Principal minors and rhombus tilings

R. Kenyon and R. Pemantle

Abstract

The algebraic relations between the principal minors of an $n \times n$ matrix are somewhat mysterious, see e.g. [LS09]. We show, however, that by adding in certain *almost* principal minors, the relations are generated by a single relation, the so-called hexahedron relation, which is a composition of six cluster mutations.

We give in particular a Laurent-polynomial parameterization of the space of $n \times n$ matrices, whose parameters consist of certain principal and almost principal minors. The parameters naturally live on vertices and faces of the tiles in a rhombus tiling of a convex $2n$-gon. A matrix is associated to an equivalence class of tilings, all related to each other by Yang-Baxter-like transformations.

By specializing the initial data we can similarly parametrize the space of Hermitian symmetric matrices over $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ the quaternions. Moreover by further specialization we can parametrize the space of *positive definite* matrices over these rings.

1 Introduction

A *principal minor* of a complex $n \times n$ matrix M is the determinant of a submatrix M_A^A where A is a subsets of $[n] := \{1, \ldots, n\}$ and M_A^B denotes the submatrix of M obtained by restricting rows to A and columns to B. There are 2^n principal minors of M if one includes the trivial minor $\det M_\emptyset^\emptyset := 1$. Introducing an indeterminate x_A for each minor, one may ask what polynomial relations hold among the minors, that is, what polynomials in $\mathbb{C}[x_A : A \subseteq [n]]$ always hold. The algebraic relations between these principal minors are somewhat mysterious. For example, when $n = 4$, Lin and Sturmfels [LS09] show that the ideal of all polynomial relations is minimally generated by 65 polynomials of degree 12.

Say that $\det M_A^B$ is an *almost-principal minor* if $A, B \subseteq [n]$ with $|A| = |B|$ and if the sets differ by precisely one element: $|A \triangle B| = 1$. Divide the almost-principal minors into two classes, say *odd* and *even*, by putting M_A^B in the odd class if $A = S \cup \{i\}$ and $B = S \cup \{j\}$ with $(i - j)(-1)^{|S|} > 0$. In other words, if the extra row index is greater than the extra column index then the parity of the minor is the same as the parity of $S := A \cap B$, but when the extra column index is greater than the extra row index, then the parity of the minor is opposite to the parity of S.

Our first result concerns the relations that hold among the principal minors and the odd almost-principal minors (by symmetry we could use even almost-principal minors instead). We show that these are generated by a single polynomial relation, the so-called hexahedron relation of [KP13]. The hexahedron relation is a set of four polynomial relations holding among fourteen variables indexed by the eight vertices and six faces of a cube. For any Boolean interval $[S, S \cup D]$ of rank three in the Boolean lattice $\mathcal{B}_n$ of rank n, the vertices and faces may be naturally associated with the eight principal and six odd almost principal minors of the form $\{\det M_{S\cup A}^{S\cup B} : A, B \subseteq D\}$. As $[S, S \cup D]$ vary over rank-3 Boolean intervals in $\mathcal{B}_n$, the corresponding hexahedron relations generate the ideal of all polynomial relations among these minors. This is Theorem 4.3 below.

The hexahedron relation is a composition of six cluster mutations. This allows us explicitly to parameterize the variety of all possible collections of principal and odd almost principal minors. One must first pick a set of variables x_A^B, call these the *initial conditions*, to specify, and then describe the remaining variables in terms of the initial conditions. There are many ways of choosing the set of variables for the initial conditions. It turns out that these correspond naturally to the rhombus tilings of a $2n$-gon. For any fixed tiling, the matrix entries and all principal and odd almost principal minors turn out to be Laurent polynomials in the initial variables associated with the chosen tiling. This is Theorem 4.4 below. The reason such a result should hold is that, in the language of cluster algebras, the hexahedron relation is a composition of six cluster mutations. As one varies the tiling, the associated variables are related by Yang-Baxter-like transformations preserving the Laurent property.

In the last part of the paper we specialize the initial data to subclasses, obtaining parametrizations for certain subclasses of matrices. We paramterize the class of Hermitian matrices, and restricting to $\mathbb{R}$, the class of real symmetric matrices. This is Theorem 5.2 below. We also extend to a slightly non-commutative setting and parametrize the quaternion-Hermitian matrices (Theorem 6.1 below). Moreover by further specialization we can parametrize the space of *positive definite* matrices over these rings (Theorem 5.7 below). This is a positive description in the sense that the entries are positive Laurent polynomials in the parameters, satisfying interval constraints.

2 Background

Let a be a function on the set of vertices and faces of a cube. Label the vertices and faces of a cube by indices 0 thorugh 9 and $0^*, 1^*, 2^*$ and 3^* so that the values of a, denoted by $a_0, \ldots, a_9, a_0^*, \ldots, a_3^*$ are arranged on the cube as in Figure 1.

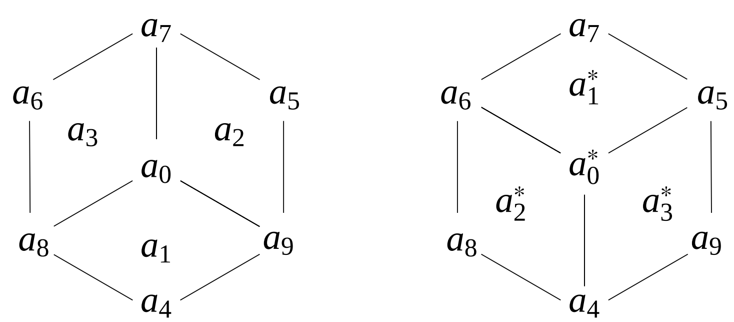

Figure 1: The variables in the hexahedron relation.

