ASYMPTOTICS OF MULTIVARIATE SEQUENCES II. MULTIPLE POINTS OF THE SINGULAR VARIETY.

ROBIN PEMANTLE AND MARK C. WILSON

ABSTRACT. Let $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$ be a multivariate generating function which is meromorphic in some neighborhood of the origin of $\mathbb{C}^d$, and let $\mathcal{V}$ be its set of singularities. Effective asymptotic expansions for the coefficients can be obtained by complex contour integration near points of $\mathcal{V}$.

In the first article in this series, we treated the case of smooth points of $\mathcal{V}$. In this article we deal with multiple points of $\mathcal{V}$. Our results show that the central limit (Ornstein-Zernike) behavior typical of the smooth case does not hold in the multiple point case. For example, when $\mathcal{V}$ has a multiple point singularity at $(1, \dots, 1)$, rather than $a_{\mathbf{r}}$ decaying as $|\mathbf{r}|^{-1/2}$ as $|\mathbf{r}| \to \infty$, $a_{\mathbf{r}}$ is very nearly polynomial in a cone of directions.

1. INTRODUCTION

In [PW02b], we began a series of articles addressing the general problem of computing asymptotic expansions for a multivariate sequence whose generating function is known. Such problems are encountered frequently in combinatorics and probability; see for instance Examples 2 – 8 in Section 1 of [Pem], which collects examples from various sources including [LL99, FM77, Wil94, Com74]. Our aim is to present methods which are as general as possible, and lead to effective computation. Our apparatus may be applied to any function whose dominant singularities are poles. Among other things, we showed in [PW02b] that for all nonnegative bivariate sequences of this type, our method is applicable (further work may be required in one degenerate case).

The article [PW02b] handled the case when the pole variety was smooth. The present work is motivated by a collection of applications where the pole variety is composed of intersecting branches. Several examples are as follows.

(1)
$$F(x,y) = \frac{2}{(1-2x)(1-2y)} \left[2 + \frac{xy-1}{1-xy(1+x+y+2xy)} \right]$$
arises in Markov modeling [Kar02];

(2)
$$F(\mathbf{z}) = \frac{G(\mathbf{z})}{\prod_{j=1}^k L_j(\mathbf{z})}$$
where the L_j are affine, arises in queueing theory [BM93];

(3)
$$F(x,y,z) = \frac{x^a y^b z^c}{(1-x)(1-y)(1-z)(1-xz)(1-yz)}$$

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is a typical generating function arising in enumeration of integer solutions to unimodular linear equations, and is the running example used in [dLS03];

(4)
$$F(x,y) = \frac{1}{(1-(1/3)x-(2/3)y)(1-(2/3)x-(1/3)y)}$$
counts winning plays in a dice game (see Example 1.3).

(5)
$$F(x,y,z) = \frac{z/2}{(1-yz)P(x,y,z)}$$
arises in the analysis of random lattice tilings [CEP96]. In this case, in addition to the factorization of the denominator, there is an isolated singularity in the variety defined by the polynomial P, for which reason the analysis is not carried out in the present paper but in [CP03].

These examples serve as our motivation to undertake a categorical examination of generating functions with self-intersections of the pole variety. As in the above cases, one may imagine that this comes about from a factorization of the generating function, although we show in Example 3.2 that the method works as well for irreducible functions with the same "multiple point" local geometry.

Our approach is analytic. For simplicity, we restrict to the two-variable case in this introduction, though our methods work for any number of variables. Given a sequence a_{rs} indexed by the 2-dimensional nonnegative integer lattice, we seek asymptotics as $r,s \to \infty$. Form the generating function $F(z,w) = \sum_{r,s} a_{rs} z^r w^s$ of the sequence; we assume that F is analytic in some neighborhood of the origin.

The iterated Cauchy integral formula yields
$$a_{rs} = \frac{1}{(2\pi i)^2} \int_{\mathcal{C}'} \int_{\mathcal{C}} \frac{F(z,w)}{z^{r+1} w^{s+1}} \, dw \, dz,$$
where $\mathcal{C}$ and $\mathcal{C}'$ are circles centered at 0 and F is analytic on a polydisk containing the torus $\mathcal{C} \times \mathcal{C}'$. Expand the torus, by expanding (say) $\mathcal{C}$, slightly beyond a *minimal singularity* of F (that is, a point (z_0, w_0) at which the expanding torus first touches the singular set $\mathcal{V}$ of F). The difference between the corresponding inner integrals is then computed via residue theory. Thus a_{rs} is represented as a sum of a residue and an integral on a large torus. One hopes that the residue term is dominant, and gives a good approximation to a_{rs}. The residue term is itself an integral (since the residue is taken in the inner integral only). A stationary phase analysis of the residue integral shows when our hope is realized, and yields an asymptotic expansion for a_{rs}.

In [PW02b] we considered the case when the minimal singularity in question is a smooth point of $\mathcal{V}$. The present article deals with the case where the minimal singularity is a *multiple point*: locally, the singular set is a union of finitely many graphs of analytic functions. This case includes the smooth point analysis of [PW02b], and Theorem 3.9 with n set to 0 essentially generalizes the analysis of that article). In the two-variable case, we know from Lemma 6.1 of [PW02b] that every minimal singularity must have this form or else be a cusp (see also [Tsi93, Lemma 3.1]), though more complicated singularities may arise in higher dimensions.

The rest of this article is organized as follows. In the remainder of this section we describe in more detail the program begun in [PW02b] and continued here and in future articles in this series. Section 2 deals with notation and preliminaries required for the statement of our main results. Those results, along with illustrative examples, are listed in Section 3. Proofs are given in Section 4. We discuss some further details and outline future work in Section 5.

Details of the Program. Our notation is similar to that in [PW02b]. For clarity we shall reserve the names of several objects throughout. We use **boldface** to indicate a (row or column) vector. The number of variables will be denoted $d+1$. The usual multi-index notation is in use: $\mathbf{z}$ denotes a vector $(z_1, \ldots, z_{d+1})^T \in \mathbb{C}^{d+1}$, and if $\mathbf{r}$ is an integer vector then $\mathbf{z}^{\mathbf{r}} = \prod_j z_j^{r_j}$. We also use the convention that a function, ostensibly of 1 variable, applied to an element of $\mathbb{C}^{d+1}$ acts on each coordinate separately — for example $e^{\mathbf{x}} = (e^{x_1}, \ldots, e^{x_{d+1}})$. Throughout, G and H denote functions analytic in some polydisk about $\mathbf{0}$ and $F = G/H = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. The set where H vanishes will be called the *singular variety* of F and denoted by $\mathcal{V}$. Let $\mathbf{D}(\mathbf{z}), \mathbf{T}(\mathbf{z})$ denote respectively the polydisk and torus (both centered at the origin) on which $\mathbf{z}$ lies.

A crude preliminary step in approximating $a_{\mathbf{r}}$ is to determine its exponential rate; in other words, to estimate $\log |a_{\mathbf{r}}|$ up to a factor of $1 + o(1)$. Let $\mathcal{D}$ denote the (open) domain of convergence of F and let $\log\mathcal{D}$ denote the logarithmic domain in $\mathbb{R}^{d+1}$, that is, the set of $\mathbf{x} \in \mathbb{R}^{d+1}$ such that $e^{\mathbf{x}} \in \mathcal{D}$. If $\mathbf{z}^* \in \mathcal{D}$ then Cauchy's integral formula

$$a_{\mathbf{r}} = \left(\frac{1}{2\pi i}\right)^{d+1} \int_{\mathbf{T}(\mathbf{z}^*)} \frac{F(\mathbf{z})}{\mathbf{z}^{\mathbf{r}+\mathbf{1}}} \, d\mathbf{z} \tag{1.1}$$

shows that $a_{\mathbf{r}} = O(|\mathbf{z}^*|^{-\mathbf{r}})$. Letting $\mathbf{z}^* \to \partial\mathcal{D}$ gives

$$\log |a_{\mathbf{r}}| \leq -\mathbf{r} \cdot \log |\mathbf{z}^*| + o(|\mathbf{r}|),$$

and optimizing in $\mathbf{z}^*$ gives $\log |a_{\mathbf{r}}| \leq \gamma(\mathbf{r}) + o(|\mathbf{r}|)$ where

$$\gamma(\mathbf{r}) := - \sup_{\mathbf{x} \in \log\mathcal{D}} \mathbf{r} \cdot \mathbf{x} \, . \tag{1.2}$$

The cases in which the most is known about $a_{\mathbf{r}}$ are those in which this upper bound is correct, that is, $\log |a_{\mathbf{r}}| = \gamma(\mathbf{r}) + o(|\mathbf{r}|)$. To explain this, note first that the supremum in (1.2) is equal to $\mathbf{r} \cdot \mathbf{x}$ for some $\mathbf{x} \in \partial\log\mathcal{D}$. The torus $\mathbf{T}(e^{\mathbf{x}})$ must contain some minimal singularity $\mathbf{z}^* \in \mathcal{V} \cap \partial\mathcal{D}$. Asking that $\log |a_{\mathbf{r}}| \sim -\mathbf{r} \cdot \mathbf{z}^*$ is then precisely the same as requiring the Cauchy integral (1.1) — or the residue integral mentioned above — to be of roughly the same order as its integrand. This is the situation in which it easiest to estimate the integral.

Our program may now be summarized as follows. Associated to each minimal singularity $\mathbf{z}^*$ is a cone $\mathbf{K}(\mathbf{z}^*) \subseteq (\mathbb{R}^+)^{d+1}$. Given $\mathbf{r}$, we find one or more $\mathbf{z}^* = \mathbf{z}^*(\mathbf{r}) \in \mathcal{V} \cap \partial\mathcal{D}$ where the upper bound is least. We then attempt to compute a residue integral there. This works only if $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*)$ and if the residue computation is of a type we can handle. Our program is guaranteed to succeed in some cases, and conjectured to succeed in others. It is known to fail only in some cases where the $a_{\mathbf{r}}$ are not nonnegative reals (not the most important case in combinatorial or probabilistic applications) and even then a variant seems to work.

To amplify on this, define a point $\mathbf{z}^* \in \mathcal{V}$ to be *minimal* if $\mathbf{z}^* \in \partial\mathcal{D}$ and each coordinate of $\mathbf{z}^*$ is nonzero. There are only three possible types of minimal singularities [PW02b, Lemma 6.1]), namely smooth points of $\mathcal{V}$, multiple points and cone points (all defined below). It is conjectured that for all three types of points, and any $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*)$, we indeed have

$$\log |a_{\mathbf{r}}| = \gamma(\mathbf{r}) + o(|\mathbf{r}|) = -\mathbf{r} \cdot \log |\mathbf{z}^*| + o(|\mathbf{r}|) \, .$$

This is proved for smooth points in [PW02b] via residue integration, and the complete asymptotic series obtained. It is proved in the present work for multiple points under various assumptions; the fact that these do not cover all cases seems due more to taxonomical problems rather than the inapplicability of the method. The problem remains open for cone points, along with the problem of computing asymptotics.

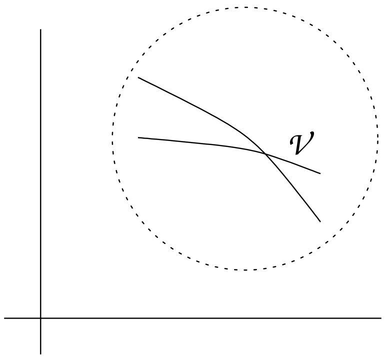

FIGURE 1. Local picture of a double pole

It is also shown in [PW02b] that when $a_{\mathbf{r}}$ are all nonnegative, then $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*(\mathbf{r}))$, and therefore that a resolution of the above problem yields a complete analysis of nonnegative sequences with pole singularities.

When the hypothesis that $a_{\mathbf{r}}$ be real and nonnegative is removed, it is not always true that $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*(\mathbf{r}))$. In all examples we have worked, there have been points $\mathbf{z}^* \notin \overline{\mathcal{D}}$ for which $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*)$ and a residue integral near $\mathbf{z}^*$ may be proved a good approximation to $a_{\mathbf{r}}$; the method is to contract the torus of integration to the origin in some way other than simply iterating the contraction of each coordinate circle to a point. Thus a second open question is to settle whether there is always such a point $\mathbf{z}^*$ in the case of mixed signs (see the discussion in [PW02b, Section 7]).

