LOCAL CHARACTERISTICS, ENTROPY AND LIMIT THEOREMS FOR SPANNING TREES AND DOMINO TILINGS VIA TRANSFER-IMPEDANCES

Running Head: LOCAL BEHAVIOR OF SPANNING TREES

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ABSTRACT:
Let G be a finite graph or an infinite graph on which $\mathbb{Z}^d$ acts with finite fundamental domain. If G is finite, let $\mathbf{T}$ be a random spanning tree chosen uniformly from all spanning trees of G; if G is infinite, methods from [Pem] show that this still makes sense, producing a random *essential spanning forest* of G. A method for calculating local characteristics (i.e. finite-dimensional marginals) of $\mathbf{T}$ from the *transfer-impedance* matrix is presented. This differs from the classical matrix-tree theorem in that only small pieces of the matrix (n-dimensional minors) are needed to compute small (n-dimensional) marginals. Calculation of the matrix entries relies on the calculation of the Green's function for G, which is not a local calculation. However, it is shown how the calculation of the Green's function may be reduced to a finite computation in the case when G is an infinite graph admitting a Z^d-action with finite quotient. The same computation also gives the entropy of the law of $\mathbf{T}$.

These results are applied to the problem of tiling certain lattices by dominos – the so-called *dimer problem*. Another application of these results is to prove modified versions of conjectures of Aldous [Al2] on the limiting distribution of degrees of a vertex and on the local structure near a vertex of a uniform random spanning tree in a lattice whose dimension is going to infinity. Included is a generalization of moments to tree-valued random variables and criteria for these generalized moments to determine a distribution.

Keywords: spanning tree, transfer-impedance, domino, dimer, perfect matching, entropy

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1 Introduction

We discuss spanning trees and domino tilings (perfect matchings) of periodic lattices. To define these terms, let $S = \{1, \ldots, k\}$ be a generic k-element set and let G be a graph whose vertex set is $\mathbb{Z}^d \times k$, i.e. its vertices are all pairs (x, i) where $x = (x_1, \ldots, x_d)$ is a vector of integers of length d and i is an integer between 1 and k. We will usually allow G to denote the vertex set $\mathbb{Z}^d \times S$, since this causes no ambiguity. We say that G is *periodic* if its edge set is invariant under the natural $\mathbb{Z}^d$-action; in other words we require that (x, i) is connected to (y, j) (written $(x, i) \sim (y, j)$) if and only if $(0, i) \sim (y - x, j)$. Assume throughout that G is connected and locally finite. By periodicity there is a maximal degree D of vertices of G. Graphs that we consider may have parallel edges, in other words more than one edge may connect the same two vertices. It is convenient to add self-edges – edges connecting a vertex to itself – until all vertices have degree D; such a graph is called *D-regular.* Adding self-edges does not alter any of the problems we address, so we assume throughout that all vertices of G have degree D. It will also be convenient to assume that simple random walk on G is aperiodic. Since this is true whenever G has a self-edge, we assume the presence of at least one self-edge.

A spanning tree of any graph is subcollection of the edges having no loops, but such that every pair of vertices is connected within the subcollection. A loopless subgraph that is not necessarily connected is called a forest, and a forest in which every vertex is connected to infinitely many others is called an essential spanning forest. It is shown in [Pem] that the uniform measures on spanning trees of a cube of finite size n in the integer lattice $\mathbb{Z}^d$ converge weakly as $n \to \infty$ to a measure $\mu_{\mathbb{Z}^d}$ on essential spanning forests of $\mathbb{Z}^d$. This measure chooses a spanning tree with probability one if $d \leq 4$ and with probability zero if $d \geq 5$. The first purpose of the present work is to show how the finite dimensional marginals and the entropy of the limiting measure $\mu_{\mathbb{Z}^d}$ may be effectively computed. Since the computations may be carried out in the more general setting of periodic lattices, and since some of these (e.g. the hexagonal lattice in the plane) seem as interesting as $\mathbb{Z}^d$ from the point of view of physical modelling, we treat the problem in this generality. The methods of [Pem] extend to the case of arbitrary periodic graphs to show that μ_G chooses a spanning tree with probability one if $d \leq 4$ and zero if $d \geq 5$. We will not re-prove this result in the more general setting, since that would involve a completely staight-forward but lengthy redevelopment of the theory of loop-erased random walks [La2] for periodic graphs.

A domino tiling of a graph is a partition of the vertices into sets of size two, each set containing two adjacent vertices. Domino tilings on $\mathbb{Z}^2$ have been studied [Kas] and the exponential growth rates of the number of tilings of large regions with various boundary conditions have been calculated. The growth rates for domino tilings are different for different boundary conditions [Kas, TF, Elk]. The second purpose of this work is to exhibit the domino tiling of maximal entropy for each periodic lattice in a special class (that includes $\mathbb{Z}^2$), and to compute the entropy. We exploit a general version of a known connection between domino tilings and spanning forests, so that this follows more or less immediately from the results on spanning forests. The correspondence also gives a way to calculate probabilities of various contours arising in a uniform random domino tiling. The first few of these are calculated by Fisher [Fi1, Fi2] using Pfaffians.

The method we use to calculate the f.d.m.'s of μ_G is to calculate the Green's function for G and then to write the f.d.m.'s as determinants of the *transfer-impedance matrix*, which is a matrix of differences of the Green's function. The main result on transfer impedance matrices is stated and proved in Section 4 (Theorem 4.2) in the general setting of periodic lattices. Since the result is interesting in itself and is for the rest of the paper *sine qua non*, we state here a simplified version for finite graphs.

Theorem 1.1 *Let G be a finite D-regular graph. Fix an arbitrary orientation of the edges of G and for edges $e = \vec{xy}$ and f of G define $H(e, f)$ to be the expected signed number of transits of f by a random walk started at x and stopped when it hits y (this can be written as a difference of Green's functions). For edges $e_1, \ldots, e_k$ let $M(e_1, \ldots, e_k)$ denote the matrix whose i, j-entry is $H(e_i, e_j)$. If $\mathbf{T}$ is a random spanning tree, uniformly distributed among all spanning trees of G, then*

$$\mathbf{P}(e_1, \ldots, e_k \in \mathbf{T}) = \det M(e_1, \ldots, e_k).$$

$\square$

The idea of a transfer-impedance matrix is not new, the terminology being taken from [Wei]. We have not, however, been able to find the key result (Theorem 4.2) on determinants of the transfer-impedance matrix stated anywhere. Furthermore, extending results about transfer-impedances from finite graphs to infinite graphs is not immediate (at least when simple random walk on the infinite graph is transient) and requires an argument based on triviality of the Poisson boundary. For these reasons, we include a derivation of all results on transfer-impedances from scratch.

The rest of the paper is organized as follows. The next section contains the notation used in the rest of the paper and a derivation of the Green's function for a periodic lattice. Section 3 contains lemmas, such as a discrete Harnack's principle, about simple random walks on periodic lattices. Rather than providing detailed proofs, we include in an appendix the outline of a standard proof for the case $G = \mathbb{Z}^d$

and indicate the necessary modifications for arbitrary periodic lattices. The connection between simple random walks and spanning trees is documented in [Pem]; the main result that will be used from there is that $\mathbf{P}(e \in \mathbf{T})$ is determined by certain hitting probabilities, but the reader desiring more details may also consult [Al2] or [Bro]. Section 4 uses these lemmas to show that the Green's function is the unique limit of Green's functions on finite subgraphs and that it in fact determines the f.d.m.'s for μ_G via determinants of the transfer-impedance matrix. Section 5 considers two examples. The first is the case $G = \mathbb{Z}^2$, which is special because the Green's function for general periodic lattices is given by a definite integral which is only explicitly evaluable when $G = \mathbb{Z}^2$. The second is the high dimensional limit of $G = \mathbb{Z}^d, d \to \infty$, which converges in a sense to be defined later to a critical Galton-Watson Poisson(1) branching process, in accordance with a conjecture of Aldous [Al1]. Section 6 calculates the entropy of μ_G. Section 7 discusses the connection between spanning trees of a lattice and domino tilings of the join of a lattice and its dual. From this follows a determination of the topological entropy for domino tilings of graphs that are joins of a periodic lattice and its dual. It is also possible from this to exhibit f.d.m.'s of the maximal entropy domino tiling in a few special cases.

Certain characterizations of the Green's function and Harnack principles are required that are essentially adaptations of known results on $\mathbb{Z}^d$ to arbitrary periodic lattices. These adaptations are treated briefly in the appendix. Also given in the appendix are criteria for determining limits of probability distributions on trees from knowledge of certain functionals which act as generalized moments.

2 Notation and a Green's Function

Let G be a periodic lattice with the assumptions of connectedness, D-regularity and there being at least one loop, as in the previous section. Let μ_G denote the weak limit as $n \to \infty$ of the uniform measures on spanning trees of the induced subgraph on G with vertices $\{(x,i) : \| x \|_\infty \leq n\}$. The arguments in [Pem] show that this limit exists because the probability of the elementary event of a finite set of edges all being in the tree is always decreasing in n and because the measures of these elementary events determine μ_G. It follows from Corollary 3.4, the random walk construction of spanning trees and the first equality of Lemma 4.3 that, as in the case where $G = \mathbb{Z}^d$, the limit can be taken independent of the boundary conditions, i.e. the limit for induced subgraphs is the same as for tori. For an edge e, we often write $\mathbf{P}(e \in \mathbf{T})$ for $\mu\{\mathbf{T} : e \in \mathbf{T}\}$.

For any finite set of edges $e_1, \ldots, e_k$ that form no loop among them there is a graph $G/e_1, \ldots, e_k$

called the contraction of G by $e_1, \ldots, e_k$. Its vertices are the vertices of G modulo the equivalence relation of being connected by edges in $\{e_1, \ldots, e_k\}$. Let the projection from vertices in G to vertices of $G/e_1, \ldots, e_k$ be called π. Then the edges of $G/e_1, \ldots, e_k$ are precisely one edge connecting $\pi(x)$ to $\pi(y)$ for each edge connecting x to y in G. If W is a subset of the vertices of G, the *induced subgraph* of G on W is the graph with vertex set W and an edge $\vec{xy}$ for each edge $\vec{xy} \in G$ with $x, y \in W$. For use in Section 7 we include the dual notion to contraction, namely deletion. If $e_1, \ldots, e_k$ are edges of connected graph G whose removal does not disconnect G then the deletion of G by $e_1, \ldots, e_k$, denoted $G - e_1, \ldots, e_k$ is simply G with $e_1, \ldots e_k$ removed. The salient point is that deletion and contraction by different edges commute, so that $G/e_1, \ldots, e_k - e'_1, \ldots, e'_l$ is well-defined when $e_i \neq e'_j$ for all i, j.

Let SRW_x^G denote simple random walk on G starting from x. More precisely, a path in G is a function f from the nonnegative integers to the vertices of G such that $f(i+1)$ is always adjacent to $f(i)$; SRW_x^G makes all possible intial segments of a given length equally likely. Write $\mathbf{P}(SRW_x^G(i) = y)$ for the probability that a simple random walk on G started from x is at y at time i. Either G or x may be suppressed in the notation when no ambiguity arises. For a subset B of the vertices of G, let ∂B denote the boundary of B, namely those vertices $x \in B$ that have neighbors in B^c. For any $x \notin B$, let $\tau_x^B = \inf\{j : SRW_x(j) \in B\}$ denote the (possibly infinite) hitting time of B from x.

Define the vector spaces $\mathcal{V}_S, \mathcal{V}_{\mathbb{Z}^d}$ and $\mathcal{V}_G$ to be the set of complex-valued functions on S, $\mathbb{Z}^d$ and $\mathbb{Z}^d \times S$ respectively, with pointwise addition, scalar multiplication, and the topology of pointwise convergence. For $u \in \mathcal{V}_S$ and $g \in \mathcal{V}_{\mathbb{Z}^d}$ let $u \otimes g$ denote the $f \in \mathcal{V}_G$ for which $f(x,i) = g(x)u(i)$. Let T^d denote the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, written as d-tuples of elements of $(-1/2, 1/2]$. For $\alpha \in T^d$ and $x \in \mathbb{Z}^d$, the inner product $\alpha \cdot x \stackrel{def}{=} \sum_{i=1}^d \alpha_i x_i$ is a well-defined element of $\mathbb{R}/\mathbb{Z}$; for fixed $\alpha \in T^d$, let $\xi^\alpha \in \mathcal{V}_{\mathbb{Z}^d}$ be defined by $\xi^\alpha(x) = e^{2\pi i \alpha \cdot x}$. Define the adjacency operator $A : \mathcal{V}_G \to \mathcal{V}_G$ by $(Af)(x,i) = D^{-1} \sum_{(y,j) \sim (x,i)} f(y,j)$. Here, and in all subsequent such summations, the element (y,j) is to be counted as many times as there are edges from (x,i) to (y,j). A function $f \in \mathcal{V}_G$ is called harmonic if $Af = f$ and harmonic at (x,i) if $(Af)(x,i) = f(x,i)$. We define a family of k by k adjacency matrices $\{R^x : x \in \mathbb{Z}^d\}$ by letting $R^x(i,j)$ equal one if $(0,i) \sim (x,j)$ and zero otherwise. Observe that $R^x = 0$ for all but finitely many $x \in \mathbb{Z}^d$ and that $R^{-x} = (R^x)^T$. For $\alpha \in T^d$, define the k by k matrix $Q(\alpha)$ by the essentially finite sum $D^{-1} \sum_{x \in \mathbb{Z}^d} e^{2\pi i \alpha \cdot x} R^x$.

