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ZEROS OF A RANDOM ANALYTIC FUNCTION APPROACH PERFECT SPACING UNDER REPEATED DIFFERENTIATION

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Abstract. We consider an analytic function whose zero set forms a unit intensity Poisson process on the real line. We show that repeated differentiation causes the zero set to converge in distribution to a random translate of the integers.

1. Introduction

Study of the relation of the zero set of a function f to the zero set of its derivative has a rich history. The Gauss-Lucas theorem (see, e.g., [Mar49, Theorem 6.1]) says that if f is a polynomial, then the zero set of f' lies in the convex hull of the zero set of f. Another property of the differentiation operator is that it is *complex zero decreasing*: the number of nonreal zeros of f' is at most the number of nonreal zeros of f. This property is studied by [CC95] in the more general context of *Pólya-Schur* operators, which multiply the coefficients of a power series by a predetermined sequence. Much of the recent interest in such properties of the derivative and other operators stems from proposed attacks on the Riemann Hypothesis involving behavior of zeros under these operators [LM74, Con83]. See also [Pem12, section 4] for a survey of combinatorial reasons to study locations of zeros such as log-concavity of coefficients [Bre89] and negative dependence properties [BBL09].

The vague statement that differentiation should even out the spacings of zeros is generally believed, and a number of proven results bear this out. For example, a theorem attributed to Riesz (later rediscovered by others) states that the minimum distance between zeros of certain entire functions with only real zeros is increased by differentiation; see [FR05, section 2] for a history of this result and its appearance in [Sto26] and subsequent works of J. v. Sz.-Nagy and of P. Walker.

The logical extreme is that repeated differentiation should lead to zeros that are as evenly spaced as possible. If the original function f has real zeros, then all derivatives of f also have all real zeros. If the zeros of f have some long-run density on the real line, then one might expect the zero set under repeated differentiation to approach a lattice with this density. A sequence of results leading up to this was proved in [FR05]. The authors show that the gaps between zeros of $f' + af$ are bounded between the infimum and supremum of gaps between consecutive zeros of f and generalize this to a local density result that is applicable to the Riemann zeta function. They claim a result [FR05, Theorem 2.4.1] that implies the convergence

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of spacings of zeros to a constant (their Theorem 2.4.2), but a key piece of their proof, Proposition 5.2.1, has a hole (D. Farmer, personal communication) for which the present authors see no easy fix.

The central object of this paper is a random analytic function f whose zeros form a unit intensity point process. We construct such a function and prove translation invariance in Theorem 2.1. Our main result is that as $k \to \infty$, the zero set of the k^{th} derivative of f approaches a random translate of the integers. Thus we provide, for the first time, a proof of the lattice convergence result in the case of a random zero set.

The remainder of the paper is organized as follows. In the next section we give formal constructions and statements of the main results. We also prove preliminary results concerning the construction, interpretation and properties of the random function f. At the end of the section we state an estimate on the Taylor coefficients of f, Theorem 2.7 below, and show that Theorem 2.6 follows from Theorem 2.7 without too much trouble. In section 3 we begin proving Theorem 2.7, that is, estimating the coefficients of f. It is suggested in [FR05] that the Taylor series for f might prove interesting, and indeed our approach is based on determination of these coefficients. We evaluate these via Cauchy's integral formula. In particular, in Theorem 3.2, we locate a saddle point σ_k of $z^{-k}f$. In section 4.2 we prove some estimates on f, allowing us to localize the Cauchy integral to the saddle point and complete the proof of Theorem 2.7. We conclude with a brief discussion.

2. Statements and preliminary results

We assume there may be readers interested in analytic function theory but with no background in probability. We therefore include a couple of paragraphs of formalism regarding random functions and Poisson processes, with apologies to those readers for whom it is redundant.

2.1. **Formalities.**

A random object X taking values in a set S endowed with a σ-field $\mathcal{S}$ is a map $X : (\Omega, \mathcal{F}) \to (S, \mathcal{S})$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. We will never need explicitly to name the σ-field $\mathcal{S}$ on $\mathcal{S}$, nor will we continue to say that maps must be measurable, though all maps are assumed to be. If S is the space of analytic functions, the map X may be thought of as a map $f : \Omega \times \mathbb{C} \to \mathbb{C}$. The statement "$f$ is a random analytic function" means that for any fixed $\omega \in \Omega$, the function $z \mapsto f(\omega, z)$ is an analytic function. The argument ω is always dropped from the notation; thus, e.g., one may refer to $f'(z)$ or $f(\lambda z)$, and so forth, which are also random analytic functions.

A unit intensity Poisson process on the real lines is a random counting measure N on the measurable subsets of $\mathbb{R}$ such that for any disjoint collection of sets $\{A_1, \ldots, A_n\}$, of finite measure, the random variables $\{N(A_1), \ldots, N(A_n)\}$ are a collection of independent Poisson random variables with respective means $|A_1|, \ldots, |A_n|$ (here $|B|$ denotes the measure of B). The term "counting measure" refers to a measure taking values in the nonnegative integers. There is a random countable set E such that the measure of any set A is the cardinality of $A \cap E$. We informally refer to the set $E := \{x \in \mathbb{R} : N(\{x\}) = 1\}$ as the "points of the Poisson process."

Let Ω henceforth denote the space of counting measures on $\mathbb{R}$, equipped with its usual σ-field $\mathcal{F}$, and let $\mathbb{P}$ denote the law of a unit intensity Poisson process. This simplifies our notation by allowing us to construct a random analytic function

$f : \Omega \times \mathbb{C} \to \mathbb{C}$ by a formula for the value of $f(N, z)$, guaranteeing that the random function f is determined by the locations of the points of the Poisson process N.

For $N \in \Omega$ and $\lambda \in \mathbb{R}$, let $\tau_\lambda N$ denote the shift of the measure N that moves points to the right by λ; in other words, $\tau_\lambda N(A) := N(A - \lambda)$ where $A - \lambda$ denotes the leftward shift $\{x - \lambda : x \in A\}$. A unit intensity point process is translation invariant. This means formally that $\mathbb{P} \circ \tau_\lambda = \mathbb{P}$ for any λ. If X is a random object in a space S admitting an action $*$ of the group $(\mathbb{R}, +)$, we say that X is constructed in a translation invariant manner from N if $X(\tau_\lambda N) = \lambda * X(N)$. This condition is sufficient (but not necessary) for the law of X to be invariant under the $(\mathbb{R}, +)$-action. In what follows we will construct a random analytic function f which is translation invariant up to constant multiple. Formally, for any function g let $[g]$ denote the set of functions $\{\lambda g : g \in \mathbb{R}\}$. Let $(\mathbb{R}, +)$ act on the set of analytic functions by translation in the domain: $\lambda * g(z) := g(z - \lambda)$. This commutes with the projection $g \mapsto [g]$. Our random analytic function f will have the property that $[f]$ is constructed in a translation invariant manner from N.

2.2. **Construction of f.** Various quantities of interest will be defined as sums and products over the set of points of the Poisson process N. The sum of g evaluated at the points of the counting measure N is more compactly denoted $\int g\, dN$. If $\int |g|\, dN < \infty$, then this is an absolutely convergent sum and its meaning is clear. Because many of these infinite sums and products are not absolutely convergent, we introduce notation for some symmetric sums that are conditionally convergent.

Let $g : \mathbb{R} \to \mathbb{C}$ be any function. Let N_M denote the restriction of N to the interval $[-M, M]$. Thus, $\int g\, dN_M$ denotes the sum of $g(x)$ over those points of the process N lying in $[-M, M]$. Define the symmetric integral $\int_* g\, dN$ to be equal to $\lim_{M\to\infty} \int g\, dN_M$ when the limit exists. It is sometimes more intuitive to write such an integral as a sum over the points x of N. Thus we denote

$$\sum_* g(x) := \int_* g(x)\, dN(x) = \lim_{M\to\infty} \int g(x)\, dN_M(x)$$

when this limit exists.

Similarly for products, we define the symmetric limit by

$$\prod_* g(s) := \lim_{M\to\infty} \exp\left(\int \log g\, dN_M\right) .$$

Note that although the logarithm is multi-valued, its integral against a counting measure is well defined up to multiples of $2\pi i$, whence such an integral has a well defined exponential.

Theorem 2.1. *Except for a set of values of N of measure zero, the symmetric product*

$$f(z) := \prod_* \left(1 - \frac{z}{x}\right) \tag{2.1}$$

exists. The random function f defined by this product is analytic and translation invariant. In particular,

$$f(\tau_\lambda N, z) = \frac{f(N, z - \lambda)}{f(N, -\lambda)}, \tag{2.2}$$

which implies $[f(\tau_\lambda N, \cdot)] = [f(N, \cdot - \lambda)]$.

We denote the k^{th} derivative of f by $f^{(k)}$. The following is an immediate consequence of Theorem 2.1.