The scope of the present article is as follows. We define multiple points and carry out the residue integral arising near a strictly minimal, multiple point. Unlike the case for smooth points, the integral is not readily recognizable as a standard multivariate Fourier-Laplace integral, and a key result of this article is a more manageable representation of the residue to be integrated (Corollary 4.3 and Lemma 4.4). The Fourier-Laplace integrals arising fall just outside the scope of the standard references, and we have been led to develop generalizations of known results. These results, which would take too much space here, are included in [PW02a].

The asymptotics arising from multiple point singularities differ substantially from asymptotics in the smooth case. In the remainder of this introduction, we give examples to illustrate this.

Example 1.1 (simplest possible multiple point). Let $F(z,w)$ be a two-variable generating function and suppose that the point $(1,1)$ is a double pole of F (thus $F = G/H$ with $G(z,w) \neq 0$ and $H(z,w)$ vanishing to order 2 at $(1,1)$). If F has no other poles (z,w) with $|z|,|w| \leq 1$, and if the two branches of the singular variety $\mathcal{V}$ meet transversely at $(1,1)$ as in Figure 1, then for some positive constants c and C,

$$a_{rs} = C + O(e^{-c|(r,s)|}) \tag{1.3}$$

for all (r,s) in a certain cone, $\mathbf{K}$, in the positive integer quadrant. This is proved in Theorem 3.1 below, and the constant C computed. Exact statement of the transversality hypothesis requires some discussion of the geometry of $\mathcal{V}$. The constant C is computed in terms of some algebraic quantities derived from F. The cone $\mathbf{K}$ in which this holds is easily described in terms of the tangents to the branches of the double pole. The need for some preliminary algebraic and geometric analysis to define transversality and to compute C and $\mathbf{K}$ motivates our somewhat lengthy Section 2.

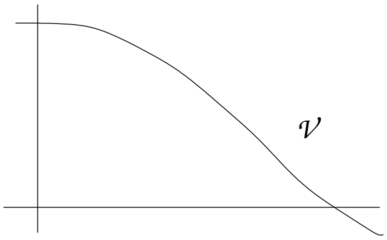

FIGURE 2. When $\mathcal{V}$ has only smooth points

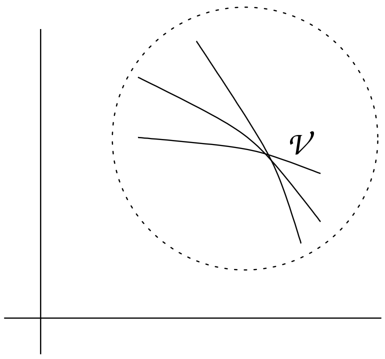

FIGURE 3. A pole of order 3

Example 3.2 below shows the details of this computation for a particular F. The exponential bound on the error follows from results in [Pem00] which are cited in Section 4. If the multiple pole is moved to a point (z^*, w^*) other than $(1,1)$, a factor of $(z^*)^{-r}(w^*)^{-s}$ is introduced.

Compare this with the case where $\mathcal{V}$ is smooth, intersecting the positive real quadrant as shown in Figure 2. In this case one has Ornstein-Zernike (central limit) behavior. Suppose, as above, that $(1,1) \in \mathcal{V}$ and F has no other poles (z,w) with $|z|,|w| \leq 1$. Then as $(r,s) \to \infty$ with r/s fixed, a_{rs} is rapidly decreasing for all but one value of r/s; for that distinguished direction, $a_{rs} \sim C|(r,s)|^{-1/2}$ for some constant, C, and the error terms may be developed in a series of decreasing powers of $|(r,s)|$. The two main differences between the double and single pole cases are thus the existence of a plateau in the double pole case, and the flatness up to an exponentially small correction versus a correction of order $|(r,s)|^{-3/2}$ for a single pole. $\square$

Example 1.2 (pole of greater order)**.** Alter the previous example so that $\mathcal{V}$ has a pole of some order $n+1$ at $(1,1)$, as in Figure 3. Then the formula (1.3) becomes instead

$$a_{rs} = P(r,s) + O(e^{-c|(r,s)|})$$

where P is a piecewise polynomial of degree $n-1$. In Example 3.7 below, P is explicitly computed in the case $n+1=3$. Again, we see the chief differences from the smooth case being a cone of non-exponential decay, and an exponentially small correction to the polynomial $P(r,s)$ in the interior of the cone. $\square$

In higher dimensions there are more possible behaviors, but the same sorts of results hold. There is a cone on which the exponential rate has a plateau; under some conditions the correction terms within this cone are exponentially small. The following table summarizes the results proved in this paper; transversality assumptions have been omitted.

Theorem number	**Hypotheses**	**Asymptotic behavior**
Theorem 3.1	two curves in 2-space	$C +$ exponentially small
Theorem 3.3	$d+1$ sheets in $(d+1)$-space	$C +$ exponentially small
Theorem 3.6	more sheets than the dimension	polynomial + exponentially small
Theorem 3.9	fewer sheets than the dimension	asymptotics start with $\|\mathbf{r}\|^{n/2-d/2}$
Theorem 3.11	all sheets tangent	asymptotics start with $\|\mathbf{r}\|^{n-d/2}$

We conclude the introduction by giving a combinatorial application of Example 1.1.

Example 1.3 (combinatorial application)**.** An independent sequence of random numbers uniform on $[0,1]$ is used to generate biased coin-flips: if p is the probability of heads then a number $x \leq p$ means heads and $x > p$ means tails. The coins will be biased so that $p = 2/3$ for the first n flips, and $p = 1/3$ thereafter. A player desires to get r heads and s tails and is allowed to choose n. On average, how many choices of $n \leq r+s$ will be winning choices?

The probability that n is a winning choice for the player is precisely

$$\sum_{a+b=n} \binom{n}{a} (2/3)^a (1/3)^b \binom{r+s-n}{r-a} (1/3)^{r-a} (2/3)^{s-b} .$$

Let a_{rs} be this expression summed over n. The array $\{a_{rs}\}_{r,s\geq 0}$ is just the convolution of the arrays $\binom{r+s}{r}(2/3)^r(1/3)^s$ and $\binom{r+s}{r}(1/3)^r(2/3)^s$, so the generating function $F(z,w) := \sum a_{rs} z^r w^s$ is the product

$$F(z,w) = \frac{1}{(1 - \frac{1}{3}z - \frac{2}{3}w)(1 - \frac{2}{3}z - \frac{1}{3}w)} .$$

Applying Theorem 3.1 with $G \equiv 1$ and $\det \mathbf{H} = -1/9$, we see that $a_{rs} = 3$ plus a correction which is exponentially small as $r, s \to \infty$ with $r/(r+s)$ staying in any subinterval of $(1/3, 2/3)$. A purely combinatorial analysis of the sum may be carried out to yield the leading term, 3, but says nothing about the correction terms. The diagonal extraction method of [HK71] yields very precise information for $r = s$ but nothing more general in the region $1/3 < r/(r+s) < 2/3$. $\square$

2. Preliminary definitions and notation

For each $\mathbf{z} \in \mathbb{C}^{d+1}$, the truncation $(z_1, \ldots, z_d)$ will be denoted $\widehat{\mathbf{z}}$, and the last coordinate z_{d+1} simply by z. We do not specify the size for constant vectors — for example $\mathbf{1}$ denotes the vector $(1, \ldots, 1)^T$ of whatever size is appropriate. Thus we write $\widehat{\mathbf{1}} = \mathbf{1}$. A minimal singularity $\mathbf{z}^*$ is *strictly minimal* if $\mathcal{V} \cap \mathbf{D}(\mathbf{z}^*) = \{\mathbf{z}^*\}$. When a minimal point is not strictly minimal, one must add (or integrate) contributions from all points of $\mathcal{V} \cap \mathbf{T}(\mathbf{z}^*)$. This step is routine and will be carried out in a future article; we streamline the exposition here by assuming strict minimality.

This article deals entirely with minimal points $\mathbf{z}^*$ of $\mathcal{V}$ near which $\mathcal{V}$ decomposes as a union of sheets $\mathcal{V}_j$, each of which is a graph of an analytic function $z = u_j(\widehat{\mathbf{z}})$. The algebraic description of this situation is as follows. By the Weierstrass Preparation Theorem [GH78] there is a neighborhood of $\mathbf{z}^*$ in which we may write $H(\mathbf{z}) = \chi(\mathbf{z}) W(\mathbf{z})$ where χ is analytic and nonvanishing and W is a *Weierstrass polynomial.* This means that

$$W(\mathbf{z}) = z^{n+1} + \sum_{j=0}^{n} \chi_j(\widehat{\mathbf{z}}) z^j$$

where the multiplicity $n+1$ is at least 2 if $\mathbf{z}^*$ is not a smooth point, and the analytic functions χ_j vanish at $\widehat{\mathbf{z}^*}$.

Now suppose that $\mathbf{z}^*$ has all coordinates nonzero. Recalling from Lemma 6.1 of [PW02b] that any such minimal point of $\mathcal{V}$ is locally homogeneous, we see in fact that χ_j vanishes to homogeneous degree $n+1-j$ at $\widehat{\mathbf{z}^*}$ and that χ_0 has nonvanishing pure z_i^{n+1} terms for each $1 \leq i \leq d$. Then $\mathcal{V}$ is locally the union of smooth sheets if the degree $n+1$ homogeneous part of W (the leading term) factors completely into *distinct* linear factors, while if $\mathcal{V}$ is locally the union of smooth sheets then we have a factorization

$$H(\mathbf{z}) = \chi(\mathbf{z}) \prod_{j=0}^{n} [z - u_j(\widehat{\mathbf{w}})] \tag{2.1}$$

for analytic (*not necessarily distinct*) functions u_j mapping a neighborhood of $\widehat{\mathbf{z}^*}$ to a neighborhood of z^*.

Remark. If the expansion of H near $\mathbf{z}^*$ vanishes to order $n+1$ in z, then there are always $n+1$ solutions (counting multiplicity) to $H(\widehat{\mathbf{z}}, z) = 0$ for $\widehat{\mathbf{z}}$ near $\widehat{\mathbf{z}^*}$. These vary analytically, but may be parametrized by $n+1$ analytic functions u_j only if there is no monodromy, that is, if one stays on the same branch moving around a cycle in the complement of the singular set. Thus a multiple point singularity is one whose monodromy group is trivial. We remark also that in two variables, local homogeneity of the minimal point $\mathbf{z}^*$ nearly implies it is a multiple point. The only other possibility is a cusp whose tangents are all equal; such a singularity turns out not to affect the leading order asymptotics and to affect the lower order asymptotics only in a single direction.

It turns out to be more convenient to deal with the reciprocals $v_j = 1/u_j$. The basic setup throughout the rest of this article is as follows.

Definition 2.1. The point $\mathbf{z}^*$ of $\mathcal{V}$ is a *multiple point* if there are analytic functions $v_0, \ldots, v_n, \phi$ and a local factorization

$$F(\mathbf{z}) = \frac{\phi(\mathbf{z})}{\prod_{j=0}^{n} (1 - z v_j(\widehat{\mathbf{z}}))}, \tag{2.2}$$

which we call the *factored form* of F, such that

- $(z^*)_{d+1} v_j(\widehat{\mathbf{z}^*}) = 1$ for all j;
- each $\dfrac{\partial v_j}{\partial z_k}(\widehat{\mathbf{z}^*}) \neq 0$;
- $z_{d+1} v_j(\widehat{\mathbf{z}}) = 1$ for some j if and only if $\mathbf{z} \in \mathcal{V}$.

Remark. Let $\mathcal{V}_j$ denote the local hypersurface parametrized by $z = u_j(\widehat{\mathbf{z}})$. Then the first of the conditions says that each $\mathcal{V}_j$ passes through $\mathbf{z}^*$: there are no extraneous factors in the denominator representing surface elements not passing through $\mathbf{z}^*$. The last condition says that the zeros of the denominator are exactly the poles of F: there are no extraneous factors in the denominator vanishing at $\mathbf{z}^*$ and cancelling a similar divisor in the numerator.