Now we begin building Green's functions for simple random walk on G. For a transient walk, the Green's function $H(x,y)$ can be defined as the expected number of visits to y starting from x. This is symmetric, and harmonic in each argument except on the diagonal. Here, we construct a function g^f that is harmonic except at a finite number of points x, at which $(I - A)g^f(x)$ is equal to some specified

f. Later, a uniqueness theorem will show that when $f = \delta_x$ for some $x \in G$, then g^f specializes to $H(x, \cdot)$. In dimension two, simple random walk is recurrent. In this case, although the Green's function may still be defined classically by subtracting the expected number of visits from x to itself, the integral defining g^f will blow up for $f = \delta_x$. It will, however, be finite for $f = \delta_x - \delta_y$.

When G is just $\mathbb{Z}^d$ with the usual nearest-neighbor edges, the eigenfunctions in $\mathcal{V}_G$ for the adjacency operator are just the functions ξ^α. The first lemma uses these to construct eigenfunctions for A.

Lemma 2.1 *Suppose $u \in \mathcal{V}_S$ satisfies $Q(\alpha)(u) = \lambda u$ for some real λ. Then $A(u \otimes \xi^\alpha) = \lambda u \otimes \xi^\alpha$.*

Proof:

$$\begin{aligned}
A(u \otimes \xi^\alpha)(x,i) &= \frac{1}{D} \sum_{(y,j)\sim(x,i)} (u \otimes \xi^\alpha)(y,j) \\
&= \frac{1}{D} \sum_{z \in \mathbb{Z}^d, 1 \le j \le k} R^z(i,j) u_j e^{2\pi i \alpha \cdot (x+z)} \\
&= e^{2\pi i \alpha \cdot x} \sum_{1 \le j \le k} (\frac{1}{D} \sum_{z \in \mathbb{Z}^d} e^{2\pi i \alpha \cdot z} R^z)(i,j) u_j \\
&= e^{2\pi i \alpha \cdot x} \sum_{1 \le j \le k} Q(\alpha)(i,j) u_j \\
&= e^{2\pi i \alpha \cdot x} \lambda u_i \\
&= \lambda (u \otimes \xi^\alpha)(x,i) \square
\end{aligned}$$

This lemma tells us how to invert $(I - A)$ on elements of $\mathcal{V}_G$ of the form $u \otimes \xi^\alpha$ where u is an eigenvector of $Q(\alpha)$ with eigenvalue not equal to one. By representing general elements of $\mathcal{V}_G$ as integrals of eigenfunctions we can then invert $(I - A)$ on these integrals since $(I - A)^{-1}$ commutes with the integral, at least when absolute integrability conditions are satisfied. A preliminary observation is that for any $u \in \mathcal{V}_S$,

$$\int u \otimes \xi^\alpha \, d\alpha = u \otimes \delta_0,$$

where $d\alpha$ is the usual Haar measure on T^d. To see this, note that the integrand is bounded in magnitude by $|u|$, so the integral makes sense and integrating pointwise gives

$$\int u \otimes \xi^\alpha(x,i)\, d\alpha = u_i \int e^{2\pi i \alpha \cdot x} d\alpha = u_i \delta_0(x).$$

Inverting $(I-A)$ on elements of $\mathcal{V}_G$ with finitely many nonzero coordinates is easily reduced to inverting $(I-A)$ on things of the form $u \otimes \delta_0$ and their translates by elements of $\mathbb{Z}^d$. The above representation shows that these are integrals of $u \otimes \xi^\alpha$ which are sums of eigenfunctions (the eigenfunctions given by letting u be an eigenvector of $Q(\alpha)$). So the above representation solves the problem as long as the inverted integrand, $(1-\lambda)^{-1} u \otimes \xi^\alpha$ remains integrable. With this in mind, observe that $Q(\alpha)$ is Hermitian for each α. It is therefore diagonalizable with real eigenvalues and has a unitary basis of eigenvectors. Let $\{v(\alpha,i) : \alpha \in T^d, 1 \leq i \leq k\}$ denote a measurable selection of ordered eigenbasis for $Q(\alpha)$ and let $\lambda(\alpha,i)$ be the eigenvalue corresponding to $v(\alpha,i)$. Now for $u \in \mathcal{V}_S$, let $c^u(\alpha,i) = \ll u, v(\alpha,i) \gg$ denote the coefficients of u in the chosen eigenbasis, in other words

$$\sum_i c^u(\alpha,i) v(\alpha,i) = u$$

for each $\alpha \in T^d$. Then we have the following theorem.

Theorem 2.2 *For $u \in \mathcal{V}_S$, let*

$$g^u = \int \sum_{i=1}^{k} Re \left\{ c^u(\alpha,i)(1-\lambda(\alpha,i))^{-1} v(\alpha,i) \otimes \xi^\alpha \right\} \, d\alpha \tag{1}$$

and for $j \leq d$, define

$$g^{u,j} = \int \sum_{i=1}^{k} Re \left\{ c^u(\alpha,i)(1-\lambda(\alpha,i))^{-1}(1-e^{-2\pi i \alpha_j}) v(\alpha,i) \otimes \xi^\alpha \right\} \, d\alpha. \tag{2}$$

The integrals are meant pointwise, i.e. as defining $g^u(x)$ and $g^{u,j}(x)$ for each $x \in G$. Then

(i) The integrand in (2) is always integrable and the integrand in (1) is integrable when $d \geq 3$ or when $d = 1$ or 2 and $\sum_i u_i = 0$.

(ii) $(I-A)g^u = u \otimes \delta_0$ and $(I-A)g^{u,j} = u \otimes (\delta_0 - \delta_{e_j})$ whenever the integrals exist, e_j being the j^{th} standard basis vector in $\mathbb{Z}^d$.

(iii) g^u and $g^{u,j}$ are bounded whenever the defining integrals exist.

Remark: When $d \geq 2$, it is not necessary to take the real part in (1) and (2) since the imaginary part integrates to zero. When $d = 1$ however, the imaginary part fails to be integrable.

For $f, g \in \mathcal{V}_G$, say f is a translate of g if $f(x,i) = g(x+a,i)$ for some $a \in \mathbb{Z}^d$. If $f \in \mathcal{V}_G$ has finitely many nonzero coordinates, then it can be represented as a finite sum of translates of elements of the form $u \otimes \delta_0$. Further, if the sum of $f(x,i)$ is zero, then f can be represented as the sum of translates of elements $u \otimes \delta_0$ for which $\sum u_j = 0$ together with translates of elements $u \otimes (\delta_0 - \delta_{e_j})$. Thus the following is an immediate corollary.

Corollary 2.3 *Let $f \in \mathcal{V}_G$ have finitely many nonzero coordinates. If $d \geq 3$ or $d = 1$ or 2 and $\sum_{x \in G} f(x) = 0$, then the previous two theorems can be used to construct a bounded solution g to $(I - A)g = f$.* □

The proof of Theorem 2.2 depends on the following lemma which bounds the eigenvalues of $Q(\alpha)$ away from 1 in terms of $|\alpha|$ in order to get the necessary integrability results.

Lemma 2.4 *There is a constant $K = K(G)$ for which $\max_i |\lambda(\alpha,i)| \leq 1 - K|\alpha|^2$, where for specificity we take $|\alpha| = \max_j |\alpha_j|$.*

Proof: The eigenvalues of a matrix are continuous functions of its entries [Kat], so it suffices to show that this is true in a neighborhood of 0 and to show that $\lambda(\alpha,i) \neq 1$ for $\alpha \neq 0$. For the first of these, it suffices to find for each $s \leq d$ a constant K_s for which the eigenvalues of $Q(\alpha)$ are bounded in magnitude by $1 - K_s|\alpha_s|^2$. So fix an $s \leq d$.

Begin with a description of the entries of $Q(\alpha)^r$. The quantity $Q(\alpha)(i,j)$ is the sum over edges of G connecting $(0,i)$ to (x,j) for $x \in \mathbb{Z}^d$ of complex numbers of modulus $1/D$. Furthermore, as i or j varies with the other fixed, there are precisely D of these paths. It follows that $Q(\alpha)^r(i,j)$ is the sum over paths of length r connecting $(0,i)$ to some (x,j) of complex numbers of modulus D^{-r} and that there are D^r of these contributions in every row and every column.

Suppose we can find an $r = r(s)$ such that for every $i \leq k$ there is a path of length r from $(0,i)$ to (e_s,i) where e_s is the s^{th} standard basis vector. Since there is a self-edge at every vertex, there is perforce a path of length r from $(0,i)$ to $(0,i)$. These two paths represent summands in the above decompositon of $Q(\alpha)^r(i,i)$ whose arguments differ by α_s. By the law of cosines, the sum of these two terms has magnitude $D^{-r}(2+2\cos(\alpha_s))^{1/2} \leq 2D^{-r}(1-c\alpha_s^2)$ for any $c < 1/8$ and α in an appropriate neighborhood of zero. Adding in the rest of the terms in the i^{th} row of $Q(\alpha)^r$ and using the triangle inequality shows that the sum of the magnitudes of the entries in the i^{th} row is most $1-2cD^{-r}\alpha_s^2$ for α_s in a neighborhood of zero. The usual Perron-Frobenius argument then shows that no eigenvalue has greater modulus than the maximal row sum of moduli. This implies that in an appropriate neighborhood of zero, no eigenvalue of $Q(\alpha)$ has modulus greater than $(1-2cD^{-r}\alpha_s)^{1/r} \leq 1-K_s\alpha_s^2$ where $K_s = 2cD^{-r(s)}/r(s)$, which is the bound we wanted.

Finding such an r is easy. Since G is connected there is for each for each $i \leq k$ a path of some length l_i from $(0,i)$ to (e_s,i). These can be extended to paths of any greater length by including some self-edges, so r can be chosen as the maximum over i of l_i. All that remains is to show that the only time $Q(\alpha)$ has an eigenvalue of 1 is when $\alpha = 0$. This is essentially the same argument. Picking $r' \geq \max_s r(s)$, the row sums of the moduli of of the entries of $Q(\alpha)^{r'}$ are strictly less than one and the Perron-Frobenius argument shows that no eigenvalue has modulus one or greater. $\square$

Proof of Theorem 2.2: First we establish when the integrands in (1) and (2) are integrable. For each α the vectors $v(\alpha,i)$ form a unitary basis, hence the coefficients $c^u(\alpha,i)$ are bounded in magnitude by $|u|$. Since $v(\alpha,i)$, ξ^α and $1-e^{-2\pi\alpha_j}$ all have unit modulus, integrability of $|(1-\lambda(\alpha,i))^{-1}|$ is certainly sufficient to imply integrability of (1) and (2). For $d \geq 3$, this now follows immediately from $|(1-\lambda(\alpha,i))^{-1}| \leq K|\alpha^{-2}|$.

For $d = 2$, $|\alpha|^{-2}$ is not integrable, so we will find another factor in the integrands that is $O(|\alpha|)$. By aperiodicity of SRW on G, we also have that the projected SRW on S is aperiodic, and therefore that the eigenvalue of 1 when $\alpha = 0$ is simple with eigenvector $v(0,1) = D^{-1/2}(1,\ldots,1)$. (Here we assume without loss of generality that the eigenvectors have been numbered so that for α in a neighborhood of zero, $v(\alpha,1)$ is an eigenvector whose eigenvalue has maximum modulus.) The assumption $\sum_i u_i = 0$ in (1) then implies that $c^u(0,1) = 0$. By analyticity of the eigenbasis with respect to the entries of the matrix (at least away from multiple eigenvalues) [Kat], $c^u(\alpha,1) = O(|\alpha|)$. Thus $|c^u(\alpha,i)(1-\lambda(\alpha,i))^{-1}| \leq K'|\alpha|^{-1}$ which implies integrabilty of (1) when $d = 2$. Similarly, $(1-e^{2\pi i\alpha_j}) = O(|\alpha|)$, which implies integrability of (2) when $d = 2$.

When $d = 1$, we will show that the real parts of

$$[c^u(\alpha,1)v(\alpha,1) \otimes \xi^\alpha](x,i) \tag{3}$$

for $\sum_i u_i = 0$ and

$$[c^u(\alpha,1)v(\alpha,1)(1-e^{-2\pi i\alpha}) \otimes \xi^\alpha](x,i) \tag{4}$$

for any u are both $O(|\alpha|^2)$ as $\alpha \to 0$ for fixed $(x,i) \in G$. Clearly, this is enough to imply integrability of (1) and (2). Observe that (3) and (4) are both zero when $\alpha = 0$. By analyticity of $v(\alpha,1)$ at zero, it suffices to show that derivatives of (3) and (4) with respect to α at zero are purely imaginary. Taking (4) first, we have

$$\begin{aligned} & \frac{d}{d\alpha}\left(c^u(\alpha,1)v(\alpha,1)(1-e^{-2\pi i\alpha}) \otimes \xi^\alpha(x,i)\right)|_{\alpha=0} \\ = \quad & 2\pi i c^u(0,1)v(0,1) \otimes \xi^0(x,i) = 2\pi i \ll u, v(0,1) \gg v(0,1)_i \in \sqrt{-1}\mathbb{R}, \end{aligned}$$

since the factor of $1-e^{-2\pi i\alpha}$ kills all the other terms in the derivative when $\alpha = 0$. For (3), we get

$$\begin{aligned} & \frac{d}{d\alpha}\left(c^u(\alpha,1)v(\alpha,1) \otimes \xi^\alpha(x,i)\right)|_{\alpha=0} \\ = \quad & [v(0,1) \otimes \xi^0](x,i)\frac{d}{d\alpha}c^u(\alpha,1)|_{\alpha=0} \\ = \quad & v(0,1)_i \ll u, (d/d\alpha)|_{\alpha=0} v(\alpha,1) \gg \end{aligned}$$

so it suffices to show that $v(\alpha,1)$ has imaginary derivative at zero.

