Resistance Bounds for First-Passage Percolation and Maximum Flow

by Russell Lyons, Robin Pemantle and Yuval Peres

Abstract. Suppose that each edge e of a network is assigned a random exponential passage time with mean r_e. Then the expected first-passage time between two vertices is at least the effective resistance between them for the edge resistances $\langle r_e \rangle$. Similarly, suppose each edge is assigned a random exponential edge capacity with mean c_e. Then the expected maximum flow between two vertices is at least the effective conductance between them for the edge conductances $\langle c_e \rangle$. These inequalities are dual to each other for planar graphs and the second is tight up to a factor of 2 for trees; this has implications for a herd of gnus crossing a river delta.

§1. Introduction.

There are well-known connections between random walks and electrical networks. (For terminology and basic results concerning electrical networks, see Doyle and Snell (1984).) For example, the commute time between two vertices in a finite network is the effective resistance between them, multiplied by twice the sum of the edge conductances (see Chandra et al. (1989)). In this paper, we show that effective resistance yields good bounds in some other probabilistic problems, namely, first-passage percolation and maximum flow with random edge capacities. Our proofs illustrate that it can be useful to interpret a potential function on a graph as a measure on cutsets, in analogy with the interpretation of a flow as a measure on paths.

Let $G = (\mathbb{V}, \mathbb{E})$ be a connected finite graph. Suppose that each edge $e \in \mathbb{E}$ is assigned an independent random passage time t_e that is exponentially distributed with mean r_e. Let $a \in \mathbb{V}$ and let $Z \subset \mathbb{V}$ be a set not containing a. A random variable of wide interest (see, e.g., the review by Kesten (1987)) is the first-passage time $\mathsf{Time}(a \leftrightarrow Z; \langle t_e \rangle)$ between a and Z, defined as the minimum of $\sum_{e\in\mathcal{P}} t_e$, where $\mathcal{P}$ ranges over the paths connecting a

1991 Mathematics Subject Classification. Primary 90B15. Secondary 68R10, 60K35.
Key words and phrases. Electrical network, resistance, percolation, passage time, maximum flow, tree.
Research partially supported by the Institute for Advanced Studies, Jerusalem (Lyons), by a Presidential Faculty Fellowship (Pemantle), and NSF grant DMS-9404391 (Peres).

and Z. Now if each edge e is regarded as a resistor with resistance t_e, then the effective resistance of a path $\mathcal{P}$ equals $\sum_{e\in\mathcal{P}} t_e$. Thus by Rayleigh's monotonicity principle, the effective resistance $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle t_e\rangle)$ between a and Z satisfies

$$\mathsf{Time}\,(a \leftrightarrow Z;\, \langle t_e\rangle) \geq \mathsf{Resis}\,(a \leftrightarrow Z;\, \langle t_e\rangle)\,. \tag{1.1}$$

Although the two sides of the inequality (1.1) are equal if G consists of several edges connected in series, this inequality is far from tight even if G consists of n edges in parallel between a and z: in this case, $\mathsf{Time}\,(a \leftrightarrow z;\, \langle t_e\rangle) = \min_e\{t_e\}$, while $\mathsf{Resis}\,(a \leftrightarrow z;\, \langle t_e\rangle) = \left(\sum_e t_e^{-1}\right)^{-1}$. Our first result is that by averaging the left-hand side and the random times in the right-hand side, (1.1) can be sharpened to a general inequality:

$$\mathbf{E}\big[\mathsf{Time}\,(a \leftrightarrow Z;\, \langle t_e\rangle)\big] \geq \mathsf{Resis}\,(a \leftrightarrow Z;\, \langle r_e\rangle)\,; \tag{1.2}$$

equality holds both for pure series networks and for pure parallel networks. Thompson's principle, which expresses effective resistance as a minimum energy, implies that $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle t_e\rangle)$ is a concave function of its arguments (see Doyle and Snell (1984) or Lyons and Peres (1997)). Hence the right-hand side of (1.2) is at least $\mathbf{E}[\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle t_e\rangle)]$, and is usually much larger. For this reason, (1.2) is better than simply taking expectations in (1.1). Moreover, both sides of (1.2) equal $\left(\sum r_e^{-1}\right)^{-1}$ if G consists of n edges in parallel. To substantiate our claim that (1.2) is a fairly good inequality, we prove that for a class of networks, namely, for the planar duals of trees, the opposite inequality with a factor of 2 holds; see Theorem 2.1.