It may appear that some generality has been lost in imposing the second condition, since we are assuming that each sheet of $\mathcal{V}$ projects diffeomorphically onto any coordinate hyperplane. This latter property is in fact guaranteed by the non-vanishing of the pure z_i^{n+1} terms of W.

To compute ϕ directly from G and H, differentiate (2.1) $n+1$ times in the $z := z_{d+1}$ coordinate at the point $\mathbf{z}^*$ to write

$$\left(\frac{\partial}{\partial z}\right)^{n+1} H(\mathbf{z}^*) = (n+1)!\, \chi(\mathbf{z}^*)\,.$$

We may then write

$$\frac{\phi}{\prod_{j=0}^n (1 - z v_j \widehat{\mathbf{z}})} = \frac{G}{H} = \frac{G}{\chi \prod_{j=0}^n (-u_j) \prod_{j=0}^n (1 - z v_j(\widehat{\mathbf{z}}))}$$

and solve for ϕ at $\mathbf{z} = \mathbf{z}^*$ to obtain

$$\phi(\mathbf{z}^*) = \frac{(n+1)!}{(-(z^*)_{d+1})^{n+1}} \frac{G(\mathbf{z}^*)}{\left(\frac{\partial}{\partial z_{d+1}}\right)^{n+1} H(\mathbf{z}^*)}\,. \tag{2.3}$$

□

For the remaining definitions, fix a strictly minimal element $\mathbf{z}^* \in \mathcal{V}$ which is a multiple point, and let $v_0, \ldots, v_n$ be as above.

Definition 2.2 (Cone of directions corresponding to multiple point)**.** For each sheet $\mathcal{V}_j$ and multiple point $\mathbf{z}^*$, let $\mathbf{dir}_j(\mathbf{z}^*)$ be the vector defined by

$$\mathbf{dir}_j(\mathbf{z}^*) := \left(\frac{z_1}{z_{d+1}} \frac{\partial v_j}{\partial z_1}, \ldots, \frac{z_d}{z_{d+1}} \frac{\partial v_j}{\partial z_d}, 1\right)\Bigg|_{\mathbf{z}=\mathbf{z}^*}.$$

We denote by $\mathbf{K}(\mathbf{z}^*)$ the positive hull of all the $\mathbf{dir}_j(\mathbf{z}^*)$ and by $\mathbf{K}_0(\mathbf{z}^*)$ their convex hull, in other words the intersection of $\mathbf{K}(\mathbf{z}^*)$ with the hyperplane $z_{d+1} = 1$. Geometrically, $\mathbf{K}$ is precisely the collection of outward normal vectors to support hyperplanes of the logarithmic domain of convergence of F at the point $(\log |z_1^*|, \ldots, \log |z_{d+1}^*|)$; see [PW02b] for details.

Let $\mathbf{C}(\mathbf{z}^*)$ be the matrix whose jth row is $\mathbf{dir}_j(\mathbf{z}^*)$. We say that $\mathbf{z}^*$ is **nondegenerate** if the rank of $\mathbf{C}$ is $d+1$, **transverse** if the rank is $n+1$, and **completely nondegenerate** if it is both transverse and nondegenerate; in this case necessarily $n = d$ and the multiplicity of each sheet is 1.

If K is a subset of $\mathbf{K}$ or $\mathbf{K}_0$ consisting of vectors whose directions are bounded away from the walls (equivalently the image of K under the natural map $\bar{}: \mathbb{R}^{d+1} \setminus \{\mathbf{0}\} \to \mathbb{RP}^d$ is a compact subset of the interior of $\overline{\mathbf{K}}$) then we shall say that K is **boundedly interior** to $\mathbf{K}$ (or $\mathbf{K}_0$). □

Remark. The importance of $\mathbf{K}$ is that analysis near $\mathbf{z}^*$ will yield asymptotics for $\mathbf{r}$ in precisely the directions in $\mathbf{K}(\mathbf{z}^*)$. In the smooth case, $n = 0$ and so $\mathbf{K}(\mathbf{z}^*)$ reduces to a single ray. The point $\mathbf{z}^*$ is transverse if and only if the normals there to the surfaces $\mathcal{V}_j$ span a space of dimension $n+1$. Alternatively, the $n+1$ tangent hyperplanes intersect transversally. When $n > d$, transversality must be violated; nondegeneracy means that there is as little violation as possible. □

3. Main theorems and illustrative examples

All of the results in this section are ultimately proved by reducing the problem to the computation of asymptotics for a Fourier-Laplace integral (the proofs are presented in later sections). Owing to the large number of possibilities arising in this analysis, constructing a complete taxonomy of cases is rather challenging, and the number of potential theorems is enormous.

We have chosen to present a series of theorems of varying complexity and generality. Taken together, they completely describe asymptotics associated with nondegenerate multiple points. Our analysis of other types of multiple points requires additional (mild) hypotheses, which will almost always be satisfied in applications. The most important case is that of transversal points, but we also treat various types of tangencies and degeneracies. However, it is always possible that a practical problem involving a meromorphic generating function may not fit neatly into our classification scheme. We hope to convince the reader that in such a situation our basic method will yield Fourier-Laplace integrals from which asymptotics can almost certainly be extracted in a systematic way.

In two variables, our results specialize to the following cases. If $\mathcal{V}$ is locally the union of $n+1$ analytic graphs, then either at least two tangents are distinct, in which case Theorem 3.6 applies, or all tangents coincide. This latter case is more complicated and we require some extra hypotheses; see Theorem 3.11.

We begin with the simplest case, in which the result admits a relatively self-contained statement, with as little extra notation as possible.

Theorem 3.1 (2 curves meeting transversally in 2-space). *Let F be a meromorphic function of two variables, not singular at the origin, with $F(z,w) = G(z,w)/H(z,w) = \sum_{r,s} a_{rs} z^r w^s$.*

Suppose that (z^*, w^*) is a strictly minimal, double point of $\mathcal{V}$. Let $\mathbf{H}(z^*, w^*)$ denote the Hessian of H at (z^*, w^*) and suppose that $\det \mathbf{H}(z^*, w^*) \neq 0$.

Then

(i) *for each boundedly interior subset K of $\mathbf{K}(z^*, w^*)$, there is $c > 0$ such that*

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left(\frac{G(z^*,w^*)}{\sqrt{-(z^*)^2(w^*)^2 \det \mathbf{H}(z^*,w^*)}} + O(e^{-c|(r,s)|})\right) \qquad \textit{uniformly for } (r,s) \in K.$$

(ii) *if $\delta = r/s$ lies on the boundary of $\overline{\mathbf{K}(z^*, w^*)}$, then in direction δ there is a complete asymptotic expansion*

$$a_{rs} \sim (z^*)^{-r}(w^*)^{-s} \sum_{k \geq 0} b_k s^{-(k+1)/2}$$

where $b_0 = \frac{G(z^*,w^*)}{2\sqrt{-(z^*)^2(w^*)^2 \det \mathbf{H}(z^*,w^*)}}$.

The geometric significance of the nonsingularity of $\mathbf{H}(z^*, w^*)$ is that this is equivalent to the curves $\mathcal{V}_0, \mathcal{V}_1$ intersecting transversally. If, on the other hand, $\det \mathbf{H}(z^*, w^*) = 0$ (equivalently the curves are tangent), then higher order information is required in order to compute the relevant asymptotics. We treat the latter situation in Theorem 3.11.

Note that in the situation of the above theorem, the asymptotic exponential rate of a_{rs} is constant on the interior of the cone $\mathbf{K}(z^*, w^*)$, that is, $a_{rs} \sim \exp(-(r,s) \cdot \mathbf{v})$ where $\mathbf{v} = (\log z^*, \log w^*)$ is constant on $\mathbf{K}(z^*, w^*)$; this differs considerably from the asymptotics previously derived for smooth points [PW02b].

Example 3.2 (a lemniscate). Consider $F = 1/H$, where $H(z,w) = 19 - 20z - 20w + 5z^2 + 14zw + 5w^2 - 2z^2w - 2zw^2 + z^2w^2$. The real points of $\mathcal{V}$ are shown in Figure 4.

The only common zero of H and ∇H is $(1,1)$, and so this is the only candidate for a double point. All other strictly minimal elements of $\mathcal{V}$ must be smooth. The second order part of the Taylor expansion of H near $(1,1)$ is $4(w-1)^2 + 10(z-1)(w-1) + 4(z-1)^2 = 4[w - (3-2z)][w - (3/2 - z/2)]$. The Hessian determinant at $(1,1)$ is therefore -36. The two tangent lines have slopes $-1/2$ and -2, so that asymptotics in directions $r/s = \kappa \in [1/2, 2]$ cannot be obtained by smooth points. By Theorem 3.1, for slopes in the interior of this interval, the asymptotic $a_{rs} \sim 1/6$ holds, whereas on the boundary of the cone (slopes $1/2$ or 2), $a_{rs} \sim 1/12$. Directions corresponding to slopes outside the interval $[1/2, 2]$ may be treated by the methods of [PW02b]. The exponential order is nonzero in these directions.

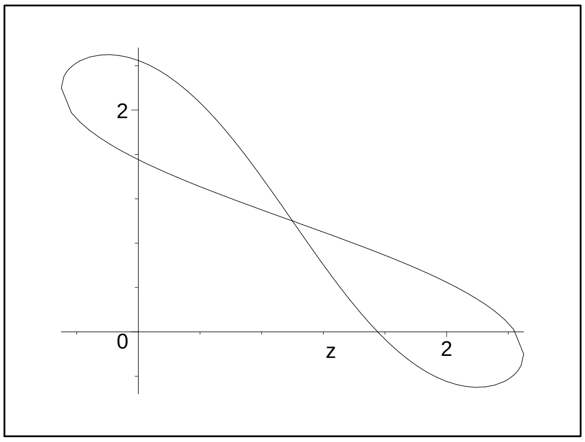

FIGURE 4. Figure-eight shape

This example demonstrates that the local behavior of H at the strictly minimal point $(1,1)$ determines asymptotics. In fact $H(1+z, 1+w)$ is homogeneous and irreducible in $\mathbb{C}[z,w]$ and hence does not factor globally (in the power series ring $\mathbb{C}[[z,w]]$).

□

The previous theorem treated the simplest case, when $d = n = 1$, and our subsequent results will be labelled in a similar way. We begin with some theorems involving nondegenerate points. The simplest case is when the point is completely nondegenerate (a generalization of the hypothesis of Theorem 3.1).

Theorem 3.3 ($d = n$; completely nondegenerate). *Let F be a function of $d+1$ variables with $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}^*$ is a strictly minimal, completely nondegenerate multiple point of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z}^*)$.*

Then there is C such that for each boundedly interior subset K of $\mathbf{K}(\mathbf{z}^*)$, there is $c > 0$ satisfying

$$a_{\mathbf{r}} \sim {\mathbf{z}^*}^{-\mathbf{r}} \left(C + O(e^{-c|\mathbf{r}|}) \right) \text{ uniformly for } \mathbf{r} \in K.$$

The constant C is given by

$$C = \frac{d!\, \phi(\mathbf{z}^*)}{|\det \mathbf{C}(\mathbf{z}^*)|}.$$

Remark. The picture to keep in mind here is of $d+1$ complex hypersurfaces in $\mathbb{C}^{d+1}$ intersecting at a single point. Notice that we again obtain asymptotic constancy on the interior of the cone of allowable directions, provided $G(\mathbf{z}^*) \neq 0$. If $\mathbf{r}$ belongs to the boundary of $\mathbf{K}$, many different asymptotics are possible. □

In case $d < n$, a modification of the last result is required, involving some linear algebra which we now introduce. Let $\Delta = \Delta_n$ denote the standard n-simplex in $\mathbb{R}^{n+1}$,

$$\Delta := \left\{ \boldsymbol{\alpha} \in \mathbb{R}^{n+1} \;\middle|\; \sum_{j=0}^{n} \alpha_j = 1 \text{ and } \alpha_j \geq 0 \text{ for all } j \right\}.$$

The subspace $\mathbf{1}^\perp$ parallel to Δ will be denoted by $\mathcal{H}$. Assuming nondegeneracy of a minimal point $\mathbf{z}^*$, the rows of $\mathbf{C}(\mathbf{z}^*)$ are $n+1$ vectors spanning $\mathbb{R}^{d+1}$, whence for any $\boldsymbol{\delta} \in \mathbb{R}^{d+1}$ the space of solutions $\mathcal{A} := \mathcal{A}(\boldsymbol{\delta}) := \{\boldsymbol{\alpha} \mid \boldsymbol{\alpha}\mathbf{C}(\mathbf{z}^*) = \boldsymbol{\delta}\}$ will always be $(n-d)$-dimensional. The set $\mathcal{A} \cap \Delta$

is just the set of coefficients of convex combinations of rows of $\mathbf{C}$ that yield the given vector $\boldsymbol{\delta}$. If $\boldsymbol{\delta} \in \mathbf{K}_0$ then $\mathcal{A}$ is a subset of the translate of $\mathcal{H}$ containing Δ, thus if $\boldsymbol{\delta}$ is in the interior of $\mathbf{K}_0$ then $\mathcal{A} \cap \Delta$ is $(n-d)$-dimensional. Let $\mathcal{A}^\perp$ denote the orthogonal complement in $\mathcal{H}$ of $\mathcal{A}_0$, the set $\mathcal{A}$ translated to the origin. The dimension (later denoted ρ) of $\mathcal{A}^\perp$ is d. The space $\mathcal{A}^\perp$ does not depend on $\boldsymbol{\delta}$, since changing $\boldsymbol{\delta}$ only translates $\mathcal{A}$.

