ON PATH INTEGRALS FOR THE HIGH-DIMENSIONAL BROWNIAN BRIDGE

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Abstract

Let v be a bounded function with bounded support in R^d, $d \geq 3$. Let $x, y \in R^d$. Let $Z(t)$ denote the path integral of v along the path of a Brownian bridge in R^d which runs for time t, starting at x and ending at y. As $t \to \infty$, it is perhaps evident that the distribution of $Z(t)$ converges weakly to that of the sum of the integrals of v along the paths of two independent Brownian motions, starting at x and y and running forever. Here we prove a stronger result, namely convergence of the corresponding moment generating functions and of moments. This result is needed for applications in physics.

1 Introduction

Suppose that $(X_s, s \geq 0)$ is a stochastic process taking values in R^d, and v is a Real-valued function on R^d. Define the path integral $Z(t)$ by

$$Z(t) = \int_0^t v(X_s)ds. \tag{1}$$

If v is the indicator function of a set $Z(t)$ is the occupation time of that set by (X_s) up to time t.

If (X_s) is a suitable time-homogeneous Markov process, such as a Feller process, then the distribution of $Z(t)$ can be studied via the Feynman-Kac formula. See for example Williams [8]. For example, if (X_s) is Brownian motion and v is the indicator function of the unit ball, the distribution of Z_∞ was obtained by Cieselski and Taylor [3] (see also [8]). In the Markov process literature the path integral $Z(t)$ is known as an *additive functional*.

We study here the path integral $Z(t)$ given by (1), where the process $(X_s, 0 \leq s \leq t)$ is a Brownian bridge, and therefore is *not* time-homogeneous. Loosely speaking, a Brownian bridge $(X_s, 0 \leq s \leq t)$ is a Brownian motion conditioned to take some fixed value y at time t.

Our motivation for this work came from the study of the heat equation with added potential $v(\cdot)$; that is the equation

$$\partial f / \partial t = [(1/2) \bigtriangledown^2 - v(x)]f(t;x) \quad (t > 0, x \in R^d) \tag{2}$$

(where the Laplacian $\bigtriangledown^2$ acts on functions of x at fixed t). This equation is closely related to the Schrödinger equation. In quantum Statistical Mechanics, in [6] and [7] for example, one studies the second virial coefficient for a gas consisting of particles with a two-body interaction potential v, via the fundamental solution $G(t;x,y) = G_y(t;x)$ to the associated Bloch equation (2); that is, the solution $f = G_y$ to (2) with the initial condition $G_y(0;x) = \delta(x - y)$.

By the Feynman-Kac formula, one can show $G(t;x,y)$ is equal to the Brownian transition density $(2\pi t)^{-d/2}e^{-|y-x|^2/(2t)}$, multiplied by the moment generating function (m.g.f.) of a random variable of the form $Z(t)$ given by (1) where $(X_s, 0 \leq s \leq t)$ is a Brownian bridge running from x to y in time t (see [7], equation (29) for details). It is this m.g.f. which we study here.

Another motivation arises in the simulation of molecular dynamics for a chemical reaction. See Clifford *et al.* [2]. Suppose two particles (molecules)

execute independent Brownian motions, so that one particle executes a Brownian motion (W_s) relative to the other. Suppose the particles react (for example by mutual annihilation) at a rate given by a function v of their separation vector W_s; that is, if τ denotes the reaction time,

$$P[\tau \leq t] = E \exp\{-\int_0^t v(W_s)dx\}. \tag{3}$$

In a simulation of the system of two particles (maybe as part of a much larger system), it is natural to simulate the value of W_s, say at integer values of s, and to interpolate between successive integer times n and $n+1$ by taking the path of (W_s) between those times to be a Brownian bridge with endpoints given by the simulated values of W_n and W_{n+1}. Given that no reaction takes place before time n, and given the simulated values of W_n and W_{n+1}, the probability of reaction before time $n+1$ is again the m.g.f. of a path integral of v along the path of a Brownian bridge.