Observe first that $Q(\alpha)$ has imaginary derivative at $\alpha = 0$ since the entries of $Q(\alpha)$ are all sums of $e^{2\pi i\alpha x}$ for various $x \in \mathbb{Z}$. Call this imaginary derivative R. Secondly, observe that the derivative of $\lambda(\alpha,1)$ vanishes at $\alpha = 0$ since $\lambda(\alpha,1)$ is real and attains its maximum at $\alpha = 0$. Then letting w denote the derivative of $v(\alpha,1)$ at $\alpha = 0$,

$$[Q + \epsilon R + O(\epsilon^2)](v + \epsilon w + O(\epsilon^2)) = (1 + O(\epsilon^2))(v + \epsilon w + O(\epsilon^2))$$

from which it follows that

$$Rv = (I - Q)w + O(\epsilon).$$

Letting $\epsilon \to 0$ gives $Rv = (I-Q)w$. Since R is imaginary and I, Q and v are real, it follows that w is imaginary. This shows that the real part of (3) is $O(|\alpha|^2)$ and completes the proof of (i).

The above argument actually also establishes boundedness of g^u and $g^{u,j}$ when $d \geq 2$, but we give a different probabilistic argument since it is necessary to do so anyway for the case $d = 1$. Let $s \in (0,1)$ be a real parameter and consider the functions g^u_s and $g^{u,j}_s$ gotten by replacing $\lambda(\alpha,i)$ by $s\lambda(\alpha,i)$ in (1) and (2). The integrands are *a fortiori* absolutely integrable, being bounded in magnitude by $|1-s|^{-1}$ times a possible factor of 2 for the $1-e^{-2\pi i\alpha_j}$. Thus in fact $g^u_s \in l^\infty(G)$. Now taking the real part is no longer necessary, since the imaginary part is an odd function of α and must integrate to zero. We have then

$$
\begin{aligned}
g_s^u &= \int \sum_{i=1}^{k} \sum_{n=0}^{\infty} s^n \lambda(\alpha,i)^n c^u(\alpha,i) v(\alpha,i) \otimes \xi^\alpha \, d\alpha \qquad (5)\\
&= \sum_{n=0}^{\infty} s^n \int \sum_{i=1}^{k} \lambda(\alpha,i)^n c^u(\alpha,i) v(\alpha,i) \otimes \xi^\alpha \, d\alpha \\
&= \sum_{n=0}^{\infty} s^n \int \sum_{i=1}^{k} c^u(\alpha,i) A^n (v(\alpha,i) \otimes \xi^\alpha) \, d\alpha \\
&= \sum_{n=0}^{\infty} s^n A^n \left(\int \sum_{i=1}^{k} c^u(\alpha,i)(v(\alpha,i) \otimes \xi^\alpha) \right) d\alpha \\
&= \sum_{n=0}^{\infty} s^n A^n (u \otimes \delta_0).
\end{aligned}
$$

The reason A may be commuted with the sum and integral is that $Af(x)$ is a finite linear combination of terms $f(y)$ for $y \in G$, and each of these terms is integrable. In a similar manner, we get

$$g_s^{u,j} = \sum_{n=0}^{\infty} s^n A^n (u \otimes (\delta_0 - \delta_{e_j})). \qquad (6)$$

Since A^n gives the transition probabilities for an n-step simple random walk on G, this means that

$$g_s^u(x) = \sum_{n=0}^{\infty} \mathbf{E} u \otimes \delta_0(X_n)$$

where X_n is a SRW_x^G killed with probability $1-s$ at each step. Similarly,

$$g_s^{u,j}(x) = \sum_{n=0}^{\infty} \mathbf{E} u \otimes (\delta_0 - \delta_{e_j})(X_n).$$

It follows from this that $g_s^u, g_s^{u,j} \to 0$ as $x \to \infty$ for fixed s, u, j. From the forward equation for the random walk (or by direct calculation from (5) and (6)), $g_s^u = sAg_s^u + u \otimes \delta_0$ and $g_s^{u,j} = sAg_s^{u,j} + u \otimes (\delta_0 - \delta_{e_j})$, whence it follows that g_s^u (resp. $g_s^{u,j}$) cannot have a maximum or minimum except on the support of $u \otimes \delta_0$ (resp. $u \otimes (\delta_0 - \delta_{e_j})$). Since $u \otimes \delta_0$ (resp. $u \otimes (\delta_0 - \delta_{e_j})$) has finite support, say $W \subset G$, this implies that for all $y \in G$,

$$\min_{x \in W} g_s^u(x) \le g_s^u(y) \le \max_{x \in W} g_s^u(x), \qquad (7)$$

and similarly for $g_s^{u,j}$. The proof of integrability of (1) and (2) shows that $g_s^u \to g^u$ and $g_s^{u,j} \to g^{u,j}$ as $s \to 1$. Taking the limit of (7) gives

$$\min_{x \in W} g^u(x) \le g^u(y) \le \max_{x \in W} g^u(x)$$

and similarly for $g^{u,j}$, establishing (iii).

Finally to show (ii), we have from (6) that

$$(I - sA) g_s^u = u \otimes \delta_0.$$

Since $g_s^u \to g^u$ pointwise as $s \to 1$ and since $A g_s^u(x)$ is a finite sum of values $g_s^u(y)$, the limit of the LHS as $s \to 1$ exists and is equal to $(I - A) g^u$. A similar argument for $g^{u,j}$ completes the proof of (ii) and of the theorem. □

3 Simple Random Walk on G

Define the cube B_n of size n in G to be those vertices (x, i) for which $|x_j| \le n : 1 \le j \le d$. For $x \in B_n$, define the hitting distribution on the boundary, $\nu_x^{B_n}$, to be the law of $SRW_x(\tau_x^{B_n})$. Thus for example, if $x \in \partial B_n$, then $\nu_x^{B_n} = \delta_x$. For $x \notin B_n$, define the hitting distribution on the boundary to be the same, but conditioned on the SRW hitting the boundary; thus $\nu_x^{B_n}(C) = \mathbf{P}(SRW_x(\tau_x^{B_n}) \in C) / \mathbf{P}(\tau_x^{B_n} < \infty)$. For $n < m$ and $x \in \partial(B_m^c)$, let $\rho_x^{B_n B_m}$ be the hitting distribution on ∂B_n of a SRW_x conditioned never to return to B_m^c; for $m < n$ and $x \in \partial B_m$, let $\rho_x^{B_n B_m}$ be the hitting distribution on ∂B_n of SRW_x conditioned not to return to B_m. The following lemma is an adaptation of the discrete Harnack inequalities on $\mathbb{Z}^d$ for general periodic lattices. The proof is merely an adaptation of the proof for $\mathbb{Z}^d$ and will be sketched in the appendix.

Lemma 3.1 (Harnack principles) *Let G be a periodic graph satisfying the assumptions of the first section. Fix a positive integer n. Then as $m \to \infty$,*

$$(i) \qquad \max_{x,y \in B_n, z \in \partial B_m} \nu_x^{B_m}(\{z\}) / \nu_y^{B_m}(\{z\}) \to 1$$

$$(ii) \qquad \max_{x,y \in \partial B_n, z \in \partial B_m} \rho_x^{B_m B_n}(\{z\}) / \rho_y^{B_m B_n}(\{z\}) \to 1$$

$$(iii) \qquad \max_{z \in \partial B_n, x,y \in B_m^c} \nu_x^{B_n}(\{z\}) / \nu_y^{B_n}(\{z\}) \to 1$$

$$(iv) \qquad \max_{z\in\partial B_n, x,y\in\partial(B_m^c)} \rho_x^{B_nB_m}(\{z\})/\rho_y^{B_nB_m}(\{z\}) \to 1$$

(v) $\quad$ *$(i)-(iv)$ hold when G is replaced by a finite contraction of G*

$\square$

Corollary 3.2 *Bounded harmonic functions on finite contractions of periodic graphs are constant.*

Proof: For any vertex x, Let $X_0, X_1, \ldots$ be a simple random walk starting from x. If g is harmonic then $\{g(X_i)\}$ is a martingale, and if g is bounded and $x \in B_m$ then optional stopping gives $g(x) = \mathbf{E}g(X(\tau_x^{B_m})) = \int g(z)d\nu_x^{B_m}(z)$. By (i) of the previous lemma, $\nu_x^{B_m}(z) = (1+0(1))\nu_y^{B_m}(z)$ as $m \to \infty$, hence $g(x) = g(y)$ and g is constant. $\square$

Corollary 3.3 *Let G be a finite contraction of a periodic graph. Then for $f \in \mathcal{V}_G$ there is, up to an additive constant at most one bounded solution g to $(I-A)g = f$.*

Proof: If g_1 and g_2 are two solutions then $g_1 - g_2$ is a bounded harmonic function. $\square$

Corollary 3.4 *Let G be a finite contraction of a periodic graph and let G_n be the induced subgraph on B_n. For $x, y, z \in G$ with $x \sim y$ define $h(x,y,z,n) = \mathbf{P}(SRW_z^{G_n}(\tau_z^y - 1) = x)$ to be the probability that $SRW_z^{G_n}$ first hits y by coming from x (with $h(x,y,y,n) \stackrel{def}{=} 0$). Then $\lim_{n\to\infty} h(x,y,z,n)$ exists for all x, y, z.*

Proof: Fix L such that $x, y \in B_L$. For $w \in B_L$ define

$$\phi_1(x,y,w) = \mathbf{P}(\tau_w^y < \infty \text{ and } SRW_w(\tau_w^y - 1) = x)$$

and

$$\phi_2(x,y,w) = \mathbf{P}(\tau_w^y < \infty \text{ and } SRW_w(\tau_w^y - 1) \neq x).$$

For $i = 1, 2$, let $\phi_i(x,y,w,n)$ be $\phi_i(x,y,w)$ with the clause $\tau_w^y < \infty$ replaced by $\tau_w^y < \tau_w^{B_n^c}$ and observe that $\phi_i(x,y,w,n) \to \phi_i(x,y,w)$ as $n \to \infty$. From the Harnack lemma we know that $\rho_z^{B_LB_m}$ approaches a limiting measure ρ on ∂B_L as $m \to \infty$. We claim that

$$\begin{aligned} h(x,y,z,n) \quad &\to \quad \mathbf{P}(\tau_z^y < \infty \text{ and } SRW_z(\tau_z^y - 1) = x) \\ &+ \quad \mathbf{P}(\tau_z^y = \infty) \int \phi_1(w)/(\phi_1(w) + \phi_2(w))\, d\rho(w). \end{aligned} \tag{8}$$

To see this, write $h(x,y,z,n)$ as $\mathbf{P}(SRW_z^G(\tau_z^y - 1) = x \text{ and } \tau_z^y < \tau_z^{G_n}) + \mathbf{P}(\tau_z^y > \tau_z^{G_n})\mathbf{P}(SRW_z^{G_n}(\tau_z^y - 1) = x \,|\, \tau_z^y > \tau_z^{G_n})$. The first of these terms is clearly converging to the first term in (8), while the first factor of the second is converging to $\mathbf{P}(\tau_z^y = \infty)$; the second factor is a mixture over $u \in \partial B_n$ of $\mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) = x)$, so it suffices to show that this is converging to the integral in (8) uniformly in u and $n \to \infty$. Consider the sequence of times $\tau_1, \sigma_1, \tau_2, \sigma_2, \ldots$ where τ_1 is the first time that SRW_u hits B_L, σ_1 is the nest time it hits ∂B_n, τ_2 is the next time it hits B_L, and so forth. The first hitting time τ_u^y of y must satisfy $\tau_i \leq \tau_u^y < \sigma_i$ for some i. Now write

$$\mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) = x) =$$

$$\frac{\sum_{i=1}^\infty \mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) = x; \tau_i \leq \tau_u^y < \sigma_i)}{\sum_{i=1}^\infty \mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) \neq x; \tau_i \leq \tau_u^y < \sigma_i) + \mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) = x; \tau_i \leq \tau_u^y < \sigma_i)}.$$

The sum in the denominator is of course 1, but the point of writing it this way is to illustrate that for each i the ratio is approximately the integral in (8). More precisely, for fixed i the Markov property gives that $\mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) = x; \tau_i \leq \tau_u^y < \sigma_i)$ is equal to $\mathbf{P}(\tau_u^y > \sigma_{i-1})$ times a mixture over $v \in \partial B_n$ (corresponding to the last exit from ∂B_n before τ_i) of $\int \phi_1(x,y,w,n)\, d\rho_v^{B_L B_n}$. Similarly, $\mathbf{P}(SRW_u^{G_n}(\tau_u^y - 1) \neq x; \tau_i \leq \tau_u^y < \sigma_i)$ is equal to $\mathbf{P}(\tau_u^y > \sigma_{i-1})$ times a mixture over $v \in \partial B_n$ of $\int \phi_2(x,y,w,n)\, d\rho_v^{B_L B_n}$. Since $\phi_i(x,y,w,n) \to \phi_i(x,y,w)$ and $\rho_v^{B_L B_n} = (1 + o(1))\rho$ as $n \to \infty$, this shows that the ratio of the numerator to the denominator in the sum is $(1 + o(1))$ times the integral in (8) and proves the corollary. $\square$

4 Transfer Impedance

Before going into the definition of transfer impedance, it is worth pausing to remark that the functions g^u constructed in Theorem 2.2 really are versions of the Green's function. This is not essential to any of the arguments below, so the proofs are relegated to the appendix. Define the usual Green's function $H(\cdot,\cdot)$ on pairs of vertices of a periodic graph G by

$$H(x,y) = \sum_{n=0}^\infty \mathbf{P}(SRW_x(n) = y)$$

when $d \geq 3$, and

$$H(x,y) = \sum_{n=0}^\infty [\mathbf{P}(SRW_x(n) = y) - \mathbf{P}(SRW_x(n) = x)]$$

when $d = 1$ or 2. It is easy to see that the sums are finite and that $H(x, y)$ is harmonic in y except at $y = x$; it will also be shown that H is symmetric. [Later, we will use the above definition of H for finite graphs as well; see the appendix.] Now let $\mathcal{V}_0 \subseteq \mathcal{V}_G$ be the subspace of all functions with finite support. Think of the vertices of G as embedded in $\mathcal{V}_0$ by $x \mapsto \delta_x$ and the oriented edges $\vec{xy}$ as embedded n $\mathcal{V}_0$ by $\vec{xy} \mapsto \delta_x - \delta_y$. Now extend H to a bilinear map on $\mathcal{V}_0 \times \mathcal{V}_0$. Similarly, think of the functions g^u from Theorem 2.2, or in general the solution g^f to $(I - A)g = f$ from Corollary 2.3 as defining a bilinear form g on $\mathcal{V}_0 \times \mathcal{V}_0$ (or when $d \leq 2$, on part of $\mathcal{V}_0 \times \mathcal{V}_0$) by letting $g(f, \delta_x) = g^f(x)$ and extending linearly. We then have

Theorem 4.1 *$g = H$ whenever g is defined. Consequently, g is symmetric.* □

Define the *transfer impedance* of two oriented edges e and f to be $g(e, f)$. For any finite set $e_1, \ldots, e_k$ of edges, define their *transfer impedance matrix* $M = M(e_1, \ldots, e_k)$ to be the k by k matrix with $M(i, j) = D^{-1} g(e_i, e_j)$, where D is the degree of the graph G. Observe that the determinant of the transfer impedance matrix is independent of the orientation of the edges, since changing the orientation of e_i has the effect of multiplying both the i^{th} row and the i^{th} column of M by -1.