The role that $\mathcal{A}$ will play is this: there will be an integral over Δ whose stationary points are essentially the points of $\mathcal{A}$, each contributing the same amount; thus the total contribution will be the measure of $\mathcal{A}$ in the *complementary measure* which we denote by σ. The proper normalization in the definition of σ is

$$\sigma(S) := \sigma(S \cap \mathcal{A} \cap \Delta) := \frac{\mu_{n-d}(S \cap \mathcal{A} \cap \Delta)}{\mu_n(\Delta)}$$

where μ_k denotes k-dimensional volume in $\mathbb{R}^{n+1}$. One may think of this as: $\sigma \times \mu_d =$ the normalized volume measure, μ, on Δ (*Warning*: when $\mathcal{A}$ is a single point, it is tempting to assume that $\sigma(\mathcal{A}) = 1$, but in fact in this case $\sigma(\mathcal{A}) = \mu_n(\Delta)^{-1}$.)

Proposition 3.4. *Assume that the columns of* $\mathbf{C}$ *are linearly independent. Then the projection of the column space of* $\mathbf{C}$ *onto* $\mathcal{A}^\perp$ *has dimension* d.

Proof. The matrix $\mathbf{C}$ is the sum of three columnwise projections, namely one onto $\mathcal{A}_0$, one onto $\mathcal{A}^\perp$ and one onto the span of $\mathbf{1}$. The sum of the first two projections annihilates the last column of $\mathbf{C}$, mapping the column space to a space of dimension d. By definition of $\mathcal{A}$, the space $\mathcal{A}\mathbf{C}$ is a single point, hence $\mathcal{A}_0\mathbf{C} = 0$, meaning that the first projection is null. Therefore, the second projection has rank d. □

Definition 3.5. When the columns of $\mathbf{C}$ are linearly independent, let $\overline{\mathbf{C}}$ denote the matrix representing the linear transformation $\mathbf{v} \mapsto \mathbf{v}\mathbf{C}$ on $\mathcal{A}^\perp$ with respect to some basis of $\mathcal{A}^\perp$. Note that $\overline{\mathbf{C}}$ is independent of $\boldsymbol{\delta}$ since $\mathcal{A}^\perp$ is, and that $\det \overline{\mathbf{C}}$ is independent of the choice of basis of $\mathcal{A}^\perp$. For general $\mathbf{C}$, we extend this definition so that $\overline{\mathbf{C}}$ is the projection of the column space of $\mathbf{C}$ onto the space $\mathcal{A}^\perp$, then represented in a (fixed but arbitrary) orthonormal basis of $\mathcal{A}^\perp$. □

In stating subsequent results, we shall rely increasingly on derived data such as $\overline{\mathbf{C}}$. It is possible in principle to give formulae for these asymptotics in terms of the original data G and H, but such expressions rapidly become too cumbersome to be useful.

Theorem 3.6 ($\rho = d \le n$; nondegenerate)**.** *Let* F *be a function of* $d+1$ *variables, with* $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. *Suppose that* $\mathbf{z}^*$ *is a strictly minimal, nondegenerate multiple point of degree* $n+1$ *of* $\mathcal{V}$ *and that* F *is meromorphic in a neighborhood of* $\mathbf{T}(\mathbf{z}^*)$.

Then $\mathbf{K}(\mathbf{z}^*)$ *is a finite union of cones* $\mathbf{K}_j$ *such that for each boundedly interior subset* K *of each* $\mathbf{K}_j$, *there are* $c > 0$ *and a polynomial* P *of degree at most* $n-d$, *such that*

$$a_{\mathbf{r}} = \mathbf{z}^{*-\mathbf{r}} \left(P(\mathbf{r}) + O(e^{-c|\mathbf{r}|}) \right) \qquad \textit{uniformly for } \mathbf{r} \in K.$$

Here

$$P(\mathbf{r}) = \frac{\phi(\mathbf{z}^*)\sigma(\mathcal{A}(\boldsymbol{\delta}) \cap \Delta)}{|\det \overline{\mathbf{C}(\mathbf{z}^*)}|} (r_{d+1})^{n-d} + O((r_{d+1})^{n-d-1}),$$

where $\mathcal{A}(\boldsymbol{\delta})$ *is the solution set to* $\boldsymbol{\alpha}\mathbf{C}(\mathbf{z}^*) = \boldsymbol{\delta}$ *with* $\boldsymbol{\delta} = \mathbf{r}/r_{d+1} \in \mathbf{K}_0(\mathbf{z}^*)$, *and* σ *is the complementary measure.*

Remark. The approximation to $a_{\mathbf{r}}$ is actually piecewise polynomial and is asymptotically valid throughout the interior of $\mathbf{K}$. The correction term may, however, fail to be exponentially small on some lower-dimensional surfaces in the interior of $\mathbf{K}$ where the piecewise polynomial is pieced together.

We have broken the coordinate symmetry in the formula for the leading term by parametrizing $\mathbf{r}$ in terms of r_{d+1} and $\boldsymbol{\delta}$. See section 5 for more comments. $\square$

Example 3.7 (3 curves in 2-space)**.** The simplest interesting case to which Theorem 3.6 applies is that where $\mathcal{V}$ is the union of 3 curves in 2-space which intersect at a strictly minimal point. Suppose that the strictly minimal singularity in question is at $(1,1)$. In this case since $d=1$ and $n=2$, Theorem 3.6 shows that

$$a_{rs} \sim P(r,s)$$

for some piecewise polynomial P of degree at most 1, whenever $\delta := r/s$ lies in the interior of the interval formed by the slopes $c_j = v_j'(1)$. We shall obtain a more explicit expression for P.

Without loss of generality, we suppose that $0 < c_0 \le c_1 \le c_2$, at least one of the two inequalities being strict. Assume for now that both inequalities are strict: $c_0 < c_1 < c_2$. Let $\mathcal{A}_\delta = \{(\alpha_0,\alpha_1,\alpha_2) \in \Delta \mid \alpha_0 c_0 + \alpha_1 c_1 + \alpha_2 c_2 = \delta\}$. When δ belongs to the convex hull of c_0, c_1 and c_2, the affine set $\mathcal{A}_\delta$ is a line segment whose endpoints are on the boundary of the simplex Δ. Since $\mathcal{A}$ is orthogonal to $\mathbf{c}$, the set $\mathcal{A}^\perp$ is parallel to the projection $\bar{\mathbf{c}}$ of $\mathbf{c}$ onto $\mathcal{H}$. Letting μ denote the mean of the c_j, $\sigma/\sqrt{3}$ the standard deviation, we see that the projection is $\mathbf{c} - \mu\mathbf{1}$ and its euclidean length is Σ.

If $c_0 < \delta < c_1$, then one endpoint of the line segment is on the face $\alpha_2 = 0$, with $\alpha_0 = \frac{c_1-\delta}{c_1-c_0}, \alpha_1 = \frac{\delta-c_0}{c_1-c_0}$. The other endpoint is when $\alpha_1 = 0$, with $\alpha_0 = \frac{c_2-\delta}{c_2-c_0}, \alpha_2 = \frac{\delta-c_0}{c_2-c_0}$. The squared euclidean length of this line segment simplifies to

$$3\frac{(\delta-c_0)^2\Sigma^2}{(c_1-c_0)^2(c_2-c_0)^2}.$$

A similar argument gives the answer when $c_1 < \delta < c_2$. The squared euclidean length of the line segment is then

$$3\frac{(\delta-c_2)^2\Sigma^2}{(c_2-c_1)^2(c_2-c_0)^2}.$$

Both answers agree, and by continuity give the correct answer, when $\delta = c_1$. Since the area of Δ_2 is $\sqrt{3}/2$, the complementary measure σ of $\mathcal{A}_\delta$ is $2/\sqrt{3}$ times its euclidean length. Thus we obtain $P(r,s) = sf(r/s)$, where

$$f(\delta) = \begin{cases} 2\frac{\delta-c_0}{(c_1-c_0)(c_2-c_0)}, & \text{if } c_0 \le \delta \le c_1; \\ 2\frac{c_2-\delta}{(c_2-c_1)(c_2-c_0)}, & \text{if } c_1 \le \delta \le c_2. \end{cases}$$

Thus

$$P(r,s) = \begin{cases} 2\frac{r-c_0 s}{(c_1-c_0)(c_2-c_0)}, & \text{if } c_0 \le r/s \le c_1; \\ 2\frac{c_2 s-r}{(c_2-c_1)(c_2-c_0)}, & \text{if } c_1 \le r/s \le c_2. \end{cases} \tag{3.1}$$

Note that $f=0$ on the boundary of the cone, namely when $\delta = c_0$ or $\delta = c_2$. The above formula extends to the case $c_0 = c_1$ or $c_1 = c_2$ in the obvious way. This example illustrates the piecewise polynomial nature of the asymptotics in $\mathbf{K}$. The approximation $a_{rs} \sim P(r,s)$ is exponentially close for δ in compact subintervals of $(c_0,c_1) \cup (c_1,c_2)$, but requires polynomial correction when $\delta \to c_1$ as well as at the endpoints c_0 and c_2. $\square$

More general formulae for P in this case are described in the forthcoming work [BP03].

We move on to investigate situations which are degenerate according to our terminology. We begin with the simplest case, namely when the minimal point is transverse. Note that in this case, the set $\mathcal{A}$ of linear combinations of the rows of $\mathbf{C}$ that yield $\boldsymbol{\delta}$ is always a single point since the rows of $\mathbf{C}$ are linearly independent.

Definition 3.8. If $\mathbf{z}^*$ is a strictly minimal multiple point and $\boldsymbol{\alpha}$ is an element of Δ, we define the matrix $Q = Q(\mathbf{z}^*, \boldsymbol{\alpha})$ to be the Hessian matrix of the function

$$\widehat{\boldsymbol{\theta}} \mapsto -\log \boldsymbol{\alpha}\mathbf{v}(\widehat{\mathbf{z}^* e^{i\boldsymbol{\theta}}}) .$$

at the point $\widehat{\boldsymbol{\theta}} = \mathbf{0}$. We define the matrix M by

$$M := M(\mathbf{z}^*, \boldsymbol{\alpha}) := \begin{pmatrix} 0 & -i\overline{\mathbf{C}}(\mathbf{z}^*) \\ -i\overline{\mathbf{C}}^T(\mathbf{z}^*) & Q(\mathbf{z}^*, \boldsymbol{\alpha}) \end{pmatrix}. \tag{3.2}$$

□

Theorem 3.9 ($\rho = n \le d$; transverse). *Let F be a function of $d+1$ variables, with $F(\mathbf{z}) = G(\mathbf{z})/H(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}^*$ is a strictly minimal, transverse multiple point of degree $n+1$ of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z}^*)$.*

Let K be a boundedly interior subset of $\mathbf{K}_0(\mathbf{z}^*)$ such that $\det M(\mathbf{z}^*, \boldsymbol{\alpha}(\boldsymbol{\delta})) \ne 0$ for $\boldsymbol{\delta} \in K$, where $\boldsymbol{\alpha}(\boldsymbol{\delta})$ is the unique point in $\mathcal{A}(\boldsymbol{\delta})$. Then there is a complete asymptotic expansion

$$a_{\mathbf{r}} \sim \mathbf{z}^{*-\mathbf{r}} \sum_{k \ge 0} b_k (r_{d+1})^{\frac{n-d}{2} - k} \quad \textit{uniformly for } \boldsymbol{\delta} \in K.$$

If $G(\mathbf{z}^*) \ne 0$, the leading coefficient is given by

$$b_0 = \frac{n!}{\sqrt{n+1}} \frac{(2\pi)^{\frac{n-d}{2}} \phi(\mathbf{z}^*)}{\det M(\mathbf{z}^*, \boldsymbol{\alpha}(\boldsymbol{\delta}))^{1/2}}$$

where the square root is the product of principal square roots of the eigenvalues.

