Sharpness of second moment criteria for branching and tree-indexed processes

Robin Pemantle [1, 2]

ABSTRACT:

A class of branching processes in varying environments is exhibited which become extinct almost surely even though the means M_n grow fast enough so that $\sum M_n^{-1}$ is finite. In fact, such a process is constructed for every offspring distribution of infinite variance, and this establishes the converse of a previously known fact: that if a distribution has finite variance then $\sum M_n^{-1} = \infty$ is equivalent to almost sure extinction. This has as an immediate consequence the converse to a theorem on equipolarity of Galton-Watson trees.

Keywords: Galton-Watson, branching, tree, tree-indexed, equipolar.

[1]Department of Mathematics, University of Wisconsin-Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706

[2]Research supported in part by National Science Foundation Grant # DMS 9300191, by a Sloan Foundation Fellowship and by a Presidential Faculty Fellowship

This note provides a class of examples of branching processes in varying environments (BPVE's) that die out almost surely even though the means grow relatively fast.

Theorem 1 *Let f be any offspring generating function with $f'(1) = m > 1$ and $f''(1) = \infty$. Then there is a sequence of positive numbers $p_n \leq 1$ such that*

(i) $\displaystyle\sum_{n=1}^{\infty} \prod_{k=1}^{n} (m \cdot p_k)^{-1} < \infty$ *and*

(ii) *The BPVE with offspring generating functions $f_n(z) = f(1 - p_n + p_n z)$ dies out almost surely.*

The expected size of the n^{th} generation of the BPVE in (ii) is given by

$$\mathbf{E}Z_n = \prod_{k=1}^{n} (m \cdot p_k)$$

which is why condition (i) is a growth condition on the means. The proof of this theorem is easy, and the exposition will focus mostly on saying why the theorem is interesting. There are two reasons, one having to do with branching processes and one to do with tree-indexed processes. The motivation coming from branching processes is more straightforward.

A branching process in a varying environment (BPVE) is defined by a sequence of offspring generating functions

$$f_n(z) = \sum_{k=0}^{\infty} q_{n,k} z^k$$

where for each n, the nonnegative real numbers $\{q_{n,k}\}$ sum to 1. From the sequence $\{f_n\}$ a random tree Γ is constructed as follows. The root has a random number Z_1 of children, where $\mathbf{P}(Z_1 = k) = q_{1,k}$. Each of these first-generation individuals has a random number of children, these random numbers $X_1, \ldots, X_{Z_1}$ being IID given Z_1 and satisfying $\mathbf{P}(X_1 = k) = q_{2,k}$. This continues in the same manner, so that if Z_n is the total number of individuals in generation n, then the numbers of children of each of these Z_n individuals are IID, being equal to k with probability $q_{n+1,k}$.

The mean number of children of an individual in generation n is $f_n'(1)$ and therefore the expected number of individuals in generation n is given by

$$M_n = \prod_{k=1}^n f_k'(1).$$

A *Galton-Watson* or simple branching process is one where the environment does not vary, i.e., $f_n = f$ for all n. In this case $f'(1) \leq 1$ is necessary and sufficient for almost sure extinction:

$$f'(1) \leq 1 \Longleftrightarrow \mathbf{P}(Z_n \to 0) = 1.$$

In the varying case, it is possible for M_n to grow without growing exponentially, and one may ask whether the growth rate of M_n determines whether $\mathbf{P}(Z_n \to 0)$ is equal to 1. Under the assumption that $\liminf_{n\to\infty} 1 - f_n(0) - f_n'(0) > 0$ (a weak nondegeneracy condition saying the probability of at least two children is bounded away from zero), Agresti (1975) shows that

$$\sum M_n^{-1} = \infty \quad \text{implies almost sure extinction .}$$

On the other hand, a second moment condition is needed for the converse. The second moment of the number of children in generation n is $f_n''(1) + f_n'(1)$. Agresti shows that if $\sup_n f_n''(1) < \infty$ then

$$\sum M_n^{-1} < \infty \quad \text{implies a positive probability of non-extinction.} \tag{1}$$

See also Theorem 4.14 of Lyons (1992).