The function a is said to satisfy the *hexahedron relation* on this cube if the following four polynomial identities hold.

$$\begin{aligned} a_1^* a_1 a_0 &= a_1 a_2 a_3 + a_7 a_8 a_9 + a_0 a_4 a_7 && (1)\\ a_2^* a_2 a_0 &= a_1 a_2 a_3 + a_7 a_8 a_9 + a_0 a_5 a_8 && (2)\\ a_3^* a_3 a_0 &= a_1 a_2 a_3 + a_7 a_8 a_9 + a_0 a_6 a_9 && (3)\\ a_0^* a_0^2 a_1 a_2 a_3 &= (a_1 a_2 a_3)^2 + a_1 a_2 a_3 (2 a_7 a_8 a_9 + a_0 a_4 a_7 + a_0 a_5 a_8 + a_0 a_6 a_9) + \\ &\quad + (a_8 a_9 + a_0 a_4)(a_9 a_7 + a_0 a_5)(a_7 a_8 + a_0 a_6). && (4) \end{aligned}$$

Note that the relation is symmetric under cyclic rotation around the $a_0 a_0^*$ axis; one can check that this relation is also "top-down" symmetric: symmetric under the reversal

$$a_0^* \leftrightarrow a_0,\ a_1^* \leftrightarrow a_1,\ a_2^* \leftrightarrow a_2,\ a_3^* \leftrightarrow a_3,\ a_4 \leftrightarrow a_7,\ a_5 \leftrightarrow a_8,\ a_6 \leftrightarrow a_9.$$

This relation was introduced in [KP13], where the cube was taken to vary over cells of the cubic lattice $\mathbb{Z}^3$ and the hexadron relations taken to define translation invariant relations on a function on vertices and faces of the cubic lattice. The relations were shown there to be compositions of six cluster mutations. Initial conditions in this case correspond to stepped surfaces in the cubic lattice and the cluster structure implies that all variables are Laurent polynomials in any set of initial variables.

In the present work, we show that the hexahedron relation is the relation satisfied by the minors of a matrix. This requires placing the hexahedron relations on the Boolean lattice $\mathcal{B}_n$ (the n-cube $\{0,1\}^n$ with its natural partial order) in place of the cubic lattice $\mathbb{Z}^3$. We do so by allowing the cube in Figure 1 to vary over Boolean intervals of rank 3 in the rank-n Boolean lattice. We do this in a way that obtains the picture in Figure 2, which we now explain.

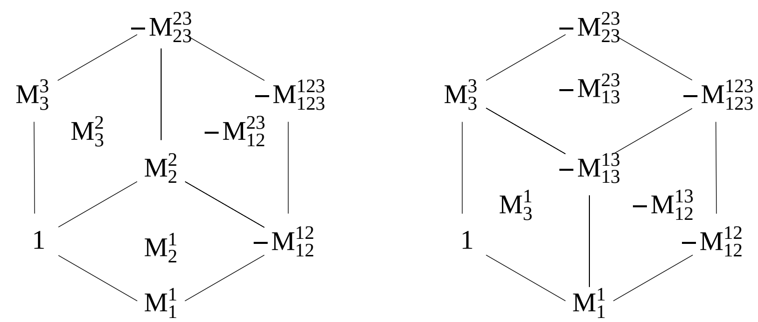

Figure 2: Arrangement of principal and near principal minors on $\mathcal{B}_3$

Fix n and an interval $\mathcal{I} := [S, S \cup D]$ of rank three in $\mathcal{B}_n$. The order preserving bijection α between $[3]$ and D induces a bijection between $\mathcal{B}_3$ and $\mathcal{I}$, explictly $\alpha_*(A) = (S \cup \alpha[A])$. The notation for matrix minors becomes less cluttered if we use M_A^B in place of $M_{\alpha*(A)}^{\alpha_*(B)}$ when the set S can be understood. Using this abbreviation, write the principal minor M_A^A at the element $A \in \mathcal{B}_3$. Interpreting the Hasse diagram of $\mathcal{B}_3$ as a cube, each of the six faces is a rank-2 interval. If A and B are the two middle-rank elements of such an interval, then associate with the corresponding face the almost principal minor M_A^B or M_B^A, choosing whichever one of these is odd. We now have a set of eight principal and six odd almost principal minors of M associated with the eight vertices and six faces of $\mathcal{B}_3$. We need to change the signs of seven of these, namely the vertices of rank 2 and 3 and the upper faces (spanning rank 1 to 3). Invoking the hexahedron recurrence is now a matter of matching to Figure 1, which we do in a slightly non-intuitive manner, matching $M_\emptyset^\emptyset$ to a_8, M_1^1, M_2^2 and M_3^3 to a_4, a_0 and a_6 respectively, and so on (there is only one way to extend this graph isomorphism to the whole cube). The result is Figure 2.

Lemma 2.1. *Under the correspondence between the diagrams in Figures 1 and 2, the minors of M satisfy the hexahedron relation.*

Proof. When $n = 3$, the only choice for S is $S = \emptyset$ and the abbreviation and actual notation M_A^B coincide. In this case the proof is a quick algebraic verification. Muir's law of extensible minors [Mui83], states that "a homogeneous determinantal identity for the minors of a matrix remains valid when all the index sets are enlarged by the same disjoint index set." See [BB08] for this wording and [BS83, Section 7] for a proof). Here, homogeneity means that every monomial in the identity is a product of determinants of degrees summing to the same value. In the first three hexahedron identities (1)–(3) every monomial has degree 4, while in (4), every monomial has degree 8. The conclusion of Muir's law is the conclusion of the lemma. □

3 $2n$-gon networks

On the cubic lattice, initial conditions are stepped surfaces, with moves from one stepped surface to another corresponding to the addition or removal of a cube. The Boolean lattice is a cell complex and although its dimension is not 3, addition and removal of a 3-cube still represents a well defined family of moves between 2-chains in a family of 2-chains sharing a common boundary. These two-chains, which correspond to initial conditions, are described by tilings of a $2n$-gon, as we now describe. One of these tilings is called the *standard tiling* and is shown in Figure 3 (ignore the blue for now).

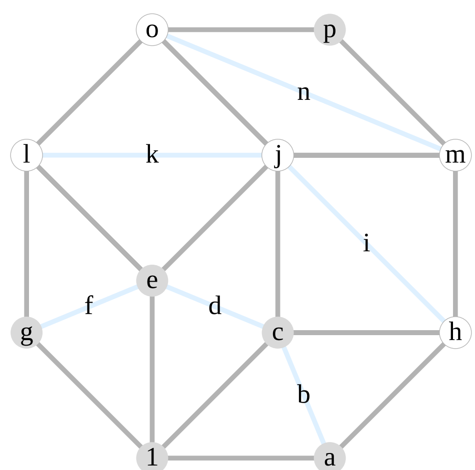

Figure 3: The standard network. White vertices have $\sigma(v) = -1$.