Example 3.10 (2 planes in 3-space). Consider the trivariate sequence $(a_{r,s,t})$ whose terms are zero if any index is negative, and is otherwise given by the boundary condition $a_{0,0,0} = 1$ and the recurrence

$$\begin{aligned} 16 a_{r,s,t} &= 12 a_{r-1,s,t} + 12 a_{r,s-1,t} + 8 a_{r,s,t-1} - 5 a_{r-1,s-1,t} - 3 a_{r-1,s,t-1} - 3 a_{r,s-1,t-1} \\ &\quad - 2 a_{r-2,s,t} - 2 a_{r,s-2,t} - a_{r,s,t-2}. \end{aligned}$$

Let $F(x,y,z) = \sum_{r,s,t} a_{r,s,t} x^r y^s z^t$ be the associated generating function. Then we obtain directly the explicit form

$$F(x,y,z) = \frac{16}{(4-2x-y-z)(4-x-2y-z)}.$$

The point $(1,1,1)$ is a multiple point of $\mathcal{V}$ which is readily seen to be strictly minimal. In fact this point is on the line segment of multiple points $\{(1,1,1) + \lambda(-1,-1,3) : -1/3 < \lambda < 1\}$. We focus here on the point $(1,1,1)$ in particular because it is the only point giving asymptotics which do not decay exponentially. Near $(1,1,1)$, the set $\mathcal{V}$ is parametrized by $z = 4 - 2x - y$ and $z = 4 - x - 2y$. The cone $\mathbf{K}$ is the positive hull of $(2,1,1)$ and $(1,2,1)$, with $\mathbf{K}_0$ being the line segment between these. Other multiple points on the line segment govern other two dimensional cones. For example, the point $(1/3, 1/3, 3)$ corresponding to $\lambda = 2/3$ governs the cone $\mathbf{K}_{2/3}$ which is positive hull of $(2/3, 1/3, 3)$ and $(1/3, 2/3, 3)$, or in other words, $\mathbf{K}_{2/3}$ is the set of (r,s,t) with $r + s = t/3$ and $r, s \ge t/9$. These other cones sweep out, as λ varies, a polyhedral cone with non-empty interior. There are still directions outside the cone, namely directions (r,s,t) with $\min\{r,s\} \le (r+s)/3$; asymptotics in these directions are governed by smooth points on one of the two hyperplanes. We return to the examination of the non-exponentially decaying asymptotic directions.

The multiple point $(1,1,1)$ yields asymptotics for a_{rst} with $r+s=3t$ and $r,s\geq t$. Given $\boldsymbol{\delta}=(2-\alpha,1+\alpha,1)\in\mathbf{K}_0$, the set $\mathcal{A}$ is the single point $(1-\alpha,\alpha)$. The complement $\mathcal{A}^\perp$ is always all of $\mathcal{H}$, which has an orthonormal basis $\{\mathbf{x}\}$, $\mathbf{x}:=(\sqrt{1/2},-\sqrt{1/2})$.

The matrix $\overline{\mathbf{C}}$ is equal to $((2,1)\cdot\mathbf{x},(1,2)\cdot\mathbf{x})=\mathbf{x}$. This leads to $\det M=(A+B+C+D)/2$, where A,B,C and D are the entries of the restricted Hessian, Q. A routine computation shows that $A+B+C+D=12$, leading to $\det M=6$ (as is always the case for transverse intersections of the pole set, the determinant of M does not vary with direction inside the cone of a fixed multiple point). Computing $\phi(\mathbf{1})$ from (2.3) then gives

$$\phi(\mathbf{1})=\frac{2!}{(-1)^2}\frac{16}{2}=16\,.$$

Plugging this into Theorem 3.9 we find (to first order, as $t\to\infty$) that

$$a_{r,s,t}\sim\frac{16}{\sqrt{24\pi t}}\qquad\text{if } r+s=3t \text{ and } r/s\in(1/2,2).$$

The above first order approximation differs from the true value of a_{rst} by less than 0.3% when $(r,s,t)=(90,90,60)$. □

As mentioned in the introduction, instead of producing enough variants of these theorems for a complete taxonomy, we will stop with just one more result. The case we describe is the most degenerate, namely where all the sheets $\mathcal{V}_j$ are tangent at $\mathbf{z}^*$, so that $\mathbf{K}(\mathbf{z}^*)$ is a single ray. In this case $\det M$ varies with $\boldsymbol{\alpha}$ and the resulting formula is an integral over $\boldsymbol{\alpha}$. The methods used here may be adapted to prove results when the degeneracy is somewhere between this extreme and the nondegenerate cases.

Theorem 3.11 ($\rho=0$). *Let F be a function of $d+1$ variables, not singular at the origin, with $F(\mathbf{z})=G(\mathbf{z})/H(\mathbf{z})=\sum_{\mathbf{r}}a_{\mathbf{r}}\mathbf{z}^{\mathbf{r}}$. Suppose that $\mathbf{z}^*$ is a strictly minimal, multiple point of degree $n+1$ of $\mathcal{V}$ and that F is meromorphic in a neighborhood of $\mathbf{T}(\mathbf{z}^*)$. Further suppose that $\mathbf{K}(\mathbf{z}^*)$ is a single ray.*

If $\det Q(\mathbf{z}^*,\boldsymbol{\alpha})\neq 0$ on Δ, then for $\mathbf{r}\in\mathbf{K}(\mathbf{z}^*)$ we have

$$a_{\mathbf{r}}={\mathbf{z}^*}^{-\mathbf{r}}[b_0(r_{d+1})^{n-d/2}+O(r_{d+1})^{n-d/2-1}]\,.$$

The value of b_0 is given by

$$b_0=\frac{\phi(\mathbf{z}^*)}{(2\pi)^{d/2}}\int_\Delta\det Q(\mathbf{z}^*,\boldsymbol{\alpha})^{-1/2}\,d\mu(\boldsymbol{\alpha}).$$

Remark. Theorem 3.11 covers the situation where all the sheets coincide, as opposed to merely being tangent. That situation can also be analyzed in a different way by a slight modification of the proof of the smooth case, as discussed in [PW02b]. The methods of [PW02b] yield a complete asymptotic expansion. □

Example 3.12. The simplest case illustrating Theorem 3.11 is that of two curves $w=u_j(z)$ in $\mathbb{C}^2$, intersecting tangentially at the strictly minimal point $\mathbf{z}=(1,1)$. Suppose that $\operatorname{Re}\log v_j(e^{i\theta})=-d_j\theta^2+\cdots$ with $d_j>0$. The simplex in question is one-dimensional and we identify it with the interval $0\leq t\leq 1$. Then for $0\leq t\leq 1$ the matrix $M(\mathbf{z},t)$ is just the 1×1 matrix with entry $d_t=(1-t)d_0+td_1$. Hence when $d_0\neq d_1$ we obtain asymptotics in the unique direction $\boldsymbol{\delta}$ of the theorem:

$$
\begin{aligned}
a_{rs} &\sim \frac{\phi(1,1)\sqrt{s}}{\sqrt{2\pi}} \int_0^1 (d_t)^{-1/2}\, dt \\
&= \frac{\phi(1,1)\sqrt{s}}{\sqrt{2\pi}} \frac{1}{d_1 - d_0} \int_{d_0}^{d_1} y^{-1/2}\, dy \\
&= \frac{2\phi(1,1)\sqrt{s}}{\sqrt{2\pi}} \frac{\sqrt{d_1} - \sqrt{d_0}}{d_1 - d_0} \\
&= \frac{\phi(1,1)\sqrt{s}}{\sqrt{2\pi}} \frac{2}{\sqrt{d_0} + \sqrt{d_1}}.
\end{aligned}
$$

This final formula is easily seen to hold also in the case $d_0 = d_1$, in which case the formula $\phi(1,1)\sqrt{s}/\sqrt{2\pi d_0}$ agrees with the formula in [PW02b]. □

Remark. Since analysis in 1 variable is considerably easier than in general, it is possible to derive a result for plane curves even when d_j vanishes. We omit the details here, but note that such a derivation is possible because (by minimality) we must have $\log wv_j(ze^{i\theta}) = ic_j\theta - d\theta^m + \cdots$, where $c_j \geq 0$, m is even and $\operatorname{Re} d > 0$.

If all tangents are equal at (z^*, w^*), then $\mathbf{K}_0(z^*, w^*)$ is a singleton. If all $\operatorname{Re}\log wv_j(ze^{i\theta})$ vanish to the same exact order m and $(r,s) \in \mathbf{K}(z^*, w^*)$, then

$$
a_{rs} = (z^*)^{-r}(w^*)^{-s}[b_0 s^{n-1/m} + O(s^{n-2/m})]
$$

with b_0 given by

$$
b_0 = \frac{\phi(z^*, w^*)\Gamma(1/m)}{2\pi(1 - 1/m)} \frac{d_1^{1-1/m} - d_0^{1-1/m}}{d_1 - d_0}.
$$

The above formula for b_0 also holds in the limit when $d_0 = d_1$, when it becomes $b_0 = \dfrac{\phi(z,w)\Gamma(1/m)}{2\pi(d_0)^{1/m}}$.
□

4. Proofs

Throughout, we assume that $\mathbf{z}^*$ is a fixed strictly minimal multiple point of $\mathcal{V}$, of multiplicity $n+1$, and $F(\mathbf{z}) = \phi(\mathbf{z})/\prod_{j=0}^n (1 - zv_j(\widehat{\mathbf{z}}))$ near $\mathbf{z}^*$ as in Definition 2.1. Recall our notational conventions $s := r_{d+1}, z := z_{d+1}$. To ease notation we shall later assume that $\mathbf{z}^* = \mathbf{1}$.

The proofs follow a 4-step process. First we use the residue theorem in one variable to reduce the dimension by 1 and to restrict attention to a neighbourhood of $\widehat{\mathbf{z}^*}$. Second, the residue sum resulting is rewritten in a form more amenable to analysis. Third, the Cauchy-type integral is recast in the Fourier-Laplace framework by the usual substitution $\widehat{\mathbf{z}} = \widehat{\mathbf{z}^*} e^{i\boldsymbol{\theta}}$. Finally, detailed analysis of these Fourier-Laplace integrals (including some generalizations of known results we have included separately in [PW02a] to save space here) yields our desired results.

Restriction to a neighborhood of $\mathbf{z}^*$. We first apply the residue theorem in one variable: the proof of the following proposition is entirely analogous to that of [PW02b, Lemma 4.1].