Corollary 2.2. *For each k, the law of the zero set of $f^{(k)}(z)$ is translation invariant.* □

Translation invariance of f is a little awkward because it only holds up to a constant multiple. It is more natural to work with the logarithmic derivative

$$h(z) := \sum_* \frac{1}{z-x} .$$

Lemma 2.3. *The random function h is meromorphic, and its poles are precisely the points of the process N, each being a simple pole. Also h is translation invariant and is the uniform limit on compact sets of the functions*

$$h_M(z) := \int \frac{1}{z-x}\, dN_M(x) .$$

Proof. Let $\Delta_M := h_{M+1}(0) - h_M(0)$. It is easily checked that
(i) $\sum_{M=1}^\infty \mathbb{P}(\Delta_M > \varepsilon) < \infty$;
(ii) $\mathbb{E}\Delta_M = 0$;
(iii) $\sum_{M=1}^\infty \mathbb{E}\Delta_M^2 < \infty$.
By Kolmogorov's three series theorem, it follows that $\lim_{M\to\infty} h_M(0)$ exists almost surely.

To improve this to almost sure uniform convergence on compact sets, define the M^{th} tail remainder by $T_M(z) := h(z) - h_M(z)$ if the symmetric integral h exists. Equivalently,

$$T_M(z) := \lim_{R\to\infty} \int \frac{1}{z-x} d(N_R - N_M)(x)$$

if such a limit exists. Let K be any compact set of complex numbers. We claim that the limit exists and that

$$G(M) := \sup_{z\in K} |T_M(z) - T_M(0)| \to 0 \text{ almost surely as } M \to \infty. \tag{2.3}$$

To see this, assume without loss of generality that $M \geq 2\sup\{|\Re\{z\}| : z \in K\}$. Then

$$T_M(z) - T_M(0) = \lim_{R\to\infty} \int \left(\frac{1}{z-x} - \frac{1}{-x} \right) d(N_R - N_M)(x) . \tag{2.4}$$

Denote $C_K := \sup_{z\in K} |z|$. As long as $z \in K$ and $|x| \geq M$, the assumption on M gives

$$\left| \frac{1}{z-x} - \frac{1}{-x} \right| = \left| \frac{z}{x(z-x)} \right| \leq \frac{2C_K}{x^2} . \tag{2.5}$$

This implies that the integral in (2.4) is absolutely integrable with probability 1. Thus, almost surely, $T_M(z) - T_M(0)$ is defined by the convergent integral

$$T_M(z) - T_M(0) = \int \left(\frac{1}{z-x} - \frac{1}{-x} \right) d(N - N_M)(x) .$$

Plugging in (2.5), we see that $G(M) \leq 2C_K \int x^{-2}\, d(N - N_M)(x)$, which goes to zero (by Lebesgue dominated convergence) except on the measure zero event that $\int |x|^{-2}\, dN(x) = \infty$.

This proves (2.3). The triangle inequality then yields $\sup_{z\in K}|T_M(z)| \le G(M) + |T_M(0)|$, both summands going to zero almost surely. By definition of T_M, this means $h_M \to h$ uniformly on K. The rest is easy. For fixed K and M, $h = h_M + \lim_{R\to\infty}(h_R - h_M)$. When M is sufficiently large and $R > M$, the functions $h_R - h_M$ are analytic on K. Thus h is the sum of a meromorphic function with simple poles at the points of N in K and a uniform limit of analytic functions. Such a limit is analytic. Because K was arbitrary, h is meromorphic with simple poles exactly at the points of N.

The final conclusion to check is that h is translation invariant. Unraveling the definitions gives

$$h(\tau_\lambda N, z) = \int_{*\lambda} \frac{1}{(z-\lambda)-x}\, dN(x)$$

where $\int_{*\lambda}$ is the limit as $M\to\infty$ of the integral over $[-\lambda - M, -\lambda + M]$. Translation invariance then follows from checking that $\int_{M-\lambda}^{M} \frac{1}{z-x}\, dN(x)$ and $\int_{-M-\lambda}^{-M} \frac{1}{z-x}\, dN(x)$ both converge almost surely to zero. This follows from the large deviation bound

$$\mathbb{P}\left(\left|\int_{M-\lambda}^{M} \frac{1}{z-x}\, dN(x)\right| \ge \varepsilon\right) = O\left(e^{-cM}\right)$$

and Borel-Cantelli. □

Proof of Theorem 2.1. The antiderivative of the meromorphic function h is an equivalence class (under addition of constants) of functions taking values in $\mathbb{C} \mod (2\pi i)$. Choosing the antiderivative of h_M to vanish at the origin and exponentiating gives the functions f_M, whose limit as $M\to\infty$ is the symmetric product f. Analyticity follows because f is the uniform limit of analytic functions. Translation invariance up to constant multiple follows from translation invariance of h. The choice of constant (2.2) follows from the definition, which forces $f(0) = 1$. □

Before stating our main results, we introduce a few properties of the random analytic function f.

Proposition 2.4. *$f(\overline{z}) = \overline{f(z)}$ and $|f(a+bi)|$ is increasing in $|b|$.*

Proof. Invariance under conjugation is evident from the construction of f. For $a, b\in\mathbb{R}$,

$$\begin{aligned} \log|f(a+bi)| &= \sum_* \log\left|1 + \frac{a+bi}{x}\right| \\ &= \frac{1}{2}\sum_* \log\left[\left(1+\frac{a}{x}\right)^2 + \left(\frac{|b|}{x}\right)^2\right]. \end{aligned}$$

Each term of the sum is increasing in $|b|$. □

The random function f, being almost surely an entire function, almost surely possesses an everywhere convergent power series

$$f(z) = \sum_{n=0}^{\infty} e_n z^n .$$

By construction $f(0) = 1$, hence $e_0 = 1$. The function f is the uniform limit on compact sets of $f_M := \exp\left(\int \log(1 - z/x)\, dN_M(x)\right)$. The Taylor coefficients $e_{M,n}$ of f_M are the elementary symmetric functions of the negative reciprocals of the points of N_M:

$$e_{M,k} = e_k\left(\{-1/x : N_M(x) = 1\}\right) .$$

It follows that $e_{M,k} \to e_k$ as $M \to \infty$ for each fixed k. Thus we may conceive of e_k as the k^{th} elementary symmetric function of an infinite collection of values, namely the negative reciprocals of the points of the Poisson process. The infinite sum defining this symmetric function is not absolutely convergent but converges conditionally in the manner described above.

We do not know a simple form for the marginal distribution of e_k except in the case $k = 1$. To see the distribution of e_1, observe that the negative reciprocals of the points of a unit intensity Poisson process are a point process with intensity dx/x^2. Summing symmetrically in the original points is the same as summing the negative reciprocals, excluding those in $[-\varepsilon, \varepsilon]$, and letting $\varepsilon \to 0$. By a well known construction of the stable laws (see, e.g., [Dur10, section 3.7]), this immediately implies:

Proposition 2.5. *The law of e_1 is a symmetric Cauchy distribution.*

While we have not before seen a systematic study of symmetric functions of points of an infinite Poisson process, symmetric functions of IID collections of variables have been studied before. These were first well understood in Rademacher variables (plus or minus one with probability 1/2 each). It was shown in [MS82, Theorem 1] that the marginal of e_k, suitably normalized, is the value of the k^{th} Hermite polynomial on a standard normal random input. This was extended to other distributions, the most general result we know of being the one in [Maj99].

2.3. Main result and reduction to coefficient analysis.

The random analytic function f is the object of study for the remainder of the paper. Our main result is as follows, the proof of which occupies most of the remainder of the paper.

Theorem 2.6 (Main result). *As $k \to \infty$, the zero set of $f^{(k)}$ converges in distribution to a uniform random translate of the integers.*

We prove the main result via an analysis of the Taylor coefficients of f, reducing Theorem 2.6 to the following result.

Theorem 2.7 (Behavior of coefficients of the derivatives). *Let $a_{k,r} := [z^r]f^{(k)}(z)$. There are random quantities $\{A_k\}_{k\geq 1}$ and $\{\theta_k\}_{k\geq 1}$ such that*

$$a_{k,r} \quad = \quad A_k \left[\cos\left(\theta_k - \frac{r\pi}{2}\right) + o_k(1)\right] \cdot \frac{\pi^r}{r!} \text{ in probability,} \tag{2.6}$$

$$\sum_{r=1}^{\infty} M^r \frac{|a_{k,r}|}{A_k} \quad < \quad \infty \text{ with probability } 1 - o(1), \tag{2.7}$$

for any $M > 0$. The use of the term "in probability" in the first statement means that for every $\varepsilon > 0$ the quantity

$$\mathbb{P}\left(\left|\frac{r!}{\pi^r A_k} a_{k,r} - \cos\left(\theta_k - \frac{r\pi}{2}\right)\right| > \varepsilon\right)$$

goes to zero for fixed r as $k \to \infty$.

