Sojourn Times of Brownian Sheet

D. Khoshnevisan*
University of Utah

R. Pemantle†
Ohio State University

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This paper is dedicated to Professor Endré Csáki
on the occasion of his 65th birthday.

1 Introduction

Let B denote the standard Brownian sheet. That is, B is a centered Gaussian process indexed by $\mathbb{R}_+^2$ with continuous trajectories and covariance structure

$$\mathbb{E}\{B_s B_t\} = \min\{s_1, t_1\} \times \min\{s_2, t_2\}, \qquad s = (s_1, s_2),\ t = (t_1, t_2) \in \mathbb{R}_+^2.$$

In a canonical way, one can think of B as "two-parameter Brownian motion".

In this article, we address the following question: "*Given a measurable function $\upsilon : \mathbb{R} \to \mathbb{R}_+$, what can be said about the distribution of $\int_{[0,1]^2} \upsilon(B_s)\ ds$?*" The one-parameter variant of this question is both easy-to-state and well understood. Indeed, if b designates standard Brownian motion, the Laplace transform of $\int_0^1 \upsilon(b_s+x)\ ds$ often solves a Dirichlet eigenvalue problem (in x), as prescribed by the Feynman–Kac formula; cf. Revuz and Yor [6], for example. While analogues of Feynman-Kac for B are not yet known to hold, the following highlights some of the unusual behavior of $\int_{[0,1]^2} \upsilon(B_s)\ ds$ in case $\upsilon = \mathbf{1}_{[0,\infty)}$ and, anecdotally, implies that finding explicit formulæ may present a challenging task.

Theorem 1.1
There exists a $c_0 \in (0,1)$, such that for all $0 < \varepsilon < \frac{1}{8}$,

$$\exp\Big\{ -\frac{1}{c_0} \log^2(1/\varepsilon)\Big\} \leqslant \mathbb{P}\Big\{\int_{[0,1]^2} \mathbf{1}_{\{B_s>0\}}\ ds < \varepsilon\Big\} \leqslant \exp\Big\{ -c_0 \log^2(1/\varepsilon)\Big\}.$$

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Remark 1.2
By the arcsine law, the one-parameter version of the above has the following simple form: given a linear Brownian motion b,

$$\lim_{\varepsilon\to 0^+} \varepsilon^{-1/2}\mathbb{P}\left\{\int_0^1 \mathbf{1}_{\{b_s>0\}}\, ds < \varepsilon\right\} = \frac{2}{\pi};$$

see [6, Theorem 2.7, Ch. 6]. □

Remark 1.3
R. Pyke (personal communication) has asked whether $\int_{[0,1]^2} \mathbf{1}\{B_s > 0\}\, ds$ has an arcsine-type law; see [5, Section 4.3.2] for a variant of this question in discrete time. According to Theorem 1.1, as $\varepsilon \to 0$, the cumulative distribution function of $\int_{[0,1]^2} \mathbf{1}_{\{B_s>0\}}\, ds$ goes to zero faster than any power of ε. In particular, the distribution of time (in $[0,1]^2$) spent positive does not have any simple extension of the arsine law. □

Theorem 1.4
Let $\upsilon(x) := \mathbf{1}_{[-1,1]}(x)$, or $\upsilon(x) := \mathbf{1}_{(-\infty,1)}(x)$. Then, there exists a $c_1 \in (0,1)$, such that for all $\varepsilon \in (0, \frac{1}{8})$,

$$\exp\left\{ -\frac{\log^3(1/\varepsilon)}{c_1\varepsilon}\right\} \leqslant \mathbb{P}\left\{\int_{[0,1]^2} \upsilon(B_s)\, ds < \varepsilon\right\} \leqslant \exp\left\{ -c_1\frac{\log(1/\varepsilon)}{\varepsilon}\right\}.$$

For a refinement, see Theorem 2.2 below.