Proposition 4.1 (Local residue formula). *Let $R(s, \widehat{\mathbf{z}}, \varepsilon)$ denote the (finite) sum of the residues of $g : z \mapsto z^{-s-1} F(\widehat{\mathbf{z}}, z)$ at $z = u_j(\widehat{\mathbf{z}})$ inside the ball $|z - z^*| < \varepsilon$.*

Let $\mathcal{N}'$ be a neighborhood of $\widehat{\mathbf{z}^*}$ in $\mathbf{T}(\widehat{\mathbf{z}^*})$. Then for sufficiently small $\varepsilon > 0$,

$$
(4.1) \quad |\mathbf{z}^{\mathbf{r}}| \left| a_{\mathbf{r}} - (2\pi i)^{-d} \int_{\mathcal{N}'} -\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}} - \mathbf{1}} R(s, \widehat{\mathbf{z}}, \varepsilon)\, d\widehat{\mathbf{z}} \right| \text{ is exponentially decreasing in } s \text{ as } s \to \infty.
$$

This is uniform in $\mathbf{r}$ as $\mathbf{r}/s$ varies over some neighborhood of $\mathbf{K}_0(\mathbf{z}^*)$. □

Rewriting the residue sum. After using the local residue formula (4.1) we must perform a d-dimensional integration of the residue sum R. The following lemma, whose proof may be found in [DL93, p. 121, Eqs 7.7 & 7.12], will yield a more tractable form for R. Recall from Section 3 the relevant facts about simplices. In the following, the notation $\boldsymbol{\alpha}\mathbf{v}$ denotes the scalar product $\sum_{j=0}^{n} \alpha_j v_j$, and $h^{(n)}$ the nth derivative of h.

Lemma 4.2. *Let h be a function of one complex variable, analytic at 0, and let μ be the normalized volume measure on Δ_n. Then*

$$\sum_{j=0}^{n} \frac{h(v_j)}{\prod_{r \neq j}(v_j - v_r)} = \int_{\Delta_n} h^{(n)}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha})$$

both as formal power series in $n+1$ variables $v_0, \ldots, v_n$ and in a neighborhood of the origin in $\mathbb{C}^{n+1}$. □

Corollary 4.3 (Residue sum formula)**.** *Let $R(s, \widehat{\mathbf{z}}, \varepsilon)$ be the sum of the residues of the function $g : z \mapsto z^{-s-1} F(\mathbf{z})$ inside the ball $|z - z^*| < \varepsilon$. Define $h_{s,\widehat{\mathbf{z}}}(y) = y^{s+n}\phi(\widehat{\mathbf{z}}; 1/y)$.*

Then for sufficiently small ε, there is $\delta > 0$ such that for $|\widehat{\mathbf{z}} - \widehat{\mathbf{z}^*}| < \delta$,

$$R(s, \widehat{\mathbf{z}}, \varepsilon) = \int_{\Delta} h_{s,\widehat{\mathbf{z}}}^{(n)}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha}). \tag{4.2}$$

Proof. First suppose that the functions $v_0, \ldots, v_n$ are distinct. Choose ε sufficiently small and $\delta > 0$ such that g has exactly $n+1$ simple poles in $|z - z^*| < \varepsilon$ whenever $|\widehat{\mathbf{z}} - \widehat{\mathbf{z}^*}| < \delta$. Then there are $n+1$ residues, the jth one being

$$\frac{v_j(\widehat{\mathbf{z}})^{s+n}\phi(\widehat{\mathbf{z}}, 1/v_j(\widehat{\mathbf{z}}))}{\prod_{r \neq j}(v_r(\widehat{\mathbf{z}}) - v_j(\widehat{\mathbf{z}}))} = \frac{h_{s,\widehat{\mathbf{z}}}(v_j(\widehat{\mathbf{z}}))}{\prod_{r \neq j}(v_r(\widehat{\mathbf{z}}) - v_j(\widehat{\mathbf{z}}))}.$$

The result in this case now follows by summing over j and applying Lemma 4.2.

In the case when $v_0, \ldots, v_n$ are not distinct, let v_j^t be functions approaching v_j as $t \to 0$, such that v_j^t are distinct for t in a punctured neighborhood of 0, and let $g^t(z) = z^{-s-1}\phi(\mathbf{z}) / \prod_{j=0}^{n}(z - 1/v_j^t(\widehat{\mathbf{z}}))$, so that $g^t \to g$ as well. The sum of the residues of g^t may be computed by integrating g^t around the circle $|z - z^*| = \varepsilon$, and since $g^t \to g$, this sum approaches the sum of the residues of g. Since the expression in (4.2) is continuous in the variables v_j, this proves the general case. □

The residue sum formula and Proposition 4.1 combine to show that for some neighborhood $\mathcal{N}'$ of $\widehat{\mathbf{z}^*}$ in $\mathbf{T}(\mathbf{z}^*)$,

(4.3)

$$|(\mathbf{z}^*)^{\mathbf{r}}| \left| a_{\mathbf{r}} - (2\pi i)^{-d} \int_{\mathcal{N}'} -\widehat{\mathbf{z}}^{-\widehat{\mathbf{r}}-\mathbf{1}} \int_{\Delta} h_{s,\widehat{\mathbf{z}}}^{(n)}(\boldsymbol{\alpha}\mathbf{v})\, d\mu(\boldsymbol{\alpha})\, d\widehat{\mathbf{z}} \right| \quad \text{is exponentially decreasing in } s,$$

where, as usual, we have let s denote r_{d+1}.

Recasting the problem in the Fourier-Laplace framework. In this subsection we assume throughout (purely in order to ease notational complexity) that $\mathbf{z}^* = \mathbf{1}$.

Lemma 4.4 (Reduction to Fourier-Laplace integral)**.** *For $0 \leq k \leq n$, $e^{i\widehat{\boldsymbol{\theta}}} \in \mathcal{N}'$ and $\boldsymbol{\alpha} \in \mathbb{R}^{n+1}$, define*

$$\begin{aligned}
p_k(s) &:= \frac{n!(s+n)!}{k!(n-k)!(s+k)!} \\
f(\widehat{\boldsymbol{\theta}}, \boldsymbol{\alpha}) &:= \frac{i\widehat{\mathbf{r}}}{s} - \log(\boldsymbol{\alpha}\mathbf{v}(e^{i\widehat{\boldsymbol{\theta}}})) \\
\psi_k(\widehat{\boldsymbol{\theta}}, \boldsymbol{\alpha}) &:= \left(\frac{d}{dy}\right)^k \phi(e^{i\widehat{\boldsymbol{\theta}}}, 1/y)\bigg|_{y=\mathbf{v}(e^{i\widehat{\boldsymbol{\theta}}})} \\
I(s; f, \psi_k) &:= \int_{\mathcal{E}} e^{-sf} \psi_k \, d\mu_{n+d}.
\end{aligned}$$

Then there is a neighborhood $\mathcal{N}$ of $\mathbf{0}$ in $\mathbb{R}^d$, a product of compact intervals, such that

$$\left| a_{\mathbf{r}} - (2\pi)^{-d} \frac{n!}{\sqrt{n+1}} \sum_{k=0}^{n} p_k(s) I(s; f, \psi_k) \right| \quad \textit{is exponentially decreasing in } s\,, \tag{4.4}$$

where $\mathcal{E} := \mathcal{N} \times \Delta_n$.

Remark. Note for later that each ψ_k is independent of $\boldsymbol{\alpha}$, as is $f(\mathbf{0}, \boldsymbol{\alpha})$.

Proof. Recall that $\sigma(\mathcal{A}) = n!/\sqrt{n+1} = V_n(\Delta_n)^{-1}$. Applying Leibniz' rule for differentiating h yields

$$\begin{aligned}
h^{(n)}_{s,\widehat{\mathbf{z}}}(y) &= \sum_{k=0}^{n} \binom{n}{k} \left(\frac{d}{dy}\right)^{n-k} y^{s+n} \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{z}}, 1/y) \\
&= \sum_{k=0}^{n} \frac{n!(s+n)!}{k!(n-k)!(s+k)!} y^{s+k} \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{z}}, 1/y) \\
&= y^s \sum_{k=0}^{n} p_k(s) y^k \left(\frac{d}{dy}\right)^k \phi(\widehat{\mathbf{z}}, 1/y)\,.
\end{aligned}$$

Substituting this into (4.3) with $\widehat{\mathbf{z}} = e^{i\widehat{\boldsymbol{\theta}}}$, shrinking $\mathcal{N}'$ if necessary and letting $\mathcal{N}$ be the inverse image under $\widehat{\boldsymbol{\theta}} \mapsto e^{i\widehat{\boldsymbol{\theta}}}$ completes the proof and the third step. □

Analysis of the Fourier-Laplace integral. The proofs of all the theorems in Section 3 proceed in parallel while we compute the asymptotics for (4.4) with the aid of the complex Fourier-Laplace expansion. To that end, observe that $\mathcal{E}$ is a compact product of simplices and intervals of dimension $n+d$ inside $\mathbb{R}^{n+d+1}$, f and ψ_k are complex analytic functions on a neighbourhood of $\mathcal{E}$ in $\mathcal{H} \times \mathbb{R}^d$ (this requires a sufficiently small choice of ε in Proposition 4.1), and $\operatorname{Re} f \geq 0$ on $\mathcal{E}$.

To apply the standard theory, we must locate the critical points of f on $\mathcal{E}$. Recall from Definition 2.2 the matrix $\mathbf{C} = \mathbf{C}(\mathbf{z}^*)$ whose rows are extreme rays in the cone of outward normals to support hyperplanes to the logarithmic domain of convergence of F.

Proposition 4.5 (Critical points of f)**.** *For $\boldsymbol{\delta} := \widehat{\mathbf{r}}/s \in \mathbf{K}_0$, let $\mathcal{A}(\boldsymbol{\delta})$ denote the solution set $\{\boldsymbol{\alpha} \in \mathbb{R}^{n+1} \mid \boldsymbol{\alpha}\mathbf{C} = \boldsymbol{\delta}\}$. Let $\mathcal{S}$ denote the set of critical points of f on $\mathcal{E}$. Then $\mathcal{S}$ consists precisely of those points $(\mathbf{0}, \boldsymbol{\alpha})$ with $\boldsymbol{\alpha} \in \mathcal{A}(\boldsymbol{\delta}) \cap \Delta$.*

Proof. By strict minimality of $\mathbf{z}^*$, for each j the modulus of $v_j(\widehat{\mathbf{z}})$ achieves its maximum only when $\widehat{\mathbf{z}} = \widehat{\mathbf{z}^*}$. Thus any convex combination of $v_j(\widehat{\mathbf{z}})$ with $\widehat{\mathbf{z}} \neq \widehat{\mathbf{z}^*}$ has modulus less than $|v_j(\widehat{\mathbf{z}^*})|$. In other words, when $\widehat{\boldsymbol{\theta}} \neq \mathbf{0}$, the real part of f is strictly positive, from which it follows that all critical points of f are of the form $(\mathbf{0}, \boldsymbol{\alpha})$ for $\boldsymbol{\alpha} \in \Delta$.

In fact f is somewhat degenerate: $f(\mathbf{0}, \boldsymbol{\alpha}) = 0$ for all $\boldsymbol{\alpha} \in \Delta$, so not only does the real part of f vanish when $\widehat{\boldsymbol{\theta}} = \mathbf{0}$, but also the $\boldsymbol{\alpha}$-gradient of f vanishes there. We compute the $\widehat{\boldsymbol{\theta}}$-derivatives at $\widehat{\boldsymbol{\theta}} = \mathbf{0}$ as follows. For $1 \leq j \leq d$,

$$\frac{\partial f}{\partial \theta_j} = i \left(\frac{r_j}{s} - \frac{z_j}{z_{d+1}} \boldsymbol{\alpha} \left(\frac{\partial}{\partial z_j} \right) \mathbf{v}(\widehat{z}) \right) \Big|_{\widehat{\mathbf{z}} = \widehat{\mathbf{z}^*}} .$$

Recalling the definition of $\mathbf{C}$, we see that these vanish simultaneously if and only if $\boldsymbol{\delta} = \boldsymbol{\alpha}\mathbf{C}$, which finishes the proof. $\square$

The set of critical points of f on $\mathcal{E}$ is thus the intersection of f with an affine subspace of $\mathbb{R}^{n+d+1}$. We now wish to use known results on asymptotics of Fourier-Laplace integrals. Note that in the situation of Theorems 3.1, 3.3 and 3.9, f has a unique critical point on $\mathcal{E}$, whereas in Theorem 3.6, $\mathcal{S}$ is essentially an affine subspace of Δ, and in Theorem 3.11 $\mathcal{S}$ is essentially just Δ. Though the existing literature on asymptotics of integrals is extensive, we have been unable to find in it results applicable to all cases of interest of us here. We have therefore derived the extra asymptotics we need in [PW02a] and cite the relevant ones below in Lemmas 4.7 and 4.8.