A surprising consequence of this result is that the signs of the coefficients $\{e_k\}$ are periodic with period 4. In particular, e_k and e_{k+2} have opposite signs with probability approaching 1 as $k \to \infty$. It is interesting to compare this with simpler models, such as the Rademacher model in [MS82] in which a polynomial g has n zeros, each of them at ± 1, with signs chosen by independent fair coin flips. The number of positive roots will be some number $b = n/2 + O(\sqrt{n})$. Once n and b are determined, the polynomial g is equal to $(z-1)^b(z+1)^{n-b}$. The coefficients of g are the elementary symmetric functions of b ones and $n-b$ negative ones. The signs of these coefficients have 4-periodicity as well ([MS82, Remark 4]). An analogue of Theorem 2.7 in the case of IID variables with a reasonably general common distribution appears in [Maj99] (see also [Sub14] for extensions). The proofs, in that case as well as in the present paper, are via analytic combinatorics. We know of no elementary argument for the sign reversal between e_k and e_{k+2}.

Proof of Theorem 2.6 *from Theorem* 2.7. We assume the conclusion of Theorem 2.7 holds and establish Theorem 2.6 in the following steps. Let θ_k and A_k be as in the conclusion of Theorem 2.7.

Step 1 (Convergence of the iterated derivatives on compact sets).[1] Let $\psi_k(x) := \cos(\pi x - \theta_k)$. Fix any $M > 0$. Then

$$\sup_{x\in[-M,M]} \left| \frac{f^{(k)}(x)}{A_k} - \psi_k(x) \right| \to 0 \text{ in probability as } k \to \infty\,. \tag{2.8}$$

To prove this, use the identity $\cos(\theta_k - r\pi/2) = (-1)^j \cos(\theta_k)$ when $r = 2j$ and $(-1)^j \sin(\theta_k)$ when $r = 2j+1$ to write

$$\begin{aligned} \psi_k(x) &= \cos(\theta_k)\cos(\pi x) + \sin(\theta_k)\sin(\pi x) \\ &= \cos(\theta_k)\left[1 - \frac{\pi^2 x^2}{2!} + \cdots\right] + \sin(\theta_k)\left[\pi x - \frac{\pi^3 x^3}{3!} + \cdots\right] \\ &= \sum_{r=0}^{\infty} \cos\left(\theta_k - \frac{r\pi}{2}\right)\frac{\pi^r}{r!}x^r\,. \end{aligned}$$

This last series is uniformly convergent on $[-M, M]$. Therefore, given $\varepsilon > 0$ we may choose L large enough so that

$$\sup_{x\in[-M,M]} \left| \psi_k(x) - \sum_{r=0}^{L} \cos\left(\theta_k - \frac{r\pi}{2}\right)\frac{\pi^r}{r!}x^r \right| < \frac{\varepsilon}{3}\,. \tag{2.9}$$

By (2.7), we may choose L larger if necessary in order to ensure that

$$\left| \sum_{r=L+1}^{\infty} \frac{a_{k,r}}{A_k}x^r \right| < \frac{\varepsilon}{3} \tag{2.10}$$

for all $x \in [-M, M]$. Fix such an L and use the power series for $f^{(k)}$ to write

$$\frac{f^{(k)}(x)}{A_r} - \psi_k(x) = \left(\sum_{r=0}^{L} \frac{a_{k,r}}{A_k}x^r - \psi_k(x) \right) + \sum_{r=L+1}^{\infty} \frac{a_{k,r}}{A_k}x^r\,. \tag{2.11}$$

[1]This step is analogous to [FR05, Theorem 2.4.1], the correctness of which is unknown to us at this time.

Putting (2.9) together with (2.6) shows that the first term on the right-hand side of (2.11) is at most $\varepsilon/3 + \sum_{r=0}^L \xi_r$ where ξ_r is the term of (2.6) that is $o_k(1)$ in probability. By (2.6) we may choose k large enough so that $\varepsilon/3 + \sum_{r=0}^L \xi_r < 2\varepsilon/3$ with probability at least $1 - \varepsilon/2$. Thus, we obtain

$$\sup_{x \in [-M,M]} \left| \frac{f^{(k)}(x)}{A_k} - \psi_k(x) \right| \le \varepsilon$$

with probability at least $1 - \varepsilon$, establishing (2.8).

Step 2 (The $k + 1^{st}$ derivative as well). Let $\eta_k(x) := -\pi \sin(\pi x - \theta_k)$. Fix any $M > 0$. Then

$$\sup_{x \in [-M,M]} \left| \frac{f^{(k+1)}(x)}{A_k} - \eta_k(x) \right| \to 0 \text{ in probability as } k \to \infty \,. \tag{2.12}$$

The argument is the same as in Step 1, except that we use the power series $f^{(k+1)}(x) = \sum_{r=1}^\infty r a_{k,r} x^{r-1}$ in place of $f^{(k)}(x) = \sum_{r=0}^\infty a_{k,r} x^r$ and $\eta_k(x) = \sum_{r=1}^\infty \cos(\theta_k - r\pi/2) \frac{\pi^r}{(r-1)!} x^{r-1}$.

Step 3 (Convergence of the zero set to some lattice). On any interval $[-M, M]$, the zero set of $f^{(k)}$ converges to the zero set of ψ_k in probability. More precisely, for each $\varepsilon > 0$, if k is large enough, then except on a set of probability at most ε, for each zero of $f^{(k)}$ in $[-M + 2\varepsilon, M - 2\varepsilon]$ there is a unique zero of ψ_k within distance 2ε and for each zero of ψ_k in $[-M+2\varepsilon, M-2\varepsilon]$ there is a unique zero of $f^{(k)}$ within distance 2ε.

This follows from Steps 1 and 2 along with the following fact applied to $\psi = \psi_k$, $\tilde\psi = f^{(k)}$, $I = [-M, M]$ and $c = 1/2$. ☐

Lemma 2.8. *Let ψ be any function of class C^1 on an interval $I := [a, b]$. Suppose that the quantity $\min\{\,|\psi(x)|, |\psi'(x)|\}$ is never less than some $c > 0$ when $x \in I$. For any $\varepsilon > 0$, let I^ε denote $[a + \varepsilon, b - \varepsilon]$. Let $\varepsilon < c^2$ be positive, and suppose that a C^1 function $\tilde\psi$ satisfies $|\tilde\psi - \psi| \le \varepsilon$ and $|\tilde\psi' - \psi'| \le c/2$ on I. Then the zeros of ψ and $\tilde\psi$ on I are in correspondence as follows.*

- (*i*) *For every $x \in I^{\varepsilon/c}$ with $\psi(x) = 0$ there is an $\tilde x \in I$ such that $\tilde\psi(\tilde x) = 0$ and $|\tilde x - x| \le \varepsilon/c$. This $\tilde x$ is the unique zero of $\tilde\psi$ in the connected component of $\{|\psi| < c\}$ containing x.*
- (*ii*) *For every $\tilde x \in I^{\varepsilon/c}$ with $\tilde\psi(\tilde x) = 0$ there is an $x \in I$ with $\psi(x) = 0$. This x is the unique zero of ψ in the connected component of $\{|\psi| < c\}$ containing x.*

Proof. For (*i*), pick any $x \in I^{\varepsilon/c}$ with $\psi(x) = 0$. Assume without loss of generality that $\psi'(x) > 0$ (the argument when $\psi'(x) < 0$ is completely analogous). On the connected component of $|\psi| \le c$ one has $\psi' > c$. Consequently, moving to the right from x by at most ε/c finds a value x_2 such that $\psi(x_2) \ge \varepsilon$; moving to the left from x by at most ε/c finds a value x_2 such that $\psi(x_2) \le -\varepsilon$, and ψ' will be at least c on $[x_1, x_2]$. We have $|\tilde\psi - \psi| \le \varepsilon$, whence $\tilde\psi(x_1) \le 0 \le \tilde\psi(x_2)$, and by the Intermediate Value Theorem $\tilde\psi$ has a zero $\tilde x$ on $[x_1, x_2]$. To see uniqueness, note that if there were two such zeros, then there would be a zero of $\tilde\psi'$, contradicting $|\tilde\psi' - \tilde\psi| < c/2$ and $|\psi'| > c$.

To prove (ii) pick $\tilde{x} \in I^{\varepsilon/c}$ with $\tilde{\psi}(\tilde{x}) = 0$. Then $|\psi(\tilde{x})| \le \varepsilon \le c$ whence $|\psi'(\tilde{x})| > c$. Moving in the direction of decrease of $|\psi(\tilde{x})|$, $|\psi'|$ remains at least c, so we must hit zero within a distance of ε/c. Uniqueness follows again because another such zero would imply a critical point of ψ in a region where $|\psi| < c$. $\square$

Step 4 (Uniformity of the random translation). Because convergence in distribution is a weak convergence notion, it is equivalent to convergence on every $[-M, M]$. We have therefore proved that the zero set of $f^{(k)}$ converges in distribution to a random translate of the integers. On the other hand, Corollary 2.2 showed that the zero set of $f^{(k)}$ is translation invariant for all k. This implies convergence of the zero set of $f^{(k)}$ to a uniform random translation of $\mathbb{Z}$ and completes the proof of Theorem 2.6 from Theorem 2.7. $\square$

3. Estimating coefficients

3.1. **Overview.** The coefficients $e_k := [z^k]f(z)$ will be estimated via the Cauchy integral formula

$$e_k = \frac{1}{2\pi i} \int z^{-k} f(z) \frac{dz}{z} . \tag{3.1}$$

Denote the logarithm of the integrand by $\phi_k(z) := \log f(z) - k \log z$. Saddle point integration theory requires the identification of a saddle point σ_k and a contour of integration Γ, in this case the circle through σ_k centered at the origin, with the following properties.