This approach was used by Clifford and Green [1], although their reaction mechanism is different from the one discussed here. The Brownian bridge could arise similarly in simulations of an integrated cost function in Control theory.

In the physical examples discussed above, the process (X_s) of (1) is typically a Brownian bridge in three dimensions. In three or more dimensions, Brownian motion is transient, and if v has bounded support, denoted A say, then for large t one might expect the Brownian motion to spend most of its time outside A, leaving A shortly after time 0 and returning shortly before time t (where 'shortly' is relative to t). Since the time-reversed process is also a Brownian bridge, for large t one might expect the distribution of $Z(t)$ to approximate to that of

$$\int_0^\infty v(W_x(s))ds + \int_0^\infty v(W_y'(s))ds, \tag{4}$$

where W_x and W_y' are independent unconditioned Brownian motions starting at x and y (the endpoints of the bridge). Theorem 1 below makes this intuition precise; we go beyond convergence in distribution, proving convergence of m.g.f.s and of moments.

2 Statement of results

Assumptions. We assume throughout that $d \geq 3$ and that v is a bounded,

Borel measurable function on R^d with bounded support. Let $x(t) \in R^d$ and $y(t) \in R^d$ be defined for $t > 0$. For each $t > 0$ let the random variable $Z(t)$ be given by the path integral (1) of v, along the path $(X_s, 0 \le s \le t)$ of a d-dimensional Brownian bridge running for time t with $X_0 = x(t)$ and $X_t = y(t)$. We view $\{Z(t), t \ge 0\}$ not as a stochastic process, merely as a family of probablity distributions indexed by t.

For $x \in R^d$, let $(W_x(s), s \ge 0)$ be a Wiener process (unconstrained Brownian motion) with $W_x(0) = x$. For $t \in [0, \infty]$, define

$$Y_x(t) = \int_0^t v(W_x(s))ds.$$

Write Y_x as an abbreviation for $Y_x(\infty)$.

Before stating our main results on Brownian bridge path integrals, we state some facts about the Brownian motion path integral X_x.

Lemma 1 *There exists $\alpha_0 > 0$ such that for $|\alpha| < \alpha_0$,*

$$E \exp\{\alpha \int_0^\infty |v(W_x(s))|ds\} < \infty, \qquad x \in R^d. \tag{5}$$

Lemma 2 *For every real α, $Ee^{\alpha Y_x}$ is either finite for all x or infinite for all x. In the former case, $Ee^{\alpha Y_x}$ is a locally bounded, continuous function of x. Also, for all nonnegative integers k, $E(Y_x)^k$ is continuous in x.*

Lemma 3 *Suppose v is nonnegative and not almost everywhere zero. Then there exists $\alpha_1 \in (0, \infty)$ such that*

$$Ee^{\alpha Y_x} < \infty, \quad \alpha < \alpha_1, \ x \in R^d; \tag{6}$$

$$Ee^{\alpha Y_x} = \infty, \quad \alpha > \alpha_1, \ x \in R^d. \tag{7}$$

We can now state our main results. All limits below are taken as $t \to \infty$. This limiting procedure is motivated by problems arising in [7]. Using the scaling property of Brownian motion, one can re-state the results in terms of a limiting regime where the time for which the Brownian bridge runs remains fixed, and the range (support) of v shrinks.

Convergence in distribution is implied by the statements (8) and (9) below on convergence of m.g.f.s on a neighborhood of 0 (see [5]). Our results on convergence of m.g.f.s are stronger than convergence in distribution, and are used in [7].