Remark 1.5
The one-parameter version of Theorem 1.4 is quite simple. For example, let $\Gamma = \int_0^1 \mathbf{1}_{[-1,1]}(b_s)\, ds$, where b is linear Brownian motion. In principle, one can compute the Laplace transform of Γ by means of Kac's formula and invert it to calculate its distribution function. However, direct arguments suffice to show that the two-parameter Theorem 1.4 is more subtle than its one-parameter counterpart:

$$-\infty < \liminf_{\varepsilon\to 0^+} \varepsilon \ln \mathbb{P}\{\Gamma < \varepsilon\} \leqslant \limsup_{\varepsilon\to 0^+} \varepsilon \ln \mathbb{P}\{\Gamma < \varepsilon\} < 0, \tag{1.1}$$

where ln denotes the natural logarithm function. We will verify this later on in the Appendix. □

Remark 1.6
The arguments used to demonstrate Theorem 1.4 can be used to also estimate the distribution function of additive functionals of form, e.g., $\int_{[0,1]^2} \upsilon(B_s)\, ds$, as long as $\alpha\mathbf{1}_{[-r,r]} \leqslant \upsilon \leqslant \beta\mathbf{1}_{[-R,R]}$, where $0 < r \leqslant R$ and $0 < \alpha \leqslant \beta$. Other formulations are also possible. For instance, when $\alpha\mathbf{1}_{(-\infty,r)} \leqslant \upsilon \leqslant \beta\mathbf{1}_{(-\infty,R)}$. □

2 Proof of Theorems 1.1 and 1.4

Our proof of Theorem 1.1 rests on a lemma that is close in spirit to a Feynman–Kac formula of the theory of one-parameter Markov processes.

Proposition 2.1
There exists a finite and positive constant c_2, such that for all measurable $D \subset \mathbb{R}$ and all $0 < \eta, \varepsilon < \frac{1}{8}$.

$$\mathbb{P}\Big\{\int_{[0,1]^2} \mathbf{1}_{\{B_s \notin D\}}\, ds < \varepsilon\Big\} \leqslant \mathbb{P}\Big\{\forall s \in [0,1]^2 :\ B_s \in D_{\varepsilon^{\frac{1}{4}-2\eta}}\Big\} + \exp\{-c_2 \varepsilon^{-\eta}\},$$

where D_δ denotes the δ-enlargement of D for any $\delta > 0$. That is,

$$D_\delta := \big\{x \in \mathbb{R} :\ \mathrm{dist}(x; D) \leqslant \delta\big\},$$

where 'dist' denotes Hausdorff distance.

Proof For all $t \in [0,1]^2$, let $|t| := \max\{t_1, t_2\}$. Then, it is clear that for any $\varepsilon, \delta > 0$, whenever there exists some $s_0 \in [0,1]^2$ for which $B_{s_0} \notin D_\delta$, either

1. $\sup_{|t-s| \leqslant \varepsilon^{1/2}} |B_t - B_s| > \delta$, where the supremum is taken over all such choices of s and t in $[0,1]^2$; or
2. for all $t \in [0,1]^2$ with $|t - s_0| \leqslant \varepsilon^{1/2}$, $B_t \in D$, in which case, we can certainly deduce that $\int_{[0,1]^2} \mathbf{1}_{D^\complement}(B_t)\, dt > \varepsilon$.

Thus,

$$\begin{aligned}\mathbb{P}\big\{\exists s_0 \in [0,1]^2 :\ B_{s_0} \notin D_\delta\big\} \quad &\leqslant \quad \mathbb{P}\big\{\sup_{|t-s| \leqslant \varepsilon^{1/2}} |B_t - B_s| > \delta\big\} + \\ &\qquad + \mathbb{P}\Big\{\int_{[0,1]^2} \mathbf{1}_{D^\complement}(B_t)\, dt > \varepsilon\Big\}.\end{aligned}$$

By the general theory of Gaussian processes, there exists a universal positive and finite constant c_2 such that

$$\mathbb{P}\Big\{\sup_{|t-s| \leqslant \varepsilon^{1/2}} |B_t - B_s| > \delta\Big\} \leqslant \exp\big\{-c_2 \delta^2 \varepsilon^{-1/2}\big\}. \tag{2.1}$$