Theorem 4.2 *Let G be any periodic graph satisfying the assumptions of the first section. For $e_1, \ldots, e_k$ edges of G, pick an orientation for each edge and let M denote their transfer impedance matrix, so $DM(i, j) = g(e_i, e_j) = g^{\delta_x - \delta_y}(z) - g^{\delta_x - \delta_y}(w)$, where $e_i = \vec{xy}$ and $e_j = \vec{zw}$. (Here g may be defined by Corollary 2.3 or by extending H linearly, if H is already known.) If $\mathbf{T}$ is a uniform essential spanning forest for G, then*

$$\mathbf{P}(e_1, \ldots, e_k \in \mathbf{T}) = \det(M).$$

Here is an outline of why Theorem 4.2 is true. For an oriented edge $e = \vec{xy}$, the function $(1/D)g(e, \cdot)$ gives the voltages at vertices of G when each edge is a one-ohm resistor and one amp of current is run from x to y. A straight-forward application of Cramer's rule then produces a function $h(z)$ that is a linear combination of $g(e_i, \cdot)$ for $i = 1, \ldots, k$ and which computes the voltage for a unit current across e_k in the graph $\mathbb{Z}^d / e_1, \ldots, e_{k-1}$. By the equivalences between electrical networks, random walks and uniform spanning trees [Pem], $h(x) - h(y)$ computes the probability $\mathbf{P}(e_k \in \mathbf{T} \mid e_1, \ldots, e_{k-1} \in \mathbf{T})$. The conditional probability turns out to be $\det(M(e_1, \ldots, e_k)) / \det(M(e_1, \ldots, e_{k-1}))$ and multiplying these together gives $\mathbf{P}(e_1, \ldots, e_k \in \mathbf{T}) = \det(M)$.

Lemma 4.3 *Let G be a periodic graph and for edges $e_1, \ldots, e_k$ forming no loop let $G' = G/e_1, \ldots, e_{k-1}$. Let $e_k = \vec{xy}$ in G. Let ϕ be a bounded function on the vertices of G' harmonic everywhere except at x and y, with excess $1/D$ at x and $-1/D$ at y. If $\mathbf{T}$ is the uniform random essential spanning forest of G and $\mathbf{T}'$ is the uniform random essential spanning forest of G', then*

$$\mathbf{P}(e_k \in \mathbf{T}') = \mathbf{P}(e_k \in \mathbf{T} \,|\, e_1, \ldots e_{k-1} \in \mathbf{T}) = \phi(x) - \phi(y).$$

Proof: The first equality is standard [Pem]. For the other one, recall from [Pem] that $\mathbf{P}(e_k \in \mathbf{T}')$ is defined as the limit of $\mathbf{P}(e_k \in \mathbf{T}'_n)$ where $\mathbf{T}'_n$ is the uniform random spanning tree on $G'_n = G_n/e_1, \ldots, e_{k-1}$. This probability is just the probability that $SRW_x^{G'_n}$ first hits y by moving from x. Now Corollary 3.4 shows that the probability $h(x, y, \cdot, n)$ of $SRW^{G'_n}_{\cdot}$ first hitting y from x converges as $n \to \infty$ to some function $h(x, y, z)$. Since $h(x, y, z, n)$ is harmonic in z except at x and y, so is the limit. (In this notation, the probability we are after is $h(x, y, x)$.) Also, it is easy to see that the excess of $h(x, y, \cdot, n)$ at x is $1/D$ (use the forward equation). Thus the excess of $h(x, y, \cdot, n)$ at y must be -1 and the limit satisfies $(I - A)h(x, x, \cdot) = (\delta_x - \delta_y)/D$. By Corollary 3.3, there is only one such function up to an additive constant, so ϕ must equal $h(x, y, \cdot)$, thus $\phi(x) - \phi(y) = h(x, y, x) - h(x, y, y) = h(x, y, x)$. $\square$

Proof of Theorem 4.2: Proceed by induction on k. When $k = 1$, $\det M = M(1, 1) = [g^{\delta_x - \delta_y}(x) - g^{\delta_x - \delta_y}(y)]/D$, where $e_1 = \vec{xy}$. Denote this by $(1/D)g^{e_1}$. According to Theorem 2.3, $(1/D)g^{e_1}$ is bounded and solves $(I - A)g = [\delta_x - \delta_y]/D$, so by the previous lemma, $M(1, 1)$ calculates the probability of $e_1 \in \mathbf{T}$.

Now assume for induction that the theorem is true for $k - 1$. The easy case to dispose of is when $M' \stackrel{def}{=} M(e_1, \ldots, e_{k-1})$ has zero determinant. Then by induction $\mathbf{P}(e_1, \ldots, e_{k-1} \in \mathbf{T}) = 0$ and it follows (e.g. from the random walk construction of $\mathbf{T}$ in [Al2, Bro, Pem]) that the edges $e_1, \ldots, e_{k-1}$ form some loop. Suppose without loss of generality that the loop is given by $e_1, \ldots, e_r$ and that all edges are oriented forward along the loop. Then $\sum_{i=1}^r g^{e_i} = 0$, hence M is singular and $\det(M) = \mathbf{P}(e_1, \ldots, e_k \in \mathbf{T}) = 0$.

In the case where $\det(M') \neq 0$, the inductive hypothesis says that $\mathbf{P}(e_1, \ldots, e_{k-1} \in \mathbf{T}) = \det(M')$ and it therefore suffices to show

$$\mathbf{P}(e_k \in \mathbf{T} \,|\, e_1, \ldots, e_{k-1} \in \mathbf{T}) = \det(M)/\det(M'). \tag{9}$$

To show (9) we construct the function ϕ of Lemma 4.3. Write $x_i\vec{y}_i$ for e_i. Electrically, what we will be doing is starting with the function $(1/D)g^{e_k}$, which is the voltage function for one unit of current put in at x_k and taken out at y_k, and adjusting it by adding a linear combination of functions g^{e_j} for

$j < k$ in order to exactly cancel the current through each $e_j, j < k$. This is then the voltage function for $G/e_1, \ldots, e_{k-1}$ when one unit of current is run across $\pi(e_k)$, and thus its difference across e_k computes $\mathbf{P}(e_k \in \mathbf{T}')$.

To do this formally, let α_i for $i = 1, \ldots, k-1$ be real numbers for which $M_{ik} + \sum_{i=1}^{k-1} \alpha_i M_{ij} = 0$ for $1 \leq j \leq k-1$. These equations uniquely define the α_i because the columns of M' are linearly independent and hence there is a unique linear combination of them summing to v where $v_i = -M_{ik}$. Let N be the k by k matrix for which $N_{ij} = M_{ij}$ for $j < k$ and $N_{ik} = M_{ik} + \sum_{j=1}^{k-1} \alpha_j M_{ij}$; in other words, the first $k-1$ columns are used to zero the nondiagonal elements of the last column of M and N is the resulting matrix. Then $\det M = \det N = N_{kk} \det(N_{ij} : i,j \leq k-1) = N_{kk} \det M'$, whence $N_{kk} = \det M / \det M'$. Now define a bounded element J of $\mathcal{V}_G$ by

$$J = D^{-1} \sum_{i=1}^{k} \alpha_i g^{e_i},$$

with $\alpha_k \stackrel{def}{=} 1$. We verify that

(i) For $j < k$, $J(x_j) - J(y_j) = 0$;
(ii) $J(x_k) - J(y_k) = \det(M)/\det(M')$.

Indeed, $J(x_j) - J(y_j) = D^{-1} \sum_{i=1}^{k} \alpha_i g^{e_i}(x_j) - g^{e_i}(y_j) = D^{-1} \sum_{i=1}^{k} \alpha_i D M_{ij}$ so (i) follows from the definition of α_i and (ii) follows from the determination of N_{kk} above.

Now we have just shown that J is constant on the pre-image of any vertex under the contraction map π, and hence there is a well-defined function ϕ on the vertices of G for which $J = \phi \circ \pi$. The excess of ϕ at a point z is just the sum over edges $\vec{zw}$ in G of $\phi(z) - \phi(w)$. This is just the sum of $J(u) - J(v)$ over edges $\vec{uv}$ for which $\pi(\vec{uv}) = \vec{zw}$, which is just the sum of excess of J at u over $u \in \pi^{-1}(z)$. The excess of J at u, $(I-A)J(u)$, is just $\sum_i \alpha_i (I-A) D^{-1} g^{e_i}(u)$ which is just $D^{-1} \sum_i \alpha_i (\delta_{x_i}(u) - \delta_{y_i}(u))$. This must be summed over $\pi^{-1}(z)$, which, for $i < k$ contains x_i if and only if it contains y_i. Then the only possible nonzero contribution to the sum is $\alpha_k(\delta_{x_k}(u) - \delta_{y_k}(u))$ and summing this over $u \in \pi^{-1}(z)$ gives 1 if $z = \pi(x_k)$, -1 if $z = \pi(y_k)$ and zero otherwise. Thus ϕ satisfies the conditions of Lemma 4.3 and hence $\mathbf{P}(e_k \in \mathbf{T} \,|\, e_1, \ldots, e_{k-1} \in \mathbf{T}) = \phi(\pi(x_k)) - \phi(\pi(y_k)) = J(x_k) - J(y_k) = \det(M)/\det(M')$ by property (ii) above. This finishes the induction and the proof of the theorem. □

For ease of calculation, we derive a corollary to Theorem 4.2.

Corollary 4.4 *With $\mathbf{T}, M$ and $e_1, \ldots, e_k$ as in Theorem 4.2, pick an integer r with $0 \leq r \leq k$. Define a k by k matrix $M^{(r)}$ by $M^{(r)}(i,j) = M(i,j)$ if $i > r$ and $\delta_{ij} - M(i,j)$ if $i \leq r$. Then $\mathbf{P}(e_1, \ldots, e_r \notin \mathbf{T}, e_{r+1}, \ldots, e_k \in \mathbf{T}) = \det(M^{(r)})$.*

Proof: The assertion for $r = 0$ is just the previous theorem; now assume for induction it is true for $r - 1$. By linearity of the determinant in each row of a matrix, we have $\det M^{(r)} + \det M^{(r-1)} = \det P$ where $P(i,j) = M^{(r)}(i,j) = M^{(r-1)}(i,j)$ if $i \neq r$ and $P(i,j) = \delta_{ij}$ if $i = r$. Expanding by minors along the r^{th} row of P gives $\det P = \det M^{(r-1)}(e_1, \ldots, e_{r-1}, e_{r+1}, \ldots, e_k)$ which is equal to $\mathbf{P}(e_1, \ldots, e_{r-1} \notin \mathbf{T}, e_{r+1}, \ldots, e_k \in \mathbf{T})$ by induction. Also by induction, $\det M^{(r-1)} = \mathbf{P}(e_1, \ldots, e_{r-1} \notin \mathbf{T}, e_r, \ldots, e_k \in \mathbf{T})$, whence by subtraction, $\det M^{(r)} = \mathbf{P}(e_1, \ldots, e_r \notin \mathbf{T}, e_{r+1}, \ldots, e_k \in \mathbf{T})$, as desired. □

We end this section with a discussion of the case $d = 0$, or in other words, finite graphs. Let G be a connected aperiodic (i.e. non-bipartite) graph on the vertices $S = \{1, \ldots, k\}$, where as usual self-edges have been added to make the graph D-regular. The transition matrix A has a simple eigenvalue of 1, so it is immediate that for any u such that $\sum_i u_i = 0$ there is a solution $g^u \in \mathcal{V}_S$ to $(I - A)g^u = u$ and it is unique up to an additive constant. Defining the transfer impedance matrix by $M(\vec{xy}, \vec{zw}) = g^{\delta_x - \delta_y}(z) - g^{\delta_x - \delta_y}(w)$ as before, Lemma 4.3 shows again that $M(1,1)$ calculates $\mathbf{P}(e_1 \in \mathbf{T})$ and the induction is completed as before, showing that $\mathbf{P}(e_1, \ldots, e_k \in \mathbf{T}) = \det M(e_1, \ldots, e_k)$. Now add another self-edge to each vertex so the degrees are all $D + 1$. If A' is the new transition matrix then $I - A' = (1 - (D+1)^{-1})(I - A)$ and hence the solution $g^{u\prime}$ to $(I - A')g^{u\prime} = u$ is just $(1 + D^{-1})g^u$. Thus the new transfer impedance matrix is the same as the old one, and hence the transfer impedance matrix is independent of the degree D at which we choose to equalize the loops. The electrical explanation for this is that $M(e,f)$ is the induced voltage across f for a unit current with source x and sink y, where $e = \vec{xy}$ and every edge of G is a one ohm resistor. (To prove this just add self-edges to regularize the degree; this leaves M and the electrical properties unchanged and they now solve the same boundary value problem.) The random walk interpretation [DS] is that $M(e,f)$ is the expected number of signed transits across f of SRW_x stopped when it hits y.