Let P_n be the regular $2n$-gon with unit length edges, oriented so that it has a horizontal edge. Let v_0 be the vertex of P_n which is the left endpoint of the lower horizontal edge. Place P_n so that v_0 is at the origin in $\mathbb{R}^2$. The polygon P_n is the projection to the plane of the n-cube $[0,1]^n$ with the property that for each $j \in [n]$, the basis vector $\mathbf{e}_j$ projects to the vector $e_j := e^{\pi i(j-1)/n}$.

The tilings of P_n we consider are tilings by translations of the set $\mathcal{W}_n$ of tiles, where $\mathcal{W}_n$ is the set of rhombi R_{jk} with unit edges parallel to two distinct edges e_j and e_k of P_n. The set $\mathcal{W}_n$ has cardinality $\binom{n}{2}$. Each tile in $\mathcal{W}_n$ occurs precisely once in each tiling. It may not be obvious that there exist such tilings (or even that the areas of tiles in $\mathcal{W}_n$ sum to the area of P_n) but the following construction of the standard tiling shows there to be at least one such tiling. Define the *standard tiling* T_0 by placing all rhombi $R_{i,i+1}$ with their lowest point at the origin (in the case of R_{12}, the leftmost lowest point). In the $n-2$ gaps between the uppermost extensions of these, place the rhombi $R_{i,i+2}$, and continue this way until the rhombus R_{1n} is placed, filling the last hole in P_{2n}. This tiling has the property that the vertices are precisely the points $v_0 + \sum_{j \in G} e_j$ for some set G of consecutive elements of $[n]$.

A "cube move" consists in taking three tiles of T whose union is a hexagon and replacing them with the three same tiles in the other order, effectively "pushing" the tiling across a 3-cube. Lifting back to $\mathcal{B}_n$, one sees that all the 2-chains have the same boundary, which is the lifting of the boundary of P_n to $\mathcal{B}_n$. Each vertex v of the tiling lifts to a lattice point in $\mathcal{B}_n$, which is the sum of e_j for all j such that e_j is on the path from the origin to v using edges of the tiling. The space of all tilings of P_n by $\mathcal{W}_n$ is connected under cube moves: see [Ken93].

Labeled tilings

A $2n$*-gon network* is a labeled $2n$-gon tiling. Formally, this means it is a pair (T, F) where T is a tiling and F is a real or complex function on $U(T)$, the set of faces and vertices of T. (In the last section we consider quaternionic networks and matrices.)

Two networks are *equivalent* if one can be obtained from the other by a sequence of cube moves, in which the tiles are replaced by a cube move and the vertex and face values undergo a hexahedron transformation, meaning that the values on the center vertex a_0 and the faces a_1, a_2, a_3 are transformed to $a_0^*, a_1^*, a_2^*, a_3^*$ on the new network or vice versa. We also allow as an equivalence move multiplication of all values by a single nonzero constant; the hexahedron relations are homogeneous, hence always preserved by such scaling. We say that a $2n$-gon network is *generic* if it and all equivalent networks have only nonzero labels.

Proposition 3.1. *The equivalence class of a generic network contains precisely one network for each tiling such that $F(v_0) = 1$.*

In other words, if two sequences of cube moves lead to the same tiling, then the resulting network does not depend on the sequence of cube moves leading to it. The proof will follow from Theorem 4.2 below; see the remarks after the proof of that theorem.

4 Correspondence between matrices and networks

Let $M_n^*(\mathbb{C})$ denote the set of generic $n \times n$ complex matrices, meaning those with only nonzero minors. Let $\mathcal{N}$ denote the set of generic $2n$-gon networks. In this section we describe a map β_T of the form $A \mapsto (T, F_{A,T})$ and a map $\Psi : \mathcal{N} \to M_n^*(C)$ that together establish a bijection between $M_n^*(\mathbb{C})$ and equivalence classes in $\mathcal{N}$.

4.1 Matrices to networks

Let $A \in M_n^*(\mathbb{C})$ be a matrix and T a tiling of the$2n$-gon. For a vertex v of T, define $\sigma(v) = (-1)^{\lfloor d/2 \rfloor}$ where d is the graph distance in the tiling from v to v_0. Recall that $U(T)$ denotes the union of the vertices and faces of T. Define a function $F = F_{A,T}$ on $U(T)$ as follows. Each vertex v of T is naturally associated with a point in $\mathcal{B}_n$, that is, a subset $S \subseteq [n]$. Let $F(v) = \sigma(v) \det A_S^S$ where A_S^S is the principal minor of A indexed by S. On a rhombus R_{ij} with vertices $v, v + e_i, v + e_i + e_j, v + e_j$ and $i < j$ we assign the value

$$F(R_{ij}) = \sigma(v) \det K_{S\cup\{j\}}^{S\cup\{i\}} \quad \text{or} \quad \sigma(v) \det K_{S\cup\{i\}}^{S\cup\{j\}}\,, \text{ whichever is the odd minor;} \tag{5}$$

here again S is the subset of $[n]$ corresponding to v.

Theorem 4.1. *For any tilings T and T' the networks $(T, F_{A,T})$ and $(T', F_{A,T'})$ are equivalent. Consequently the map $(A, T) \mapsto (T, F_{A,T})$ induces a function Φ mapping each matrix $A \in M_m^*(\mathbb{C})$ to the equivalence class of $(T, F_{A,T})$, which does not depend on T.*

Proof. Suppose T and T' differ by a cube move. The functions $F_{A,T}$ and $F_{A,T'}$ label the vertices and faces according to the diagrams in Figure 2 (the matrix is now named A rather than M and we use the convention that $[S, S \cup D]$ is mapped in the order preserving way to $\mathcal{B}_3$). By Lemma 2.1, these obey the hexahedron relations and are thus by definition equivalent. Any two tilings are connected by a sequence of cube moves, hence $(T, F_{A,T})$ and $(T', F_{A,T'})$ are equivalent for any T, T'. Genericty of A implies $F_{A,T}$ is nowhere zero, which proves genericity of the equivalence class of $(T, F_{A,T})$. ∎

Remark. This implies Proposition 3.1 for networks in the range of Φ.