Theorem 1 *Suppose $x(t) \to x \in R^d$, and $y(t) \to y \in R^d$. Then each moment of $Z(t)$ converges to the corresponding moment of $Y_x + Y'_y$, where Y'_y is defined to be a random variable, independent of Y_x, with the same distribution as Y_y. Also, for α_0 given by Lemma 1,*

$$Ee^{\alpha Z(t)} \to Ee^{\alpha(Y_x+Y'_y)} = Ee^{\alpha Y_x} Ee^{\alpha Y_y}, \quad \alpha \in (-\alpha_0, \alpha_0). \tag{8}$$

If v is nonnegative, then the convergence in (8) holds for every $\alpha \in (-\infty, \alpha_1)$, where α_1 is given by Lemma 3.

Theorem 2 *Suppose $x(t) \to x \in R^d$, while $|y(t)| \to \infty$, and $|y(t)|^2/t$ remains bounded. Then every moment of $Z(t)$ converges to the corresponding moment of Y_x, and for $|\alpha| < \alpha_0$,*

$$Ee^{\alpha Z(t)} \to Ee^{\alpha Y_x}, \quad \alpha \in (-\alpha_0, \alpha_0). \tag{9}$$

If v is nonnegative, then (9) holds for every $\alpha \in (-\infty, \alpha_1)$.

To prove the Theorems, we need the following extension of the continuity results of Lemma 3.

Lemma 4 *(a) Suppose (x_n) and (t_n) are sequences in R^d and $(0, \infty)$ respectively, with $x_n \to x \in R^d$ and $t_n \to \infty$ as $n \to \infty$. Then each moment of $Y_{x_n}(t_n)$ converges to the corresponding moment of Y_x. Also,*

$$E \exp(\alpha Y_{x_n}(t_n)) \to E \exp(\alpha Y_x), \quad \alpha \in (-\alpha_0, \alpha_0). \tag{10}$$

If $v \geq 0$ on R^d, then (10) holds for all $\alpha \in (-\infty, \alpha_1)$.
(b) Suppose now that $|x_n| \to \infty$. Then $Ee^{\alpha Y_{x_n}(t_n)} \to 0$ as $n \to \infty$, for any fixed $\alpha \in (-\alpha_0, \alpha_0)$ and for any fixed $\alpha \in (-\infty, \alpha_1)$ if v is nonnegative.

3 Proof of Lemmas

Notation. Let $q(t; x)$ ($t > 0, x \in R^d$) denote the transition density of d-dimensional Brownian motion; that is, $q(t; x) = (2\pi t)^{-d/2} \exp(-|x|^2/(2t))$. For $\mathbf{s} = (s_1, \ldots, s_k) \in (0, \infty)^k$ and $\mathbf{z} = (z_1, \ldots, z_k) \in (R^d)^k$, let $q_k^x(\mathbf{s}, \mathbf{z})$ denote the joint probability density function at $(z_1, \ldots, z_k)$ for the positions at times $(s_1, \ldots, s_k)$ of a d-dimensional Brownian motion starting at x. Let

$q_{t,k}^{x,y}(\mathbf{s}, \mathbf{z})$ denote the corresponding joint density of a d-dimensional Brownian bridge $(X_s, 0 \le s \le t)$ starting at x and finishing at y.

For $t \in (0, \infty]$, write $T(k;t)$ for the 'triangular' set $\{\mathbf{s} = (s_1, \ldots, s_k) : 0 < s_1 < \cdots < s_k < t\}$. For $\mathbf{s} \in T(k; \infty)$, we have

$$q_k^x(\mathbf{s}, \mathbf{z}) = \prod_{j=0}^{k-1} q(\Delta_j s; \Delta_j z). \tag{11}$$

where we set $s_0 = 0$, $z_0 = x$, $\Delta_j s = s_{j+1} - s_j$ and $\Delta_j z = z_{j+1} - z_j$ $(0 \le j \le k-1)$. Similarly for the Brownian bridge; for $\mathbf{s} \in T(k; t)$,

$$q_{t,k}^{x,y}(\mathbf{s}, \mathbf{z}) = [\prod_{j=0}^{k} q(\Delta_j s; \Delta_j z)]/q(y - x; t), \tag{12}$$

where we set $s_{k+1} = t$, and $x_{k+1} = y$.