(i) σ_k is a critical point of ϕ, that is, $\phi'(\sigma_k) = 0$.

(ii) The contribution to the integral from an arc of Γ of length of order $\phi''(\sigma_k)^{-1/2}$ centered at σ_k is asymptotically equal to $e^{\phi(\sigma_k)}\sqrt{2\pi/\phi''(\sigma_k)}$.

(iii) The contribution to the integral from the complement of this arc is negligible.

In this case we have a real function f with two complex conjugate saddle points σ_k and $\overline{\sigma_k}$. Accordingly, there will be two conjugate arcs contributing two conjugate values to the integral while the complement of these two arcs contributes negligibly. One therefore modifies (i)–(iii) to:

(i') σ_k and $\overline{\sigma_k}$ are critical points of ϕ, on the circle Γ, centered at the origin, of radius $|\sigma_k|$.

(ii') The contribution to the integral from arcs of Γ of length of order $\phi''(\sigma_k)^{-1/2}$ centered at σ_k and $\overline{\sigma}_k$ is asymptotically equal to $e^{\phi(\sigma_k)}\sqrt{2\pi/\phi''(\sigma_k)}$ and the conjugate of this.

(iii') The contribution to the integral from the complement of the two conjugate arcs is negligible compared to the contribution from either arc.

Note that (iii') leaves open the possibility that the two contributions approximately cancel, leaving the supposedly negligible term dominant.

3.2. **Locating the dominant saddle point.** The logarithm of the integrand in (3.1), also known as the phase function, is well defined up to multiples of $2\pi i$. We denote it by

$$\phi_k(z) := -k \log z + \sum_* \log\left(1 - \frac{z}{x}\right) .$$

When $k = 0$ we denote $\sum_* \log(1 - z/x)$ simply by $\phi(z)$.

Proposition 3.1. *For each k, r, the r^{th} derivative $\phi_k^{(r)}$ of the phase function ϕ_k is the meromorphic function defined by the almost surely convergent sum*

$$\phi_k^{(r)}(z) = (-1)^{r-1}(r-1)! \left[-\frac{k}{z^r} + \sum_* \frac{1}{(z-x)^r} \right] . \tag{3.2}$$

Thus in particular,

$$\phi_k'(z) = -\frac{k}{z} + \sum_* \frac{1}{z-x} .$$

Proof. When $r = 1$, convergence of (3.2) and the fact that this is the derivative of ϕ is just Lemma 2.3 and the subsequent proof of Theorem 2.1 in which f is constructed from h. For $r \geq 2$, with probability 1 the sum is absolutely convergent. $\square$

The main work of this subsection is to prove the following result, locating the dominant saddle point for the Cauchy integral.

Theorem 3.2 (Location of saddle)**.** *Let $E_{M,k}$ be the event that ϕ_k has a unique zero, call it σ_k, in the ball of radius $Mk^{1/2}$ about ik/π. Then $\mathbb{P}(E_{M,k}) \to 1$ as $M, k \to \infty$ with $k \geq 4\pi^2 M^2$.*

This is proved in several steps. We first show that $\phi_k'(ik/\pi)$ is roughly zero, then use estimates on the derivatives of ϕ and Rouché's Theorem to bound how far the zero of ϕ_k' can be from ik/π.

The function ϕ_k' may be better understood if one applies the natural scale change $z = ky$. Under this change of variables,

$$\phi_k'(z) = -\frac{1}{y} + \sum_* \frac{1/k}{y - x/k} .$$

Denote the second of the two terms by

$$h_k(y) := \sum_* \frac{1/k}{y - x/k} .$$

We may rewrite this as $h_k(Y) = \displaystyle\int \frac{1}{y-x} dN^{(k)}(x)$ when $N^{(k)}$ denotes the rescaled measure defined by $N^{(k)}(A) = k^{-1}N(kA)$. The points of the process $N^{(k)}$ are k times as dense and $1/k$ times the mass of the points of N. Almost surely as $k \to \infty$ the measure $N^{(k)}$ converges to Lebesgue measure. In light of this it is not surprising that $h_k(y)$ is found near $\displaystyle\int \frac{1}{z-y}\, dy$. We begin by rigorously confirming this, the integral being equal to $-\pi \operatorname{sgn} \Im\{z\}$ away from the real axis.

Lemma 3.3. *If z is not real, then*

$$\mathbb{E} \int_* \frac{1}{|z-x|^m} dN(x) = \lim_{M \to \infty} \mathbb{E} \int \frac{1}{|z-x|^m} dN_M(x),$$

for $m \geq 2$, and

$$\mathbb{E} \int_* \frac{1}{(z-x)^m} dN(x) = \lim_{M \to \infty} \mathbb{E} \int \frac{1}{(z-x)^m} dN_M(x),$$

for $m \geq 1$.

Proof. The first equality holds trivially by the Monotone Convergence Theorem. Next, write $\mathcal{R}_M$ as the number of points of the process N within $[-M, M]$, and $L = 2\Im(z)$. Then, for $m = 2$,

$$\begin{aligned}\mathbb{E}\int \frac{1}{|z-x|^2} dN_M(x) &= \mathbb{E}\sum_{j:|X_j|\le M} \frac{1}{\Im(z)^2 + (\Re(z) - X_j)^2} \\ &\le \mathbb{E}\left(\frac{\mathcal{R}_L}{\Im(z)^2}\right) + \mathbb{E}\sum_{j:L\le|X_j|\le M} \frac{4}{|X_j|^2} \\ &\le \frac{2L}{\Im(z)^2} + \frac{4}{L}.\end{aligned}$$

Therefore, as $\Im(z) \ne 0$, $\mathbb{E}\int_* \frac{1}{|z-x|^2} dN(x) < \infty$, and moreover, $\mathbb{E}\int_* \frac{1}{|z-x|^m} dN(x) < \infty, \forall m \ge 2$. Thus, by the Dominated Convergence Theorem,

$$\mathbb{E}\int_* \frac{1}{(z-x)^m} dN(x) = \lim_{M\to\infty} \mathbb{E}\int \frac{1}{(z-x)^m} dN_M(x)$$

holds for $m \ge 2$. We shall now show the above to hold true for $m = 1$.

Note that

$$\mathbb{E}\left[\left|\int \frac{1}{z-x} dN_M(x)\right|^2\right] = \mathbb{E}\int \frac{1}{|z-x|^2} dN_M(x) + \mathbb{E}\sum_{j\ne k:|X_j|,|X_k|\le M} \frac{1}{(z-X_j)(\overline{z}-X_k)}.$$

The first term in the above equation converges to $\mathbb{E}\int_* \frac{1}{|z-x|^2} dN(x)$ as $M \to \infty$. As for the second part,

$$\mathbb{E}\sum_{j\ne k:|X_j|,|X_k|\le M} \frac{1}{(z-X_j)(\overline{z}-X_k)} = \mathbb{E}\left[\mathcal{R}_M(\mathcal{R}_M - 1)\cdot \mathbb{E}\left(\frac{1}{(z-\mathcal{U}_2)(\overline{z}-\mathcal{U}_2)}\right)\right],$$

where $\mathcal{U}_1$ and $\mathcal{U}_2$ are i.i.d. Uniform$(-M, M)$ random variables. So,

$$\begin{aligned}\mathbb{E}\sum_{j\ne k:|X_j|,|X_k|\le M} \frac{1}{(z-X_j)(\overline{z}-X_k)} &= \left(\int_{-M}^{M} \frac{1}{z-u} du\right)^2 \\ &= \left[-\log\left|\frac{M-z}{M+z}\right| - i\arctan\left(\frac{M-\Re(z)}{\Im(z)}\right)\right. \\ &\quad \left. + i\arctan\left(\frac{-M-\Re(z)}{\Im(z)}\right)\right]^2 \\ &\longrightarrow -\pi^2, \text{ as } M \to \infty.\end{aligned}$$

Thus the quantities $\left\{\mathbb{E}\left[\left|\int \frac{1}{z-x} dN_M(x)\right|^2\right], M > 0\right\}$ have a uniform upper bound; let us call it $B(z)$. Then, given $\varepsilon > 0$,

$$\begin{aligned}\mathbb{E}\left[\left|\int \frac{1}{z-x} dN_M(x)\right| \cdot \mathbf{1}_{\left|\int \frac{1}{z-x} dN_M(x)\right| \ge K}\right] &\le \frac{1}{K}\cdot \mathbb{E}\mathbb{E}\left[\left|\int \frac{1}{z-x} dN_M(x)\right|^2\right] \\ &\le \frac{B(z)}{K} < \varepsilon,\end{aligned}$$

for $K > \frac{B(z)}{\varepsilon}$. Thus, if z is not real, $\left\{\mathbb{E}\left[\int \frac{1}{z-x} dN_M(x)\right], M > 0\right\}$ is a uniformly integrable collection and, hence, converges in L_1. □