4.2 Networks to matrices

Conversely, let us explain how to go from an equivalence class of generic networks to a matrix in $M_n^*(\mathbb{C})$. A network (T, F) is called *standard* if $T = T_0$ and $F(v_0) = 1$. The key step is to construct the map Ψ taking a standard network (T_0, F) to a matrix A such that $F_{T_0,A}$ agrees with F on $U(T_0)$.

Our strategy will be to assign matrix entries A_{ij} in a particular order so that we can check inductively that the assigned entries force F_{A,T_0} to agree with F on ever larger subsets of T_0 no matter what the values of the yet unassigned entries of A. In this way we both construct A and verify that $F_{A,T_0} = F$. Visualization is easy when working with T_0 because each vertex and rhombus corresponds to a contiguous subdeterminant, the values of F_{A,T_0} at vertices being principal minors of the form $\det A_{i,i+1,\ldots,j}^{i,i+1,\ldots,j}$ and the values at rhombi being odd almost-principal minors of the form $A_{i,\ldots,j}^{i\pm1,\ldots,j\pm1}$.

Before giving a formal description we illustrate with an example where $n = 4$. Figure 3 shows the standard tiling of P_8 with vertices and rhombi labeled by indeterminates. If A is a matrix with $F_{T_0,A} = F$ then reading values of F on $U(T_0)$ along successive blue paths, starting from the lower right, determines successive minors of A as follows. The first blue path contains 1×1 minors, therefore dictating the entries $A_{11}, A_{21}, A_{22}, A_{32}, A_{33}, A_{43}$ and A_{44}. The second blue path, read right to left, gives the negatives of the minors $A_{12}^{12}, A_{23}^{12}, A_{23}^{23}, A_{34}^{23}$ thereby determining $A_{12}, A_{23}, A_{34}, A_{31}$ and A_{42}. The third blue path, read right to left, gives the negatives of the minors $A_{123}^{123}, A_{234}^{123}$ and A_{234}^{234}, thereby determing A_{13}, A_{24} and A_{41}. The last value is $\det A$, which now determines A_{14}, all other entries of A already having been determined. Explicitly, in terms of the indeterminates labeling the vertices and faces in Figure 3, the matrix is given by

$$\begin{pmatrix} a & \frac{ac}{b} + \frac{h}{b} & \frac{he}{bd} + \frac{ace}{bd} + \frac{hj}{bcd} + \frac{aj}{bd} + \frac{hj}{ci} + \frac{m}{i} & X \\ b & c & \frac{ce}{d} + \frac{j}{d} & \frac{jg}{df} + \frac{ceg}{df} + \frac{jl}{def} + \frac{cl}{df} + \frac{jl}{ek} + \frac{o}{k} \\ \frac{bd}{c} + \frac{i}{c} & d & e & \frac{eg}{f} + \frac{l}{f} \\ \frac{if}{ce} + \frac{bdf}{ce} + \frac{ik}{cde} + \frac{bk}{ce} + \frac{ik}{dj} + \frac{n}{j} & \frac{df}{e} + \frac{k}{e} & f & g \end{pmatrix}$$

where $X =$

$$\frac{aceg}{bdf} + \frac{acl}{bdf} + \frac{ajl}{bdef} + \frac{agj}{bdf} + \frac{ajl}{bek} + \frac{ao}{bk} + \frac{hjl}{bcdef} + \frac{ghj}{bcdf} + \frac{hjl}{bcek} + \frac{ho}{bck} + \frac{egh}{bdf} + \frac{hl}{bdf} + \frac{dhjl}{ceik} + \frac{dho}{cik} + \frac{hjl}{cefi} + \frac{ghj}{cfi} + \frac{dlm}{eik} + \frac{dmo}{ijk} + \frac{lm}{efi} + \frac{gm}{fi} + \frac{mo}{jn} + \frac{p}{n}.$$

To see why this works in general, divide $U(T_0)$ into disjoint paths, each alternating between vertices and rhombi. The zeroth path is the vertex v_0; the first path contains the vertices at distance 1 from v_0 and the rhombi $R_{i,i+1}$ between them. The j^{th} path contains the vertices at distance j from v_0 and the rhombi between them. This partition is illustrated by the blue paths in Figure 3.

Vertices on the j^{th} path, $j \geq 1$, will induce assignments of elements of A on the $(j-1)^{st}$ superdiagonal, where the zeroth superdiagonal is the main diagonal. Rhombi on the j^{th} path will induce assignments of elements of A on the j^{th} subdiagonal. The zeroth path always contains the element 1, so provides no new information and does not induce an assignment.

Inductively, we check that for each vertex or rhombus, the equation that F_{A,T_0} agrees with F at each new face orvertex, which is the equation $\det M_A^B = c$ for some $A, B \subseteq [n]$ and some number c, is a multilinear linear equation with precisely one unassigned variable. Indeed, for vertices in the j^{th} path it is a specification of a contiguous subdeterminant spanning from the diagonal to the $(j-1)^{st}$ superdiagonal while for rhombi on this path it is a specification of a contiguous subdeterminant spanning from the first subdiagonal down to the jth subdiagonal.

One of these linear equations is degenerate if and only if the cofactor of that determinant vanishes. The cofactor is the value of F at a position one row closer to the main diagonal. Genericity of (T_0, F) implies that this is nonzero. This completes the induction. We conclude there is a unique matrix A for which F_{A,T_0} agrees with F; we call this $\Psi(T_0, F)$. We have now proved the first and only nontrivial statement in the following theorem.