Although it is well known, we include a brief derivation of this inequality for completeness. Indeed, we recall C. Borell's inequality from Adler [1, Theorem 2.1]: if $\{g_t;\ t \in T\}$ is a centered Gaussian process such that $\|g\|_T = \mathbb{E}\{\sup_{t \in T} |g_t|\} < \infty$ and whenever T is totally bounded in the metric $d(s,t) = \sqrt{\mathbb{E}\{(g_t - g_s)^2\}}$ $(s,t \in T)$,

$$\mathbb{P}\{\sup_{t \in T} |g_t| \geqslant \lambda + \|g\|_T\} \leqslant 2 \exp\Big\{-\frac{\lambda^2}{2\sigma_T^2}\Big\},$$

where $\sigma_T^2 = \sup_{t \in T} \mathbb{E}\{g_t^2\}$. Eq. (2.1) follows from this by letting $T = \{(s,t) \in (0,1)^2 \times (0,1)^2 :\ |s-t| \leqslant \varepsilon^{1/2}\}$, $g_{t,s} = B_t - B_s$ and by making a few lines of standard calculations. Having derived (2.1), we can let $\delta := \varepsilon^{\frac{1}{4} - \frac{\eta}{2}}$ to obtain the proposition. $\square$

Proof of Theorem 1.1 Let $D = (-\infty, 0)$ and use Proposition 2.1 to see that

$$\mathbb{P}\big\{\int_{[0,1]^N} \mathbf{1}_{\{B_s>0\}} < \varepsilon\big\} \leqslant \mathbb{P}\big\{\sup_{s\in[0,1]^2} B_s \leqslant \varepsilon^{\frac{1}{4}-2\eta}\big\} + \exp\{-c_2\varepsilon^{-\eta}\}.$$

Thus, the upper bound of Theorem 1.1 follows from Li and Shao [4], which states that

$$\limsup_{\varepsilon\to 0^+} \frac{1}{\log^2(1/\varepsilon)} \log \mathbb{P}\big\{\sup_{s\in[0,1]^2} B_s \leqslant \varepsilon\big\} < -\infty.$$

(An earlier, less refined version, of this estimate can be found in Csáki et al. [2].) To prove the lower bound, we note that

$$\begin{aligned}
&\mathbb{P}\big\{\int_{[0,1]^2} \mathbf{1}_{\{B_s>0\}}\, ds < 2\varepsilon - \varepsilon^2\big\} \\
&\quad \geqslant \mathbb{P}\big\{\sup_{s\in[\varepsilon,1]^2} B_s < 0\big\} \\
&\quad = \mathbb{P}\big\{\forall (u,v) \in [0, \ln(\tfrac{1}{\varepsilon})]^2 :\ e^{(u+v)/2}\, B(e^{-u}, e^{-v}) < 0\big\},
\end{aligned}$$

and observe that the stochastic process $(u,v) \mapsto B(e^{-u}, e^{-v})/e^{-(u+v)/2}$ is the 2-parameter Ornstein–Uhlenbeck sheet. All that we need to know about the latter process is that it is a stationary, positively correlated Gaussian process whose law is supported on the space of continuous functions on $[0,1]^2$. We define $c_3 > 0$ via the equation

$$e^{-c_3} := \mathbb{P}\Big\{\forall (u,v) \in [0,1]^2 :\ \frac{B(e^{-u}, e^{-v})}{e^{-(u+v)/2}} < 0\Big\}.$$

By the support theorem, $0 < c_3 < \infty$; this is a consequence of the Cameron-Martin theorem on Gauss space; cf. Janson [3, Theorem 14.1]. Moreover, by stationarity and by Slepian's inequality (cf. [1, Corollary 2.4]),

$$\begin{aligned}
&\mathbb{P}\big\{\int_{[0,1]^2} \mathbf{1}_{\{B_s<0\}}\, ds < \varepsilon\big\} \\
&\quad \geqslant \prod_{0\leqslant i,j\leqslant \ln(1/\varepsilon)+1} \mathbb{P}\Big\{\forall (u,v) \in [i,i+1]\times[j,j+1] :\ \frac{B(e^{-u}, e^{-v})}{e^{-(u+v)/2}} < 0\Big\} \\
&\quad = \exp\Big\{-c_3 \ln^2(e^2/\varepsilon)\Big\}.
\end{aligned}$$

This proves the theorem. □

Next, we prove Theorem 1.4.