In particular, suppose G is a finite, connected graph but not necessarily having vertices of the same degree. We have seen that the transfer impedances for G may be unambiguously defined as the transfer impedances for any graph that extends G to a D-regular graph for some D by addition of self-edges. Write $deg_*(x)$ for the number edges incident to v that are not self-edges. Then deg_* is invariant under degree equalization. The relevance of deg_* to transfer impedances is that for many graphs $M(e,f)$ is approximately equal to $\sum_{x \in e \cap f} deg_*(x)^{-1}$. In other words, $M(\vec{xy}, \vec{xy})$ is aproximately

$deg_*(x)^{-1} + deg_*(y)^{-1}$, $M(e,f)$ is approximately $deg_*(x)^{-1}$ if e and f intersect at x, and $M(e,f)$ is approximately zero in other cases.

To see why this should be true, consider the condition

$$deg_*(x) \geq 2\epsilon^{-1} \qquad \text{and}$$
$$(1-\epsilon)\frac{deg_*(x)}{deg_*(x)+deg_*(y)} \leq \mathbf{P}(SRW_z \text{ hits } x \text{ before } y) \leq (1+\epsilon)\frac{deg_*(x)}{deg_*(x)+deg_*(y)} \tag{10}$$

for all distinct x, y, z in some subset W of G. This says that a SRW from any point other than x or y in W will get "lost" with high probability before hitting x or y and will in fact end up choosing which to hit first in proportion to the stationary measures at the two vertices. There are many families of graphs G_k for which (10) holds with $W = G$ for a sequence ϵ_k converging to zero as $k \to \infty$; examples include the complete graph on k vertices and the k-cube.

Assume condition (10) for some $\epsilon > 0$. For some vertex x, all of whose neighbors are in W, enumerate its neighbors $y_1, \ldots, y_{deg_*(x)}$ (allow repeated neighbors if there are parallel edges). Let $e_i = \vec{xy}_i$ for $i \leq deg_*(x)$. Use the interpretation of $M(e_i, e_j)$ as the expected number of signed transits across e_j for a random walk started at x and stopped at y_i. Reversibility implies that the expected number of signed transits is zero over all times before the last visit to x, so conditioning on the first step of SRW_x and using the "craps" principle shows that for fixed i and varying j, $M(e_i, e_j)$ is proportional to $\mathbf{P}(SRW_{y_j} \text{ hits } y_i \text{ before } x)$. Applying condition (10) together with the fact that $\mathbf{P}(SRW_{y_i} \text{ hits } y_i \text{ before } x) = 1$ shows that, when $j \neq i$,

$$\begin{aligned} & \frac{(1-\epsilon)deg_*(y_i)/(deg_*(x)+deg_*(y_i))}{1+(deg_*(x)-1)(1+\epsilon)deg_*(y_i)/(deg_*(x)+deg_*(y_i))} \\ \leq M(e_i, e_j) \quad \leq \quad & \frac{(1+\epsilon)deg_*(y_i)/(deg_*(x)+deg_*(y_i))}{1+(deg_*(x)-1)(1-\epsilon)deg_*(y_i)/(deg_*(x)+deg_*(y_i))}. \end{aligned}$$

Since $deg_*(x), deg_*(y) \geq 2\epsilon^{-1}$, $deg_*(x)deg_*(y)/(deg_*(x)+deg_*(y)) \geq \epsilon^{-1}$ and the addition of 1 in the denominator of the first term in the above inequality loses no more than a factor of $1-\epsilon$, and we may rewrite the inequalities as

$$(1-\epsilon)^3 deg_*(x)^{-1} \leq M(e_i, e_j) \leq (1-\epsilon)^{-2} deg_*(x)^{-1}.$$

Similarly,

$$(1-\epsilon)^3(deg_*(x)^{-1} + deg_*(y_i)^{-1}) \leq M(e_i, e_i) \leq (1-\epsilon)^{-2}(deg_*(x)^{-1} + deg_*(y_i)^{-1}).$$

Finally, for $e = \vec{xy}$ and $f = \vec{zw}$ such that x, y, z, w and all the neighbors of x are in W, we have $M(e,f) = M(e,e)[\mathbf{P}(SRW_z \text{ hits } x \text{ before } y) - \mathbf{P}(SRW_w \text{ hits } x \text{ before } y)] \leq 2\epsilon(1-\epsilon)^{-2}(deg_*(x)^{-1} + deg_*(y)^{-1})$. This follows from the electrical interpretation of $M(e,f)$, since a unit current flow puts a voltage difference of $M(e,e)$ across e, after which the voltages elsewhere are $M(e,e)$ times the probability from there of SRW hitting x before y. Thus the mixing condition (10) does indeed imply that

$$M(e,f) = \sum_{x \in e \cap f} deg_*(x)^{-1} + O(\epsilon) \sum_{x \in e} deg_*(x)^{-1} \tag{11}$$

for $x, y, z, w \in W$ with all neighbors of x inside W.

5 An Example and a High Dimensional Limit

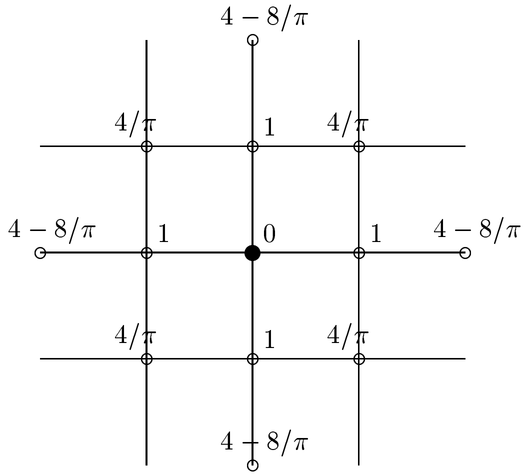
The case where G is the nearest neighbor graph for $\mathbb{Z}^2$ is special because the Green's function can be explicitly evaluated as a polynomial in π^{-1}. Following [Spi], we have that the Green's function is given by

$$H(0,x) = (2\pi)^{-2} \int \frac{1 - \cos(x \cdot \alpha)}{1 - (1/2)\cos(\alpha_1) - (1/2)\cos(\alpha_2)} \, d\alpha$$

and for $x = (n,n)$ a change of variables from (α_1, α_2) to $(\alpha_1 + \alpha_2, \alpha_1 - \alpha_2)$ yields

$$H((0,0),(n,n)) = 4\pi^{-1} \left[1 + \frac{1}{3} + \cdots + \frac{1}{2n-1} \right].$$

These values, along with the symmetries of the lattice and the fact that H is harmonic, allow H to be determined recursively, the first few values being



Let w_1, w_2, w_3, w_4 denote respectively the edges connecting the origin to $(1,0),(0,1),(-1,0),(0,-1)$. The above values for H then yield the following circulant for M:

$$M(w_1,\ldots,w_4) = \begin{bmatrix} 1/2 & 1/2-\pi^{-1} & 2\pi^{-1}-1/2 & 1/2-\pi^{-1} \\ 1/2-\pi^{-1} & 1/2 & 1/2-\pi^{-1} & 2\pi^{-1}-1/2 \\ 2\pi^{-1}-1/2 & 1/2-\pi^{-1} & 1/2 & 1/2-\pi^{-1} \\ 1/2-\pi^{-1} & 2\pi^{-1}-1/2 & 1/2-\pi^{-1} & 1/2 \end{bmatrix}.$$

Theorem 4.2 and Corollary 4.4 assert for example, that $\mathbf{P}(w_1, w_2, w_3, w_4 \in \mathbf{T})$ and $\mathbf{P}(w_1, w_2, w_3 \notin \mathbf{T},\, w_4 \in \mathbf{T})$ are given respectively by $\det M$ and $\det M^{(3)}$ respectively, where

$$M^{(3)} = \begin{bmatrix} 1/2 & -1/2+\pi^{-1} & -2\pi^{-1}+1/2 & -1/2+\pi^{-1} \\ -1/2+\pi^{-1} & 1/2 & -1/2+\pi^{-1} & -2\pi^{-1}+1/2 \\ -2\pi^{-1}+1/2 & -1/2+\pi^{-1} & 1/2 & -1/2+\pi^{-1} \\ 1/2-\pi^{-1} & 2\pi^{-1}-1/2 & 1/2-\pi^{-1} & 1/2 \end{bmatrix}.$$

The determinants of M and $M^{(3)}$ are respectively $(4\pi^{-1}-1)(2\pi^{-1}-1)^2$ and $2\pi^{-2}-4\pi^{-3}$, so the probability of all four edges incident to 0 being in $\mathbf{T}$ is $(4\pi^{-1}-1)(2\pi^{-1}-1)^2 \approx .0361$, while the probability that the origin is a leaf of $\mathbf{T}$ (i.e. has degree one in $\mathbf{T}$) is $8\pi^{-2}-16\pi^{-3} \approx .2945$.

The remainder of this section corrects, proves and generalizes some conjectures of Aldous about spanning trees for graphs as the graphs tend to infinity *locally*, in the sense that the minimum number of neighbors of a vertex all grow without bound. We first quote Conjecture 11 from [Al2]. To do this, let G_k denote a sequence of finite graphs, each with a distinguished vertex v_k. Let $r_k = deg_*(v_k)$, let A_k be the set of neighbors of v_k in G_k and for $w \in A_k$, let $\psi_k(w) = \mathbf{P}(SRW_w^{G_k}$ hits $A_k \setminus \{w\}$ before $v_k)$. Aldous then conjectures the following [Al2, Conjecture 11].

Let $1 + D_k$ denote the random degree of v_k in the uniform random spanning tree on G_k. Suppose that

$$r_k \to \infty; \sup_w |r_k(1 - \psi_k(w)) - 1| \to 0.$$

Then D_k converges in distribution to a Poisson with mean 1.

Here is a counterexample to the conjecture. Let the vertices of G_k other than v_k be $\{x_i, y_i, z_{ij} : 1 \leq i \leq k; 1 \leq j \leq 4k\}$, with edges connecting v_k to each x_i and y_i and for every i, an edge connecting x_i to each z_{ij} and an edge connecting y_i to each z_{ij}. Then $r_k = 2k$. By symmetry, we have $\psi_k(w) = \psi_k(x_1)$ for any $w \in A_k$. This is equal to $\mathbf{P}(SRW_{x_1}$ hits y_1 before $v_k) = 2k/(1 + 2k)$, hence $r_k(1 - \psi_k(w)) = 2k/(2k + 1) \to 1$ and the hypothesis in the conjecture is satisfied. But for each i, any spanning tree contains either an edge connecting v_k to x_i or an edge connecting v_k to y_i, so the degree of v_k is at least k.

Evidently, a condition different from $r_k(1 - \psi_k(w))$ converging uniformly to 1 is required for the conjecture to be true. Aldous' condition is trying to capture two aspects of the graph: some sort of mixing (SRW from any neighbor of v_k returns to v_k before A with the same probability) and the correct total probabilities (these probabilities should all be about r_k^{-1} so they can sum to 1). The mixing part of the condition as stated in the conjecture is too weak, as illustrated by the counterexample, and needs to be replaced by a condition that equalizes the individual return probabilities of SRW_w hitting v_k before z for any neighbors w, z of v_k. The most natural such condition from our viewpoint is (10) with $x = v_k$ and $\epsilon = \epsilon_k$ for some $\{\epsilon_k\}$ going to zero. On the other hand, there is no need to require all the neighbors of v_k to have the same degree. Once a sufficient amount of independence has been achieved, it suffices for the expected number of such edges in the tree to be converging to a constant, $1 + \lambda$. The conjecture may thus be revised to yield the following theorem.

Theorem 5.1 *Let G_k be a sequence of finite graphs. Let $\epsilon_k \to 0$ be a sequence of positive numbers and let v_k, A_k, D_k and r_k be as above. Assume that (11) holds with $W = A_k$ and $\epsilon = \epsilon_k$; this is implied for* *example by (10). If in addition, $\sum_{w \in A_k} 1/deg_*(w) \to 1 + \lambda$ as $k \to \infty$ for some positive λ, then D_k converges to a Poisson with mean λ.*

Remark: Since the theorem gives local behavior at v_k, it can easily be extended to the case where G_k are infinite graphs on which there is a Harnack principle: simply take G'_k to be a large enough finite piece of G_k so that the hypotheses of the theorem are true (this is possible by the Harnack principle).

The following lemma will be necessary when calculating determinants of transfer impedance matrices.

Lemma 5.2 *Let $a_1, \ldots, a_{k+1}$ be positive real constants and let $e_1, \ldots e_k$ be the edges of a spanning tree whose vertices are $\{1, \ldots, k+1\}$. Define a k by k matrix M by letting $M(i,j)$ be $a_r + a_s$ if $i = j$ and e_i connects r to s; a_r if e_i and e_j are distinct edges meeting at r; and zero otherwise. Then* $\det M = (\prod a_i)(\sum a_i^{-1})$.