Proposition 3.4. *If z is not real, then*

$$\mathbb{E}\left[\int_* \frac{1}{z-x}\, dN(x)\right] = \mp i\pi \tag{3.3}$$

with the negative sign if z is in the upper half plane and the positive sign if z is in the lower half plane. If z is not real and $m \geq 2$, then

$$\mathbb{E}\left[\int_* \frac{1}{(z-x)^m}\, dN(x)\right] = 0\,. \tag{3.4}$$

Proof. If $\mathcal{R}_M$ denotes the number of Poisson points in $[-M, M]$, then conditioning on $\mathcal{R}_M$, the poisson points X_j that are contained in $[-M, M]$ are identically and independently distributed as Uniform$[-M, M]$. Then,

$$\mathbb{E}\left[\int \frac{1}{z-x}\, dN_M(x) \middle| \mathcal{R}_M\right] = \mathcal{R}_M \cdot \mathbb{E}\left(\frac{1}{z-\mathcal{U}}\right),$$

where $\mathcal{U} \sim$ Uniform$[-M, M]$. Writing $z = re^{i\theta}$, we get

$$\begin{aligned}
\mathbb{E}\left[\int \frac{1}{z-x}\, dN_M(x) \middle| \mathcal{R}_M\right] &= \frac{\mathcal{R}_M}{2M} \int_{x\in[-M,M]} \frac{1}{r\cos\theta + ir\sin\theta - x} dx \\
&= \mathcal{R}_M \left[\frac{-1}{2M} \log\left|\frac{M-z}{M+z}\right| - \frac{i}{2M} \arctan\left(\frac{M - r\cos\theta}{r\sin\theta}\right)\right. \\
&\quad \left. + \frac{i}{2M} \arctan\left(\frac{-M - r\cos\theta}{r\sin\theta}\right)\right] \\
\implies \mathbb{E}\left[\int \frac{1}{z-x}\, dN_M(x)\right] &= -\log\left|\frac{M-z}{M+z}\right| - i\arctan\left(\frac{M - r\cos\theta}{r\sin\theta}\right) \\
&\quad + i\arctan\left(\frac{-M - r\cos\theta}{r\sin\theta}\right)
\end{aligned}$$

since $\mathcal{R}_M \sim$ Poisson$(2M)$. Taking $M \to \infty$, by Lemma 3.3 we get

$$\mathbb{E}\left[\int_* \frac{1}{z-x}\, dN(x)\right] = -\pi i,$$

for z in the upper half plane, and

$$\mathbb{E}\left[\int_* \frac{1}{z-x}\, dN(x)\right] = \pi i,$$

for z in the lower half plane, where the interchange of limits and expectation is by Lemma 3.3.

Now fix $m \geq 2$ and $z \notin \mathbb{R}$:

$$\begin{aligned}
\mathbb{E}\left[\int \frac{1}{(z-x)^m}\, dN_M(x) \middle| \mathcal{R}_M\right] &= \mathcal{R}_M \cdot \mathbb{E}\left[\frac{1}{(z-\mathcal{U})^m}\right] \\
&= \frac{\mathcal{R}_M}{2M} \cdot \frac{1}{m-1} \left\{\frac{1}{(z-M)^{m-1}} - \frac{1}{(M+z)^{m-1}}\right\} \\
\implies \mathbb{E}\left[\int \frac{1}{(z-x)^m}\, dN_M(x)\right] &= \frac{1}{m-1} \left\{\frac{1}{(z-M)^{m-1}} - \frac{1}{(M+z)^{m-1}}\right\}.
\end{aligned}$$

Thus, using Lemma 3.3,

$$\mathbb{E}\left[\int_* \frac{1}{(z-x)^m}\, dN(x)\right] = \lim_{M\to\infty} \frac{1}{m-1}\left\{\frac{1}{(z-M)^{m-1}} - \frac{1}{(M+z)^{m-1}}\right\} = 0.$$

□

The next proposition and its corollaries help us to control how much the functions ϕ_k and h_k can vary. These will be used first in Lemma 3.10, bounding h_k over a ball, then in section 4.1 to estimate Taylor series involving Φ_k. We begin with a general result on the variance of a Poisson integral.

Proposition 3.5. *Let $\psi : \mathbb{R} \to \mathbb{C}$ be any bounded function with $\int |\psi(x)|^2\, dx < \infty$. Let Z denote the compensated Poisson integral of ψ, namely*

$$Z := \lim_{M\to\infty}\left[\int \psi(x)\, dN_M(x) - \int_{-M}^{M} \psi(x)\, dx\right].$$

Then Z is well defined and has finite variance given by

$$\mathbb{E}|Z|^2 = \int |\psi(x)|^2\, dx\,.$$

Proof. This is a standard result, but the proof is short so we supply it. Let

$$Z_M := \int \psi(x)\, dN_M(x) - \int_{-M}^{M} \psi(x)\, dx$$

and let $\Delta_M := Z_M - Z_{M-1}$ denote the increments. We apply Kolmogorov's Three Series Theorem to the independent sum $\sum_{M=1}^{\infty} \Delta_M$, just as in the proof of Lemma 2.3. Hypothesis (i) is satisfied because $\int_M^{M+1} |\psi|$ goes to zero. Hypothesis (ii) is satisfied because $\mathbb{E}\Delta_M = 0$ for all M. To see that hypothesis (iii) is satisfied, observe that $\mathbb{E}|\Delta_M|^2 = \int |\psi(x)|^2 \mathbf{1}_{M-1\le|x|\le M}\, dx$, the summability of which is equivalent to our assumption that $\psi \in L^2$. We conclude that the limit exists almost surely. By monotone convergence as $M \to \infty$, $\mathrm{Var}\,(Z) = \int |\psi|^2$. □

Define

$$W_r(z) := \int_* (z-x)^{-r}\, dN(x)\,.$$

If $\alpha > 1$ and λ is real, the integral $\int |z-x|^{-\alpha}\, dx$ is invariant under $z \mapsto z+\lambda$ and scales by $\lambda^{1-\alpha}$ under $z \mapsto \lambda z$. Plugging in $\psi(x) = (z-x)^{-r}$ therefore yields the following immediate corollary.

Corollary 3.6. *Let z have nonzero imaginary part and let $r \ge 2$ be an integer. Then $W_r(z)$ is well defined and there is a positive constant γ_r such that*

$$\mathbb{E}|W_r(z)|^2 = \frac{\gamma_r}{|\Im\{z\}|^{2r-1}}.$$

□

In the case of $r = 1$ we obtain the explicit constant $\gamma_1 = 1$:

$$\mathbb{E}|W_1(z) \mp \pi i|^2 = \frac{\pi}{|\Im(z)|}.$$

To see this, compute

$$\mathbb{E}\left[\int \frac{1}{|z-x|^2}\,dN_M(x)\middle|\mathcal{R}_N\right] = \mathcal{R}_N \cdot \frac{1}{2N}\int_{x\in[-N,N]} \frac{1}{(z-x)\cdot(\bar{z}-x)}dx$$

$$= \frac{\mathcal{R}_N}{2N(\bar{z}-z)}\left[\int_{x\in[-N,N]} \frac{1}{z-x}dx - \int_{x\in[-N,N]} \frac{1}{\bar{z}-x}dx\right]$$

$$= \frac{1}{\bar{z}-z}\left\{\mathbb{E}\left[\int \frac{1}{z-x}\,dN_M(x)\middle|\mathcal{R}_N\right] - \mathbb{E}\left[\int \frac{1}{\bar{z}-x}\,dN_M(x)\middle|\mathcal{R}_N\right]\right\}.$$

Thus, taking expectations and by Lemma 3.3,

$$\mathbb{E}\left[\int_* \frac{1}{|z-x|^2}\,dN(x)\right] = \frac{1}{\bar{z}-z}\left\{\mathbb{E}\left[\int_* \frac{1}{z-x}\,dN(x)\right] - \mathbb{E}\left[\int_* \frac{1}{\bar{z}-x}dN(x)\right]\right\}.$$

Proposition 3.4 shows the difference of expectations on the right-hand side to be $-2i\pi$, yielding $\gamma_1 = \pi$.

Corollary 3.7. *For y with nonzero imaginary part and $r \geq 1$, $W_r(ky)$ has variance $\mathbb{E}[\Re\{W-\overline{W}\}^2 + \Im\{W-\overline{W}\}^2] = k^{-1/2}\gamma_r(y)$. It follows (with $\delta_{1,r}$ denoting the Kronecker delta) that*

$$\phi_k^{(r)}(ky) = -i\pi\delta_{1,r} + (r-1)!k^{1-r}\left(\frac{-1}{y}\right)^r + O\left(k^{1/2-r}\right) \tag{3.5}$$

in probability as $k \to \infty$.