Now suppose that each edge $e \in \mathbb{E}$ is assigned an independent random capacity κ_e that is exponentially distributed with mean c_e. Denote by $\mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)$ the effective conductance between a and Z when each edge e has conductance κ_e. To consider flows on G, it will be convenient to fix one orientation for each edge $e \in \mathbb{E}$. A **flow** θ from a to Z is a function $\theta : \mathbb{E} \to \mathbb{R}$ so that at each vertex $v \notin a \cup Z$, Kirchhoff's node law is satisfied, i.e., the incoming flow to v equals the outgoing flow from v. The net outgoing flow from a, which is assumed to be be nonnegative, is called the **strength** of θ. We consider flows θ from a to Z that are **feasible**, i.e., so that $|\theta(e)| \leq \kappa_e$ for all $e \in \mathbb{E}$. The maximal strength of such a flow, denoted $\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)$, equals the minimal cutset sum of the capacities by Ford and Fulkerson's (1962) Max-Flow Min-Cut Theorem. Analogous to (1.1) is the simple inequality

$$\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle) \geq \mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)\,. \tag{1.3}$$

Indeed, the current flow corresponding to unit voltage drop from a to Z is bounded by κ_e on each edge e, and its strength is $\mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)$ by definition. However, the inequality (1.3) is again far from tight: if G consists of n edges in series between a and z, then $\mathsf{Max\,Flow}\,(a \leftrightarrow z;\, \langle \kappa_e \rangle) = \min\{\kappa_e\}$, while $\mathsf{Conduc}\,(a \leftrightarrow z;\, \langle \kappa_e \rangle) = \left(\sum \kappa_e^{-1}\right)^{-1}$. Again, we get a better inequality by averaging the random function on the left and the random arguments of the concave function on the right:

$$\mathbf{E}\big[\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e \rangle)\big] \ge \mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e \rangle)\,. \tag{1.4}$$

Furthermore, when G is a tree, a is its root, and Z is its set of leaves, then the opposite inequality holds with a factor of 2; see Theorem 3.1.

For planar networks, the inequalities (1.2) and (1.4) are actually equivalent. To be precise, suppose that G is a planar network that can be embedded in an infinite strip, with a and Z on opposite sides of the strip (see Fig. 1). Let G^* be the planar dual of G in the strip, i.e., the vertices of G^* are the faces of G. To every edge e in G corresponds a dual edge e^* in G^* that connects the two faces touching e. This edge can be represented by a smooth path that crosses e orthogonally; we assign e^* an orientation by rotating the given orientation of e by an angle of $\pi/2$ clockwise. Locate the source a^* and sink z^* of G^* in the two unbounded faces of G. Assign the same number (whether a passage time or a capacity) to an edge e as to its dual edge e^*. The Max-Flow Min-Cut Theorem says that $\mathsf{Time}\,(a \leftrightarrow z;\, \langle x_e \rangle) = \mathsf{Max\,Flow}\,(a^* \leftrightarrow z^*;\, \langle x_e \rangle)$. Furthermore, the following well-known lemma shows that $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle x_e \rangle) = \mathsf{Conduc}\,(a^* \leftrightarrow z^*;\, \langle x_e \rangle)$, providing the asserted equivalence between (1.2) and (1.4). Our proofs for general networks reflect a more abstract duality.

