and infinite games

Let G be a finite bipartite oriented graph. In other words G is a system $\langle P, Q, E \rangle$, where P and Q are finite disjoint sets and $E \subseteq (P \times Q) \cup (Q \times P)$ is called the set of arrows. We assume moreover that for each $(a, b) \in E$ there exists c such that $(b, c) \in E$. A function $\varphi\colon E \to \mathbb{R}$ is given and a point $p_{\text{first}} \in P$ is fixed. The players I and II pick alternately $p_0 = p_{\text{first}}$, $q_0 \in Q$, $p_1 \in P$, $q_1 \in$

$Q, \ldots$ such that $(p_i, q_i) \in E$ and $(q_i, p_{i+1}) \in E$, thereby defining a zig-zag path composed of arrows.

We define three PI-games.

G_1: player II pays to player I the value

$$\limsup_{n\to\infty} \frac{1}{2n} \sum_{i=0}^{n-1} (\varphi(p_i, q_i) + \varphi(q_i, p_{i+1})) .$$

G_2: player II pays to player I the value

$$\liminf_{n\to\infty} \frac{1}{2n} \sum_{i=0}^{n-1} (\varphi(p_i, q_i) + \varphi(q_i, p_{i+1})) .$$

G_3: the game ends as soon as a closed loop arises in the path defined by the players, i.e., as soon as I picks any $p_n \in \{p_0, \ldots, p_{n-1}\}$ or II picks $q_n \in \{q_0, \ldots, q_{n-1}\}$, whichever happens earlier. Then II pays to I the "loop average" v defined as follows. In the first case $p_n = p_m$ for some $m < n$, and then

$$v = \frac{1}{2(n-m)} \sum_{i=m}^{n-1} (\varphi(p_i, q_i) + \varphi(q_i, p_{i+1})) ;$$

in the second case $q_n = q_m$ with $m < n$ and then

$$v = \frac{1}{2(n-m)} \sum_{i=m}^{n-1} (\varphi(q_i, p_{i+1}) + \varphi(p_{i+1}, q_{i+1})) .$$

Thus in all three games the players are competing to minimize or maximize the means of some numbers which they encounter on the arrows of the graph.

Since the game G_3 is finite, by Proposition 2.1, it has a value V. Given a strategy σ of one of the players which secures V in G_3, each of the infinite games G_1 and G_2 can be played according to σ, by forgetting the loops (which necessarily arise). This also secures V [see Ehrenfeucht and Mycielski (1979) for details]. So it follows that *the games G_1 and G_2 are determined, and have the same value V as G_3.*

A strategy a for player I is called *positional* if $a(q_0, \ldots, q_n)$ depends only only on q_n. In a similar way a strategy b for II is *positional* if $b(p_0, \ldots, p_n)$ depends only on p_n.

Theorem 6.1. *Both players have positional strategies a_0 and b_0 which secure V for each of the games G_1, G_2 and G_3.*

This theorem was shown in Ehrenfeucht and Mycielski (1979). We shall not reproduce its proof here but only mention that it was helpful to use the infinite games G_1 and G_2 to prove the claim about the finite game G_3 and vice versa. In fact no direct proof is known. So, there is at least one example where infinite PI-games help us to analyze some finite PI-games.

An open problem related to the above games is the following: Is an appropriate version of Theorem 6.1, where P and Q are compact spaces and φ is continuous, still true?

7. Continuous PI-games

In this section we extend the theory of PI-games with countable sequences to a theory with functions over the interval $[0, \infty)$. R. Isaacs in the United States and H. Steinhaus and A. Zieba in Poland originated this development. Here are some examples of continuous games.

Two dogs try to catch a hare in an unbounded plane, or one dog tries to catch a hare in a half-plane. The purpose of the dogs is to minimize the time of the game and the purpose of the hare is to maximize it. We assume that each dog is faster than the hare and that only the velocities are bounded while the accelerations are not. There are neat solutions of those two special games: at each moment t the hare should run full speed toward any point a_t such that a_t is the most distant from him among all points which he can reach prior to any of the dogs (here "prior" is understood in the sense of $\leq$). And the dogs, at each instant t, should run full speed toward that same point a_t. (To achieve the best result the hare does not have to change the point a_t during the game.)

Now, how to turn the above statements into mathematical theorems? Notice that the sets of strategies have not been defined so we have not constructed any games in the sense of Section 2. The main point of this section is to build such definitions which may be useful for a wide variety of games. The literature of this subject is rich [see, for example, Behrand (1987), Hájek (1975), Kuhn and Szegö (1971), Mycielski (1988), and Rodin (1987)], but the games are rarely defined with full precision. The fundamentals of this theory presented in this section will not use the concepts of differentiation or integration.

Let P and Q be arbitrary sets and $F_{\mathrm{I}} \subseteq P^{[0,\infty)}$ and $F_{\mathrm{II}} \subseteq Q^{[0,\infty)}$ two sets of functions from $[0, \infty)$ to P or Q, respectively. We assume that F_X, for $X = \mathrm{I}, \mathrm{II}$, are *closed* in the following sense: if f is a function with domain $[0, \infty)$ such that for all $T > 0$ the restriction $f \restriction [0, T)$ has an extension in F_X, then $f \in F_X$.

We will say that F_X is *saturated* if it is closed under the following operations. For every $\delta > 0$ if $f \in F_X$, then $f_\delta \in F_X$, where

$$f_\delta(t) = \begin{cases} f(0) & \text{for } 0 \leqslant t < \delta, \text{ and} \\ f(t-\delta) & \text{for } t \geqslant \delta \ . \end{cases}$$

Let a function $\psi: F_{\mathrm{I}} \times F_{\mathrm{II}} \to \bar{\mathbb{R}}$ be given. We will define in terms of ψ the payoff functions of two PI-games G^+ and G^-. In order for those games to be convincing models of continuous games (like the games of the above examples with dogs and a hare) we will need that ψ satisfies at least one of the following two conditions of semicontinuity.