Proof. Let $N^{(k)}$ denote a Poisson law of intensity k, rescaled by k^{-1}. In other words, $N^{(k)}$ is the average of k independent Poisson laws of unit intensity. Under the change of variables $u = x/k$, the Poisson law $dN(x)$ becomes $kdN^{(k)}(u)$. Therefore,

$$\begin{aligned} W_r(ky) &= \int_* \frac{1}{(ky-x)^r}\,dN(x) \\ &= k^{1-r}\int_* \frac{1}{(y-u)^r}\,dN^{(k)}(u) \\ &= k^{1-r}\left(\frac{1}{k}\sum_{j=1}^k W_r^{[j]}\right) \end{aligned}$$

where $\{W_r^{[1]}, \ldots, W_r^{[k]}\}$ are k independent copies of $W_r(y)$. Because $W_r(y)$ has mean $-i\pi\delta_{1,r}$ and variance $\gamma_r(y)$, the variance of the average is $k^{-1/2}\gamma_r(y)$. The remaining conclusion follows from the expression (3.2) for $\phi_k^{(r)}$ and the fact that a random variable with mean zero and variance V is $O(V^{1/2})$ in probability. $\square$

3.3. Uniformizing the estimates.

At some juncture, our pointwise estimates need to be strengthened to uniform estimates. The following result is a foundation for this part of the program.

Lemma 3.8. *Fix a compact set K in the upper half plane and an integer $r \geq 1$. There is a constant C such that for all integers $k \geq 1$,*

$$\mathbb{E}\sup_{z\in K}\left|h_k^{(r)}(z)\right| \leq Ck^{-1/2}.$$

Proof. Let $F^{(k)}$ denote the CDF for the random compensated measure $N^{(k)} - dx$ on $\mathbb{R}^+$; thus $F(x) = N^{(k)}[0,x] - x$ when $x > 0$ and $F(x) = x - N^{(k)}[x,0]$ when $x < 0$. We have

$$h_k^{(r)}(z) = \int_* C(z-x)^{-r-1}\, dN(x) = \int_* C(z-x)^{-r-1}\, dF^{(k)}(x)$$

because $\int_* (z-x)^{-r-1}\, dx = 0$. This leads to

$$\begin{aligned}&\mathbb{E} \sup_{z\in K} \left|h_k^{(r)}(z)\right| \\ \leq \quad & \lim_{M\to\infty} \mathbb{E} \sup_{z\in K} \left| \int_0^M \frac{1}{(z-x)^r}\, dF^{(k)}(x) \right| + \mathbb{E} \sup_{z\in K} \left| \int_{-M}^0 \frac{1}{(z-x)^r}\, dF^{(k)}(x) \right| .\end{aligned}$$

The two terms are handled the same way. Integrating by parts,

$$\int_0^M (z-x)^{-r}\, dF^{(k)}(x) = (z-x)^{-r} N[0,M] - \int -r(z-x)^{-r-1}\, F^{(k)}(x)\, dx.$$

This implies that

$$\begin{aligned}&\mathbb{E} \sup_{z\in K} \left|h_k^{(r)}(z)\right| \\ \leq \quad & \lim_{M\to\infty} \left[\mathbb{E}|F^{(k)}(M)| \sup_{z\in K} |z-x|^{-r} + \int_0^M \sup_{z\in K} r|z-x|^{-r-1} |F^{(k)}(x)| \right] dx \\ \leq \quad & C_K \lim_{M\to\infty} \left(M^{-r+1/2} + k^{-1/2} \right) .\end{aligned}$$

Sending M to infinity gives the conclusion of the lemma. $\square$

Corollary 3.9. *The following hold:*

- (i) $\sup_{z\in K} |h_k^{(r)}(z)| = O(k^{-1/2})$ *in probability.*
- (ii) h_k *and its derivatives are Lipschitz on* K *with Lipschitz constant* $O(k^{-1/2})$ *in probability.*
- (iii) *For* $r \geq 2$, *the* $O(k^{-1/2})$ *term in the expression* (3.5) *for* $\phi_k^{(r)}(ky)$ *is uniform as* y *varies over compact sets of the upper half plane.*

Proof. Conclusion (i) is Markov's inequality. Conclusion (ii) follows because any upper bound on a function $|g'|$ is a Lipschitz constant for g. Conclusion (iii) follows from the relation between h_k and ϕ_k. $\square$

Lemma 3.10. *For any* $c > 0$,

$$\mathbb{P}\left[\sup \left\{ |h_k(y) + i\pi| \,:\, |y - \frac{i}{\pi}| \leq Mk^{-1/2} \right\} \geq cMk^{-1/2} \right] \to 0$$

as $M \to \infty$ *uniformly in* $k \geq 4\pi^2 M^2$.

Proof. Fix $c, \varepsilon > 0$. Choose $L > 0$ such that the probability of the event G is at most $\varepsilon/2$, where G is the event that the Lipschitz constant for some h_k on the ball $B(i\pi, 1/(2\pi))$ is greater than L. Let B be the ball of radius $Mk^{-1/2}$ about i/π. The assumption $k \geq 4\pi^2 M^2$ guarantees that B is a subset of the ball $B(i\pi, 1/(2\pi))$ over which the Lipschitz constant was computed. Let y be any point in B. The ball of radius $\rho := cMk^{-1/2}\varepsilon/(2L)$ about y intersects B in a set whose area is at least $\rho^2\sqrt{3}/2$, the latter being the area of two equilateral triangles of side ρ. If $|h_k(y)+i/\pi| \geq cMk^{-1/2}$ and G does not occur, then $|h_k(u)+i/\pi| \geq (1/2)cMk^{-1/2}$ on the ball of radius ρ centered at y.

Now we compute in two ways the expected measure $\mathbb{E}|S|$ of the set S of points $u \in B$ such that $|h_k(u) + i\pi| \geq \rho$. Firstly, by what we have just argued,

$$\mathbb{E}|S| \geq \frac{\sqrt{3}}{2}\rho^2\left(Q - \frac{\varepsilon}{2}\right) = \left(Q - \frac{\varepsilon}{2}\right)\sqrt{\frac{3c^2\varepsilon^2}{16L^2}\frac{M^2}{k}} \tag{3.6}$$

where Q is the probability that there exists a $y \in B$ such that $|h_k(y) + i/\pi| \geq cMk^{-1/2}$. Secondly, by Proposition 3.4 and the computation of γ_1, for each $u \in B$, $\mathbb{E}h_k(u) + i/\pi = 0$ and $\mathbb{E}|h_k(u)|^2 = \pi/k$, leading to $\mathbb{E}|h_k(u) + i/\pi| \leq \sqrt{2\pi/k}$ and hence

$$\begin{aligned}\mathbb{P}\left(\left|h_k(u) + \frac{i}{\pi}\right| \geq \rho\right) &\leq \frac{\sqrt{2\pi/k}}{\rho} \\ &= \frac{\sqrt{2\pi/k}}{cMk^{-1/2}\varepsilon/(2L)} \\ &= \sqrt{\frac{32\pi L^2}{c^2}}M^{-1/2}.\end{aligned}$$

By Fubini's theorem,

$$\mathbb{E}|S| \leq |B| \sup_{u\in B}\mathbb{P}\left(\left|h_k(u) + \frac{i}{\pi}\right| \geq cMk^{-1/2}\right) \leq \pi\frac{M^2}{k}\sqrt{\frac{32\pi L^2}{c^2}}M^{-1/2}. \tag{3.7}$$

Putting together the inequalities (3.6) and (3.7) gives

$$Q - \frac{\varepsilon}{2} \leq \sqrt{\frac{512\pi^3L^4}{3c^4\varepsilon^2}}M^{-1/2}.$$

Once M is sufficiently large so that the radical is at most $\varepsilon/2$ implies that $Q \leq \varepsilon$. Because $\varepsilon > 0$ was arbitrary, we have shown that $Q \to 0$ as $M \to \infty$ uniformly in k, as desired. □

Proof of Theorem 3.2. Using Lemma 3.10 for $c < 1$, we know that

$$\mathbb{P}\left[\sup\left\{|h_k(y) + i\pi| \,:\, |y - \frac{i}{\pi}| \leq Mk^{-1/2}\right\} \leq cMk^{-1/2}\right] \longrightarrow 1, \text{ as } k \to \infty.$$

Writing

$$A_{M,k} = \left\{\omega : \sup\left\{|h_k(y) + i\pi| \,:\, |y - \frac{i}{\pi}| \leq Mk^{-1/2}\right\} \leq cMk^{-1/2}\right\},$$

for all $\omega \in A_{M,k}$ and all y such that $|y - \frac{i}{\pi}| = Mk^{-1/2}$, we get

$$\begin{aligned}\left|\phi_k(y)(\omega) - \left(-i\pi - \frac{1}{y}\right)\right| &= |h_k(y)(\omega) + i\pi| \\ &\leq cMk^{-1/2} \\ &= c\left|y - \frac{i}{\pi}\right| \\ &< \left|y - \frac{i}{\pi}\right|\end{aligned}$$

for k sufficiently large. Thus, by Rouche's theorem, $\phi_k(y)(\omega)$ and $y - \frac{i}{\pi}$ have the same number of zeros inside the disc centered at i/π of radius $Mk^{-1/2}$, i.e. exactly one. This implies that $\mathbb{P}(E_{M,k}) \to 1$ as $M, k \to \infty$ with $k \geq 4\pi^2 M^2$. □

4. The Cauchy integral

4.1. **Dominant arc: saddle point estimate.** We sum up those facts from the foregoing subsection that we will use to estimate the Cauchy integral in the dominant arc near σ_k.