Theorem 4.2. *If (T_0, F) is generic then there is a unique $A \in M_n^*(\mathbb{C})$ such that $F_{A,T_0} = F$ on $U(T_0)$. The map $A \mapsto (T_0, F_{A,T_0})$ and the map Ψ mapping (T_0, F) to A are two-sided inverses.*

Proof. The construction always produces a matrix A such that F_{A,T_0} agrees with the given F. If (T, F) and (T', F') are related by a cube move and $F_{A,T} = F$ then $F_{A,T'} = F'$ because the hexahedron hold for the minors of A. Therefore, if (T, F) is equivalent to (T, F') then $F = F'$. This proves there is only one network (T, F) in each equivalence class, implying Proposition 3.1, ensuring that Ψ is well defined, and proving that $A \mapsto (T_0, F_{A,T_0})$ and Ψ are inverses. □

4.3 Further properties of the correspondence

Each matrix $M \in M_n(\mathbb{C})$ has a vector of 2^n principal minors. Let V be the variety in $\mathbb{C}^{2^n}$ consisting of all vectors of principal minors of matrices in $M_n(\mathbb{C})$. The ideal in $\mathbb{C}[x_S : S \subseteq [n]]$ of polynomials vanishing on V is denoted $J(V)$. Similarly, let V' be the variety of all vectors of principal and odd almost-principal minors of matrices in $M_n(\mathbb{C})$. Its ideal $J(V')$ lives in the ring of polynomials in variables corresponding to all vertices and faces of $\mathcal{B}_n$.

Theorem 4.3. *The ideal $J(V')$ is the radical of the ideal J generated by the hexahedron relations on rank-3 Boolean intervals. The ideal $J(V)$ is the intersection of $\mathbb{C}[x_A : A \subseteq [n]]$ with $J(V')$.*

Proof. Suppose there is a polynomial $p \notin \sqrt{J}$ that vanishes on V'. Then the variety defined by J would be bigger than the Zariski closure of V'. We have seen, however, that every point in the variety defined by J is in the Zariski closure of the generic points of V', hence by contradiction the ideal defined by V' is $\sqrt{J}$. The second fact is quite general: any polynomial in the subring vanishing on V is a polynomial in the big ring vanishing on the cylinder of V, and is therefore in $\sqrt{J}$ as well as the small ring. □

We have seen that the entries of A are determined by equations in the network variables, each being linear in the new variable, hence producing a rational function of the initial variables. In fact a Laurent property holds.

Theorem 4.4. *Let $A = \Psi(T_0, F)$ be the matrix such that $F_{T_0,A} = F$. Then the entries of A are Laurent polynomials in the standard network variables, with coefficients* 1. *The monomials in M_{ij} are in bijection with domino tilings of the half-aztec diamond.*

The bijection is illustrated in Figure 4. A half-aztec diamond is a region as in Figure 4 for the case $n = 4$. It is a triangular stack of squares; the bottom row consists of $2n$ squares (numbered 1 through $2n$), and successive rows have two fewer squares. To get the ij-entry of M for $i \leq j$, delete the squares on the bottom row at locations $2i - 1$ and $2j$. The M_{ij} entry enumerates the domino tilings of the resulting figure using the formula of Figure 4.

To get the ij-entry of M for $i > j$, delete the squares on the bottom row at locations $2i - 1$ and $2j$ *and* the outer layer of squares from the left and right sides of previous figure. The M_{ij} entry enumerates the domino tilings of the resulting figure (a smaller half-aztec diamond) using again the formula of Figure 4.

These tilings are also in a natural bijection with Schröder paths.

Proof. The proof uses a few facts about the combinatorics of Dodgson condensation, see e.g. [Spe07]. Recall how Dodgson condensation works. Define $m_{ij}^{(0)} = 1$. Starting from an $n \times n$ array of numbers $(m_{ij}^{(1)})$ representing a matrix M. Define an pyramidal array $m_{ij}^{(k)}$ where i, j are integers for k odd and half-integers for k even, by the (signed) octahedron recurrence

$$m_{i,j}^{(n+1)} = \frac{m_{i-1/2,j-1/2}^{(n)} m_{i+1/2,j+1/2}^{(n)} - m_{i-1/2,j+1/2}^{(n)} m_{i+1/2,j-1/2}^{(n)}}{m_{ij}^{(n-1)}}.$$

Here the defined values $m_{ij}^{(k)}$ form a pyramid called the *Dodgson pyramid* of the matrix M. Its apex value is the determinant of M.

The (consecutive-index) principal minors of M occur on a slice of the pyramid: the slice in the $x = y$ plane (here we are thinking of the x-axis as the row coordinate and the y axis as the column coordinate). The subprincipal minors occur on the parallel plane $x = y + 1$.

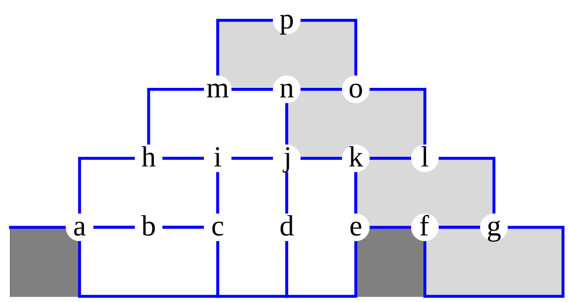

Figure 4: A half-aztec diamond of order 4, and one of the domino tilings contributing to M_{13}. In the corresponding monomial, each variable occurs with power $d-3$, where d is the local degree at the corresponding vertex in the domino tiling. In this example, the monomial is $\frac{aj}{bd}$. Note that the (light gray) tiles above the line of slope -1 containing the right removed square (dark gray) are all horizontal; removing these, we have a tiling of a truncated order-2 diamond (the top half of an aztec diamond along with the top row of its bottom half; white in the figure).

It follows from the above that the matrix M associated to the standard network has entries which form the base of the Dodgson pyramid of M.

Typically the octahedron recurrence (for which Dodgson condensation is a special case) is defined by taking initial data on the $z=0$ and $z=1$ planes, and working upwards. We can, however, instead take our initial data on the $x=y$ and $x=y+1$ planes, and use the recurrence to successively define values on planes $x-y=2,3,\ldots$ and $-1,-2,\ldots$. Because the entries of the Dodgson pyramid satisfy the *signed* octahedron recurrence when going upward (increasing z), they satisfy the *unsigned* octahedron recurrence when going in these horizontal directions.