LEMMA 1.1. *Let G be a planar network and G^* its dual in a strip, as defined above. Then* $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle x_e \rangle) = \mathsf{Conduc}\,(a^* \leftrightarrow z^*;\, \langle x_e \rangle)$ *for any $x_e > 0$. [Recall that x_e denotes an edge resistance in* $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle x_e \rangle)$*, but an edge conductance in* $\mathsf{Conduc}\,(a^* \leftrightarrow z^*;\, \langle x_e \rangle)$*].*

This can be deduced from results in Brooks et al. (1940), but we give a direct proof for the convenience of the reader.

Proof. Let $\langle I(e) \rangle$ be the unit current flow from a to Z corresponding to the resistances $\langle x_e \rangle$. For any dual edge e^* in G^*, orient it as described above, and define $J(e^*) := x_e I(e)$. Kirchhoff's cycle law for I ($\sum_{e \in \gamma} x_e I(e) = 0$ along any cycle γ) implies Kirchhoff's node law for J ($\sum_{e^*} J(e^*) = 0$, summing over edges incident to a vertex $v^* \notin \{a^*, z^*\}$); similarly, Kirchhoff's node law for I yields Kirchhoff's cycle law for J along elementary cycles, whence along all cycles. Thus J is a current flow from a^* to z^*. Since I is a unit flow, J induces a unit voltage difference between a^* and z^*, whence the strength of J is $\mathsf{Conduc}\,(a^* \leftrightarrow z^*;\, \langle x_e \rangle)$. On the other hand, the strength of J is also the voltage difference between a

and Z induced by I. Since I is a unit current flow, this difference is $\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle x_e \rangle)$, whence the lemma. ∎

Finally, a context in which the maximum flow in a network with random exponential capacities arose naturally was the study of directed fractal percolation; see Chayes, Pemantle and Peres (1997).

§2. Passage Times.

THEOREM 2.1. *Let G be a finite network and t_e be independent exponentially-distributed random variables with mean r_e. Then*

$$\mathbf{E}\big[\mathsf{Time}\,(a \leftrightarrow Z;\, \langle t_e \rangle)\big] \geq \mathsf{Resis}\,(a \leftrightarrow Z;\, \langle r_e \rangle)\,. \tag{2.1}$$

Furthermore, if $G = T^*$ is the planar dual of a tree T in a strip, with a^* on the left and z^* on the right of T (see Fig. 1), then

$$\mathbf{E}\big[\mathsf{Time}\,(a^* \leftrightarrow z^*;\, \langle t_e \rangle)\big] \leq 2\,\mathsf{Resis}\,(a^* \leftrightarrow z^*;\, \langle r_e \rangle)\,. \tag{2.2}$$

Figure 1.

REMARK. In the inequalities (2.1) above and (3.1) below, one can replace the vertex a by a set of vertices A, since gluing together all vertices in A reduces to the case where A is a singleton.

To prove the theorem, we represent the voltage function by a random cutset. We shall use the following notation: for a function $f : \mathbb{E} \to \mathbb{R}$ and $e = (x, y) \in \mathbb{E}$, write $df(e) := f(y) - f(x)$.

LEMMA 2.2. *Let $f : \mathbb{V} \to [0, 1]$ be a function that is 0 at a and 1 on Z. Then there exists a probability measure ν on the collection of cutsets Π separating a from Z so that*

$$\forall e \in \mathbb{E} \qquad \sum_{\{\Pi\,;\,e\in\Pi\}} \nu(\Pi) = |df(e)|\,. \tag{2.3}$$

Furthermore, suppose that every vertex $x \notin a \cup Z$ that is a local extremum for f satisfies $f(y) = f(x)$ for all vertices y adjacent to x. Then there exists a measure ν satisfying (2.3) that is supported on minimal cutsets.

Proof. Choose a random cutset Π as follows: let U be a uniform random variable on $[0, 1]$, and set Π to be the collection of edges where f crosses the value U. The distribution ν of Π clearly satisfies (2.3).