Proof of Theorem 1.4 Let $\mathcal{D}_\varepsilon$ denote the collection of all points $(s,t) \in [0,1]^2$, such that $st \leqslant \varepsilon$. Note that

1. Lebesgue's measure of $\mathcal{D}_\varepsilon$ is at least $\varepsilon \ln(1/\varepsilon)$; and
2. if $\sup_{s\in\mathcal{D}_\varepsilon} |B_s| \leqslant 1$, then $\int_{[0,1]^2} \mathbf{1}_{(-1,1)}(B_s)\, ds > \varepsilon \ln(1/\varepsilon)$.

Thus,

$$\mathbb{P}\Big\{ \int_{[0,1]^2} \mathbf{1}_{(-1,1)}(B_s)\ ds < \varepsilon \ln(1/\varepsilon) \Big\} \leqslant \mathbb{P}\Big\{ \sup_{s\in\mathcal{D}_\varepsilon} |B_s| > 1 \Big\}.$$

A basic feature of the set $\mathcal{D}_\varepsilon$ is that whenever $s \in \mathcal{D}_\varepsilon$, then $\mathbb{E}\{B_s^2\} \leqslant \varepsilon$. Since $\mathbb{E}\{\sup_{s\in\mathcal{D}_\varepsilon} |B_s|\} \leqslant \mathbb{E}\{\sup_{s\in[0,1]^2} |B_s|\} < \infty$, we can apply Borell's inequality to deduce the existence of a finite, positive constant $c_4 < 1$, such that for all $\varepsilon > 0$, $\mathbb{P}\{\sup_{s\in\mathcal{D}_\varepsilon} |B_s| > 1/c_4\} \leqslant \exp\{-c_4/\varepsilon\}$. We apply Brownian scaling and possibly adjust c_4 to conclude that

$$\mathbb{P}\Big\{ \sup_{s\in\mathcal{D}_\varepsilon} |B_s| > 1 \Big\} \leqslant e^{-c_4/\varepsilon}.$$

Consequently, we can find a positive, finite constant c_5, such that for all $\varepsilon \in (0, \frac{1}{8})$,

$$\mathbb{P}\{\Gamma < \varepsilon\} \leqslant \exp\Big\{ - c_5 \frac{\ln(1/\varepsilon)}{\varepsilon} \Big\}. \tag{2.2}$$

This implies the upper bound in the conclusion of Theorem 1.4. For the lower bound, we note that for all $\varepsilon \in (0, \frac{1}{8})$, Lebesgue's measure of $\mathcal{D}_\varepsilon$ is bounded above by $c_6 \varepsilon \log(1/\varepsilon)$. Thus,

$$\mathbb{P}\Big\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s)\ ds < c_6 \varepsilon \log(1/\varepsilon) \Big\} \geqslant \mathbb{P}\Big\{ \inf_{s\in[0,1]^2\setminus\mathcal{D}_\varepsilon} B_s > 1 \Big\}.$$

On the other hand, whenever $s \in [0,1]^2 \setminus \mathcal{D}_\varepsilon$, $s_1 s_2 \geqslant \varepsilon$. Thus,

$$\begin{aligned} \mathbb{P}\Big\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s)\ ds < c_6 \varepsilon \log(1/\varepsilon) \Big\} &\geqslant \mathbb{P}\Big\{ \inf_{s\in[0,1]^2\setminus\mathcal{D}_\varepsilon} \frac{B_s}{\sqrt{s_1 s_2}} > \frac{1}{\sqrt{\varepsilon}} \Big\} \\ &= \mathbb{P}\Big\{ \inf_{\substack{u,v\geqslant 0: \\ u+v\leqslant \ln(1/\varepsilon)}} O_{u,v} > \varepsilon^{-1/2} \Big\}, \end{aligned}$$

where $O_{u,v} := B(e^{-u}, e^{-v})/e^{-(u+v)/2}$ is an Ornstein–Uhlenbeck sheet. Consequently,

$$\mathbb{P}\Big\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s)\ ds < c_6 \varepsilon \log(1/\varepsilon) \Big\} \geqslant \mathbb{P}\Big\{ \inf_{0\leqslant u,v\leqslant \ln(1/\varepsilon)} O_{u,v} > \varepsilon^{-1/2} \Big\},$$