(S_1) The space F_{I} is saturated and for every $\varepsilon > 0$ there exists a $\Delta > 0$ such that for all $\delta \in [0, \Delta]$ and all $(p, q) \in F_{\mathrm{I}} \times F_{\mathrm{II}}$ we have

$$\psi(p_\delta, q) \leqslant \psi(p, q) + \varepsilon \ .$$

We consider also a dual property for ψ:

(S_2) The space F_{II} is saturated and for every $\varepsilon > 0$ there exists a $\Delta > 0$ such that for all $\delta \in [0, \Delta]$ and all $(p, q) \in F_{\mathrm{I}} \times F_{\mathrm{II}}$ we have

$$\psi(p, q_\delta) \geqslant \psi(p, q) - \varepsilon \ .$$

The system $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi \rangle$ will be called *normal* iff F_{I} and F_{II} are closed and (S_1) or (S_2) holds.

Example 1. A metric space M with a distance function $d(x, y)$ and two points $p_0, q_0 \in M$ are given and $P = Q = M$. F_{I} is the set of all functions $\boldsymbol{p}: [0, \infty) \to M$ such that $\boldsymbol{p}(0) = p_0$, and

$$d(\boldsymbol{p}(t_1), \boldsymbol{p}(t_0)) \leqslant |t_1 - t_0| \quad \text{for all } t_0, t_1 \geqslant 0 \ .$$

F_{II} is the set of all functions $\boldsymbol{q}: [0, \infty) \to Q$ such that $\boldsymbol{q}(0) = q_0$, and

$$d(\boldsymbol{q}(t_1), \boldsymbol{q}(t_0)) \leqslant v|t_1 - t_0| \quad \text{for all } t_0, t_1 \geqslant 0 \ ,$$

where v is a constant in the interval $[0, 1]$.

Now ψ can be defined in many ways, e.g.

$$\psi(\boldsymbol{p}, \boldsymbol{q}) = d(\boldsymbol{p}(1), \boldsymbol{q}(1)) \ ,$$

or

$$\psi(\boldsymbol{p}, \boldsymbol{q}) = \limsup_{t \to \infty} d(\boldsymbol{p}(t), \boldsymbol{q}(t)) \ .$$

It is easy to prove that in these cases the system $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi \rangle$ is normal.

Example 2. The spaces F_{I} and F_{II} are defined as in the previous example but with further restrictions. For example, the total length of every $\boldsymbol{p}$ and/or every $\boldsymbol{q}$ is bounded, i.e., say for all $\boldsymbol{p} \in F_{\mathrm{I}}$,

$$\lim_{n\to\infty} \sum_{i=0}^{\infty} d\left(\boldsymbol{p}\left(\frac{i}{n}\right), \boldsymbol{p}\left(\frac{i+1}{n}\right)\right) \leq L ;$$

or P and/or Q is $\mathbb{R}^n$ and the acceleration of every $\boldsymbol{p}$ and/or every $\boldsymbol{q}$ is bounded, i.e., say for all $\boldsymbol{p} \in F_{\mathrm{I}}$,

$$\left|\left|\boldsymbol{p}(t_0) - 2\boldsymbol{p}\left(\frac{t_0 + t_1}{2}\right) + \boldsymbol{p}(t_1)\right|\right| \leq A(t_1 - t_0)^2 \quad \text{for all } t_0, t_1 \geq 0 .$$

If the space F_X ($X = \mathrm{I}, \mathrm{II}$) represents the possible trajectories of a vehicle, the above conditions may correspond to limits of the available fuel or power. Conditions of this kind and functionals ψ as in the previous example are compatible with normality.

Example 3. F_{I} and F_{II} are the sets of all measurable functions $\boldsymbol{p}: [0, \infty) \to B_{\mathrm{I}}$ and $\boldsymbol{q}: [0, \infty) \to B_{\mathrm{II}}$, respectively, where B_{I} and B_{II} are some bounded sets in $\mathbb{R}^n$. And

$$\psi(\boldsymbol{p}, \boldsymbol{q}) = \left|\left|\int_0^1 (\boldsymbol{p}(t) - \boldsymbol{q}(t))\,\mathrm{d}t\right|\right| .$$

Such F_X are called spaces of *control functions*. Again it is easy to see that the system $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi\rangle$ is normal. Similar (and more complicated) normal systems are considered in the theory of differential games.

Given $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi\rangle$, with F_{I} and F_{II} closed in the sense defined above, we define two PI-games G^+ and G^-. In G^+ player I chooses some $\delta > 0$ and a path $\boldsymbol{p}_0: [0, \delta) \to P$. Then II chooses $\boldsymbol{q}_0: [0, \delta) \to Q$. Again I chooses $\boldsymbol{p}_1: [\delta, 2\delta) \to P$ and II chooses $\boldsymbol{q}_1: [\delta, 2\delta) \to Q$, etc. If $(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i) \in F_{\mathrm{I}} \times F_{\mathrm{II}}$, then I pays to II the value $\psi(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i)$. If $(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i) \not\in F_{\mathrm{I}} \times F_{\mathrm{II}}$, then there is at least an n such that $\bigcup_{i<n} \boldsymbol{p}_i$ has no extension to a function in F_{I} or $\bigcup_{i<n} \boldsymbol{q}_i$ has no extension to a function in F_{II}. If n satisfies the first alternative, I pays ∞ to II. Otherwise II pays ∞ to I.

The game G^- is defined in the same way except that now player II chooses $\delta > 0$ and $\boldsymbol{q}_0: [0, \delta) \to Q$ and then I chooses $\boldsymbol{p}_0: [0, \delta) \to P$, etc. Again if $(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i) \in F_{\mathrm{I}} \times F_{\mathrm{II}}$, then I pays to II the value $\psi(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i)$ and again, if $(\bigcup \boldsymbol{p}_i, \bigcup \boldsymbol{q}_i) \not\in F_{\mathrm{I}} \times F_{\mathrm{II}}$, the player who made the first move causing this pays ∞ to the other.