For $\mathbf{z} = (z_1, z_2, \cdots, z_k) \in (R^d)^k$, set $v_k(\mathbf{z}) = v(z_1)v(z_2) \cdots v(z_k)$.

Proof of Lemma 1. We have

$$E\left(\int_0^\infty |v(W_x(s))|\, ds\right)^k / k! = \int_{T(k;\infty)} \left(\int_{(R^d)^k} |v_k(\mathbf{z})| q_k^x(\mathbf{s}, \mathbf{z}) d\mathbf{z}\right) d\mathbf{s}. \tag{13}$$

Since v is bounded and has bounded support, there exists finite K_1 for which

$$\int_0^\infty \int_{R^d} |v(z)| q(s; z - y) dz ds < K_1, \quad y \in R^d. \tag{14}$$

This implies that the expression in (13) is at most K_1^k; hence for $|\alpha| < K_1^{-1}$, the expansion of the m.g.f. (5) as a series of moments is absolutely convergent.

Proof of Lemma 2. Set $f(x) = E e^{\alpha Y_x}$. Suppose for some $x \in R^d$ that $f(x) < \infty$. We must show that $f(y) < \infty$ for all $y \in R^d$.

Let B be a ball of radius r, centered at x. Take $s > 0$, and for all $y \in B$, set $\tau_y = \inf\{t \ge 0 : W_y(t) \notin B\}$, and $\sigma_y = \min(\tau_y, s)$. Define the measure μ_y on ∂B, by $\mu_y(E) = P[W_y(\sigma_y) \in E]$. Let $K = \sup\{|v(z)| : z \in R^d\}$. By using the Markov property for W_x at time s,

$$f(x) \ge e^{-Ks} \int_B f(z) q(s; z - x) dz \tag{15}$$

and by the strong Markov property at time σ_x,

$$f(x) \geq e^{-Ks} \int_{\partial B} f(z)\mu_x(dz). \tag{16}$$

Suppose $|y - x| \leq r/2$. By the strong Markov property at time σ_y,

$$f(y) \leq e^{Ks} \left(\int_B f(z)q(s; z - y)dz + \int_{\partial B} f(z)\mu_y(dz) \right). \tag{17}$$

Since for s fixed, $q(s; z - y)/q(s; z - x)$ is uniformly bounded on $z \in B$, and μ_y has bounded Radon-Nikodym derivative with respect to μ_x, it follows that $f(y) < \infty$, and indeed f is bounded on $\{y : |y - x| \leq r/2\}$. Since r was arbitrary, f is everywhere finite and is locally bounded.

We now show f is continuous. In the above, let the ball B have radius 1. Let M be the supremum of f over ∂B, which is finite. The second integral in (17) is at most $MP[\tau_y < s]$, so if $y_n \to x$, then by (17),

$$\limsup f(y_n) \leq e^{Ks} \left(\int_B f(z)q(s; z - x)dz + MP[\tau_x < s] \right). \tag{18}$$

Now combine (18) with (15), and make $s \to 0$, to obtain $\limsup f(y_n) \leq f(x)$. A similar argument gives us $\liminf f(y_n) \geq f(x)$.

To show $E(Y_x)^k$ is continuous in x for nonnegative integer k, use induction on k. Set

$$f_k(x) = (Ee^{\alpha Y_x} - \sum_{j=0}^{k} \alpha^j E(Y_x)^j/j!)/\alpha^{k+1}.$$

Then f_k is continuous in x by the inductive hypothesis, and as $\alpha \to 0$, $f_k(x)$ converges to $E(Y_x)^{k+1}/(k + 1)!$, uniformly in x (see the proof of Lemma 1).