By appealing to Slepian's inequality and to the stationarity of O, we can deduce that

$$\begin{aligned} &\mathbb{P}\Big\{ \int_{[0,1]^2} \mathbf{1}_{(-\infty,1)}(B_s)\ ds < c_3 \varepsilon \log(1/\varepsilon) \Big\} \\ &\qquad \geqslant \prod_{0\leqslant i,j\leqslant \ln(1/\varepsilon)} \mathbb{P}\Big\{ \inf_{i\leqslant u\leqslant i+1} \inf_{j\leqslant v\leqslant j+1} O_{u,v} > \varepsilon^{-1/2} \Big\} \\ &\qquad = \Big[\mathbb{P}\Big\{ \inf_{0\leqslant u,v\leqslant 1} O_{u,v} > \varepsilon^{-1/2} \Big\} \Big]^{\ln^2(e/\varepsilon)}. \end{aligned} \tag{2.3}$$

On the other hand, recalling the construction of O, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\inf_{0\leqslant u,v\leqslant 1} O_{u,v} > \varepsilon^{-1/2}\Big\}\\
&\quad\geqslant \mathbb{P}\Big\{\inf_{1\leqslant s,t\leqslant e} B_{s,t} \geqslant e\ \varepsilon^{-1/2}\Big\}\\
&\quad\geqslant \mathbb{P}\Big\{B_{1,1}\geqslant 2e\ \varepsilon^{-1/2}\ ,\ \sup_{1\leqslant s_1,s_2\leqslant e}\big|B_s - B_{1,1}\big|\leqslant e\ \varepsilon^{-1/2}\Big\}\\
&\quad= \mathbb{P}\Big\{B_{1,1}\geqslant 2e\ \varepsilon^{-1/2}\Big\}\cdot\mathbb{P}\Big\{\sup_{1\leqslant s_1,s_2\leqslant e}\big|B_s - B_{1,1}\big|\leqslant e\ \varepsilon^{-1/2}\Big\}\\
&\quad\geqslant c_7\mathbb{P}\Big\{B_{1,1}\geqslant 2e\ \varepsilon^{-1/2}\Big\},
\end{aligned}$$

for some absolute constant c_7 that is chosen independently of all $\varepsilon\in(0,\frac{1}{8})$. Therefore, by picking c_8 large enough, we can insure that for all $\varepsilon\in(0,\frac{1}{8})$,

$$\mathbb{P}\Big\{\inf_{0\leqslant u,v\leqslant 1} O_{u,v} > \varepsilon^{-1/2}\Big\}\geqslant \exp\big\{-c_8\varepsilon^{-1}\big\}.$$

Plugging this in to Eq. (2.3), we obtain

$$\mathbb{P}\Big\{\int_{[0,1]^2}\mathbf{1}_{(-\infty,1)}(B_s)\ ds < c_6\varepsilon\log(1/\varepsilon)\Big\}\geqslant\exp\Big\{-c_8\frac{\ln^2(1/\varepsilon)}{4\varepsilon}\Big\}. \tag{2.4}$$

The lower bound of Theorem 1.4 follows from replacing ε by $\varepsilon/\ln(1/\varepsilon)$. $\square$

The methods of this proof go through with few changes to derive the following extension of Theorem 1.4.

Theorem 2.2
Suppose $\varphi:\mathbb{R}_+\to\mathbb{R}_+$ is a measurable function such that (a) as $r\downarrow 0$, $\varphi(r)\uparrow\infty$; and (b) there exists a finite constant $\gamma>0$, such that for all $r\in(0,\frac{1}{2})$, $\varphi(2r)\geqslant\gamma\varphi(r)$. Define $J_\varphi=\int_{[0,1]^2}\mathbf{1}_{\{|B_s|\leqslant\sqrt{s_1s_2}\varphi(s_1s_2)\}}\ ds$. Then, there exist a finite constant $c_9>1$, such that for all $\varepsilon\in(0,\frac{1}{2})$,

$$\exp\Big\{-c_9\varphi^2\big(\frac{\varepsilon}{\log(1/\varepsilon)}\big)\log^2(1/\varepsilon)\Big\}\leqslant\mathbb{P}\big\{J_\varphi<\varepsilon\big\}\leqslant\exp\Big\{-\frac{1}{c_9}\varphi^2\big(\frac{\varepsilon}{\log(1/\varepsilon)}\big)\Big\}.$$