Since both G^+ and G^- are PI-games, under very general conditions about ψ (see Corollary 3.3 and Theorem 8.1 in Section 8), both games G^+ and G^- have values. We denote those values by V^+ and V^-, respectively. By a proof similar to the proof of Theorem 5.1(ii), it follows from these definitions that

$$V^+ \geqslant V^- .$$

We claim that if $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi\rangle$ is normal, then G^+ and G^- represent essentially the same game. More precisely, we have the following theorem.

Theorem 7.1. *If V^+ and V^- exist and the system $\langle F_{\mathrm{I}}, F_{\mathrm{II}}, \psi\rangle$ is normal, then*

$$V^+ = V^- .$$

Proof. Suppose the condition (S_1) of normality holds. Choose $\varepsilon > 0$. Given a strategy σ^- for I in G^- which secures a payoff $\leqslant V^- + \varepsilon$ we will construct a strategy σ^+ for I in G^+ which secures a payoff $\leqslant V^- + 2\varepsilon$. Of course this implies $V^+ \leqslant V^-$ and so $V^+ = V^-$. Let σ^+ choose δ according to (S_1), and $p_0^+(t) = p_0$ for $t \in [0, \delta)$. When II answers with some $q_0\colon [0, \delta) \to Q$, then σ^+ chooses $p_1^+(t) = p_0^-(t - \delta)$ for $t \in [\delta, 2\delta)$, where $p_0^- = \sigma^-(q_0)$. Then II chooses $q_1\colon [\delta, 2\delta) \to Q$ and σ^+ chooses $p_2^+(t) = p_1^-(t - \delta)$ for $t \in [2\delta, 3\delta)$, where $p_1^- = \sigma^-(q_0, q_1)$, etc. Now the pair $(\bigcup p_i^-, \bigcup q_i)$ is consistent with a game in G^- where I uses σ^-. Also, we have $\bigcup p_i^+ = (\bigcup p_i^-)_\delta$. Hence, by (S_1),

$$\psi(\bigcup p_i^+, \bigcup q_i) \leqslant \psi(\bigcup p_i^-, \bigcup q_i) + \varepsilon \leqslant V^- + 2\varepsilon .$$

This concludes the proof in the case (S_1). In the case (S_2) the proof is symmetric. □

The theorems about the existence of values presented in Section 8 plus the above Theorem 7.1 encompass the existential part of the theory of continuous PI-games over normal systems. However, we will consider an interesting case of continuous PI-games, called *pursuit and evasion*, which is not normal:

M, P, Q, F_{I} and F_{II} are defined as in Example 1, but now $\psi(p, q)$ is the least t such that $p(t) = q(t)$, if such a t exists, and $\psi(p, q) = \infty$ otherwise. It is easy to see that ψ violates (S_1) and (S_2). Still an interesting theory is possible. We will assume that the metric space M is complete, locally compact and connected by arcs of finite length. This is a natural assumption because under those conditions *for every two points of M there exists a shortest arc connecting them.* Then we can also assume without loss of generality that d is the geodesic metric, i.e., $d(x, y) =$ length of the shortest arc from x to y.

Now consider the game G^-. By Corollary 3.3, G^- has a value V^-. Of course I can be called the *pursuer* and II the *evader*, and G^- gives some tiny unfair advantage to the pursuer (because II has to declare first his trajectory over $[0, \delta)$, then $[\delta, 2\delta)$, etc.).

In this setting the dual game G^+ is uninteresting because, for trivial reasons, in most cases its value will be ∞. However, there exists a similar game G^{++} which gives a tiny unfair advantage to the evader. In G^{++} player II chooses first a number $\delta_0 > 0$, then I chooses δ and $\boldsymbol{p}_0$: $[0, \delta) \to M$, then II chooses $\boldsymbol{q}_0$: $[0, \delta) \to M$, and again I chooses $\boldsymbol{p}_1$: $[\delta, 2\delta) \to M$, etc. Otherwise the rules are the same as in G^+, except that now I pays to II the least value t such that the distance from $\boldsymbol{p}(t)$ to $\boldsymbol{q}(t)$ is $\leqslant v\delta_0$, where $\boldsymbol{p} = \bigcup \boldsymbol{p}_i$ and $\boldsymbol{q} = \bigcup \boldsymbol{q}_i$. Again, by Corollary 3.3, it is clear that G^{++} has a value V^{++}. It is intuitively clear that $V^- \leqslant V^{++}$. Games very similar to G^- and G^{++} have been studied in Mycielski (1988) and the methods of that paper can be easily modified to prove the following theorems.

Theorem 7.2. *If* $v < 1$, *then* $V^- = V^{++}$.

(We do not know any example where $v = 1$ and $V^- < V^{++}$.)

By Theorem 7.2, for $v < 1$, it is legitimate to denote both V^- and V^{++} by V.

Now, given (M, d), it is interesting to study V as a function of p_0, q_0 and v (we will omit the argument v when its value is fixed). The function $V(p_0, q_0)$ is useful since the best strategy for I is to choose $\boldsymbol{p}_i$: $[i\delta, (i+1)\delta) \to M$ such as to keep in F_{I} and to minimize $V(\boldsymbol{p}_i((i+1)\delta), \boldsymbol{q}_i((i+1)\delta))$, and the best strategy for II, after his choice of δ, is to choose $\boldsymbol{q}_i$:$[i\delta, (i+1)\delta) \to M$ such as to keep in F_{II} and to maximize $\inf\{V(\boldsymbol{p}_i((i+1)\delta), \boldsymbol{q}_i((i+1\delta)): \boldsymbol{p}_i$ is any choice of I$\}$.

The function $V(p, q, v)$ was studied in Mycielski (1988), where the following theorems are proved.