Proof: If $k = 1$ then $M = (a_1 + a_2)$ and the lemma is clearly true. now assume for induction that the lemma is true for $k-1$. Assume by renumbering if necessary that the vertex 1 is a leaf of the tree, there being a single edge e_1 connecting 1 to 2. Also assume that 2 is connected to $1, 3, 4, \ldots, r$ by $e_1, \ldots, e_{r-1}$ respectively, for some $r \leq k+1$. Row reduce M by subtracting $a_2/(a_1+a_2)$ times the first row from rows $2, \ldots, r$. Since edges $2, \ldots, r$ were the only edges incident to e_1, this clears the first column. The remaining entries of M are unchanged except that all appearances of a_2 get replaced by $a_1 a_2/(a_1+a_2)$. Expanding along the new first column gives $\det M = (a_1 + a_2)$ times the determinant of the $k-1$ by $k-1$ matrix gotten by taking all but the first row and column of M and replacing a_2 by $a_1 a_2/(a_1+a_2)$. By induction, the latter determinant is $(a_1/(a_1+a_2))(\prod_{i \geq 2} a_i)((a_1+a_2)/a_1 a_2 + \sum_{i \geq 3} a_i^{-1})$, so $\det M = (\prod a_i)((a_1+a_2)/a_1 a_2 + \sum_{i \geq 3} a_i^{-1}) = \prod a_i \sum a_i^{-1}$ as desired. $\square$

Proof of Theorem 5.1: Let X be a random variable for which $X - 1$ is Poisson with mean λ. Set

$$\phi(z) = \sum \mathbf{P}(X = n) z^n = z e^{\lambda(z-1)}.$$

Then the s^{th} factorial moment $\mathbf{E}(X)_s \stackrel{def}{=} \mathbf{E} X(X-1) \cdots (X - s + 1)$ of X is the s^{th} derivative of ϕ at 1, which is equal to $\lambda^s + s\lambda^{s-1}$. The factorial moments determine this distribution uniquely, since its moment generating function exists in a neighborhood of zero, from which it follows that convergence of the factorial moments of a sequence of random variables to the factorial moments of X implies convergence in distribution to X (see for example Theorem 3.10 of [Du]). It suffices therefore to prove that $\mathbf{E}(1 + D_k)_s \to \lambda^s + s\lambda^{s-1}$ as $k \to \infty$ for each s.

Write $\mathbf{E}(1+D_k)_s$ as $\sum \mathbf{P}(e_1,\ldots,e_s \in \mathbf{T})$ where the sum is over all ordered collections $(e_1,\ldots,e_s)$ of s distinct edges incident to v_k. Fixing such a collection, we have $\mathbf{P}(e_1,\ldots,e_s \in \mathbf{T}) = \det M(e_1,\ldots,e_s)$. From (11) we have that $M(i,j) = (1+O(\epsilon_k))(deg_*(x)^{-1} + \delta_{ij} deg_*(y_i)^{-1})$ where $e_i = \vec{xy}_i$. With s staying fixed, this and Lemma 5.2 give

$$\det M = (1+o(1))(\prod_{i=1}^{s} deg_*(w_i)^{-1})(1 + r_k^{-1} \sum_{i=1}^{s} deg_*(w_i)). \qquad (12)$$

Summing over all ordered collections gives

$$\det M = (1+o(1))[\sum_{(e_1,\ldots,e_s)} \prod_{i=1}^{s} deg_*(w_i)^{-1} + \sum_{(e_1,\ldots,e_{s-1})} s(r_k - s) r_k^{-1} \prod_{i=1}^{s-1} deg_*(w_i)^{-1}]$$

since each ordered collection of size $s-1$ appears $s(r_k - s)$ times in the second term of (12) and each ordered collection of size s appears once in the first term of (12). As s remains fixed with the minimum degree among x and its neighbors converging to infinity, the above expression for $\det M$ converges to $(\sum_i deg_*(w_i)^{-1})^s + s(\sum_i deg_*(w_i)^{-1})^{s-1} \to \lambda^s + s\lambda^{s-1}$, proving the convergence of factorial moments and the theorem. $\square$

Suppose now that we are interested not only in the degree of v_k but in the local structure of the essential spanning forest near v_k. For any locally finite rooted tree $\mathbf{T}$, let $\mathbf{T} \wedge r$ denote the random finite subtree of $\mathbf{T}$ consisting of vertices connected to the root by paths of length at most r in $\mathbf{T}$. We say that a sequence of tree-valued random variables converges in distribution (written $\mathbf{T}_k \stackrel{\mathcal{D}}{\to} \mathbf{T}$) if $\mathbf{T}_k \wedge r \stackrel{\mathcal{D}}{\to} \mathbf{T} \wedge r$ for every r, where the latter is defined to hold when $\mathbf{P}(\mathbf{T}_k \wedge r = t) \to \mathbf{P}(\mathbf{T} \wedge r = t)$ for every t of height at most r. Under suitable conditions on the graphs G_k, it will turn out that the component $\mathbf{T}_k$ of the uniform essential spanning forest on G_k rooted at v_k will converge in distribution to a particular tree $\mathcal{P}_1$ which we now define. Let $\mathcal{P}_1$ be a singly infinite path, $x_0, x_1, \ldots$, to which has been added at each x_i the tree of an independent Poisson (1) branching process (which is critical hence finite with probability one). Another way of describing $\mathbf{T}$ is as the tree of a Poisson (1) branching process rooted at x_0 and conditioned to survive forever. Aldous has conjectured (personal communication, though the conjecture is implicit in [Al3]) that $\mathbf{T}_k$ converges in distribution to $\mathcal{P}_1$ whenever G_k grows locally in a sufficiently regular manner. In the terminology of [Al1], $\mathbf{T}_k$ should converge to a sin-tree with the fringe distribution of a Poisson (1) branching process. This is known in the special case where $G_k = K_k$, the complete graph on k vertices [Al3, Gr].

We are now in a position to prove this. A consequence is that the probability of there existing two disjoint paths of length L_k in $\mathbf{T}_k$ from v_k goes to zero as $k \to \infty$, provided that $L_k \to \infty$. An question left open in [Pem] is whether the components of the uniform essential spanning forest have one or two ends (the possibility of more than two is ruled out by an argument in [BK1]). We believe the answer to be that all components have one end, and convergence to zero of the probability of there being two disjoint infinite paths from v_k can be viewed as a heuristic argument in favor of all components having one end.

Theorem 5.3 *Let G_k be a sequence of finite graphs with distinguished vertices v_k. Assume, by renumbering if necessary, that*

$$(1 + o(1)) \max_{x \in G} deg_*(x) = k = (1 + o(1)) \min_{x \in G} deg_*(x)$$

as $k \to \infty$. Also assume the following version of (11) uniformly in edges $e, f \in G$:

$$M(e, f) = k^{-1}(|e \cap f| + o(1)).$$

Then $\mathbf{T}_k \stackrel{\mathcal{D}}{\to} \mathcal{P}_1$.

Remark: For $r = 1$, $\mathbf{T} \wedge r$ is a star centered at v_k with $1 + D_k$ edges, while $\mathcal{P}_1 \wedge r$ is a star centered at x_0 with $1 + X$ edges, where X is Poisson with mean 1. Thus the case $r = 1$ is essentially the previous theorem. Notice also that the usual families of graphs G_k (e.g. complete graph on k vertices, k-cube, $k/2$-dimensional torus of arbitrary length) all satisfy the hypotheses of the theorem.

Proof: For a finite rooted tree t and finite rooted graph u, say that a map f from the vertices of t to the vertices of u is a tree-map if f is injective, maps the root of t to the root of u, and $f(x) \sim f(y)$ for each $x \sim y$. Let $N(u; t)$ denote the number of distinct tree maps from t to u. For example, if t and u are stars of respective sizes s and r about their roots, then $N(u; t) = (r)_s$. The proof of this theorem generalizes the proof of the preceding theorem, in the sense that $\mathbf{E}N(\mathbf{T}_k; t)$ is a sort of generalized t^{th} moment of $\mathbf{T}_k$. In the appendix, the usual tightness criteria for convergence of probability measures are extended in an obvious way to tree-valued random variables, showing in particular (Theorem 8.7) that if $\mathbf{E}N(\mathbf{T}_k; t) \to \mathbf{E}N(\mathbf{T}; t)$ for each t and the values of $\mathbf{E}N(\mathbf{T}; t)$ uniquely determine the distribution of $\mathbf{T}$ then $\mathbf{T}_k \stackrel{\mathcal{D}}{\to} \mathbf{T}$. Also proved there is the somewhat less trivial fact (Theorem 8.8) that the values of $\mathbf{E}N(\mathbf{T}; t)$ uniquely determine the distribution of $\mathbf{T}$ under the growth condition: $\mathbf{E}N(\mathbf{T}; t) \leq e^{c|t|}$ for some c. (A sharper growth condition such as an analogue to Carleman's condition could be obtained but is not needed here.) What remains then, is to show that $\mathbf{E}N(\mathbf{T}_k; t) \to \mathbf{E}N(\mathcal{P}_1; t)$ for each finite t and to verify the growth condition on $\mathbf{E}N(\mathcal{P}_1; t)$.

Begin by establishing

$$\text{For any finite rooted tree } t, \mathbf{E}N(U;t) = 1, \tag{13}$$

where U is the tree of a Poisson (1) branching process rooted at some vertex y_0. Let z_0 be the root of t. Use induction on the height of t. If t is just z_0, then $N(u;t) = 1$ for any u, so the equation is trivially true. Now suppose z_0 has s descendants $z_1, \ldots, z_s$ for some $s > 0$ and assume for induction that (13) holds for each of the subtrees $t_i, 1 \leq i \leq s$ rooted at z_i. Let $r \geq 0$ be the random number of descendants $y_1, \ldots, y_r$ of y_0. Conditional upon r, each of the subtrees u_i rooted at y_i is the tree of an independent Poisson (1) branching process. Now any tree-map from t to U maps each t_i to a distinct u_j. Thus $N(U;t) = \sum \prod_{j=1}^s N(u_{k_j};t_j)$ where the sum is over all ordered sequences of distinct $k_1, \ldots, k_s$ chosen from among $1, \ldots, r$. Then by independence conditional on r,

$$\mathbf{E}N(U;t) = \sum_r (e^{-1}/r!) \sum \prod_{j=1}^s \mathbf{E}N(u_{k_j};t_j).$$

By the induction hypothesis each expectation is one, so the sum is just the number of ways of choosing the k_j's. Thus $\mathbf{E}N(U;t) = \sum_r (e^{-1}/r!)(r)_s = \mathbf{E}(X)_s$ where X is a Poisson of mean 1. This is equal to 1, establishing (13).

Now we compute $\mathbf{E}N(\mathcal{P}_1;t)$. Recall that $\mathcal{P}_1$ is a path $x_i : i \geq 0$ to which has been added an independent Poisson (1) branching process, say U_i at each x_i. If f is a tree-map from t into $\mathcal{P}_1$, there is a greatest i for which x_i is in the range of f. Let $w(f)$ denote the vertex of t that maps to x_i for this greatest i. For each vertex z of t, we will show that the expected number of tree-maps $f : t \to \mathcal{P}_1$ for which $w(f) = z$ is one. Indeed, if $x_0 = z_0, z_1, \ldots, z_k = z$ is the path from the root of t to z, then the tree-maps f for which $w(f) = z$ are in one to one correspondence with the collections of maps $f_0, \ldots, f_k$ where f_i maps the subtree t_i of t rooted at z_i to the subtree $\mathcal{P}_1(i)$ of $\mathcal{P}_1$ rooted at X_i. Thus the number of f for which $w(f) = z$ is $\prod_{i=0}^k N(\mathcal{P}_1(i);t_i)$. But each $\mathcal{P}_1(i)$ is an independent Poisson (1) branching process, so by equation (13), the product is one. Finally, summing over the vertices z of t gives that $\mathbf{E}N(\mathcal{P}_1;t)$ is equal to $|t|$, the number of vertices of t. This verifies the growth condition on $\mathbf{E}N(\mathcal{P}_1;t)$ with miles to spare!

To calculate $\mathbf{E}N(\mathbf{T}_k;t)$, observe that any tree-map $f : t \to \mathbf{T}$ is also a tree-map from t to G_k where G_k is considered to be rooted at v_k. If $u(f)$ denotes the image of f as a subtree of G_k, then we may write

$$\mathbf{E}N(\mathbf{T}_k;t) = \sum \mathbf{P}(u(f) \subseteq \mathbf{T}_k) \tag{14}$$

where the sum is over all tree-maps $f : t \to G_k$. Fix f. Then $\mathbf{P}(u(f) \subseteq \mathbf{T}_k) = \det M$ where M is the transfer impedance matrix for $u(f)$ as a subgraph of G. By hypothesis $M(e,e') = k^{-1}(|e \cap e'| + o(1))$.

Since f is injective, f^{-1} is defined on $u(f)$ so $|e \cap e'| = |f^{-1}(e) \cap f^{-1}(e')|$, thus for any f the transfer impedance matrix for $u(f)$ in G_k may be written as $k^{-1}P$, where P is a matrix indexed by the edges of t for which $P(e,e') = (|e \cap e'| + o(1))$ as $k \to \infty$. Since the size of P remains fixed as $k \to \infty$, Lemma 5.2 with $a_i = 1$ for all i gives that $\det P$ is equal to the $|t|(1+o(1))$, and thus $\det M = k^{1-|t|}|t|(1+o(1))$. Then the probabilities in equation (14) are all equal and the identity becomes

$$\mathbf{E}N(\mathbf{T}_k;t) = N(G_k,t)k^{1-|t|}|t|(1+o(1)).$$

But $N(G_k,t) = k^{|t|-1}(1+o(1))$. An easy way to see this is to consider building a tree-map $f: t \to G_k$ starting at the root and working outwards, not worrying about injectivity. If f is defined on z then by hypothesis there are $k(1+o(1))$ neighbors of $f(z)$ and each descendant of z may be mapped to any neighbor of $f(z)$. The fraction of all maps built this way that are injective goes to one as t remains fixed and $k \to \infty$, so the total number of maps is $k^{|t|-1}(1+o(1))$. Now (14) becomes $\mathbf{E}N(\mathbf{T}_k;t) = |t|(1+o(1))$, hence $\mathbf{E}N(\mathbf{T}_k;t) \to \mathbf{E}N(\mathcal{P}_1;t)$ as claimed. Since the growth condition on $\mathbf{E}N(\mathcal{P}_1;t)$ has been verified, this shows $\mathbf{T}_k \xrightarrow{\mathcal{D}} \mathcal{P}_1$. $\square$

6 Entropy

In this section we consider the entropy of the essential spanning forest process.