Appendix: On Remark 1.5

In this appendix, we include a brief verification of the exponential form of the distribution function of Γ; cf. Eq. (1.1). Given any $\lambda>\frac{1}{2}$ and for $\zeta=(2\lambda)^{-1/2}$, we have

$$\begin{aligned}
\mathbb{E}\{e^{-\lambda\Gamma}\} &\leqslant \mathbb{E}\Big\{\exp\big(-\lambda\textstyle\int_0^\zeta v(b_s)\ ds\big)\Big\}\\
&\leqslant e^{-\lambda\zeta}+\mathbb{P}\{\sup_{0\leqslant s\leqslant\zeta}|b_s|>1\} \qquad (2.5)\\
&\leqslant e^{-\lambda\zeta}+e^{-1/(2\zeta)}\\
&= 2e^{-\sqrt{\lambda/2}}. \qquad (2.6)
\end{aligned}$$

By Chebyshev's inequality, $\mathbb{P}\{\int_0^1 \upsilon(b_s)\ ds < \varepsilon\} \leqslant 2 \inf_{\lambda>1} e^{-\sqrt{\lambda/2}+\lambda\varepsilon}$. Choose $\lambda = \frac{1}{8}\varepsilon^{-2}$ to obtain the following for all $\varepsilon \in (0, \frac{1}{2})$:

$$\mathbb{P}\{\Gamma < \varepsilon\} \leqslant 2e^{-1/(8\varepsilon)}. \tag{2.7}$$

Conversely, we can choose $\delta = (2\lambda)^{-1/2}$ and $\eta \in (0, \frac{1}{100})$ to see that

$$\begin{aligned} \mathbb{E}\{e^{-\lambda\Gamma}\} &\geqslant \mathbb{E}\Big\{\exp\big(-\lambda \textstyle\int_0^\delta \upsilon(b_s)\ ds\big);\ \inf\limits_{\delta \leqslant s \leqslant 1} |b_s| > 1\Big\} \\ &\geqslant e^{-\lambda\delta}\ \mathbb{P}\big\{|b_\delta| > 1+\eta\ ,\ \sup\limits_{\delta<s<1+\delta} |b_s - b_\delta| < \eta\big\}. \end{aligned}$$

Thus, we can always find a positive, finite constant c_{10} that only depends on η and such that

$$\mathbb{E}\{e^{-\lambda\Gamma}\} \geqslant c_{10} \exp\Big\{-\sqrt{\frac{\lambda}{2}}\ \big[1 + (1+\eta)^2(1+\psi_\delta)\big]\Big\},$$

where $\lim_{\delta\to 0^+} \psi_\delta = 0$, uniformly in $\eta \in (0, \frac{1}{100})$. In particular, after negotiating the constants, we obtain

$$\liminf_{\lambda\to\infty} \lambda^{-1/2} \ln \mathbb{E}\{e^{-\lambda\Gamma}\} \geqslant -2^{1/2}. \tag{2.8}$$

Thus, for any $\varepsilon \in (0, \frac{1}{100})$,

$$e^{-\sqrt{2\lambda}(1+o_1(1))} \leqslant \mathbb{E}\{e^{-\lambda\Gamma}\} \leqslant \mathbb{P}\{\Gamma < \varepsilon\} + e^{-\lambda\varepsilon},$$

where $o_1(1) \to 0$, as $\lambda \to \infty$, uniformly in $\varepsilon \in (0, \frac{1}{100})$. In particular, if we choose $\varepsilon = (1+\eta)\sqrt{2/\lambda}$, where $\eta > 0$, we obtain

$$\mathbb{P}\{\Gamma < (1+\eta)\sqrt{2/\lambda}\} \geqslant e^{-\sqrt{2\lambda}(1+o_2(1))},$$

where $o_2(1) \to 0$, as $\lambda \to \infty$. This, Eq. (2.7) and a few lines of calculations, together imply Eq. (1.1). $\square$

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Davar Khoshnevisan
University of Utah
Department of Mathematics
155 S 1400 E JWB 233
Salt Lake City, UT 84112–0090
davar@math.utah.edu
http://www.math.utah.edu/~davar

Robin Pemantle
Ohio State University
Department of Mathematics
231 W. 18 Ave., Columbus, OH 43210
Columbus, OH 43210
pemantle@math.ohio-state.edu
http://www.math.ohio-state.edu/~pemantle