Theorem 7.3. *If* $v < 1$, *then*

(i) $d(x, y) \leqslant V(x, y, v) \leqslant d(x, y)/(1 - v)$;

(ii) $|V(x_1, y, v) - V(x_2, y, v)| \leqslant d(x_1, x_2)/(1 - v)$;

(iii) $|V(x, y_1, v) - V(x, y_2, v)| \leqslant d(y_1, y_2)/(1 - v)$;

(iv) *if* $0 \leqslant v_1 < v_2 < 1$, *then*

$$V(x, y, v_1) \leqslant V(x, y, v_2) \leqslant \frac{1 - v_1}{1 - v_2} V(x, y, v_1) .$$

We do not know if $V(x, y, v) \to V(x, y, 1)$ for $v \uparrow 1$.

Fixing $v < 1$, the next theorem gives a characterization of $V(x, y)$ which does not depend on any game theoretic concepts.

Theorem 7.4. *The function $V: M \times M \to \mathbb{R}$ satisfies, and is the only function satisfying, the following four conditions*:

(i) $V(p, p) = 0$;

(ii) $V(p, q) - d(p, x) \leq \sup\{V(x, y): d(y, q) \leq vd(x, p)\}$;

(iii) $V(p, q) \geq d(p, q)$;

(iv) $\max(0, V(p, q) - (1/v)d(q, y)) \geq \inf\{V(x, y): d(x, p) \leq (1/v)d(q, y)\}$.

The intuitive meaning of the inequality (ii) is the following: if I moves from p to x using the time $d(p, x)$, then II has an answer y using the same time such that after those moves the value $V(p, q)$ will not decrease by more than $d(p, x)$. The intuitive meaning of (iv) is the following: if II moves from q to y using the time $(1/v)d(q, y) \leq V(p, q)$, then I has an answer x using the same time such that the value $V(p, q)$ will decrease at least by $(1/v)d(q, y)$.

The above theorem implies the Isaacs equation [see Behrand (1987) and Mycielski (1988)].

Corollary 7.5. *If M is a Riemannian manifold with boundary, e.g., an n-dimensional polytope in $\mathbb{R}^n$, x_0 and y_0 are in the interior of M, and V is differentiable at (x_0, y_0), then V satisfies the Isaacs equation*

$$\|\nabla_x V(x_0, y_0)\| = 1 + v\|\nabla_y V(x_0, y_0)\| ,$$

where ∇ is the gradient operator.

In spite of all those facts and properties of $V(x, y)$, this function is still unknown, even for some simple spaces M such as a plane with the interior of a circle removed or if M is a circular disk. Those problems are discussed in Breakwell (1989) and Mycielski (1988).

The following function $W(x, y)$ could be useful:

$$W(x, y) = \sup\{d(x, z): d(x, z) > \frac{1}{v} d(y, z)\} .$$

Problem. Is it true that $V(x_1, y) < V(x_2, y)$ if $W(x_1, y) < W(x_2, y)$?

If the answer is yes, then the best strategy for I is to minimize W, which, as a rule, is much easier to compute than V. For the two games with dogs and a hare defined at the beginning of this section the answer is yes, and this is easy to prove by means of the games G^{++}.

8. The main results of the theory of infinite PI-games

The considerations of Sections 2 and 3 suggest the following general problem.

Problem. Let $g: A \times B \to C$ be a continuous function, where A, B and C are compact spaces. Suppose that for every continuous function $f: C \to \bar{\mathbb{R}}$ the game $\langle A, B, f \circ g \rangle$ is determined. Must it be also determined for every Borel measurable f?

This problem is open already for the case when C is the Cantor space $\{0, 1\}^\omega$, and instead of all Borel measurable functions we consider only characteristic functions of sets of class $\boldsymbol{F}_\sigma$ or $\boldsymbol{G}_\sigma$.

The only known results about this problem pertain to the case of PI-games and do not assume compactness of the spaces A, B and C. In this case $C = P^\omega$ with the product topology (see Section 3), A and B are defined as in Section 2 and $g(a, b) = (p_0, p_1, \ldots)$. Let us state immediately those results (which are the deepest theorems of the theory of PI-games), and explain later the concepts and terminology used in those statements. Part (ii) of Theorem 8.1 will be proved in Section 9.

Theorem 8.1. (i) *If $X \subseteq P^\omega$ is a Borel set, then the game $\langle P^\omega, X \rangle$ is determined (assuming the usual set theory ZFC).*

(ii) *If $X \subseteq P^\omega$ is an analytic set, then the game $\langle P^\omega, X \rangle$ is determined [assuming ZFC + there exists an Erdös cardinal $\kappa \to (\omega_1)_\lambda^{<\omega}$, where $\lambda = 2^{|P| + \aleph_0}$].*

(iii) *If $X \subseteq \omega^\omega$ and $X \in L(\mathbb{R})$, then the game $\langle \omega^\omega, X \rangle$ is determined (assuming ZFC + there exists a measurable cardinal with ω Woodin cardinals below it).*

A brief history and some outstanding qualities of these results are the following. Theorem 8.1(i) is due to Martin (1975, 1985). Thereby he solved a problem already stated by Gale and Stewart (1953). This theorem is remarkable not only because of its very ingeneous proof but also because it was the first theorem *in real analysis* the proof of which required the full power of the set theory *ZFC*. Indeed, Harvey Friedman proved that Theorem 8.1(i) depends on the axiom schema of replacement, while all the former theorems of real analysis could be proved from the weaker axiom schema of comprehension. We shall not include here any proof of Theorem 8.1(i) since it is not easier than that of 8.1(ii); the conclusion of 8.1(ii) is stronger, and we feel that the refinement of *ZFC* upon which 8.1(ii) depends is very natural.