We can thus form the entries of M using the octahedron recurrence (with $+$ signs) with initial data on the planes $x=y$ and $x=y+1$, that is, with initial data consisting of the principal and subprincipal minors of M.

By a small generalization of a result of Speyer [Spe07], the entries on the plane $x=y+j$ are counted by domino tilings of truncated aztec diamonds: the entries on $z=1$ are defined by aztec diamonds truncated to remove all but the first row of the bottom half, as in the unshaded squares in Figure 4; the entries on $z=j$ are counted by domino tilings of aztec diamonds from which the bottom $n-j$ rows have been removed, that is, take the upper half and add the first j rows of the bottom half.

To see this, extend the half-aztec diamond in the $x=y$ plane to a full aztec diamond, defining parameters $\varepsilon^{-|z|}$ for vertices at negative z values, where ε is small. Now Speyer's bijection between the octahedron recurrence and tilings of the full aztec diamond shows that, in the limit $\varepsilon \to 0$, the desired term is counting tilings of the aztec diamond in which only horizontal dominos occur in all rows below $z = 0$. These are equivalent to tilings of the truncated aztec diamond. □

5 Hermitian networks

In this section we examine the image of various subsets of $M_n^*(\mathbb{C})$ under the correspondence mapping matrices to networks. In particular, we describe the images of the set of real symmetric matrices, the set of Hermitian matrices and the set of positive definite Hermitian matrices. These descriptions not only parametrize the respective sets but answer the question as to which collections of minors are possible.

Definition 5.1.

(i) A $2n$-gon network (T, F) with entries in $\mathbb{R}$ or $\mathbb{C}$ is said to be Hermitian if it satisfies the condition that $F(v)$ is real for all vertices v and for each face $f \in U(T)$ we have

$$|F(f)|^2 = F(a)F(c) + F(b)F(d) \tag{6}$$

where a, b, c, d lists the vertices of f in cyclic order.

(ii) A Hermitian network (T, F) is said to be positive if for all vertices v, the sign of $F(v)$ is $\sigma(v)$.

The following result will be proved in Section 5.2.

Theorem 5.2. *The following are equivalent.*

(i) The matrix $A \in M_n^*(\mathbb{R})$ is Hermitian;

(ii) The network $(T, F_{A,T})$ is Hermitian for some T;

(iii) The network $(T, F_{A,T})$ is Hermitian for every T.

5.1 Hermitian Kashaev relation

Values $a_0, a_4, \ldots, a_9$ on vertices and $a_1, a_2, a_3, a_1^*, a_2^*, a_3^*$ on faces of a cube (as in Figure 1), with $a_0, a_4, \ldots, a_9$ real, are said to satisfy the *Hermitian Kashaev relation* if (6) holds on every face and

$$a_1^* = \frac{a_2a_3 + \bar{a}_1a_7}{a_0} \tag{7}$$

$$a_2^* = \frac{a_3a_1 + \bar{a}_2a_8}{a_0} \tag{8}$$

$$a_3^* = \frac{a_1a_2 + \bar{a}_3a_9}{a_0} \tag{9}$$

$$a_0^* = \frac{a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9 + 2a_7a_8a_9 + a_1a_2a_3 + \overline{a_1a_2a_3}}{a_0^2}. \tag{10}$$

Lemma 5.3. *Let $a_0, \ldots, a_9$ be complex numbers making the left-hand diagram of Figure 2 satisfy the relation* (6) *on each face:*

$$a_1\bar{a}_1 = a_0a_4 + a_8a_9, \quad a_2\bar{a}_2 = a_0a_5 + a_7a_9, \quad a_3\bar{a}_3 = a_0a_6 + a_7a_8\,. \tag{11}$$

Then the values $a_0^*, a_1^*, a_2^*, a_3^*$ obtained from a cube move (the hexahedron relation) satisfy the Hermitian Kashaev relations (7)–(10)*. Furthermore, the right-hand side will also satisfy* (6) *on each face. It follows that any network equivalent to a Hermitian network is Hermitian and that the Hermitian Kashaev relations are a special case of the hexahedron relations under the constraint that* (11) *holds on any, hence every, network.*

Proof. This is a simple algebraic check: see the proof in [KP13, Section 7] for real valued networks; the same proof goes through for complex valued networks taking real values at the vertices. □

5.2 Hermitian correspondence

We begin with a proof of Theorem 5.2. By Lemma 5.3 (*ii*) and (*iii*) are equivalent. It remains to prove that (*iii*) ⇒ (*i*) ⇒ (*ii*).

(*i*) ⇒ (*ii*): Let A be any Hermitian matrix and let $(T_0, F) = (T_0, F_{T_0,A})$ be the standard network associated with A. It is immediate that $F(v)$ is real for all vertices $v \in U(T_0)$ because principal minors of Hermitian matrices are real. To check (6), let f be a face of T_0 with vertices $v, v+e_i, v+e_i+e_j$ and $v+e_j$. Let $S \subseteq [n]$ corresond to v. The values of F at the vertices of f are respectively (using the notation $M_i^j := M_{S\cup\{i\}}^{S\cup\{j\}}$, etc.),

$$\sigma(v)\det M\,,\ \ \sigma(v+e_i)\det M_i^i\,,\ \ \sigma(v+e_i+e_j)M_{ij}^{ij}\,,\ \ \sigma(v+e_j)M_j^j\,,$$

while the face value is $F(f) = \sigma(v+e_i)M_j^i$. Observe that the signs satisfy $\sigma(v)\sigma(v+e_i+e_j) = -1$. Dodgson's condensation [Dod66] states that

$$\det M_{ij}^{ij} \det M = \det M_i^i \det M_j^j - \det M_j^i \det M_i^j.$$

Because M is Hermitian $M_i^j = \overline{M_j^i}$. Thus we have

$$-F(v+e_1+e_2)F(v) = F(v+e_1)F(v+e_2) - |F(f)|^2 \tag{12}$$

which means the network is Hermitian.