Next, suppose that the assumption on local extrema of f holds. If Π is chosen as above and an edge e is in Π, then a.s. $df(e) \neq 0$; hence the assumption easily implies that e (possibly with the reverse orientation) is on a path $\mathcal{P}$ from a to Z such that f is strictly increasing along $\mathcal{P}$. Thus $\Pi \setminus \{e\}$ is not a cutset, whence Π is a minimal cutset a.s. ∎

Proof of Theorem 2.1. Let V be the unit voltage function on $\mathbb{V}$ that is 0 at a and 1 on Z. For convenience, orient each edge e so that $dV(e) \geq 0$. Since V is harmonic off $a \cup Z$, the preceding lemma provides a probability measure ν on minimal cutsets Π separating a from Z with the property that

$$\forall e \in \mathbb{E} \qquad \sum_{\{\Pi\,;\,e\in\Pi\}} \nu(\Pi) = dV(e)\,.$$

We claim that

$$\mathsf{Time}\,(a \leftrightarrow Z;\, \langle t_e \rangle) \geq \sum_{\Pi} \nu(\Pi) \min_{e\in\Pi} \frac{t_e}{dV(e)}\,. \tag{2.4}$$

To see this, let $\mathcal{P}$ be a path from a to Z; we have

$$\sum_{\Pi} \nu(\Pi) \min_{e\in\Pi} \frac{t_e}{dV(e)} \leq \sum_{\Pi} \nu(\Pi) \sum_{e\in\Pi\cap\mathcal{P}} \frac{t_e}{dV(e)}$$

$$
\begin{aligned}
&= \sum_{e \in \mathcal{P}} \frac{t_e}{dV(e)} \sum_{\Pi \ni e} \nu(\Pi) \\
&= \sum_{e \in \mathcal{P}} t_e \,.
\end{aligned}
$$

Now $\min_{e\in\Pi} t_e/dV(e)$ is the minimum of independent exponential random variables. Thus, it is also an exponential random variable; its parameter is the sum of the parameters of $t_e/dV(e)$, which means that

$$
\mathbf{E}\left[\min_{e\in\Pi} \frac{t_e}{dV(e)}\right] = \left[\sum_{e\in\Pi} \frac{dV(e)}{r_e}\right]^{-1} . \tag{2.5}
$$

When Π is a minimal cutset, $\sum_{e\in\Pi} dV(e)/r_e$ is the strength of the unit voltage flow, which equals $\mathsf{Conduc}\,(a \leftrightarrow Z;\ \langle 1/r_e\rangle)$. Thus

$$
\begin{aligned}
\mathbf{E}\big[\mathsf{Time}\,(a \leftrightarrow Z;\ \langle t_e\rangle)\big] &\geq \sum_{\Pi} \nu(\Pi)\mathbf{E}\left[\min_{e\in\Pi} \frac{t_e}{dV(e)}\right] \\
&= \sum_{\Pi} \nu(\Pi)\mathsf{Resis}\,(a \leftrightarrow Z;\ \langle r_e\rangle) = \mathsf{Resis}\,(a \leftrightarrow Z;\ \langle r_e\rangle)\,.
\end{aligned}
$$

This proves (2.1); the inequality (2.2) will follow immediately by combining Lemma 1.1 above and (3.2) below. ∎

Remark. A heuristic interpretation of Theorem 2.1 is that a good estimate of the expected time it takes the first of a herd of gnus to cross a river delta is the effective conductance of the delta, provided the time to cross each tributary is exponentially distributed. This interpretation fares better when not examined too closely.

For series-parallel networks, inequality (2.1) can be extended to concave domination: Recall that for two nonnegative random variables X and Y, we say that X **concavely dominates** Y if $\mathbf{E}[\varphi(X)] \geq \mathbf{E}[\varphi(Y)]$ for any concave nondecreasing function φ on $\mathbb{R}^+$. Say that one random network **concavely dominates** another if the minimum passage time for the first concavely dominates the minimum passage time for the second.