Theorem 8.1(ii) is also due to Martin (1970). Again its proof is very remarkable since it is the simplest application of an axiom *beyond ZFC* to a theorem in real analysis. A set $X \subseteq P^\omega$ is called *analytic* if X is a projection of a closed subset of the product space $P^\omega \times \omega^\omega$ into P^ω, where ω^ω has also the product topology (ω^ω is homeomorphic to the set of irrational numbers of the real line). We will see in Section 9 that every Borel subset of P^ω is analytic but not vice versa. So, as mentioned above, the conclusion of Theorem 8.1(ii) is stronger than that of 8.1(i) (at the cost of a stronger set theoretic assumption). The Erdös cardinal numbers will be explained in Section 9. A measurable cardinal $\kappa > |P|$ would suffice since it satisfies the condition in 8.1(ii).

Theorem 8.1(iii) was proved by Martin and Steel (1989) using a former theorem of H. Woodin (the proof of the latter is still unpublished). $L(\mathbb{R})$ denotes the least class of sets which constitutes a model of *ZF* (i.e. *ZFC* without the Axiom of Choice), contains all the ordinal numbers and all the real numbers and is such that if $x \in L(\mathbb{R})$ and $y \in x$, then $y \in L(\mathbb{R})$. Theorem 8.1(iii) solves in the affirmative the problem raised in Mycielski and Steinhaus (1962) of showing that the Axiom of Choice is necessary to prove the existence of sets $X \subseteq \{0, 1\}^\omega$ such that the game $\langle \{0, 1\}^\omega, X \rangle$ is not determined. Also it yields a very large class $\boldsymbol{F}$ of sets $X \subseteq \omega^\omega$, namely $\boldsymbol{F} = \mathcal{P}(\omega^\omega) \cap L(\mathbb{R})$, where $\mathcal{P}(S) = \{R: R \subseteq S\}$, such that all the games $\langle \omega^\omega, X \rangle$ with $X \in \boldsymbol{F}$ are determined. This family $\boldsymbol{F}$ is closed under countable unions and complementation, under the Souslin operation (see Section 9) and many other set theoretic constructions. In particular, $\boldsymbol{F}$ includes all projective subsets of ω^ω. For the case $|P| = \omega$ the conclusion of Theorem 8.1(iii) is much stronger than that of 8.1(ii). But, as we shall see in Section 10, Theorem 8.1(iii) would fail if ω^ω was replaced by P^ω with an uncountable set P.

The concept of Woodin cardinals will not be explained here since it is rather technical. But there are several possible additions to *ZFC* which are simpler, intuitively well motivated and stronger than those of Theorem 8.1(iii). For example, the existence of 1-extendible cardinals [an axiom proposed by W. Reinhardt, see Solovay et al. (1978)] implies the existence of a measurable cardinal with ω Woodin cardinals below it. Again we cannot present here enough logic and set theory to explain the above axiom, but we can state an axiom proposed by P. Vopenka which is still stronger and hence also suffices for the conclusion of Theorem 8.1(iii).

(V) *If C is a proper class of graphs, then there are two graphs in C such that one is isomorphic to an induced subgraph of the other.*

The intuitive idea supporting (V) is the following: a proper class must be so large relative to the size of a set that a proper class of graphs must be repetitive in the sense expressed in (V). The proof of Theorem 8.1(iii) [even the part published in Martin and Steel (1989)] is much harder than the proof of 8.1(ii) given in the next section.

Let us add that H. Friedman, L. Harrington, D.A. Martin and H. Woodin have shown that the set theoretic axioms in Theorem 8.1(i), (ii) and (iii) are nearly as weak as possible for proving those theorems.

9. Proof of Theorem 8.1(ii)

For any topological space S, a set $X \subseteq S$ is called *analytic* if X is a projection of a closed subset of the product space $S \times \omega^\omega$ into S. For example, if $f\colon \omega^\omega \to S$ is a continuous function, then the image $f[\omega^\omega]$ is the projection of the graph of f into S, whence $f[\omega^\omega]$ is analytic. We list some elementary facts about analytic sets.

9.1. *A union of countably many analytic sets is analytic.*

This follows immediately from the fact that ω^ω can be partitioned into ω clopen sets homeomorphic to ω^ω.

9.2. *An intersection of countably many analytic sets is analytic.*

Proof. Let $A_0, A_1, \ldots$ be analytic subsets of S. Let A_i be the projection of a closed set $C_j \subseteq S \times (\omega^\omega)_j$, where $(\omega^\omega)_j$ is a homeomorphic copy of ω^ω. Let C_j^* be the cylinder over C_j in the product space $S \times \Pi_{i<\omega} (\omega^\omega)_i$. Then C_j^* is closed and $\bigcap_{i<\omega} A_i =$ the projection of $\bigcap_{i<\omega} C_i^*$ into S. Since $\Pi_{i<\omega} (\omega^\omega)_i$ is homeomorphic to ω^ω and $\bigcap_{i<\omega} C_i^*$ is closed, it follows that $\bigcap_{i<\omega} A_i$ is analytic.

9.3. *Every closed set is analytic and every open set in P^ω is analytic.*

For closed sets the assertion is obvious and for open sets it follows from the easy fact that in the space P^ω every open set is a countable union of clopen sets, and from 9.1.

Corollary 9.4. *All Borel subsets of P^ω are analytic.*

This corollary is not true for all spaces. For example, the set ω_1 is open in the compact space $\omega_1 + 1$ with its interval topology, but ω_1 is not analytic.

If $A \subseteq S$ is of the form

$$A = \bigcup_{q \in \omega^\omega} \bigcap_{n<\omega} F_{q \restriction n}, \tag{1}$$

where $q \restriction n = (q_0, \ldots, q_{n-1})$ and $F_{q \restriction n}$ are closed subsets of S, then A is analytic. In fact, A is the projection into S of the set

$$C = \bigcap_{n<\omega} \bigcup_{q \in \omega^\omega} F_{q \restriction n} \times U(q \restriction n).$$

It is easy to check that each union in this intersection is closed, and hence C itself is closed.