Proof of Lemma 3. If v is nonnegative, then $Ee^{\alpha Y_x}$ is clearly monotone in α, and Lemmas 1 and 2 imply the existence of $\alpha_1 \in (0, \infty]$ satisfying (6) and (7). Also, α_1 is finite; indeed, a straightforward geometric series argument [4] shows that for given x, $Ee^{\alpha Y_x}$ is infinite for large enough α. Since we shall not use the fact that $\alpha_1 < \infty$ here, we omit details.

Proof of Lemma 4. (a) First suppose v is nonnegative. By Lemma 2,

$$E \exp\{\alpha Y_{x_n}(t_n)\} \leq E \exp\{\alpha Y_{x_n}\} \to E \exp\{\alpha Y_x\}. \tag{19}$$

Also, for fixed $t_0 < \infty$, the series $(E|\alpha Y_{x_n}(t_0)|^k/k!, k \geq 0)$ is dominated (for all n) by the summable series $((\alpha K t_0)^k/k!, k \geq 0)$. Each moment of $Y_{x_n}(t_0)$

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converges to the corresponding moment of $Y_x(t_0)$; to see this, write each moment as a multiple integral with respect to the transition density $q_k^{x_n}$ or q_k^x, and use a domination argument. Hence,

$$E \exp\{\alpha Y_{x_n}(t_0)\} \to E \exp\{\alpha Y_x(t_0)\}, \qquad (20)$$

and by making $t_0 \to \infty$,

$$\liminf E \exp\{\alpha Y_{x_n}(t_n)\} \geq E \exp\{\alpha Y_x\}$$

so that by comparison with (19) we have (10).

Now drop the assumption that v is nonnegative. If $|\alpha| < \alpha_0$, then (10) would hold if we replaced $v(\cdot)$ by $|v(\cdot)|$ in the definitions of $Y_{x_n}(t_n)$ and Y_x. Hence by obvious term-by-term estimation on the difference of the power series, $|Ee^{\alpha Y_x(t)} - Ee^{\alpha Y_x}| \to 0$ locally uniformly in x, and (10) follows. The proof of convergence of moments is similar.

(b) This is a routine application of the strong Markov property, using the fact that v has bounded support and $Ee^{\alpha Y_x}$ is locally bounded in x.

4   Proof of Theorems

Take an arbitrary increasing Real-valued function $u(t), t \geq 0$ with $u(0) = 0$, and with $u(t) \to \infty$ and $u(t)/t \to 0$ as $t \to \infty$. The idea of this proof is to compare the moments of $Z(t)$ with those of $Y_{x(t)}(u(t)) + Y'_{y(t)}(u(t))$.

Proof of Theorem 1. It suffices to prove the results along an arbitrary sequence (t_n) with $t_n \to \infty$. Write x_n for $x(t_n)$, y_n for $y(t_n)$, and u_n for $u(t_n)$. Suppose first that v is nonnegative.

Let $T_1(k; t_n)$ denote the set $\{\mathbf{s} \in T(k; t_n) : s_j \in (0, u_n) \cup (t_n - u_n, \infty), 1 \leq j \leq k\}$, and let $T_2(k; t_n)$ denote the set $T(k; t_n) \setminus T_1(k; t_n)$. Then

$$E(Z(t_n))^k/k! = \int_{T(k;t_n)} \int_{(R^d)^k} q_{t_n,k}^{x_n,y_n}(\mathbf{s}; \mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s} \qquad (21)$$

$$= I_1 + I_2 \qquad (22)$$

where I_i denotes the integral over $T_i(k; t_n)$ $(i = 1, 2)$.