Proposition 2.3. *Let G be a finite series-parallel network and t_e be independent exponentially-distributed random variables with mean r_e. If Y is an exponential random variable with mean* $\mathsf{Resis}\,(a \leftrightarrow Z;\ \langle r_e\rangle)$*, then G concavely dominates a single-edge network with passage time Y.*

Proof. If two edges, e and f, of G are in parallel and G' is the network with e and f combined to a single edge given an exponential passage time of mean $r_e \wedge r_f$, then the minimum passage times on G and G' have identical distributions. Thus, it suffices to show that if two edges, e and f, of G are in series and G' is the network with e and f combined to a single edge given an exponential passage time t' of mean $r' := r_e + r_f$, then G concavely dominates G'. Moreover, it suffices to prove this when we condition on the values of t_g for all edges $g \neq e, f$. Write $\psi(t_e + t_f)$ for the minimum passage time in G given all other t_g. Since this is the minimum of linear nondecreasing functions, $\psi(\bullet)$ is concave nondecreasing. Furthermore, the passage time in G' given all other t_g is $\psi(t')$. We need to show that $\mathbf{E}[\varphi(\psi(t_e + t_f))] \geq \mathbf{E}[\varphi(\psi(t'))]$ for every concave nondecreasing φ. Since $\tilde{\varphi} := \varphi \circ \psi$ is also concave nondecreasing, this is merely the standard fact that the sum of two exponentials concavely dominates a single exponential with the same mean:

$$\mathbf{E}[\tilde{\varphi}(t_e + t_f)] = \mathbf{E}\left[\tilde{\varphi}\left(\frac{r_e}{r'}\frac{r't_e}{r_e} + \frac{r_f}{r'}\frac{r't_f}{r_f}\right)\right] \geq \mathbf{E}\left[\frac{r_e}{r'}\tilde{\varphi}\left(\frac{r't_e}{r_e}\right) + \frac{r_f}{r'}\varphi\left(\frac{r't_f}{r_f}\right)\right] = \mathbf{E}[\tilde{\varphi}(t')]\,. \quad \blacksquare$$

Question. Does the concave domination in Proposition 2.3 extend to networks that are not series-parallel?

§3. Maximum Flow.

THEOREM 3.1. *Let G be a finite network and κ_e be independent exponentially-distributed random variables with mean c_e. Then*

$$\mathbf{E}\big[\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle\kappa_e\rangle)\big] \geq \mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\rangle)\,. \tag{3.1}$$

Furthermore, if G is a tree with a its root and Z its leaves, then

$$\mathbf{E}\big[\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle\kappa_e\rangle)\big] \leq 2\,\mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\rangle)\,. \tag{3.2}$$

The following two known lemmas are needed to prove the theorem. For the convenience of the reader, we include their proofs.

LEMMA 3.2. *Let θ be an acyclic flow from a to Z. Then there exists a measure on self-avoiding paths from a to Z so that*

$$\forall e \in \mathbb{E} \qquad \sum_{\{\mathcal{P}\,;\, e\in\mathcal{P}\}} \mu(\mathcal{P}) = |\theta(e)|\,.$$

Proof. Orient edges so that $\theta(e) \geq 0$ for all e. Since θ is acyclic, one easily finds a directed self-avoiding path $\mathcal{P}$ from a to Z such that $\alpha := \min_{e\in\mathcal{P}} \theta(e) > 0$. Subtracting α times the unit flow along $\mathcal{P}$ from θ, and using induction on the number of edges e such that $\theta(e) \neq 0$, completes the proof. $\blacksquare$

LEMMA 3.3. (CHAYES, PEMANTLE AND PERES 1997) *Let Y be an exponential random variable with mean $1/r$. Then for any random variable $X \geq 0$ that is independent of Y and has finite mean,*

$$\mathbf{E}\big[\min\{X,Y\}\big] \leq \frac{2\mathbf{E}X}{2+r\mathbf{E}X} .$$

Proof. For $x \geq 0$, we have

$$\mathbf{E}\big[\min\{x,Y\}\big] = \int_0^\infty \mathbf{P}[\min\{x,Y\} > y]\, dy = \int_0^x e^{-ry}\, dy = \frac{1}{r}(1-e^{-rx}) .$$