$(iii) \Rightarrow (i)$: Let T be any tiling and let f be a face incident to v_0 with sides parallel to e_i and e_j. The values of F on vertices of f are

$$1, A_{ii}, A_{ii}A_{jj} - A_{ij}A_{ji}, A_{jj}$$

while $F(f) = A_{ji}$. If $(T, F_{A,T})$ is Hermitian then applying (6) at f gives

$$A_{ij}A_{ij} = |A_{ji}|^2 .$$

This means that $A_{ij} = \overline{A_{ji}}$. For every $i \neq j$ there is at least one tiling T having such a face f. We conclude that if each $(T, F_{A,T})$ is Hermitian then so is A. □

Fixing a tiling T and assigning values of F on $U(T)$ arbitrarily (but generically) exactly parametrizes generic $n \times n$ matrices. In the Hermitian case, the same is true if one restricts to Hermitian networks (T, F); however we would like a more explicit parametriation of this subset of networks.

Proposition 5.4. *Generic Hermitian $n \times n$ matrices are parameterized by their diagonal entries and contiguous almost-principal minors.*

Proof. We have seen that generic Hermitian matrices are parametrized by standard networks (T_0, F) satisfying the Hermitian condition (6). It remains only to observe that these networks are parametrized by the face variables $\{F(f)\}$ together with the vertex variables $\{F(e_j) : 1 \leq j \leq n\}$ along the lowest blue path. (To see this note that the rhombi not adjacent to v_0 can be ordered so that each new rhombus has only one vertex not in the union of the previous rhombi.) If A is the matrix corresonding to the network then the values $F(e_j)$ are the diagonal elements of A and the face variables are the continguous minors $M_{i,\dots,j}^{i\pm1,\dots,j\pm1}$ where the sign choice in the $\pm$ is determined by the parity of $j-i$ but does not matter because the the two minors are conjugates of each other. □

There are in fact many other choices of parameters. Take any shortest path γ in the tiling from v_0 to the opposite vertex. We claim that the variables on the vertices of γ, along with all face variables, parameterize all networks. To see this it suffices to show that on either side of γ, if there is a tile (that is, if γ is not the boundary path) there is a tile having two consecutive sides, and thus three vertices, touching γ. The value at the fourth vertex is then a function of its face value and the values at the vertices along the path; pushing the path across this tile and continuing, we see that all vertex values are obtained in this way.

To find such a tile to the left (say) of γ, take any tile left of γ and follow its train tracks (continguous tiles sharing a set of parallel edges) until they cross γ; take any new tile in the triangular region delimited by γ and these two train tracks. γ with the train tracks of this new tile forms a strictly smaller triangular region. Conclude by induction.

Another interesting representation of the generic Hermitian matrix is the following Laurent parametrization, where the initial conditions are taken to be an arbitrary network.

Proposition 5.5. *The matrix entries for a standard Hermitian network are Laurent polynomials in the interior entries (face values and interior vertex values).*

Proof. This follows from the essentially same argument as in the proof of Theorem 4.2. We work outwards from the diagonal. Inductively, each new entry M_{ij} with $i < j$ is defined by an equation $\det M_A^B = c$ where M_A^B is an odd almost-principal minor. This is a multilinear linear equation in which M_{ij} is the only unassigned variable; moreover the coefficient of M_{ij} is a principal minor. Thus M_{ij} is a Laurent polynomial, which is an actual polynomial in the (previously assigned) other matrix entries, with a denominator which is the parameter assigned to a principal minor, which is an interior vertex. Finally we can define $M_{ji} = \overline{M_{ij}}$. □

Example 5.6. For the standard tiling, the 4×4 example is easy to compute. The matrix M is given in terms of the face and interior vertex variables as follows.

$$M = \begin{pmatrix} a & \bar{x} & \frac{\bar{x}\bar{y}+u}{b} & \frac{\overline{xyz}}{bc} + \frac{\bar{z}u}{bc} + \frac{\bar{x}v}{bc} + \frac{yuv}{bcf} + \frac{\bar{w}}{f} \\ x & b & \bar{y} & \frac{\overline{yz}+v}{c} \\ \frac{xy+\bar{u}}{b} & y & c & \bar{z} \\ \frac{xyz}{bc} + \frac{z\bar{u}}{bc} + \frac{x\bar{v}}{bc} + \frac{\overline{yuv}}{bcf} + \frac{w}{f} & \frac{yz+\bar{v}}{c} & z & d \end{pmatrix}. \tag{13}$$

Examination of the matrix entries in (13) leads to the following conjecture, verified in the 5×5 case as well.

Conjecture 1. *The matrix entries for a standard Hermitian network are Laurent polynomials, with coefficient* 1*, whose numerators are monomials in the face variables and denominators are monomials in the interior vertex variables. The terms are in bijection with Catalan paths on the dual network. The purported bijection is illustrated in Figure 5.*

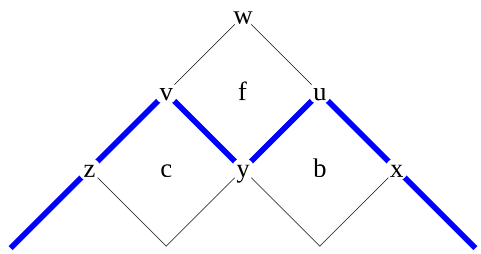

Figure 5: This grid is the dual graph to the tiling. To a Catalan path on this grid, record the variables at each local max and min. The vertex variables here (which are face variables of the network) go in the numerator; the face variables in the denominator. The given path has weight $\frac{yuv}{bcf}$.

5.3 Positive $2n$-gon networks

Recall that a positive $2n$-gon network is a Hermitian network with the additional constraint that the sign of $F(v)$ is $\sigma(v)$.

Theorem 5.7. *The network associated to a positive definite Hermitian matrix is positive. Conversely, a positive network gives rise to a positive-definite Hermitian matrix.*

Proof. Suppose a network is positive. Under the network-matrix correspondence, the values along the right-hand boundary of the network are $\sigma(v)$ times the leading (upper-left principal) minors of the matrix. Sylvester's criterion [HJ85, Theorem 7.2.5] states that positivity of the leading minors of a Hermitian matrix (the first k rows and columns, $1 \leq k \leq n$) is equivalent to positive definiteness of the matrix. Thus a positive network gives rise to a positive definite matrix.