Now consider the sum of independent unconstrained Brownian path integrals $Y_{x_n}(t_n) + Y'_{y_n}(t_n)$. By the Binomial theorem,

$$E(Y_{x_n}(t_n) + Y'_{y_n}(t_n))^k/k! = \sum_{j=0}^{k} (E[Y_{x_n}(t_n)]^j/j!)(E[Y_{y_n}(t_n)]^{k-j}/(k-j)!)$$

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$$> \sum_{j=0}^{k} \int_{T(j;t_n)} \{ \int_{T(k-j;t_n-s_j)} \int_{(R^d)^{k-j}} v_{k-j}(\mathbf{z}')q_{k-j}^{y_n}(\mathbf{s}';\mathbf{z}')d\mathbf{z}'d\mathbf{s}' \}$$

$$\times \int_{(R^d)^j} v_j(\mathbf{z})q_j^{x_n}(\mathbf{s};\mathbf{z})d\mathbf{z}d\mathbf{s}. \tag{23}$$

Indeed, the expression inside the braces is less than $E(Y_{y_n}(t_n))^{k-j}/(k-j)!$, since $\mathbf{s}'$ has been integrated over $T(k-j;t_n-s_j)$ (s_j is the last component of $\mathbf{s}$), rather than $T(k-j;t_n)$.

By the substitutions

$$z_{j+r} = z'_{k+1-(j+r)}, \quad s_{j+r} = t_n - s'_{k+1-(j+r)} \quad (r=1,2,\ldots,k-j),$$

the right hand side of (23) equals

$$\sum_{j=0}^{k} \int_{T(k;t_n)} \int_{(R^d)^k} v_k(\mathbf{z})q_j^{x_n}(s_1,\ldots,s_j;z_1,\ldots,z_j) \times$$

$$\times q_{k-j}^{y_n}(t_n-s_k, t_n-s_{k-1}, \ldots, t_n-s_{j+1}; z_k, z_{k-1}, \ldots, z_{j+1})d\mathbf{z}d\mathbf{s}. \tag{24}$$

On inspection, we find that by (12), (11) and the symmetry of $q(t;x)$ in x, the integrand in (24) equals

$$v_k(\mathbf{z})q_{t_n,k}^{x_n,y_n}(s_1,\ldots,s_k;z_1,\ldots,z_k)Q_{n,j}(\mathbf{s},\mathbf{z})$$

where the density ratio $Q_{n,j}(\mathbf{s},\mathbf{z})$ is defined by

$$Q_{n,j}(\mathbf{s},\mathbf{z}) = \frac{q(t_n;y_n-x_n)}{q(\Delta_j s; \Delta_j z)} = \left(\frac{\Delta_j s}{t_n}\right)^{d/2} \exp\left(\frac{|\Delta_j z|^2}{\Delta_j s} - \frac{|y_n-x_n|^2}{t_n}\right). \tag{25}$$

Hence,

$$E(Y_{x_n}(t_n) + Y'_{y_n}(t_n))^k/k! > \sum_{j=0}^{k} \int_{T(k;t_n)} \int_{(R^d)^k} q_{t_n,k}^{x_n,y_n}(\mathbf{s},\mathbf{z})Q_{n,j}(\mathbf{s},\mathbf{z})v_k(\mathbf{z})d\mathbf{z}d\mathbf{s} \tag{26}$$

A similar calculation for $Y_{x_n}(u_n)+Y'_{y_n}(u_n)$ gives rise to a similar formula to (23), with $\mathbf{s}'$ now integrated over $T(k-j;u_n)$ and t_n replaced by u_n elsewhere, and with equality this time, provided n is so big that $u_n < t_n/2$. By use of the *same* change of variables as before (that is, running time backwards from t_n, not u_n), we obtain

$$E(Y_{x_n}(u_n) + Y'_{y_n}(u_n))^k/k! =$$

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$$= \sum_{j=0}^{k} \int_{T_1(k;t_n)} \int_{(R^d)^k} \mathbf{1}_j(\mathbf{s}) q_{t_n,k}^{x_n,y_n}(\mathbf{s},\mathbf{z}) Q_{n,j}(\mathbf{s},\mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s}, \qquad (27)$$

where $\mathbf{1}_j$ denotes the indicator function of the set $\{\mathbf{s} \in T_1(k;t_n) : s_j < u_n < t_n - u_n < s_{j+1}\}$.