Also if $A \subseteq P^\omega$ is analytic, a projection of a closed set $C \subseteq P^\omega \times \omega^\omega$, then A is of the form (1), where

$$F_{q \restriction n} = \{p \in P^\omega : (U(p \restriction n) \times U(q \restriction n)) \cap C \neq \emptyset\} .$$

However, there are spaces S where not every analytic set is of the form (1). [The form (1) is called the *Souslin operation* or the operation (A) upon the system $\langle F_{q \restriction n} \rangle$.]

Finally, let us recall that *there exist analytic subsets of* $\{0, 1\}^\omega$ *which are not Borel* (for example, the set of all those sequences which code a subset of $\omega \times \omega$ which is not a well ordering of ω).

We now define the notion of an *Erdös cardinal* κ.

- $f \restriction X$ denotes the restriction of a function f to a subset X of its domain.
- Every ordinal number is identified with the set of all smaller ordinals.
- Cardinals are identified with initial ordinals.
- For every cardinal α, α^+ denotes the cardinal successor of α.
- For any set X, $[X]^n$ denotes the set of all subsets of X of cardinality n.

For any cardinals κ, α and λ we write

$$\kappa \to (\alpha)_\lambda^{<\omega} \tag{2}$$

iff for every function $f: \bigcup_{n<\omega} [\kappa]^n \to \lambda$ there exists a set $H \subseteq \kappa$ of cardinality α such that, for every $n < \omega$, $f \restriction [H]^n$ is constant. If (2) holds κ is called an *Erdös cardinal for* α *and* λ, and H is called a *homogeneous* set for f.

The relation (2) has many interesting properties, in particular:

Theorem 9.5. *If* α *is infinite and* κ *is the least cardinal such that* $\kappa \to (\alpha)_2^{<\omega}$, *then for every* $\lambda < \kappa$ *we have* $\kappa \to (\alpha)_\lambda^{<\omega}$, *and* κ *is strongly inaccessible.*

We refer the reader to Drake (1974, pp. 221 and 239) for the proof of the above theorem. For the proof of Theorem 8.1(ii) we need only a κ such that

$$\kappa \to (\omega_1)_\lambda^{<\omega}, \quad \text{where } \lambda = 2^{|P| + \aleph_0} . \tag{3}$$

By Theorem 9.5, if $|P|$ is less than the first strongly inaccessible cardinal, then (3) holds for the least κ such that $\kappa \to (\omega_1)_2^{<\omega}$. As mentioned above, every measurable cardinal $\kappa > |P|$ satisfies (3). The reader interested in those concepts should consult Drake (1974) and Solovay et al. (1978). Let us only recall that the condition (2) implies that κ is a very large cardinal number and its existence does not follow from the axioms of *ZFC* (not even for $\alpha = \aleph_0$ and $\lambda = 2$).

We still need a technical lemma.

Let T be the set of finite sequences of integers, i.e., $T = \bigcup_{n<\omega} \omega^n$. We define a linear ordering $<$ of T as follows: if $a, b \in T$ and a is a proper initial segment of b, then $b < a$, while, if there exists an i such that both a and b are of length $\geqslant i$ and $a_i \neq b_i$, then $a < b$ iff $a_i < b_i$ for the least such i. This is called the Brouwer–Kleene ordering of T.

Lemma 9.6. *If $T_0 \subseteq T$ and T_0 does not contain any infinite subset linearly ordered by the relation "a is an initial segment of b", then the Brouwer–Kleene ordering well orders T_0.*

Proof. We can assume without loss of generality that T_0 is saturated in the sense that it contains all the initial segments of its elements. Then T_0 partially ordered by the relation "*a is an initial segment of b*" is a tree without infinite branches, and has a root v_0. We define inductively a map $\rho\colon T_0 \to \omega_1$. We put $\rho(v) = 0$ if v is at the top of a branch. Let S_v be the set of immediate successors of v, i.e., if $v \in \omega^n$, then $S_v = \{w \in \omega^{n+1} \cap T_0 : v \subseteq w\}$. Assuming $\rho \restriction S_v$ is already defined we put

$$\rho(v) = \sup\{\rho(s) + 1 : s \in S_v\}\,.$$

It is easy to check that this defines a map ρ and that $\rho(v)$ increases as v runs towards the root along any branch. Now it is easy to prove Lemma 9.6 by induction on $\rho(v_0)$. It is clear that if the Brouwer–Kleene ordering restricted to any subtree T_s stemming from $s \in S_v$ is a well ordering, then it is also a well ordering of the subtree T_v stemming from v. □

Proof of Theorem 8.1(ii). Let $A \subseteq P^\omega$ be analytic, a projection of the closed set $C \subseteq P^\omega \times \omega^\omega$, and let (3) hold. Suppose that II does not have a winning strategy for the game $\langle P^\omega, A\rangle$. We have to show that I has a winning strategy. First we define an auxiliary PI-game G defined by a closed set and show that I has a winning strategy for G. Then we deduce that I also has a winning strategy for $\langle P^\omega, A\rangle$.

Let T be (as above) the set of all finite sequences of integers and $t_1, t_2, \ldots$ be an ω-enumeration of T without repetitions. For $q \in P^{2n}$ and $t_n \in T$ we shall say that (q, t_n) is *insecure* (for II) if q and t_n are initial segments of some $p \in P^\omega$ and $r \in \omega^\omega$, respectively, such that $(p, r) \in C$. Otherwise we say that (q, t_n) is *secure* (for II).