The set of essential spanning forests is a closed shift-invariant subset of $\{0,1\}^{E(G)}$ where $E(G)$ is the edge set of G. The topological entropy (per vertex) of the essential spanning forest is defined to be

$$H_{top} = \lim_{n\to\infty} \frac{1}{|B_n|} \log(N_{B_n})$$

where B_n is an increasing sequence of rectangular boxes (i.e. of the form $C \times S$ where C is a rectangular box in $\mathbb{Z}^d$ together with the induced edges) and N_B is the number of essential spanning forests of the induced graph B where a forest is essential if every component of the graph is required to touch the boundary of B. We may also consider boxes with boundary conditions, meaning a box B together with an equivalence relation $\equiv$ on the vertices of B that neighbor B^c. An essential spanning forest on a box with boundary conditions is one that becomes a tree under the contraction map consisting $B \mapsto B/\equiv$ (think of the boundary conditions as telling which vertices are connected by unseen edges in B^c).

Notice that if B is the union of two boxes C and D then any essential spanning forest of B restricts to essential spanning forests in both C and D. This means that $N_B \leq N_C N_D$ so that $\log(N_B)$ is subadditive and the entropy is independent of the sequence of boxes chosen. There is a variational principle for the topological entropy of the essential spanning forest.

$$H_{top} = \sup\{H(\mu) \mid \mu \text{ is an invariant probability measure }\}$$

where $H(\mu)$ is the Kolmogorov-Sinai entropy of μ per vertex with respect to the group of translations by $\mathbb{Z}^d$. See [Mis] for a short proof of this fact.

Now let μ_G be the probability measure of uniform essential spanning forests on G. Using arguments described in [Pem] it is seen that if B_n is any increasing sequence of rectangular boxes with arbitrary boundary conditions and if μ_n is the measure that gives equal weight to each forest in B_n in which each component of the forest meets the boundary of B_n the μ_n converges weakly to the translation invariant probability measure μ_G, moreover this convergence is uniform in the boundary conditions. By uniform convergence we mean the following. Suppose that we are given a cylinder set and an $\epsilon \geq 0$. Then there is a box B, containing the cylinder, so that for any box C containing B and any boundary conditions on C we have that the uniform probability measure on essential spanning forests of C takes a value on the cylinder set that agrees with that of μ_G to within ϵ.

The principle content of this section is the following theorem.

Theorem 6.1 *(a) The measure μ_G of the uniform essential spanning forest process is the unique translation-invariant measure on the set of essential spanning forests whose Komogorov-Sinai entropy is H_{top}.*
(b)

$$H_{top} = \frac{1}{k}\int_{T^d} \log(D^k\chi(Q(\alpha)(1)))d\alpha$$

where $\chi(Q(\alpha))$ is the characteristic polynomial of $Q(\alpha)$ and the integral is over the d-torus with respect to Haar measure.

Before proving the theorem we give some examples of (b) in which the entropy can essentially be read off. The case when $S = \{1\}$, i.e. $k = 1$ is especially easy to analyze because $Q(\alpha)$ is a 1×1 matrix. In these cases the entropies are the same as entropies calculated by Lind et al [LSW] of some seemingly unrelated dynamical systems that can be represented as Bernoulli shifts on certain subgroups of $(\mathbb{Z}^d)^{(\mathbb{R}/\mathbb{Z})}$ defined by periodic linear relations. We are at a loss to explain this apparent coincidence.

Suppose the origin is connected to M pairs of opposite vertices in $G = \mathbb{Z}^d$. Say the number of self-edges per vertex is l, though clearly this must drop out of the calculation. Suppose we denote representatives of these pairs by $\{x_m : 1 \leq m \leq M\}$. Then $D = 2M + l$ and

$$Q(\alpha) = D^{-1}(l + \sum_{m=1}^{M} 2\cos(2\pi\alpha \cdot x_m)).$$

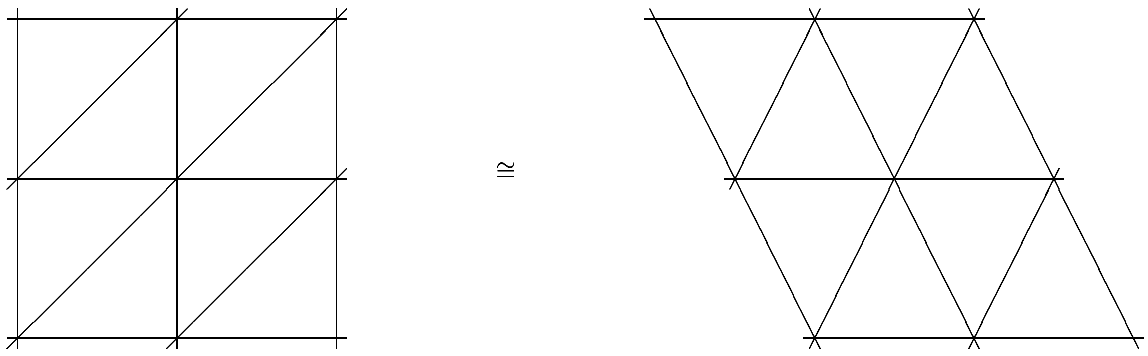
Then $\chi(Q(\alpha))(1) = 1 - Q(\alpha) = D^{-1}(\sum_{m=1}^{M} 2 - 2\cos(2\pi\alpha \cdot x_m))$ and the entropy is

$$\int_{T^d} \log(2M - \sum_{m=1}^{M} 2\cos(2\pi\alpha \cdot x_m))d\alpha.$$

One such example is $\mathbb{Z}^2$ itself with nearest neighbor edge relation. The entropy is

$$\int_0^1 \int_0^1 \log(4 - 2\cos(2\pi\alpha_1) - 2\cos(2\pi\alpha_2))d\alpha_1 d\alpha_2 \approx 1.166.$$

Another is the triangle lattice. This has a representation as the nearest neighbor lattice on $\mathbb{Z}^2$ with added edges placed in the "southwest - northeast" diagonal of each square.



In this case, ignoring self-edges, $Q(\alpha) = \frac{1}{6}(2\cos(2\pi\alpha_1) + 2\cos(2\pi\alpha_2) + 2\cos(2\pi(\alpha_1 + \alpha_2)))$ and the entropy is

$$\int_0^1 \int_0^1 \log(6 - 2\cos(2\pi\alpha_1) - 2\cos(2\pi\alpha_2) - 2\cos(2\pi(\alpha_1 + \alpha_2)))d\alpha_1 d\alpha_2 \approx 1.61.$$

These are the same as the entropies given in [LSW] for Haar measure on the subgroups of $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^2}$ consisting respectively of those configurations ϕ for which $4\phi(x) - \phi(x + (0,1)) - \phi(x + (0,-1)) - \phi(x +$

$(1,0)) - \phi(x + (-1,0)) = 0$ and those configurations ϕ for which $6\phi(x) - \phi(x + (0,1)) - \phi(x + (0,-1)) - \phi(x + (1,0)) - \phi(x + (-1,0)) - \phi(x + (1,1)) - \phi(x + (-1,-1)) = 0$.

The proof of Theorem 6.1 uses the following lemma on the stability of entropy under changes in a small percentage of the output of the process.

Lemma 6.2 *Let (Ω, μ) be a Lebesgue probability space. Suppose that $X = (X_1, X_2, \ldots, X_N)$ and $Y = (Y_1, Y_2, \ldots, Y_N)$ are binary random variables such that for all $\omega \in \Omega$, $\#\{i | X_i \neq Y_i\} \leq K$. Then*

$$|\frac{1}{N} H(X) - \frac{1}{N} H(Y)| < K \log(N)/N$$

Proof: Let $Z_i = 1_{\{X_i \neq Y_i\}}$. Then $H(X) \leq H(X, Z) = H(Y, Z) \leq H(Y) + H(Z)$. By symmetry we see that $|H(X) - H(Y)| \leq H(Z)$. But by counting $H(Z) \leq \log(N^K)$ proving the lemma. □

Proof of Theorem 6.1: Let $\tilde{B}_n$ have arbitrary boundary conditions and let B_n have the same vertex set but with unconnected boundary conditions, i.e. the equivalence relation consists of singletons. If $\tilde{\mu}_n$ gives equal probability to each spanning forest of $\tilde{B}_n$ in which each component touches the boundary then let $\tilde{\nu}_n$ be the measure concentrated on spanning trees of G obtained as follows. First partition the vertex set of G with translates of B_n and put independent copies of $\tilde{\mu}_n$ on each of these translates of B_n. Then add (at most $O(n^{d-1})$) edges in each translate of B_n to make a tree that spans this translate. Then connect each of these trees in the translates by a translation-invariant path similar to those constructed in [BK2]. This random procedure produces a random spanning tree on G with two ends whose measure is denoted $\tilde{\nu}_n$. It may be made translation-invariant by averaging the distribution over all shifts by $\mathbb{Z}^d$ shifts in B_n. By the lemma above we see that for each n

$$\frac{1}{|B_n|} H(\tilde{\mu}_n, \tilde{B}_n) \leq H(\tilde{\nu}_n) + O(\frac{n^{d-1}\log(n)}{n^d}) \leq H_{top} + O(\frac{n^{d-1}\log(n)}{n^d}).$$

Likewise it is seen that

$$\lim \frac{1}{|B_n|} H(\mu_n, B_n) = H_{top}$$

where μ_n is the uniform measure on spanning forests of B_n in which every component touches the boundary. A similar argument together with the subadditivity of entropy gives us that $H(\mu_G) = H_{top}$.

Further, if μ is any ergodic translation-invariant probability with $H(\mu) = H_{top}$ then we can show that $\mu = \mu_G$. Fix a rectangular box B. Consider a much larger C around B. Condition on the μ-outside of C and record the boundary condition $x \equiv y$ iff x and y are connected by a path in C^c. Since μ has maximal entropy the conditional distribution of μ on B is the same as the distribution of uniform essential spanning forests with these boundary conditions. (Were this not true we would be able to modify μ within such boundaries and force the entropy to be strictly larger.) If the outside box is large enough then we see (again using the arguments in [Pem]) that the conditional distribution of μ on B is very close to the distribution of μ_G on B uniformly in the boundary conditions. Integrating this μ-conditional distribution with respect to the μ-outside of the large box gives us that μ-distribution on B is very close to the μ_G-distribution on B. Taking limits gives $\mu = \mu_G$. This proves part (a) of the Theorem and leaves only the computation in (b).

Now we consider finite subgraphs with periodic boundary conditions. Consider the toral graph $(\mathbb{Z}_n)^d = \{1, 2, ..., n\}^d$ with nearest neighbor relation taken modulo n. Our vertex set will be $(\mathbb{Z}_n)^d \times S$ with incidence matrix

$$M_n((x,i),(y,j)) = R^{y-x}(i,j)$$

where the $y - x$ is taken mod n and n is assumed to be large enough that $|x| \geq n$ implies $R^x = 0$.

Let $\tilde{N}_{B_n}$ = the number of spanning trees on this graph. We have shown above that $H_{top} = \lim \frac{1}{n^d k} \log(\tilde{N}_{B_n})$.

To complete the proof of the Theorem we use the Matrix Tree Theorem to compute $\tilde{N}_{B_n}$. This theorem ([CDS, page 38]) says that if we have a D-regular connected graph with L vertices and if the eigenvalues of the incidence matrix are $\lambda_1, \lambda_2, \ldots, \lambda_{L-1}, \lambda_L = D$ then the number of spanning trees of the graph is

$$\frac{1}{L} \prod_{j=1}^{L-1} (D - \lambda_i).$$

So it is enough to compute the eigenvalues of the matrix M_n. Given $\alpha = (\frac{a_1}{n}, \ldots, \frac{a_d}{n})$ for $a_i \in \mathbb{Z}_n$ (which we may view as an element of T^d) suppose that $\lambda(\alpha)$ is an eigenvalue of $Q(\alpha)$ with eigenvector $v(\alpha)$. Then $D\lambda(\alpha)$ is an eigenvalue of M_n with eigenvector $v(\alpha) \otimes \xi^\alpha$.