Hence for any nonnegative random variable X, Jensen's inequality yields

$$\mathbf{E}\big[\min\{X,Y\}\big] = \frac{1}{r}\mathbf{E}(1-e^{-rX}) \leq \frac{1}{r}(1-e^{-r\mathbf{E}X}) . \tag{3.3}$$

Now rewrite the inequality

$$\forall x \geq 0 \quad 2+x \geq \sum_{k=0}^{\infty} \frac{2-k}{k!} x^k = (2-x)e^x$$

in the equivalent form

$$1-e^{-x} \leq \frac{2x}{2+x} .$$

Combining this with (3.3) proves the lemma. ∎

Proof of Theorem 3.1. Let $\langle I_e \rangle$ be the unit current flow from a to Z. Orient edges so that $I_e \geq 0$ for all e. By Lemma 3.2, there exists a measure μ on paths from a to Z such that

$$\forall e \in \mathbb{E} \quad \sum_{\{\mathcal{P}\,;\,e\in\mathcal{P}\}} \mu(\mathcal{P}) = I_e .$$

Since $\langle I_e \rangle$ is a unit flow, μ is a probability measure. Define a new flow

$$f \mapsto \sum_{f\in\mathcal{P}} \mu(\mathcal{P}) \min_{e\in\mathcal{P}} \frac{\kappa_e}{I_e} .$$

This flow is $\langle \kappa_e \rangle$-feasible since

$$\sum_{f\in\mathcal{P}} \mu(\mathcal{P}) \min_{e\in\mathcal{P}} \frac{\kappa_e}{I_e} \leq \sum_{f\in\mathcal{P}} \mu(\mathcal{P}) \frac{\kappa_f}{I_f} = \kappa_f .$$

Therefore,

$$\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e \rangle) \geq \sum_{\mathcal{P}} \mu(\mathcal{P}) \min_{e\in\mathcal{P}} \frac{\kappa_e}{I_e} . \tag{3.4}$$

Now, as in the proof of Theorem 2.1, we have

$$\mathbf{E}[\min_{e\in\mathcal{P}} \kappa_e/I_e] = \left[\sum_{e\in\mathcal{P}} I_e/c_e\right]^{-1} = \left[\sum_{e\in\mathcal{P}} dV(e)\right]^{-1} = \mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\rangle)\,, \tag{3.5}$$

where V denotes the voltage function corresponding to the unit current flow. Thus, taking expectation in (3.4) gives

$$\mathbf{E}\big[\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)\big] \geq \sum_{\mathcal{P}} \mu(\mathcal{P})\mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\rangle) = \mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\rangle)\,.$$

This shows (3.1).

To show (3.2), we rely on Lemma 3.3 and induction on the number of vertices in the tree. If the root a has at least 2 children, then G consists of at least 2 networks from a to Z in parallel. Since both sides of (3.2) add for networks in parallel, this part of the induction step is easy. Otherwise, the root a has only one child, a'. In this case, the edge $e' := (a, a')$ has random exponential capacity $\kappa_{e'}$ with mean $c_{e'}$, and the network G consists of e' in series with a tree T'. Therefore, we have

$$\begin{aligned}
&\mathbf{E}\big[\mathsf{Max\,Flow}\,(a \leftrightarrow Z;\, \langle \kappa_e\rangle)\big] = \mathbf{E}\big[\min\{\kappa_{e'}, \mathsf{Max\,Flow}\,(a' \leftrightarrow Z;\, \langle \kappa_e\rangle)\}\big] \\
&\quad \leq \frac{2}{2/\mathbf{E}\big[\mathsf{Max\,Flow}\,(a' \leftrightarrow Z;\, \langle \kappa_e\rangle)\big] + 1/c_{e'}} \qquad \text{by Lemma 3.3} \\
&\quad \leq \frac{2}{1/\mathsf{Conduc}\,(a' \leftrightarrow Z;\, \langle c_e\rangle) + 1/c_{e'}} \qquad \text{by the induction hypothesis} \\
&\quad = \frac{2}{\mathsf{Resis}\,(a \leftrightarrow Z;\, \langle 1/c_e\,;\ e \in G\rangle)} = 2\,\mathsf{Conduc}\,(a \leftrightarrow Z;\, \langle c_e\,;\ e \in G\rangle)\,.
\end{aligned}$$

∎

§4. Distributions with Monotone Failure Rates.