Recall that $\mathcal{A}^\perp$ is the orthogonal complement to (any and every) $\mathcal{A}(\boldsymbol{\delta})$ in $\mathcal{H}$; for each point $y = (\mathbf{0}, \boldsymbol{\alpha})$ of $\mathcal{S}$, let B_y be the product of $\mathcal{N}$ with the affine subspace of Δ containing $\boldsymbol{\alpha}$ and parallel to $\mathcal{A}^\perp$.

Then the integrals in Lemma 4.4 may be decomposed as

$$\int_{\mathcal{E}} e^{-sf} \psi_k \, d\mu_{n+d} = \int_{\mathcal{S}} \left(\int_{B_y} e^{-sf} \psi_k \, d\mu_{\rho+d} \right) d\sigma(y) , \tag{4.5}$$

where $\rho := \dim \mathcal{A}^\perp$.

We require a more explicit description of the Hessian matrix of f.

Proposition 4.6 (Hessian of f). *Let $y = (\mathbf{0}, \boldsymbol{\alpha}) \in \mathcal{S}$. At the point y, the Hessian of the restriction of f to B_y has the block form*

$$M(\mathbf{z}^*, \boldsymbol{\alpha}) = \begin{pmatrix} 0 & -i\overline{\mathbf{C}(\mathbf{z}^*)} \\ -i\overline{\mathbf{C}(\mathbf{z}^*)}^T & Q(\mathbf{z}^*, \boldsymbol{\alpha}) \end{pmatrix} . \tag{4.6}$$

In this decomposition:

- *the zero block has dimensions $\rho \times \rho$, where $\rho = \dim \mathcal{A}^\perp = \operatorname{rank} \overline{\mathbf{C}}(\mathbf{z}^*)$*
- *the jth column of $\overline{\mathbf{C}}(\mathbf{z}^*)$ is the projection of the jth column of $\mathbf{C}(\mathbf{z}^*)$ onto $\mathcal{A}^\perp$ (expressed in some orthonormal basis)*
- *$Q(\mathbf{z}^*, \boldsymbol{\alpha})$ is the Hessian at y of the restriction of f to the $\widehat{\boldsymbol{\theta}}$-directions, as defined in Definition 3.8.*

Proof. Constancy of f in the Δ directions at $\widehat{\boldsymbol{\theta}} = \mathbf{0}$ shows that the second partials in those directions vanish, giving the upper left block of zeros. Computing $(\partial / \partial \theta_j) f$ up to a constant gives

$$- \frac{i}{\boldsymbol{\alpha}\mathbf{v}(\widehat{\mathbf{z}^*})} \boldsymbol{\alpha} z_j \frac{\partial}{\partial z_j} \mathbf{v}(\widehat{\mathbf{z}^*})$$

and since $\boldsymbol{\alpha}\mathbf{v}(\widehat{\mathbf{z}^*})$ is constant when $\widehat{\boldsymbol{\theta}} = \mathbf{0}$, differentiating in the $\boldsymbol{\alpha}$ directions recovers the blocks $-i\overline{\mathbf{C}}$ and $-i\overline{\mathbf{C}}^T$. The second partials in the $\widehat{\boldsymbol{\theta}}$ directions are of course unchanged. To see that the dimension of $\mathcal{A}^\perp$ is the rank of $\overline{\mathbf{C}}$, observe that no nonzero element of $\mathcal{A}^\perp$ can be orthogonal to column space of $\mathbf{C}$, since then it would have been in $\mathcal{A}$; thus the projection of the columns of $\mathbf{C}$ onto $\mathcal{A}^\perp$ spans $\mathcal{A}^\perp$. $\square$

We may now prove Theorems 3.1, 3.3 and 3.9, in reverse order from most general to most special. In these cases the function f in the representation (4.4) has only one critical point, so the decomposition (4.5) is not really necessary.

As mentioned above, we have had to derive asymptotic expansions for some cases. We start with one of these, applicable to the situations of Theorems 3.1, 3.3, and 3.9.

Lemma 4.7 (Fourier-Laplace expansion for isolated critical point)**.** *Suppose that f has a unique critical point $\mathbf{0} \in \mathcal{E}$, at which the Hessian $\mathbf{H}$ is nonsingular.*

(i) *If $\mathbf{0}$ has a neighborhood in $\mathcal{E}$ diffeomorphic to $\mathbb{R}^d$ then as $s \to \infty$ there is an explicitly computable asymptotic expansion*

$$I(s; f, \psi_k) \sim \sum_{j=0}^{\infty} a_j s^{-d/2-j}.$$

In particular

$$a_0 = \psi_k(\mathbf{0}) \frac{(2\pi)^{d/2}}{\sqrt{\det(\mathbf{H})}}$$

where the square root is defined to be the product of the principal square roots of the eigenvalues of $\mathbf{H}$.

(ii) *If $\mathbf{0}$ has a neighborhood in $\mathcal{E}$ diffeomorphic to a d-dimensional half-space then there is an explicitly computable asymptotic expansion*

$$I(s; f, \psi_k) \sim \sum_{j=0}^{\infty} a_j s^{-d/2-j/2}.$$

Here the value of a_0 is half that of the previous case.

Proof. This is a straightforward corollary of [Hör83, Theorems 7.7.1, 7.7.5, 7.7.17 (iii)]. Details are given in [PW02a].

$\square$

Proof of Theorem 3.9. We prove the result in the case $\mathbf{z}^* = \mathbf{1}$. The result in the general case follows directly via the change of variable $\mathbf{z} \to \mathbf{z}/\mathbf{z}^*$.

The rank of $\mathbf{C}$ is $n+1$, so $\mathcal{A}(\boldsymbol{\delta})$ is a single point and f has a unique critical point, $(\mathbf{0}, \boldsymbol{\alpha})$. Hence $\mathcal{A}^\perp = \mathcal{H}$ and has dimension n. The critical point is in the interior of $\mathcal{E}$ as long as $\boldsymbol{\delta}$ is in the interior of $\mathbf{K}_0$, which occurs if and only if $\mathbf{r}$ is in the interior of $\mathbf{K}$.

Since $\mathcal{A}$ is a singleton, we may use the formula (4.4) and Lemma 4.7. The Fourier-Laplace expansion of $\int_{\mathcal{E}} e^{-sf}\psi_k$ will have terms $s^{-(n+d)/2-j}$ for all j, and when multiplied by the degree $n-k$ polynomial p_k, will contribute to the $s^{(n-d)/2-k-j}$ terms for all j. Each of these terms is explicitly computable from the partial derivatives of G and H via ψ_k, but these rapidly become messy. The leading term of $s^{(n-d)/2}$ comes only from $k=0$:

$$a_{\mathbf{r}} \sim (2\pi)^{-d} \frac{n!}{\sqrt{n+1}} s^n \frac{(2\pi)^{(n+d)/2} s^{(n-d)/2} \phi(\mathbf{z})}{\sqrt{\det M(\mathbf{z}^*; \boldsymbol{\alpha})}}$$

which proves the theorem in the case $\mathbf{z}^* = \mathbf{1}$. $\square$

Proof of Theorem 3.3. Here $\mathrm{rank}(\mathbf{C}) = d+1 = n+1$ and so the proof of Theorem 3.9 applies. We can obtain a more explicit result in this case. The block decomposition of $M(y)$ in (4.6) is into four blocks of dimensions $d \times d$, implying that $M(y)$ is nonsingular with determinant $(\det \overline{\mathbf{C}})^2$ (in performing column operations to bring $M(y)$ to block diagonal form, we pick up a factor of $(-1)^{d^2}$ which cancels the factor i^{2d}). The last column of $\mathbf{C}$ is $\mathbf{1}$, whose projection onto $\mathcal{A}^{\perp}$ is null; its projection onto the space spanned by $\mathbf{1}$ has length $\sqrt{d+1}$, whence $|\det \mathbf{C}| = \sqrt{d+1}\,|\det \overline{\mathbf{C}}|$.

Thus we obtain

$$a_{\mathbf{r}} = C + O(s^{-1})$$

where

$$C = \frac{d!\,\phi(\mathbf{z}^*)}{|\det \mathbf{C}(\mathbf{z}^*)|}\,.$$

To finish the proof, we quote [Pem00, Corollary 3.2] to see that in fact $a_{\mathbf{r}} - C$ is exponentially small on boundedly interior subcones of the interior of $\mathbf{K}$.

Remark. If $\boldsymbol{\delta}$ is on the boundary of $\mathbf{K}$, it may happen that the point $(\mathbf{0}, \boldsymbol{\alpha})$ has $\boldsymbol{\alpha}$ lying on a face of Δ of dimension $d-1$. In that case, using the second formula from Lemma 4.7 instead of the first gives a leading term with exactly half the previous magnitude. Here we do not claim exponential decay of $a_{\mathbf{r}} - C$. In Example 3.2, this occurs when $r/s \in \{1/2, 2\}$.

Proof of Theorem 3.1. The second case is covered by the previous remark.

Details for the first case follow. The analysis of Theorem 3.3 applies. We derive a more explicit result, first supposing that $(z^*, w^*) = (1,1)$.

For $j \in \{0,1\}$, write $v_j(e^{i\theta}) = ic_j\theta + O(\theta^2)$, where $c_j \geq 0$. The matrix $\mathbf{C}$ in this case is simply $\left(\begin{smallmatrix} c_0 & 1 \\ c_1 & 1 \end{smallmatrix}\right)$. It will be seen below that $\det \mathbf{C} = 0$ if and only if $\det \mathbf{H} = 0$, so for the purposes of this theorem we may and shall suppose that $c_0 \neq c_1$.

We have

$$a_{rs} \sim \frac{\phi(1,1)}{|c_0 - c_1|} \qquad \text{uniformly for } r/s \text{ boundedly interior to } [c_0, c_1].$$

This expression for the constant is quite handy, but to recover the statement of the theorem, we also express it in terms of the partial derivatives of H. Recall that $H = \chi Q$ where $Q(z,w) = (1 - wv_0(z))(1 - wv_1(z))$ and $\chi \neq 0$ near $(1,1)$. Then at $(1,1)$, $Q = Q_z = Q_w = 0$ and so $H_{zz} = \chi Q_{zz} = 2\chi c_0 c_1$. Similarly, $H_{wz} = \chi(c_0 + c_1)$ and $H_{ww} = 2\chi$. Thus c_0, c_1 are the roots of the quadratic $(H_{ww})x^2 - (2H_{wz})x + H_{zz} = 0$. Solving via the quadratic formula gives $|c_0 - c_1| = 2\sqrt{-\det \mathbf{H}}/H_{ww}$. This shows that $\det \mathbf{H} = 0$ if and only if $c_0 = c_1$, as asserted above.

Formula (2.3) then yields

$$a_{rs} \sim \frac{\phi(1,1)}{|c_0 - c_1|} = \frac{2G(1,1)}{H_{ww}(1,1)|c_0 - c_1|} = \frac{G(1,1)}{\sqrt{-\det \mathbf{H}(1,1)}}\,.$$

The general result follows by a change of scale mapping the multiple point (z^*, w^*) to $(1,1)$. Details of the computation are as follows. Make the change of variables $\widetilde{z} = z/z^*, \widetilde{w} = w/w^*$, and write $\widetilde{G}$ for G considered as a function of $\widetilde{z}, \widetilde{w}$. In an obvious notation, $\widetilde{a}_{rs} = (z^*)^r (w^*)^s a_{rs}$. Then, evaluating at $(\widetilde{z}, \widetilde{w}) = (1,1)$, we obtain $H_{\widetilde{z}} = z^* H_z$, $H_{\widetilde{w}} = w^* H_w$. Repeating this yields $H_{\widetilde{z}\widetilde{z}} = (z^*)^2 H_{zz}$, $H_{\widetilde{w}\widetilde{z}} = z^* w^* H_{wz}$ and $H_{\widetilde{w}\widetilde{w}} = (w^*)^2 H_{ww}$.