Conversely, if a matrix is positive definite, all its principal minors are positive and therefore the network is positive. $\square$

How are positive networks parametrized? On the standard network, do this as follows. We assign values on e_i arbitrarily and positively. Then on $e_i + e_{i+1}$ we assign any negative values larger than $-F(e_i)F(e_{i+1})$, so that $F(e_i)F(e_{i+1}) + F(e_i + e_{i+1}) > 0$. Then on $e_i + e_{i+1} + e_{i+2}$ assign any negative value larger than

$$F(e_i + e_{i+1})F(e_{i+1} + e_{i+2})/F(e_{i+1}).$$

Then on $e_i + e_{i+1} + e_{i+2} + e_{i+3}$ assign any positive value smaller than

$$-F(e_i + e_{i+1} + e_{i+2})F(e_{i+1} + e_{i+2} + e_{i+3})/F(e_{i+1} + e_{i+2}),$$

and so on. In each case except for the initial e_i we have a bounded positive length open interval to choose from.

Once the vertex values have been chosen, the face values are determined up to a unit real or complex number. For $\mathbb{R}$ there are 2 choices of sign for each face value. Thus the space of positive networks with nonzero face values is homeomorphic to a union of $2^{n(n-1)/2}$ open balls each of dimension $\binom{n+1}{2}$. For $\mathbb{C}$ the argument of each face value can be chosen freely so the space of positive Hermitian networks is homeomorphic to the product of a $\binom{n}{2}$-torus with a $\binom{n+1}{2}$-ball (or, if you prefer, $(\mathbb{C}^*)^{\binom{n}{2}} \times \mathbb{C}^n$).

6 The q-Hermitian case

A q-Hermitian matrix is a matrix of quaternions which satisfies $M_{ij} = (M_{ji})^*$, where $*$ denotes the quaternionic conjugate. The q-determinant of a q-Hermitian matrix is a real number defined by

$$\text{qdet}M = \sum_{\text{cycle decomps}} (-1)^{c+n} \text{tr}M_{C_1}\text{tr}M_{C_2}\dots\text{tr}M_{C_k} \tag{14}$$

where the sum is over cycle decompositions of $[n]$ (disregarding order), c is the number of cycles, and $\text{tr}M_C$ is the trace of the product of entries in cycle C (one-half the trace for cycles of length 1 or 2).

For example when a, b, c are real,

$$\text{qdet}\begin{pmatrix} a & d & e \\ d^* & b & f \\ e^* & f^* & c \end{pmatrix} = abc - aff^* - bee^* - cdd^* + \text{Tr}(dfe^*).$$

Dyson [Dys70] showed that $\text{qdet}M = \text{Pf}(Z\tilde{M})$, where Z is the block-diagonal matrix with 2×2 blocks $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\tilde{M}$ is the $2n\times 2n$ matrix obtained from M by replacing each entry $M_{ij} = a + bi + cj + dk$ with the 2×2 block $\begin{pmatrix} a+ib & c+id \\ -c+id & a-ib \end{pmatrix}$.

A q-Hermitian matrix is *positive definite* if its leading minors are all positive; equivalently if its eigenvalues are positive [Kas13].

A q-Hermitian network is a network with face values in $\mathbb{H}$; vertex values are real. In each face with vertex values a, b, c, d the face value z satisfies $zz^* = ac + bd$.

In the case of a $\mathbb{H}$-valued Hermitian matrix, almost principal minors $\mathrm{qdet} M_{S\cup\{j\}}^{S\cup\{i\}}$ can also be defined, as follows [Dys70]. Instead of summing over cycle decompositions as in (14), one sums over decompositions of the indices into configurations forming a path from i to j with the remaining indices formed into cycles. The contribution for a configuration is the product of traces over the cycles and the product of the quaternions along the path. With this definition we can define as above a q-Hermitian network associated to a q-Hermitian matrix (whereas for a general matrix over $\mathbb{H}$ no such definition can be made.)

Theorem 6.1. *The Kashaev relation (7-10) holds when a_1, a_2, a_3 are quaternions (and $a_0, a_4, \ldots, a_9$ are real), for the given order of multiplication. Theorem 5.2, Propositions 5.4, 5.5 and Theorem 5.7 hold for q-Hermitian matrices.*

Proof. The first statement is a short check. This implies Lemma 5.3 via the same proof. Theorem 5.2, Propositions 5.4 and 5.5 then follow. Sylvester's criterion also holds for q-Hermitian matrices, see [Kas13], and thus Theorem 5.7 holds as well. □

References

[BB08] A. Berliner and R. Brualdi. A combinatorial proof of the Dodgson-M-uir identity. *Int.J. Inform. and System Sciences*, 4:1–7, 2008.

[BS83] R. Brualdi and H. Schneider. Determinantal identities: Gauss, Schur, Cauchy, Sylvester, Kronecker, Jacobi, Binet, Laplace, Muir and Cayley. *Linear Alg. Appl.*, 52–53:769–791, 1983.

[Dod66] C. Dodgson. Condensation of determinants, being a new and brief method for computing their arithmetical values. *Proc. Royal Soc. London*, 15:150–155, 1866.

[Dys70] F. Dyson. Correlations between the eigenvalues of a random matrix. *Math. Phys.*, 3:1191–1198, 1970.

[HJ85] R. Horn and C. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1985.

[Kas13] A. Kassel. *Laplacians on graphs on surfaces and applications to statistical physics*. PhD thesis, Orsay, 2013.

[Ken93] R. Kenyon. Tiling a polygon with parallelograms. *Algorithmica*, 9:382–397, 1993.

[KP13] R. Kenyon and R. Pemantle. Double-dimers, the ising model and the hexahedron recurrence. *arXiv*, 2013.

[LS09] S. Lin and B. Sturmfels. Polynomial relations among principal minors of a 4×4 matrix. *J. Algebra*, 322:4121–4131, 2009.

[Mui83] Muir. The law of extensible minors in determinants. *Trans. Royal Soc. Edinburgh*, 3:1–4, 1883.

[Spe07] D. Speyer. Perfect matchings and the octahedron recurrence. *J. Alg. Comb.*, 25:309–348, 2007.