In this section, we relax the assumption that the passage times and random capacities in Theorem 2.1 and Theorem 3.1 are exponentially distributed. Say that the nonnegative random variable X has an **increasing failure rate** (IFR) if for any $t > 0$, the function $x \mapsto \mathbf{P}[x < X \leq x + t \mid X > x]$ is (weakly) increasing on $[0, \infty)$. Analogously, define **decreasing failure rate** (DFR). These notions are important in reliability theory (see Barlow and Proschan (1965)) and can be incorporated with our main results.

COROLLARY 4.1. *The inequality (2.1) holds if each of the independent passage times t_e has IFR, while the inequality (2.2) holds if each of these variables has DFR. Similarly, the inequality (3.1) holds if the independent random capacities κ_e have IFR, while (3.2) holds if they have DFR.*

Proof. The identity (2.5) is the only place that the proof of (2.1) used the assumption of exponential distribution. For distributions with IFR, the left-hand side in (2.5) is at least the right-hand side (see Cor. 4.10 in Barlow and Proschan (1965)); this suffices to complete the proof of (2.1) for such distributions. The same applies to the identity (3.5) in the proof of Theorem 3.1.

In the proof of (3.2), the assumption that the random capacities are exponentially distributed is needed only to apply Lemma 3.3 to $Y := \kappa_{e'}$. That lemma is also valid when the random variable Y has DFR, since by Theorem 4.8 in Barlow and Proschan (1965),

$$\int_0^\infty \mathbf{P}[\min\{x, Y\} > y]\, dy = \int_0^x \mathbf{P}[Y > y]\, dy \geq \int_0^x e^{-ry}\, dy$$

for Y with DFR. Finally, the extension of (2.2) to passage times with DFR follows by planar duality as before. ∎

Acknowledgement. We are grateful to L. Chayes for useful discussions.

REFERENCES

Barlow, R. and Proschan, F. (1965). *Mathematical Theory of Reliability.* Wiley, New York.

Brooks, R. L., Smith, C. A. B., Stone, A. H., and Tutte, W. T. (1940). The dissection of rectangles into squares, *Duke Math. J.* **7**, 213–340.

Chandra, A. K., Raghavan, P., Ruzzo, W. L., Smolensky, R., and Tiwari, P. (1989). The electrical resistance of a graph captures its commute and cover times, *Proc. 21st Annual ACM Sympos. Theory of Computing*, 574–586.

Chayes, L., Pemantle, R. and Peres, Y. (1997). No directed fractal percolation in zero area, *J. Stat. Phys.*, to appear.

Doyle, P. G. and Snell, J. L. (1984). *Random Walks and Electric Networks.* Mathematical Assoc. of America, Washington, DC.

Ford, L. R. Jr. and Fulkerson, D. R. (1962). *Flows in Networks.* Princeton University Press, Princeton, NJ.

Kesten, H. (1987). Percolation theory and first-passage percolation, *Ann. Probab.* **15**, 1231–1271.

Lyons, R. and Peres, Y. (1997). *Probability on Trees and Networks.* Book in preparation, draft available at URL `http://www.ma.huji.ac.il/~lyons/prbtree.html`

Department of Mathematics, Indiana University, Bloomington, IN 47405-5701, USA
`rdlyons@indiana.edu`
`http://php.indiana.edu/~rdlyons/`

Department of Mathematics, University of Wisconsin, Madison, WI 53706
`pemantle@math.wisc.edu`

Department of Statistics, University of California, Berkeley, CA 94720 and
Institute of Mathematics, The Hebrew University, Givat Ram, Jerusalem 91904, Israel
`peres@math.huji.ac.il`
`http://www.ma.huji.ac.il/~peres/`