Take the summation inside the integral in (27). For each $\mathbf{s} \in T_1(k;t_n)$, the indicator function $\mathbf{1}_j(\mathbf{s})$ is 1 for exactly one value of j. For that value of j, and for $v_k(\mathbf{z}) \neq 0$, the ratio $Q_{n,j}(\mathbf{s},\mathbf{z})$ is close to 1 when n is large, because of the assumptions on x_n, y_n, and u_n. Indeed, we have

$$E(Y_{x_n}(u_n) + Y'_{y_n}(u_n))^k/k! \sim I_1 \quad \text{as } n \to \infty, \qquad (28)$$

in the sense that the ratio of the two sides approaches 1, *uniformly* in k.

Turning to I_2, we have that for sufficiently large n, $\exp(-|y_n - x_n|^2/t_n) > 1/2$, and hence

$$\sum_{j=0}^{k} Q_{n,j}(\mathbf{s},\mathbf{z}) \geq (1/2)(k+1)^{1-d/2}, \qquad (29)$$

since by Jensen's inequality, $\sum_{j=0}^{k}(\Delta_j s/t_n)^{d/2}$ is minimised by setting $s_j = t_n/(k+1)$, for all j. So for large enough n,

$$I_2 \leq 2(k+1)^{d/2-1} \sum_{j=0}^{k} \int_{T_2(k;t_n)} \int_{(R^d)^k} q_{t_n,k}^{x_n,y_n}(\mathbf{s},\mathbf{z}) Q_{n,j}(\mathbf{s},\mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s} \qquad (30)$$

which implies (by (27) and (26)) that

$$I_2 \leq 2(k+1)^{d/2-1} D(n,k) \qquad (31)$$

where we set

$$D(n,k) = [E(Y_{x_n}(t_n) + Y'_{y_n}(t_n))^k - E(Y_{x_n}(u_n) + Y'_{y_n}(u_n))^k]/k! \qquad (32)$$

By Lemma 4, $D(n,k) \to 0$ as $n \to \infty$; combining this with (28) and (22), we obtain the desired convergence of moments.

Now consider I_2 as a function of k. Suppose $0 \leq \alpha < \alpha_1$, and choose $\alpha_2 \in (\alpha, \alpha_1)$. For a suitable choice of k_0, we have

$$\sum_{k=0}^{\infty} \alpha^k I_2(k) \leq 2 \sum_{k=0}^{k_0} \alpha^k (k+1)^{d/2-1}(D(n,k)) + \sum_{k=k_0}^{\infty} \alpha_2^k (D(n,k)). \qquad (33)$$

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The first sum converges to zero as $n \to \infty$. The second is bounded above by

$$E \exp\{\alpha_2(Y_{x_n}(t_n) + Y'_{y_n}(t_n))\} - E \exp\{\alpha_2(Y_{x_n}(u_n) + Y'_{y_n}(u_n))\}.$$

By Lemma 4, the last expression converges to zero. This, together with the uniform convergence in (28), implies the desired convergence (8) of m.g.f.s on $0 \le \alpha < \alpha_1$, for v nonnegative.

Now drop the assumption v is nonnegative. The proof of (28) is unchanged. Also, (31) would hold if v were replaced by $|v|$ in the definitions of I_2 and $D(n, k)$; hence $|I_2| \to 0$, and for $\alpha \in (-\alpha_0, \alpha_0)$, $\sum_{k \ge 0} |\alpha|^k |I_2(k)| \to 0$ as above, giving us the desired convergence (8) of m.g.f.s. Finally, this implies convergence in distribution, so (8) holds for $\alpha < 0$ when v is nonnegative.