Lemma 4.1. *The following hold:*

(*i*) $\phi'(\sigma_k) = 0$.
(*ii*) $\sigma_k^2 \phi''(\sigma_k) = k + O(k^{1/2})$ *in probability as* $k \to \infty$.
(*iii*) *If* K *is the set* $\{z : |z - \sigma_k| \leq k/2$, *then* $\sup_{z \in K} k^3 \phi^{(3)}(z) = O(k)$ *in probability.*

Proof. The first is just the definition of σ_k. For the second, using Corollary 3.9 for $r = 2$ and $y = \frac{i}{\pi}$, the estimate (3.5) is uniform, hence

$$\phi''(\sigma_k) = \phi''\left(\frac{ik}{\pi}\right) + O\left(k^{-3/2}\right) = \frac{-\pi^2}{k} + O\left(k^{-3/2}\right)$$

in probability. Multiplying by $\sigma_k^2 \sim -k^2/\pi^2$ gives (*ii*). The argument for part (*iii*) is analogous to the argument for part (*ii*). □

Definition 4.2 (Arcs, fixed value of δ)**.** For the remainder of the paper, fix a number $\delta \in (1/3, 1/2)$. Parametrize the circle Γ through σ_k in several pieces, all oriented counterclockwise, as follows (see Figure 1). Define Γ_1 to be the arc $\{z : z = \sigma_k e^{it}, -k^{-\delta} \leq t \leq k^{-\delta}\}$. Define Γ_1' to be the arc $\{z : z = \overline{\sigma}_k e^{it}, -k^{-\delta} \leq t \leq k^{-\delta}\}$, so that the arc is conjugate to Γ_1 but the orientation remains counterclockwise. Define Γ_2 to be the part of Γ in the second quadrant that is not part of Γ_1, define Γ_3 to be the part of Γ in the first quadrant not in Γ_1, and define Γ_2' and Γ_3' to be the respective conjugates. Define the phase function along Γ by

$$g_k(t) := \phi_k(\sigma_k e^{it}) \, .$$

Theorem 4.3 (Contribution from Γ_1)**.** *For any integer* $r \geq 0$,

$$\frac{\displaystyle\int_{\Gamma_1} \frac{f(z)}{z^{k+r+1}} dz}{k^{-1/2} f(\sigma_k) \sigma_k^{-k-r}} \longrightarrow i\sqrt{2\pi}$$

in probability as $k \to \infty$.

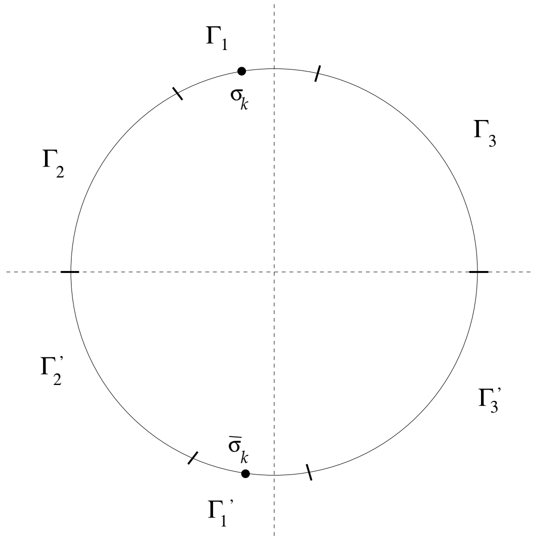

FIGURE 1. Parametrization of the circular contour Γ

Proof. For fixed k, Taylor's expansion of $g_k(t)$ gives us

$$g_k(t) = g_k(0) + t g_k'(0) + \frac{t^2}{2} g_k^{(2)}(0) + \frac{t^3}{6}\left(\Re g_k^{(3)}(t_1) + i \Im g_k^{(3)}(t_2)\right),$$

where t_1 and t_2 are points that lie between 0 and t.

By Lemma 4.1, $g_k'(0) = 0$ and

$$g_k^2(0) = k + O\left(k^{1/2}\right)$$

in probability. Thus,

$$\sup_{|t| \le k^{-\delta}} \sqrt{k}\left[\exp\left(\frac{t^2}{2} g_k^{(2)}(0)\right) - \exp\left(-\frac{kt^2}{2}\right)\right] \longrightarrow 0.$$

In addition,

$$\sup_{z \in \Gamma_1} \left|\frac{\sigma_k^r}{z^r} - 1\right| \longrightarrow 0,$$

while Lemma 4.1 also gives us

$$\sup_{|t| \le k^{-\delta}} \frac{t^3}{6} g_k^{(3)}(t) \longrightarrow 0.$$

Thus,

$$\int_{\Gamma_1} \frac{f(z)}{z^{k+r+1}} dz$$
$$= i \int_{-k^{-\delta}}^{k^{-\delta}} \sigma_k^{-r} \exp\left[g_k(0) + \frac{t^2}{2} g_k^{(2)}(0) + \frac{t^3}{6}\left(\Re g_k^{(3)}(t_1) + i \Im g_{k,N}^{(3)}(t_2)\right) - i\, rt\right] dt,$$

whence, as $k \to \infty$,

$$\sqrt{k}\frac{\int_{\Gamma_1} \frac{f(z)}{z^{k+r+1}} dz}{\sigma_k^{-r}\exp(g_k(0))} - i\sqrt{k}\int_{-k^{-\delta}}^{k^{-\delta}} \exp\left(-\frac{kt^2}{2}\right) dt \longrightarrow 0.$$

Changing variables to $t = u/\sqrt{k}$ shows that when $\delta < 1/2$, the integral is asymptotic to $\sqrt{2\pi/k}$. Plugging in $g_k(0) = f(\sigma_k)\sigma_k^{-k}$ completes the proof. $\square$

4.2. **Negligible arcs and remainder of proof of Theorem 2.7.** We now show that the Cauchy integral receives negligible contributions from $\Gamma_2, \Gamma_2', \Gamma_3$ and Γ_3'. By conjugate symmetry we need only check Γ_2 and Γ_3; the arguments are identical so we present only the one for Γ_2.

Let $R := |\sigma_k|$ and let β denote the polar argument of σ_k, that is, $\beta := \arg(\sigma_k) - \pi/2$, so that $\sigma_k = iRe^{i\beta}$. By Theorem 3.2, $\beta = O(k^{-1/2})$ in probability. We define an exceptional event G_k of probability going to zero as follows:

Let G_k be the event that either $R \notin [k/(2\pi), 2k/\pi]$ or $\beta > k^{-\delta}/2$.

If $z = iRe^{i\theta}$ is a point of Γ_2 with polar argument θ, then θ is at least $k^{-\delta} - |\beta|$, hence is at least $(1/2)k^{-\delta}$ on G_k^c. Note that the notation suppresses the dependence of R and β on k, which does not affect the proof of the in-probability result in Lemma 4.4.

Lemma 4.4.

$$\frac{\int_{\Gamma_2} \frac{f(z)}{z^{k+r+1}} dz}{k^{-1/2} f(\sigma_k)\sigma_k^{-k-r}} \longrightarrow 0 \tag{4.1}$$

in probability as $k \to \infty$.

Proof. Let $z = iRe^{i\theta} \in \Gamma_2$. Our purpose is to show that $|f(z)z^{-k}|$ is much smaller than $|f(\sigma_k)\sigma_k^{-k}|$. On Γ_2 we are worried only about the magnitude, not the argument, so we may ignore the z^{-k} and σ^{-k} terms, working with ϕ rather than with ϕ_k. This simplifies (3.5) to

$$\phi'(z) = -i\pi + O\left(k^{-1/2}\right), \tag{4.2}$$

the estimate being uniform on the part of Γ_2 with polar argument less than $\pi/2 - \varepsilon$ by part (iii) of Corollary 3.9. Let H_k be the exceptional event where the constant in the uniform $O(k^{-1/2})$ term is greater than $k^{1/2-\delta}/100$, the probability of H_k going to zero according to the corollary.