A natural class of BPVE's are those obtained from a single offspring generating function f by killing individuals in generation n independently with probabilities $1 - p_n$. Think of this as modeling a genealogy where the branching mechanism remains the same from generation to generation but the hospitality of the environment varies. Formally, $f_n(z) = f(1 - p_n + p_n z)$, so the expected generation sizes are $M_n = \prod_{k=1}^n (m \cdot p_k)$. The second moment condition $\sup_n f_n''(1) < \infty$ is equivalent in this case to $f''(1) < \infty$. Theorem 1 shows that this condition is necessary as well as sufficient for (1) to hold: for any f with infinite second moment, some BPVE of the form $f_n(z) = f(1 - p_n + p_n z)$ becomes extinct almost surely even though $\sum M_n^{-1} < \infty$.

Theorem 1 may also be viewed as a fact about tree-indexed processes. A process indexed by a tree Γ is simply a set of IID real random variables $\{X(v)\}$ indexed by the vertices of Γ. Let

$B \subseteq \mathbb{R}^\infty$ be closed in the product topology. The following notion of polar sets for tree-indexed processes was first defined by Evans (1992).

Definition: *The set B is polar for Γ (and for the common distribution of the variables $\{X(v)\}$) if and only if the probability is zero that there exists an infinite self-avoiding path $v_0, v_1, v_2, \ldots$ from the root of Γ satisfying $(X(v_1), X(v_2), \ldots) \in B$.*

Trees with the same polar sets are denoted *equipolar* by Pemantle and Peres (1994). In particular, letting $\{X(v)\}$ be uniform on the unit interval and letting $B = \{(x_1, x_2, \ldots) : \forall n \; x_n \leq p_n\}$, one sees that equipolar trees Γ_1 and Γ_2 are percolation equivalent, meaning that:

> If vertices of both trees are removed independently with the survival probability p_n of a vertex being the same for all vertices in generation n of either tree, then the root of Γ_1 has positive probability of being in an infinite component of surviving vertices if and only if the root of Γ_2 has positive probability of being in an infinite component of surviving vertices.

There is not space here for a substantial discussion of equipolarity, but the reader is referred to Pemantle and Peres (1994), wherein it is shown that equipolar trees behave similarly for a variety of common probability models (other than percolation) involving trees, including maximal displacements of branching random walks, nonextinction probabilities for branching random walks with absorption, survival of certain BPVE's, growth rates of first-passage percolation clusters, and capacities of fractal sets in Euclidean space defined by interpreting the tree as a base-b expansion of a closed subset of the unit cube. Equipolarity results for random trees lead to Peres' (1994) derivation of Fitzsimmons and Salisbury's (1989) capacity criteria for multiple points of Brownian motions and to a general capacity-theoretic framework for intersection properties of random sets. Given that equipolarity is a useful notion, the effort to understand which trees are equipolar should seem justified.

In Pemantle and Peres (1994) it is shown that trees which arise from Galton-Watson processes with respective offspring generating functions f and g are almost surely equipolar provided they have the same mean growth $f'(1) = g'(1)$, and that each has a finite variance: $f''(1) < \infty$ and $g''(1) < \infty$. It is also shown in the preprint version that the second moment assumption is almost sharp in the sense that if $f''(1) < \infty$ but the distribution defined by g fails to have a $2-\epsilon$ moment for some positive ϵ, then there is some set B which is almost surely polar for a Galton-Watson tree with offspring generating function g but almost surely nonpolar for a Galton-Watson tree with offspring generating function f. Theorem 1 improves this to a sharp result, namely that whenever g fails to have a second moment, the set

$$B = \{(x_1, x_2, \ldots) : \forall n \; x_n \leq p_n\}$$

defined from the sequence $\{p_n\}$ in the conclusion of the theorem is polar for almost every Galton-Watson tree with offspring generating function g. (The fact that it is nonpolar for almost every Galton-Watson tree with offspring generating function f follows from $\sum M_n^{-1} = \infty$.)

Having given motivation for the theorem, I now give the proof, which is based on a well-known result of Kesten, Ney and Spitzer (proved with a third moment assumption by Kolmogorov).