We define G as follows. The choices of I are still elements $p_n \in P$ but the choices of II are pairs (q_n, α_n), where $q_n \in P$ and $\alpha_n \in \kappa$. Player II wins iff,

$$\alpha_n = 0 \text{ whenever } ((p_0, q_0, \ldots, p_{n-1}, q_{n-1}), t_n) \text{ is secure,} \tag{4}$$

and

$$\alpha_n < \alpha_m \text{ whenever both } ((p_0, q_0, \ldots, p_{n-1}, q_{n-1}), t_n) \text{ and } ((p_0, q_0, \ldots, p_{m-1}, q_{m-1}), t_m) \text{ are insecure and } t_m \text{ is a proper initial segment of } t_n . \tag{5}$$

We claim that, if $p = (p_0, q_0, p_1, q_1, \ldots)$ and $p \in A$, then II must have lost in G. Indeed, if $p \in A$, there exists a $t \in \omega^\omega$ such that $(p, t) \in C$ and hence all the pairs $(p \restriction 2m, t \restriction n)$ are insecure. If $t \restriction n = t_{k(n)}$ and if II had won G, then by (5), $\alpha_{k(0)} > \alpha_{k(1)} > \cdots$ and this is impossible since κ is well ordered. Hence, since we assumed that II has no winning strategy for $\langle P^\omega, A\rangle$, II has no winning strategy for G either. By (4) and (5) the set of player I in G is closed. Hence by Proposition 3.2, G is determined and I has a winning strategy, say s, for G.

Now we will modify s to get a winning strategy s^* for I for $\langle P^\omega, A\rangle$.

Let $L = P^D$ = the set of functions from D to P, where $D = \bigcup_{n<\omega} P^{2n}$. So we have $|L| = 2^{|P|+\aleph_0}$.

We define a map $f\colon \bigcup_{m<\omega} [\kappa]^m \to L$ as in the definition of (3). For $Q \in [\kappa]^m$, $f(Q)\colon D \to P$ is defined by

$$f(Q)(p_0, q_0, \ldots, p_{n-1}, q_{n-1}) = s((q_0, \alpha_0), \ldots, (q_{n-1}, \alpha_{n-1})) ,$$

where s is the winning strategy for I for G, and $\alpha_0, \ldots, \alpha_{n-1}$ are given by the following three conditions:

$$\text{if } ((p_0, q_0, \ldots, p_{i-1}, q_{i-1}), t_i) \text{ is secure, then } \alpha_i = 0 ; \tag{6}$$

$$\text{if the set } I = \{i < n\colon ((p_0, q_0, \ldots, p_{i-1}, q_{i-1}), t_i) \text{ is insecure}\} \text{ has exactly } |Q| \text{ elements, then the map } t_i \mapsto \alpha_i \text{ for } i \in I \text{ is the unique bijection into } Q \text{ preserving the Brouwer–Kleene ordering ;} \tag{7}$$

$$\text{if } |I| \neq |Q|, \text{ then } \alpha_i = 0 \text{ for all } i < n . \tag{8}$$

Let $H \subseteq \kappa$ be a homogeneous set of order type ω_1 for f; its existence follows from the assumption (3). Then we can define a function $s^*\colon \bigcup_{n<\omega} P^n \to P$ inductively as follows:

$$s^*(q_0, \ldots, q_{n-1}) = f(Q)(p_o, q_0, \ldots, p_{n-1}, q_{n-1}) ,$$

$$p_i = s^*(q_0, \ldots, q_{i-1}) \quad \text{for } i < n ,$$

where Q is any set such that $Q \subseteq H$ and $|Q| = |I|$ and I is as in (7). [Note that $p_0 = s^*(\emptyset)$, i.e., p_0 is the first choice of I by s.] Notice that, since H is homogeneous for f, this definition of s^* is correct because we can check by induction that the value of s^* written above does not depend on the choice of Q, as long as $Q \subseteq H$ and $|Q| = |I|$.

We claim that s^* is a winning strategy for I for $\langle P^\omega, A\rangle$. Suppose to the contrary that there exists a $p = (p_0, q_0, \ldots, p_n, q_n, \ldots) \in P^\omega \setminus A$ which is obtained when I plays by means of s^*. We shall derive from this a contradiction by showing that there exists a game in G in which I plays s and II wins. To define this game, let

$$T_0 = \{t_n \colon ((p_0, q_0, \ldots, p_{n-1}, q_{n-1}), t_n) \text{ is insecure for II}, n < \omega\} .$$

Since $p \not\in A$, T_0 satisfies the assumption of Lemma 9.6, whence there exists a map $\alpha\colon T_0 \to H$ which preserves the Brouwer–Kleene ordering. Assign to p the game in G where, after each choice p_{n-1} of I, II chooses $(q_{n-1}, \alpha(t_n))$ if $t_n \in T_0$ and $(q_{n-1}, 0)$ if $t_n \not\in T_0$. Clearly, conditions (4) and (5) are satisfied, which means that II wins, and this is the desired contradiction.

This concludes the proof of Theorem 8.1(ii).

10. The Axiom of Determinacy

The results of Section 4 suggest the study of an abstract theory $T = ZF + AD + DC$. Here ZF is the set theory ZFC without the *Axiom of Choice*. AD, called the *Axiom of Determinacy*, tells that for all $X \subseteq \omega^\omega$ the game $\langle \omega^\omega, X\rangle$ is determined; DC, called the *Axiom of Dependent Choices*, tells that for any binary relation $R \subseteq Y \times Y$, if $Y \neq \emptyset$ and $\forall a \in Y\, \exists b \in Y[(a, b) \in R]$, then there exists an ω-sequence $y_0, y_1, \ldots$ such that $\forall n < \omega[(y_n, y_{n+1}) \in R]$. Notice that, by the coding described in the proof of Corollary 4.6, AD is equivalent to the statement that all games of the form $\langle 2^\omega, X\rangle$ are determined. AD was proposed by Mycielski and Steinhaus (1962). By Theorem 8.1(iii) of Martin and Steel (1989), we know that T is consistent.