Integrating the derivative of $\Re\{\phi(z)\}$ along Γ then gives

$$\log\left|\frac{f(z)}{f(\sigma_k)}\right| = \pi\left(\Im\{z\} - \Im\{\sigma_k\}\right) + O\left(k^{-1/2}|z - \sigma_k|\right). \tag{4.3}$$

The first of the two terms is $\pi R(\cos(\theta) - \cos(\beta))$, which is bounded from above by $-(R/2)(\theta^2 - \beta^2)$, which is at most $-(R/4)\theta^2$ on G_k^c. The second term is at most

$$\frac{k^{1/2-\delta}}{100} k^{-1/2}(2R\theta)$$

on $G_k^c \cap H_k^c$, provided that $\theta \le \pi/2 - \varepsilon$. Combining yields

$$\log\left|\frac{f(z)}{f(\sigma_k)}\right| \le -\frac{R}{4}\theta^2 + \frac{k^{-\delta}}{100}(2R\theta) \le -R\theta(\left(\frac{\theta}{4} - \frac{k^{-\delta}}{50}\right) \le -\frac{R\theta^2}{8}$$

on Γ_2 as long as the polar argument of z is at most $\pi/2 - \varepsilon$. Decompose $\Gamma_2 = \Gamma_{2,1} + \Gamma_{2,2}$ according to whether θ is less than or greater than $\pi/2 - \varepsilon$. On G_k^c we know that $\theta \ge (1/2)k^{-\delta}$ and $R \ge k/(2\pi)$, hence on $\Gamma_{2,1}$,

$$\log\left|\frac{f(z)}{f(\sigma_k)}\right| \le -\frac{k^{1-2\delta}}{64\pi}.$$

Using $d\theta = dz/z$ we bound the desired integral from above by

$$\left| \frac{\displaystyle\int_{\Gamma_2} \frac{f(z)}{z^{k+r+1}} dz}{k^{-1/2} f(\sigma_k) \sigma_k^{-k-r}} \right| \leq \sqrt{k} \int_{\Gamma_2} \left| \frac{f(z)}{f(\sigma_k)} \right| d\theta \, .$$

On $G_k^c \cap H_k^c$, the contribution from $\Gamma_{2,1}$ is at most

$$\sqrt{k} \, |\Gamma_2| \, \exp \left[- \frac{k^{1-2\delta}}{64\pi} \right] . \tag{4.4}$$

Finally, to bound the contribution from $\Gamma_{2,2}$, use Proposition 2.4 to deduce that $|f(z)| \leq |f(z')|$ where $\Re\{z'\} = \Re\{z\}$ and $\Im\{z'\} = k/(4\pi)$. Integrating (4.2) on the line segment between σ_k and z' now gives (4.3) again, and on $G_k^c \cap H_k^c$ the right-hand side is at most $-(k/4) + k^{-\delta} k < -k/8$ once $k \geq 8$. This shows the contribution from $\Gamma_{2,2}$ to be at most $\varepsilon R e^{-k/8}$. Adding this to (4.4) and noting that $\mathbb{P}(G_c \cup H_k) \to 0$ prove the lemma. $\square$

Theorem 4.5. *For fixed r as $k \to \infty$,*

$$e_{k+r} = 2 \, \Re \left\{ (1 + o(1)) \sigma_k^{-k-r} \, f(\sigma_k) \, \sqrt{\frac{1}{2\pi k}} \right\}$$

in probability as $k \to \infty$.

Proof. By Cauchy's integral theorem,

$$e_{k+r} = \frac{1}{2\pi i} \int_{\Gamma} f(z) z^{-k-r-1} \, dz \, .$$

By Theorem 4.3 and the fact that the contributions from Γ_1 and Γ_1' are conjugate, their sum is twice the real part of a quantity asymptotic to

$$\frac{1}{\sqrt{2\pi k}} f(\sigma_k) \sigma_k^{-k-r} \, . \tag{4.5}$$

By Lemma 4.4, the contributions from the remaining four arcs are negligible compared to (4.5). The theorem follows. $\square$

Proof of Theorem 2.7. By the definition of $a_{k,r}$, using Theorem 4.5 to evaluate e_k,

$$\begin{aligned} a_{k,r} &= e_{k+r} \frac{(k+r)!}{r!} \\ &= 2 \, k! \, \frac{(k+r)!}{k!} \, \frac{1}{r!} \, \Re \left\{ (1 + o(1)) \sigma_k^{-k-r} \, f(\sigma_k) \, \sqrt{\frac{1}{2\pi k}} \right\} . \end{aligned}$$

For fixed r as $k \to \infty$ asymptotically $(k+r)!/k! \sim k^r$. Setting

$$A_k = k! \sqrt{\frac{2}{\pi k}} \left| \sigma_k^{-k} f(\sigma_k) \right| \quad \text{and} \quad \theta_k = \arg\{\sigma_k^{-k} f(\sigma_k)\}$$

simplifies this to

$$A_k \frac{k^r}{|\sigma_k|^r} \left[\cos\left(\theta_k - r \arg(\sigma_k)\right) \right] \, .$$

Because in probability $\arg(\sigma_k) = \pi/2 + o(1)$ while $|\sigma_k| \sim k/\pi$, this simplifies finally to

$$a_{k,r} = A_k \left[\cos\left(\theta_k - \frac{r\pi}{2}\right) + o(1)\right] \cdot \frac{\pi^r}{r!} \text{ in probability,}$$

proving the first part of the theorem.

Next, from the proof of Theorem 4.3 it is clear that

$$\left| \frac{\int_{\Gamma_1} \frac{f(z)}{z^{k+r+1}} dz}{\frac{f(\sigma_k)}{\sigma_k^{k+r}}} \right| \leq \int_{\Gamma_1} |\exp(g_k(t) - g_k(0))| dt$$

is bounded above in probability, and this bound is independent of r. Also the convergence in the proof of Lemma 4.4 is independent of r. Therefore,

$$\left| \frac{a_{k,r}}{A_k} \right| = O\left(\frac{(k+r)!}{k!} \frac{1}{r! |\sigma_k|^r} \right).$$

Since, for any $M > 0$,

$$\sum_{r=1}^{\infty} \frac{(k+r)!}{k!} \frac{\pi^r}{r!} \frac{M^r}{k^r} < \infty, \forall\, k > M\pi,$$

with the convergence being uniform over $k \in [T, \infty)$, with $T > M\pi$, we have our result. $\square$

References

[BBL09] Julius Borcea, Petter Brändén, and Thomas M. Liggett, *Negative dependence and the geometry of polynomials*, J. Amer. Math. Soc. **22** (2009), no. 2, 521–567, DOI 10.1090/S0894-0347-08-00618-8. MR2476782

[Bre89] Francesco Brenti, *Unimodal, log-concave and Pólya frequency sequences in combinatorics*, Mem. Amer. Math. Soc. **81** (1989), no. 413, viii+106, DOI 10.1090/memo/0413. MR963833

[CC95] Thomas Craven and George Csordas, *Complex zero decreasing sequences*, Methods Appl. Anal. **2** (1995), no. 4, 420–441, DOI 10.4310/MAA.1995.v2.n4.a4. MR1376305

[Con83] Brian Conrey, *Zeros of derivatives of Riemann's ξ-function on the critical line*, J. Number Theory **16** (1983), no. 1, 49–74, DOI 10.1016/0022-314X(83)90031-8. MR693393

[Dur10] Rick Durrett, *Probability: theory and examples*, 4th ed., Cambridge Series in Statistical and Probabilistic Mathematics, Cambridge University Press, Cambridge, 2010. MR2722836

[FR05] David W. Farmer and Robert C. Rhoades, *Differentiation evens out zero spacings*, Trans. Amer. Math. Soc. **357** (2005), no. 9, 3789–3811, DOI 10.1090/S0002-9947-05-03721-9. MR2146650

[LM74] Norman Levinson and Hugh L. Montgomery, *Zeros of the derivatives of the Riemann zetafunction*, Acta Math. **133** (1974), 49–65. MR0417074

[Maj99] Péter Major, *The limit behavior of elementary symmetric polynomials of i.i.d. random variables when their order tends to infinity*, Ann. Probab. **27** (1999), no. 4, 1980–2010, DOI 10.1214/aop/1022677557. MR1742897

[Mar49] Morris Marden, *The Geometry of the Zeros of a Polynomial in a Complex Variable*, Mathematical Surveys, No. 3, American Mathematical Society, New York, N. Y., 1949. MR0031114

[MS82] T. F. Móri and G. J. Székely, *Asymptotic behaviour of symmetric polynomial statistics*, Ann. Probab. **10** (1982), no. 1, 124–131. MR637380

[Pem12] Robin Pemantle, *Hyperbolicity and stable polynomials in combinatorics and probability*, Current developments in mathematics, 2011, Int. Press, Somerville, MA, 2012, pp. 57–123. MR3098077

[Sto26] A. Stoyanoff, *Sur un théorem de M. Marcel Riesz*, Nouvelles Annales de Mathematique **1** (1926), 97–99.

[Sub14] Sneha Dey Subramanian, *Zeros, critical points, and coefficients of random functions*, ProQuest LLC, Ann Arbor, MI, 2014. Thesis (Ph.D.)–University of Pennsylvania. MR3251112

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