Theorem 2 (K-K-N-S) *Suppose g is an offspring generating function for a critical Galton-Watson process, i.e., $g'(1) = 1$. Let $\sigma^2 = Var(Z_1) = g''(1) \leq \infty$. Then*

$$\lim_{n \to \infty} n\mathbf{P}(Z_n > 0) = \frac{2}{\sigma^2}.$$

Proof: See Kesten, Ney and Spitzer (1966) or Lyons, Pemantle and Peres (1994). $\square$

Proof of Theorem 1: Fix an offspring generating function f with $m := f'(1) < f''(1) = \infty$. Let $g(z) = f(1 - 1/m + z/m)$ so that g is an offspring generating function satisfying the hypotheses of K-K-N-S theorem. Applying the theorem to the probabilities P_n of the critical branching process with offspring generating function g surviving n levels, we see that we may choose for every n an L_n such that for all $k \geq L_n$,

$$P_k < 4^{-n} k^{-1}.$$

Define sequences $\{t_n\}$ and $\{u_n\}$ recursively as follows. Let $u_0 = 0$ and let t_1 be the least positive integer for which $m^{t_1} > L_1$. For each $n \geq 1$, let

$$K_n = \prod_{j=1}^{n} m^{t_j - u_{j-1}},$$

let

$$u_n = t_n + \lceil 2^{-n} K_n \rceil$$

and let t_{n+1} be the least integer $k > u_n$ for which

$$m^{k-u_n} \prod_{j=1}^{n} m^{t_j - u_{j-1}} > 2^{n+1} L_{n+1}.$$

This ensures that $K_{n+1} > 2^{n+1} L_{n+1}$. Let $p_n = 1/m$ whenever $t_j \leq n < u_j$ for some j, and let $p_n = 1$ whenever $u_j \leq n < t_{j+1}$ for some j. When $t_j \leq n \leq u_j$, the n^{th} generation has expected size $\mathbf{E} Z_n = \prod_{k=1}^{n} (m \cdot p_k) = K_n$.

To verify condition (i) of Theorem 1, observe first that if b_n is the n^{th} positive integer j (in ascending order) for which $p_j = 1$, then

$$\sum_{n=1}^{\infty} \prod_{k=1}^{b_n} (m \cdot p_k)^{-1} = \sum_{n=1}^{\infty} m^{-n} < \infty.$$

Thus it suffices to show that

$$\sum_{n : p_n = \frac{1}{m}} \prod_{k=1}^{n} (m \cdot p_k)^{-1} < \infty. \tag{2}$$

Write this sum as

$$\sum_{n=1}^{\infty} \sum_{j=t_n}^{u_n - 1} \prod_{i=1}^{j} (m \cdot p_i)^{-1} = \sum_{n=1}^{\infty} (u_n - t_n) K_n^{-1}.$$

By construction, $u_n - t_n < 2^{-n} K_n + 1$, and since $K_n^{-1} < m^{-n}$, the sum in (2) is finite.

To verify condition (ii), first write

$$\mathbf{P}(Z_{u_n} > 0) = \mathbf{E}(\mathbf{P}(Z_{u_n} > 0) \mid Z_{t_n}) \leq r_{u_n - t_n} \mathbf{E} Z_{t_n} \tag{3}$$

where r_n is the probability that the critical branching process with offspring generating function g survives to the n^{th} generation. Since $u_n - t_n \geq 2^{-n} K_n > L_n$, the definition of L_n then forces

$$r_{u_n - t_n} < 4^{-n} (u_n - t_n)^{-1}.$$

Plugging this into (3) and using $\mathbf{E} Z_{t_n} = K_n$ then gives $\mathbf{P}(Z_{u_n} > 0) < 4^{-n} [2^{-n} K_n]^{-1} K_n$, and hence

$$\lim_{n \to \infty} \mathbf{P}(Z_{u_n} > 0) = 0.$$

Thus the BPVE dies out almost surely. $\square$

References

[1] Agresti, A. (1975). On the extinction times of varying and random environment branching processes. *J. Appl. Prob.* **12** 39 - 46.

[2] Evans, S. (1992). Polar and non-polar sets for a tree-indexed process. *Ann. Probab.* **20** 579 - 590.

[3] Kesten, H., Ney, P. and Spitzer, F. (1966). The Galton-Watson process with mean one and finite variance. *Theory of probability and its applications* **11** 513 - 540.

[4] Lyons, R., Pemantle, R. and Peres, Y. (1994). Conceptual proofs of $L \log L$ criteria for mean behavior of branching processes. *Ann. Probab. to appear.*

[5] Lyons, R. (1992). Random walks, capacity and percolation on trees. *Ann. Probab.* **20** 2043 - 2088.

[6] Pemantle, R. and Peres, Y. (1993). Galton-Watson trees with the same means have the same polar sets. *Ann. Probab. to appear.*

August 4, 1999