T is motivated by its deductive power, the coherence of its theorems and the interesting classes of sets which are known or conjectured to constitute models for T. For example, T proves (by the results of Section 4) that every uncountable set in a Polish space has a perfect subset and that every subset of a Polish space has the property of Baire and is measurable with respect to every Borel measure. T also yields many natural results about projective sets and projective well orderings of sets of real numbers [see Addison and Moschovakis (1968), Kechris et al. (1977, 1979, 1981, 1985), Martin (1968), and Moschovakis (1980)]. Those theorems solve problems which are not solvable in

ZFC, and admit only unnatural solutions in $ZF + (V = L)$. By Theorem 8.1(iii), T is true in $L(\mathbb{R})$ [the proof of *DC* is given in Kechris (1985)]. So all subsets of a Polish space in $L(\mathbb{R})$ which are in $L(\mathbb{R})$, in particular all projective subsets, have the above properties. The advantage of proving those theorems in T is that T may have other models. For example, the class $L(\mathcal{O}^\omega)$, where $\mathcal{O}$ is the class of all ordinal numbers. [$L(\mathcal{O}^\omega)$ is the least model of *ZF* which contains all members of its members, and all ω-sequences of ordinal numbers.] As much as we know $L(\mathcal{O}^\omega)$ may satisfy the following stronger version of *AD*. Let $X \subseteq P^\alpha$, where α is any ordinal number, and consider the game $\langle P^\alpha, X\rangle$ of length α defined in a way similar to $\langle P^\omega, X\rangle$, where player I makes all the even choices and player II all the odd choices. Perhaps, in the model $L(\mathcal{O}^\omega)$ for every $\alpha < \omega_1$ and every $X \subseteq \omega^\alpha$ the game $\langle \omega^\alpha, X\rangle$ is determined [see Mycielski (1964, p. 217)].

Let us still show *without using the Axiom of Choice* two facts which imply that the above refinement of *AD* and Theorem 8.1(iii) are the strongest possible in a certain sense.

10.1. *There exists an* $X \subseteq 2^{\omega_1}$ *such that* $\langle 2^{\omega_1}, X\rangle$ *is not determined.*

10.2. *There exists an* $X \subseteq \omega_1^\omega$ *such that* $\langle \omega_1^\omega, X\rangle$ *is not determined.*

Proof of 10.1. Assume to the contrary that all games of the form $\langle 2^{\omega_1}, X\rangle$ are determined. This implies *AD*. It follows that there is no injection $\omega_1 \to \mathbb{R}$. Indeed, if such an injection were to exist, Corollary 4.6 would imply that the image of ω_1 in $\mathbb{R}$ would have a perfect subset; hence there would be a well ordering of $\mathbb{R}$ and, again by Corollary 4.6, a non-determined game of the form $\langle \omega^\omega, X\rangle$.

Now for every ordinal number $\alpha < \omega_1$ there exists a well ordering of ω, i.e., a subset of $\omega \times \omega$, of type α. Consider the following game $\langle 2^{\omega_1}, X\rangle$. Player II wins iff I always chooses 0 or, if α being the first ordinal for which I chose 1, the ω choices of II following α constitute a sequence of 0's and 1's coding a subset of $\omega \times \omega$ which is a well ordering of ω of type α. It is clear that I has no winning strategy in that game. But the existence of a winning strategy for II implies the existence of an injection of ω_1 into $\mathbb{R}$. So we have a contradiction. □

The proof of 10.2 is quite similar to that of 10.1.

Another conjecture is the following: if $\alpha < \omega_1$, $X \subseteq \omega^\alpha$ and X is definable from an ω-sequence of ordinal numbers, then the game $\langle \omega^\alpha, X\rangle$ is determined. [The class of sets which are definable from a sequence of ordinals is definable, so the above conjecture can be expressed in the language of *ZF*. For a related conjecture see Addison and Moschovakis (1968).]

Let us return to the theory T. In 1967 R.M. Solovay showed that this theory proves that *for every partition* $\omega_1 = A \cup B$ *either* A *or* B *has a subset which is closed in* ω_1 *and cofinal with* ω_1 [see Jech (1978); see also Martin (1968)]. This implies that T yields the consistency of the theory $ZFC+$ the existence of a measurable cardinal number, and hence that T is a very strong theory. In particular it implies that Theorem 8.1(iii) could not have been proved without some additional axiom.

We will not discuss here the consequences of T except to state the following important weak form of the Axiom of Choice or Selection Principle which is useful for the theory of capacities of Choquet [see Mycielski (1972), Srebny (1984, pp. 30–47), and Busch (1979)].

Theorem 10.3 (In the theory T). *For every* $S \subseteq \mathbb{R} \times \mathbb{R}$, *there exists a function* $f \subseteq S$ *such that* $\mathrm{pr}_1(S) \setminus \mathrm{dom}(f)$ *is of Lebesgue measure zero and of the first category.*

This theorem was proved by R.M. Solovay around 1970; his proof is published in Busch (1979).

The literature about AD is large, see Kechris et al. (1977, 1979, 1981, 1985) and Moschovakis (1980), but let me add the following polemical remarks. First, Mycielski and Steinhaus (1962) overlooked that S. Ulam had already defined a game [Mauldin (1981, p. 113)] which is equivalent to their game $\langle 2^{\omega}, X \rangle$. Second, in the detailed monograph by Moschovakis (1980), the history of AD is skewed. Namely on pages 9 and 287 the role of the papers by Mycielski and Steinhaus (1962) and Mycielski (1964, 1966) is ignored, and on pages 378–379 their mathematical motivation is criticized. The reader may check that in fact the motivation expressed in Mycielski and Steinhaus (1962) and Mycielski (1964, 1966) is identical to that in Moschovakis (1980). The only difference is that Moschovakis adopts a philosophy which tells that in this area we study a pre-existing Platonic reality discovered by Cantor, while Mycielski and Steinhaus would have told that we study here some new human constructions. Be that as it may, the idea of infinite PI-games has proved to be very stimulating and still presents many challenging open problems.

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