The special case above now yields

$$\widetilde{a}_{rs} \sim \frac{\widetilde{G}(1,1)}{\sqrt{-\widetilde{D}(1,1)}} = \frac{G(z^*, w^*)}{\sqrt{-(z^*)^2 (w^*)^2 \det \mathbf{H}(z^*, w^*)}}$$

from which the result follows immediately. $\square$

To prove Theorems 3.6 and 3.11 we require a result on asymptotics of integrals that allows for non-isolated critical points. The lemma below says that, as seems intuitively plausible, we may compute a asymptotic expansions in directions transverse to $\mathcal{S}$ and then integrate their leading terms along $\mathcal{S}$ to get the leading term of the original expansion.

Lemma 4.8 (Fourier-Laplace expansion for continuum of critical points)**.** *Let $y \in \mathcal{S}$ and let $M(y)$ be the Hessian at y of the restriction of f to B_y. If $\psi_k(y) \neq 0$ and $\det M(y)$ is nonzero on $\mathcal{S}$ then*

$$\int_{\mathcal{E}} e^{-sf} \psi_k \, d\widehat{\boldsymbol{\theta}} \times d\mu \sim \left(\frac{2\pi}{s}\right)^{(\rho+d)/2} \int_{\mathcal{S}} \frac{\psi_k(y)}{\sqrt{\det M(y)}} \, d\sigma(y).$$

Proof. This is a specialization of [PW02a, Thm 2] (the proof of that result is considerably more involved than in the isolated critical point case). □

Lemma 4.8 yields the following expansion:

$$a_{\mathbf{r}} \sim (2\pi)^{-d} p_0(s) \left(\frac{2\pi}{s}\right)^{(\rho+d)/2} \int_{\mathcal{S}} \frac{\psi_0(y)}{\sqrt{\det M(y)}} \, d\sigma(y). \tag{4.7}$$

We now proceed to complete the proofs by applying (4.7) in each case. We give full details only for the case $\mathbf{z}^* = \mathbf{1}$; the general case again follows from the change of variable $\mathbf{z} \mapsto \mathbf{z}/\mathbf{z}^*$ and we leave the details to the reader.

Proof of Theorem 3.6. By the nondegeneracy assumption, $\operatorname{rank}(\mathbf{C}) = d + 1$, hence $\rho := \operatorname{rank}(\overline{\mathbf{C}}) = d$ and $\dim(B_y) = 2d$. The set $\mathcal{S}$ has dimension $n - d$; we assume this to be strictly positive since the case $n = d$ has been dealt with already in Theorem 3.3. At each stationary point $(\mathbf{0}, \boldsymbol{\alpha}) \in \mathcal{S}$ we find the Hessian determinant of f restricted to B_y to be the constant value $(\det \overline{\mathbf{C}})^2$. Now (4.7) gives

$$\begin{aligned} a_{\mathbf{r}} &\sim (2\pi)^{-d} p_0(s) s^{-d} (2\pi)^d \phi(\mathbf{1}) \int_{\mathcal{S}} \det M(y)^{-1/2} \, d\sigma(y) \\ &\sim s^{n-d} \frac{\phi(\mathbf{1}) \sigma(\mathcal{S})}{|\det \overline{\mathbf{C}}|}, \end{aligned}$$

which establishes the formula for the leading term. That the expansion is a polynomial plus an exponentially smaller correction again follows from [Pem00, Theorem 3.1].

Remark. The expansion via Leibniz' formula is not generally integrable past the leading term, and therefore does not give a method of computing the lower terms of the polynomial P. □

Proof of Theorem 3.11. Here $\mathbf{K}_0 = \{\boldsymbol{\delta}\}$, and so $\mathcal{A}(\boldsymbol{\delta}) = \Delta$. Thus $\dim \mathcal{A}^{\perp} = 0$, so that for $y = (\mathbf{0}, \boldsymbol{\alpha}) \in \mathcal{S}$, $M(y) = Q(\mathbf{z}^*, \boldsymbol{\alpha})$. Use (4.7) once more with $\dim(B_y) = d$ to get

$$a_{\mathbf{r}} \sim (2\pi)^{-d} p_0(s) s^{-d/2} (2\pi)^{d/2} \int_{\Delta} \frac{\phi(\mathbf{z})}{\sqrt{\det Q(\mathbf{z}^*, \boldsymbol{\alpha})}} \, d\mu(\boldsymbol{\alpha})$$

which simplifies to the conclusion of the theorem. □

5. Further discussion

Asymptotics in the gaps. The problem of determining asymptotics when $\widehat{\mathbf{r}}/s$ converges to the boundary of $\mathbf{K}_0$ is dual to the problem raised in [PW02b] of letting $\widehat{\mathbf{r}}/s$ converge to $\partial \mathbf{K}_0$ from the outside. Solutions to both of these problems are necessary before we understand asymptotics "in the gaps", that is, in any region asymptotic to and containing a direction in the boundary of $\mathbf{K}_0$. For example, in the case Example 3.12 with $\boldsymbol{\delta}(1,1)$, what are the asymptotics for $a_{r,r+\sqrt{r}}$ as $r \to \infty$?

In addition to the obvious discontinuity in our asymptotic expansions above at the boundary of $\mathbf{K}_0$ caused by the change from multiple to smooth point regime, in the nondegenerate case other discontinuities arise. The leading term may change degree around the boundary of $\mathbf{K}_0$. This reflects the behavior of the corresponding Fourier-Laplace integrals, as the affine subspace $\mathcal{A}$ of positive dimension moves from intersecting the interior of Δ to only intersecting a lower-dimensional face. Subtler effects occur as $\mathcal{A}$ passes a corner in the interior of $\mathbf{K}$, when the leading term is continuous but lower terms may not be.

A proper analysis of all these issues in the Fourier-Laplace integral framework requires the consideration of integrals whose phase can vary with λ. This is the subject of work in progress by M. Lladser [Lla03].

Effective computation. The greatest obstacle to making all these computations completely effective lies in the location of the minimal point $\mathbf{z}^*$ given $\mathbf{r}$. Assuming the existence of a $\mathbf{z}^*(\mathbf{r})$ with $\mathbf{r} \in \mathbf{K}(\mathbf{z}^*)$, how may we compute $\mathbf{z}^*(\mathbf{r})$ and test whether it is a minimal point? Since the moduli of the coordinates of $\mathbf{z}^*$ are involved in the definition of minimality, this is a problem in real rather than complex computational geometry and does not appear easy. For example, how easily can one prove that the double point in Example 3.2 is minimal?

Minimal points. Many of our theorems rule out analysis of a minimal point $\mathbf{z}^*$ if one of its coordinates $(z^*)_j$ is zero. The directions in $\mathbf{K}(\mathbf{z}^*)$ will always have $r_j = 0$, in which case the analysis of coefficient asymptotics reduces to a case with one fewer variable. Thus it appears no generality is lost. If we are, however, able to solve the previous problem, wherein $\mathbf{r}/s$ converges to $\partial\mathbf{K}_0$, then we may choose to let $\mathbf{r}/s$ converge to something with a zero component. The problem of asymptotics when some $r_j = o(r_k)$ now makes sense and is not reducible to a previous case. Presumably these asymptotics are governed by the minimal point $\mathbf{z}^*$ still, but it must be sorted out which of our results persist when $(z^*)_j = 0$. Certainly the geometry near $\mathbf{z}^*$ has more possibilities, since it is easier to be a minimal point (it is easier to maintain $|z_j| \geq |(z^*)_j|$ for $\mathbf{z}$ near $\mathbf{z}^*$ when $(z^*)_j = 0$).

In this article and its predecessor we have not treated toral minimal points (minimal points $\mathbf{z}^*$ that are not strictly minimal, and not isolated in $\mathbf{T}(\mathbf{z}^*)$). Nor have we analysed strictly minimal points that are cusps. The extension of our results here to those cases is in fact rather routine, and will be carried out in future work.

Local geometry of critical points. Many of our results fail when the Hessian of f unexpectedly vanishes. Surprisingly, we do not know whether this ever happens in cases of interest. Specifically, we do not know of a generating function with nonnegative coefficients, for which the Hessian of f on B_y (notation of Section 4) ever vanishes. It goes without saying, therefore, that we do not know whether cases ever arise of degeneracies of mixed orders, such as may arise in the case of the remark following Example 3.12.

Homological methods. The methods of this paper may be described as reasonably elementary though somewhat involved. The necessary complex contour and stationary phase integrals are well known, except perhaps for the "resolving lemma", Lemma 4.2. While writing down these results, we found a more sophisticated approach involving some algebraic topology (Stratified Morse Theory) and singularity theory. This approach delivers similar results in a more general framework. Among other things, it clarifies and generalizes formulae such as the expression (3.1) derived by hand above. Still, the present approach has an advantage other than its chronological priority and its independence from theory not well known by practitioners of analytic combinatorics. Homological methods, as far as we can tell, are not capable of determining asymptotics in "boundary" directions, such as are given in part (ii) of Theorem 3.1. It seems that one needs to delve into the fine details of the integral, as we do in the present work, in order to handle these cases. Much of the interest in asymptotic enumeration centers on phase boundaries, so the weight of these cases is far from negligible. We hope some day to integrate the two approaches, but at present each does something the other cannot.

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References

[BM93] A. Bertozzi and J. McKenna. Multidimensional residues, generating functions, and their application to queueing networks. *SIAM Rev.*, 35(2):239–268, 1993.
[BP03] Y. Baryshnikov and R. Pemantle. Convolutions of inverse linear functions via multivariate residues. Preprint, 2002.
[CEP96] H. Cohn, N. Elkies, and J. Propp. Local statistics for random domino tilings of the Aztec diamond. *Duke Math. J.*, 85(1):117–166, 1996.
[Com74] L. Comtet. *Advanced combinatorics.* D. Reidel Publishing Co., Dordrecht, enlarged edition, 1974.
[CP03] H. Cohn and R. Pemantle. A determination of the boundary of the fixation region for some domino tiling ensembles. In preparation, 2003.
[DL93] R. Devore and G. Lorentz. *Constructive approximation.* Springer-Verlag, 1993.
[dLS03] J. de Loera and B. Sturmfels. Algebraic unimodular counting. *Math. Program.*, 96(2):183–203, 2003.
[FM77] L. Flatto and H. P. McKean. Two queues in parallel. *Comm. Pure Appl. Math.*, 30(2):255–263, 1977.
[GH78] P. Griffiths and J. Harris. *Principles of algebraic geometry.* Wiley, 1978.
[HK71] M. Hautus and D. Klarner. The diagonal of a double power series. *Duke Math. J.*, 23:613–628, 1971.
[Hör83] L. Hörmander. *The analysis of linear partial differential operators. I.* Springer-Verlag, Berlin, 1983. Distribution theory and Fourier analysis.
[Kar02] H. Karloff. Personal communication, 2002.
[Lla03] M. Lladser. *Asymptotic enumeration via singularity analysis.* Doctoral dissertation, Ohio State University, 2003.
[LL99] M. Larsen and R. Lyons. Coalescing particles on an interval. *J. Theoret. Probab.*, 12(1):201–205, 1999.
[Pem] R. Pemantle. Asymptotics of multivariable generating functions. Unpublished lecture notes available from the author.
[Pem00] R. Pemantle. Generating functions with high-order poles are nearly polynomial. In *Mathematics and computer science (Versailles, 2000)*, pages 305–321. Birkhäuser, Basel, 2000.
[PW02a] R. Pemantle and Mark C. Wilson. Asymptotic expansions of Fourier-Laplace integrals. Preprint, 2002.
[PW02b] R. Pemantle and Mark C. Wilson. Asymptotics of multivariate sequences. I. Smooth points of the singular variety. *J. Combin. Theory Ser. A*, 97(1):129–161, 2002.
[Tsi93] A. Tsikh. Conditions for absolute convergence of series of Taylor coefficients of meromorphic functions of two variables. *Math. USSR Sbornik*, 74:336–360, 1993.
[Wil94] H. Wilf. *generatingfunctionology.* Academic Press, Boston, 1994.

Department of Mathematics, Ohio State University, Columbus OH 43210, USA
E-mail address: pemantle@math.upenn.edu
Current address: Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19104-6395, USA

Department of Computer Science, University of Auckland, Private Bag 92019 Auckland, New Zealand
E-mail address: mcw@cs.auckland.ac.nz