Proof of Theorem 2. We use notation from the previous proof. First consider the case that $|y_n| \to \infty$, with $|y_n|^2/t_n \to 0$. Then convergence in the desired sense of $Z(t_n)$ to Y_x follows by the proof of Theorem 1 (now using Lemma 4(b)).

Now consider the case that $|y_n|^2/t_n$ is bounded away from 0 and from ∞. First assume v is nonnegative. Partition the set $T(k; t_n)$ as $T_3(k; t_n) \cup T_4(k; t_n)$, where we set

$$T_3(k; t_n) = T(k; u_n), \qquad T_4(k; t_n) = T(k; t_n) \setminus T_3(k; t_n).$$

For $i = 3, 4$, let I_i denote the integral in (21), restricted to $T_i(k; t_n)$; that is,

$$I_i = \int_{T_i(k;t_n)} \int_{(R^d)^k} q_{t_n,k}^{x_n,y_n}(\mathbf{s}; \mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s}. \tag{34}$$

So $E(Z(t_n))^k/k! = I_3 + I_4$.

A similar (easier) argument to the derivation of (27) gives us

$$E(Y_{x_n}(u_n))^k/k! = \int_{T(k;u_n)} \int_{(R^d)^k} q_{t_n,k}^{x_n,y_n}(\mathbf{s}, \mathbf{z}) Q_{n,k}(\mathbf{s}, \mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s}, \tag{35}$$

where $Q_{n,k}(\mathbf{s}, \mathbf{z})$ is given by (25), with $j = k$ and $\Delta_k z = y_n - z_k$ and $\Delta_k s = t_n - s_k$ as before. Since $|y_n - z|^2/|y_n|^2 \to 1$ as $n \to \infty$, locally uniformly in z, we can write

$$Q_{n,k}(\mathbf{s}, \mathbf{z}) = (\Delta_k s/t_n)^{d/2} \exp\{(|y_n|^2/t_n)[(\frac{t_n}{\Delta_k s}) g_n(\mathbf{s}, \mathbf{z}) - h_n(\mathbf{s}, \mathbf{z})]\}$$

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where g_n and h_n converge to 1, uniformly on $\mathbf{s} \in T(k, t_n)$ and $\mathbf{z}$ in the support of v_k.

Observe that $Q_{n,k}(\mathbf{s}, \mathbf{z}) \to 1$, uniformly on $\mathbf{s} \in T(k; u_n)$ and $\mathbf{z}$ in the support of v_k. Therefore, $I_3 \sim E(Y_{x_n}(u_n))^k$ as $n \to \infty$, uniformly in k.

To estimate I_4, note that $|y_n|^2/t_n$ is assumed to be bounded away from 0 and ∞ as $n \to \infty$, so we can find $\delta > 0$ and $n_0 \in Z$ for which $Q_{n,k}(\mathbf{s}, \mathbf{z}) \geq \delta$, for all $\mathbf{s} \in T(k; t_n)$, $\mathbf{z} \in \mathrm{Supp}(v_k)$, $n \geq n_0$ and $k \geq 0$. Thus for $n \geq n_0$,

$$I_4 \leq \delta^{-1} \int_{T_4} q_{t_n,k}^{x_n,y_n}(\mathbf{s}; \mathbf{z}) Q_{n,k}(\mathbf{s}, \mathbf{z}) v_k(\mathbf{z}) d\mathbf{z} d\mathbf{s}.$$

A similar expression to (35) holds with u_n replaced by t_n, so

$$I_4 \leq \delta^{-1}[E(Y_{x_n}(t_n))^k - E(Y_{x_n}(u_n))^k]/k!$$

which implies that as $n \to \infty$, $I_4 \to 0$ and $\sum_{k \geq 0} \alpha^k I_4(k) \to 0$ ($0 \leq \alpha < \alpha_1$). Combining the estimates obtained for I_3 and I_4, we obtain the desired convergence results for v nonnegative and $\alpha \in [0, \alpha_1)$. The full result follows as in the proof of Theorem 1.

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