This is checked analogously with Lemma 2.1.

$$\begin{aligned} & \sum_{j,y} M_n((x,i),(y,j)) v(\alpha)_j \exp(2\pi i \alpha \cdot y) \\ = & \sum_{j,y} R^{y-x}(i,j) \exp(2\pi i \alpha \cdot (y-x)) v(\alpha)_j \exp(2\pi i \alpha \cdot x) \end{aligned}$$

$$= \sum_j DQ(\alpha)_{i,j} v(\alpha)_j \exp(2\pi i\alpha \cdot x)$$

Now $D \cdot Q(\alpha)$ is Hermitian, in particular it eigenvectors span $\mathbb{C}^k$ and the characters $\exp(2\pi i\alpha \cdot x)$ span $\mathbb{C}^{\mathbb{Z}_n^d}$ we see that we have found a complete contingent of eigenvalues.

$$\begin{aligned}
\tilde{N}_{B_n} &= \frac{1}{kn^d} \prod_{\lambda,\alpha} (D - D\lambda(\alpha)) \\
&= \frac{1}{kn^d} \left(\prod_{\alpha \neq 0} D^k \prod_{\lambda} (1 - \lambda(\alpha)) D^{k-1} \right) \left(\prod_{\lambda \neq 1} (1 - \lambda(0)) \right) \\
&= \frac{1}{kn^d} (\prod_{\alpha \neq 0} D^k \chi_{Q(\alpha)}(1)) D^{k-1} (\prod_{\lambda \neq 1} (1 - \lambda(0))).
\end{aligned}$$

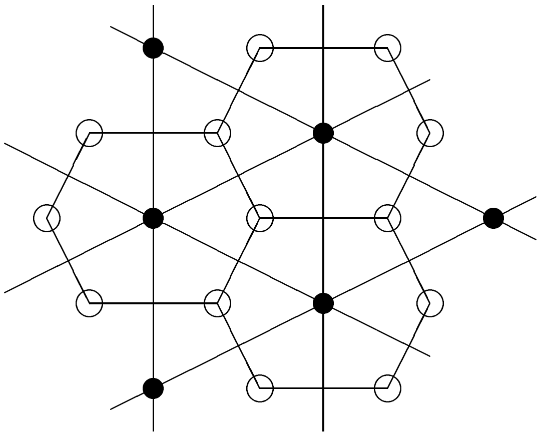
Continuing and ignoring some logarithmically insignificant terms we get

$$\begin{aligned}
H_{top} &= \lim_n \frac{1}{kn^d} \log(\tilde{N}_{B_n}) \\
&= \lim_n \frac{1}{k} \sum_{\alpha \neq 0} \log(D^k \chi_{Q(\alpha)}(1)) \frac{1}{n^d} \\
&= \frac{1}{k} \int_{T^d} \log(D^k \chi_{Q(\alpha)}(1)) d\alpha.
\end{aligned}$$

The last equality follows from approximating the integral by Riemann sums. Because of the estimate of the eigenvalues from Lemma 2.4 we see that $|\chi_{Q(\alpha)}(1)| \leq K|\alpha|^{-2k}$ so that the Lebesgue integral is finite. On $|\alpha| > \epsilon$ all the terms in the sum for all n are bounded, so by bounded convergence, the summations (qua integrals of step functions) converge to the integral. On $0 < |\alpha| < \epsilon$, all sums and the integral go to zero as $\epsilon \to 0$, implying the desired convergence. □

7 Dominoes

A *domino tiling* for a graph G, otherwise known as a perfect matching or a 1-factor, is a collection of edges of G the disjoint union of whose vertices is $V(G)$, the vertex set of G. There is a correspondence between spanning trees of a planar graph and domino tilings of a related graph which we now describe. Let G be a nice planar graph with vertex set V_G and edge set E_G, "nice" meaning here that every vertex has finite degree and every face including the exterior face is bounded by finitely many edges. Since G is planar there is a dual graph G^* with vertices V_{G^*} and edges E_{G^*}. In rough terms, G^* is obtained from G by putting a vertex at each face of G and joining two such vertices by an edge if their corresponding faces in G meet at an edge. The set V_{G^*} is identified with the faces of G and the edge sets E_G and E_{G^*} are also identified. We construct a new bipartite graph $\tilde{G}$ whose vertex set is the union of V_G, V_{G^*} and E_G. There is an edge of $\tilde{G}$ joining $v \in V_G$ and $e \in E_G$ if and only if e is incident to v. Likewise $v \in V_{G^*}$ and $e \in E_G$ are joined by an edge if v is a vertex of the edge in G^* identified with e. Both G and G^* sit inside of $\tilde{G}$ in the sense that $G = \tilde{G}/E_G - V_{G^*}$ and $G^* = \tilde{G}/E_G - V_G$ (recall this notation from Section 2 for contraction and deletion of a graph). Here is an illustration of this where G is the triangular lattice (vertices are filled circles), G^* is its hexagonal dual (open circles) and the extra vertices of $\tilde{G}$ are the crossing points.



There is a natural correspondance between subgraphs of G and subgraphs of G^* . If T is any subgraph of G let T^* be the subgraph of G^* obtained by declaring an edge e^* in T^* if and only if the corresponding edge e is not in T. Clearly this is also a dual operation so $T^{**} = T$. We record an easy lemma on dual trees.

Lemma 7.1 *(a) Let G be a finite planar graph. Then T^* is a spanning tree of G^* if and only if T is a spanning tree of G.*

(b) Let G be an infinite planar graph all of whose faces are bounded regions. Then T is a one ended spanning tree if and only if T^* is a one-ended spanning tree. Also T is an essential spanning forest if and only if T^* is an essential spanning forest. $\square$

Remark: This lemma and its soon to be described connection with domino tilings were noticed independently by Jim Propp who suggests the name "Temperleyan" for graphs that are isomorphic to $\tilde{G}$ for some G. Temperley [Tem] first used this trick in the case that $\tilde{G}$ was $\mathbb{Z}^2$. We also learned of this independently from Piet Kastelyn (personal communication).

Define a *directed* essential spanning forest to be a spanning forest together with a choice of an end for each component. Think of edges of a directed spanning forest being oriented toward this end. If T and T^* are dual essential spanning forests of a nice infinite planar graph, say the pair (T, T^*) is directed if an end has been chosen of each component of T and of T^*. For any nice infinite planar graph G, we now describe a bijection between domino tilings of $\tilde{G}$ and directed pairs of essential spanning forests of G and G^*.

If (T, T^*) is such a pair, then let $\Psi(T, T^*)$ be the domino tiling $A \subseteq E(\tilde{G})$ such that

(i) the edge from $v \in V(G)$ to $e \in E(G)$ is in A if and only if $e \in T$ and is oriented away from v, and

(ii) the edge from $v^* \in V(G^*)$ to $e \in E(G^*) = E(G)$ is in A if and only if $e \in T^*$ and is oriented away from v^*.

It is easy to verify that A is a domino tiling: each vertex $v \in V(G)$ is in precisely one edge of A, corresponding to the unique edge in T out of v; similarly each $v^* \in V(G^*)$ is in a unique edge of A; and each $e \in E(G)$ is in a unique edge of A since e is in precisely one of T, T^*. Conversely, If $A \subset E(\tilde{G})$ is a domino tiling, then each edge $f \in A$ connects some $e \in E(G)$ either to some $v \in V(G)$ or some $v^* \in G^*$. Let $\Phi(f)$ be the edge e in either G or G^* accordingly and orient it away from v or v^*. Then the collection of all $\{\Phi(f) : f \in A\}$ is the union of a subgraph G' of G and the corresponding dual subgraph G'^* of G^*. If G' has a loop, then inside the loop is a conponent of G'^*. Starting anywhere in this component of $G"^*$ and following the orientation creates a loop since the component is finite. This loop encloses a loop of G' inside the original loop. But this cannot continue forever, whence G' (and G'^*) has no loop. Thus G' and G'^* are essential spanning forests with each component directed toward an end.

Write Π for the map that takes a directed pair of ESF's (T, T^*), and forgets about T^* and about the arrows, producing the undirected ESF T. Then we have established a correspondence

$$\text{DOMINO TILINGS} \underset{\Psi}{\overset{\Phi}{\rightleftarrows}} \text{DIRECTED ESF's} \xrightarrow{\Pi} \text{ESF's}.$$

Now the projection Π from directed dual pairs of essential spanning forests to essential spanning forests that forgets the orientation and T^* is not in general one to one, but by Lemma 7.1, if T is a one-ended essential spanning tree of a nice planar graph, then so is T^*, so there is no choice to be made in orienting the components of T and T^*. Now fix a $\mathbb{Z}^2$-periodic planar graph G, so the vertex set is $\mathbb{Z}^2 \times S$ where $S = \{1, \ldots, k\}$. There is then a well defined map $\Psi \circ \Pi^{-1}$ from one-ended spanning trees of G to domino tilings of $\tilde{G}$ (which is also Z^2 periodic). The uniform spanning forest measure, ν on G is supported on the set of one-ended trees [Pem] so the above correspondence gives a transported measure $\tilde{\nu}$ on domino tilings of $\tilde{G}$.

Theorem 7.2 *The measure $\tilde{\nu}$ defined above is the unique measure of maximal entropy amongst all shift invariant probability measures on domino tilings and its entropy per vertex is $kH(\nu)/2e$ where $H(\nu)$ is the entropy of ν and e is the number of edges per fundamental domain.*

Remark: This theorem works because boundary conditions are, as we have seen in the previous section, irrelevant for trees. Since different boundary conditions for domino tilings give different entropies [Elk, Kas, TF], the excursion through trees is the only soft method we know to get uniqueness of the maximal entropy measure for domino tilings of Temperleyan graphs. Schmidt [Sch, page 58] cites an argument by Kuperberg that is supposed to prove this for $\mathbb{Z}^2$.

Proof: We have seen that $\tilde{\nu}$ is well defined and it is evidently $\mathbb{Z}^2$-invariant, so it remains to prove the assertions about its entropy. Suppose that $\tilde{\mu}$ is a translation invariant probability measure on domino tilings of $\tilde{G}$. This may be transported to a measure μ on essential spanning forests of G by $\mu(B) = \tilde{\mu}[\Psi[\Pi^{-1}[B]]]$. We show that the entropy per fundamental domain is preserved. First, note that μ is translation invariant so with probability one the components of the essential spanning forest have one or two ends [BK1]. There is only one way that a one-ended tree may be paved with dominoes and there are two ways a two-ended tree may be paved. Thus the ambiguity in determining the domino tiling is one bit for component two-ended tree in the forest in G plus one bit for each two-ended component in the dual spanning forest of G^*. Since there are $O(n)$ such components in every box of side length n which has on the order of n^2 vertices we see that the entropy of μ and $\tilde{\mu}$ are the same.

Now $H(\tilde{\mu})$ per fundamental domain $= H(\mu) \le H(\nu) = H(\tilde{\nu})$ per fundamental domain with equality only when $\mu = \nu$. But ν is concentrated on one-ended spanning trees [Pem] and hence $\tilde{\nu}$ is the only measure which transports to ν, which establishes that $\tilde{\nu}$ is the unique measure of maximal entropy on domino tilings.

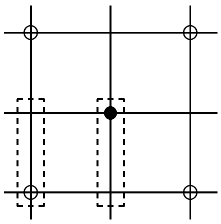
Finally, recall k is the number of vertices of G in each fundamental domain. Let e be the number of edges and f the number of faces, in the sense that a box $\{1,\ldots,n\}^d \times \{1,\ldots,k\}$ will have approximately $(1+o(1))fn^d$ faces completely contained in it as $n \to \infty$. Euler's formula applied asymptotically says that $k+f=e$. The entropy of the domino process on $\tilde{G}$ is the same as the entropy of the spanning tree process on G (and as the spanning tree process on G^*) when measured per fundamental domain. We have given the entropy formula for the spanning tree process per vertex. To convert this to the entropy of the domino process on $\tilde{G}$ per vertex we must multiply by $k/(k+e+f) = k/(2e)$. $\square$

Corollary 7.3 *There is a unique measure of maximal entropy on domino tilings of the* 2*-lattice* Z^2*. and its entropy is* $1/4$ *the entropy of the spanning tree process on* Z^2*.*

Remark: This entropy number was first calculated by Kastelyn [Kas] in 1961 as the exponential growth rate of the number of tilings of large rectangle or torus. Since these are atypical boundary conditions this does not necessarily prove that this is the largest entropy possible.

Proof: In the theorem take $G = Z^2$ so that G^* is isomorphic to Z^2 and $\tilde{G}$ is also isomorphic to Z^2. The uniform spanning tree measure ν on G induces a measure $\tilde{\nu}$ on domino tilings of $\tilde{G}$ that has a fundamental domain of four vertices. The measure on $\tilde{G}$ is invariant under the induced $\mathbb{Z}^2$ action; since there a four vertices of $\tilde{G}$ in a fundamental domain of G, this is a subgroup of index 4 in the usual group of translations of $\tilde{G}$. Actually, though it must be invariant under all graph automorphisms σ including $90°$ rotations, since otherwise $\tilde{\nu} \circ \sigma$ would be a measure on domino tilings distinct from $\tilde{\nu}$ but with the same entropy, violating the uniqueness shown in Theorem 7.2. $\square$

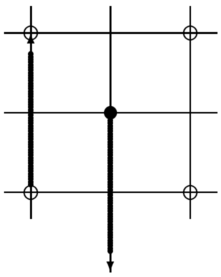
The fact that Π is not in general continuous leads to a problem in trying to compute f.d.m.'s of $\tilde{\nu}$. If $\mathcal{C} = \{T : e_1, \ldots, e_k \in \mathbf{T}\}$ is a cylinder event in the space of essential spanning forests, then $\Psi[\Pi^{-1}[\mathcal{C}]]$ is a finite union of cylinder events in the space of domino configurations. On the other hand, if $\tilde{\mathcal{C}} = \{A : e_1, \ldots, e_k \in A\}$ is a cylinder event in the space of domino configurations, then $\tilde{\mathcal{C}}$ is not necessarily $\Psi[\Pi^{-1}[\mathcal{C}]]$ for some elementary cylinder event $\mathcal{C}$ on essential spanning forests. Thus knowledge of the f.d.m.'s of $\tilde{\nu}$ would yield the f.d.m.'s of ν quite directly, but unfortunately not *vice* *versa.* To illustrate this, let $\tilde{\mathcal{C}}$ be the event of finding a square of two vertical dominos with the origin at the lower left corner.



vertices of G are open circles

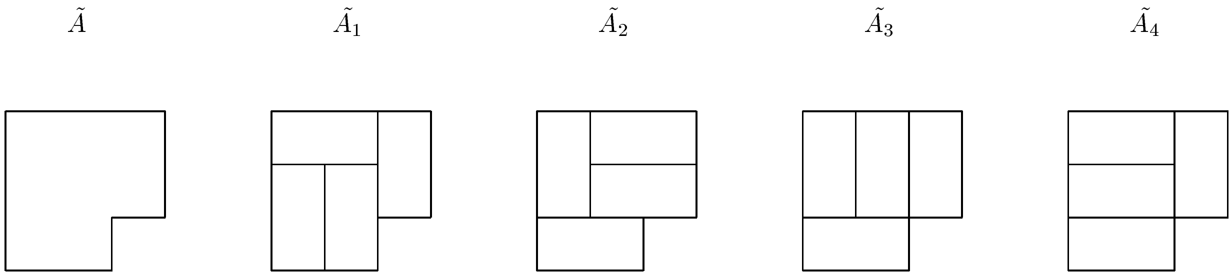
vertices of G^* are filled circles

Then $\Phi[\mathcal{C}]$ is the event that there is an oriented edge from upward from the origin in T and a dual edge in T^* oriented downward on the right of the origin.

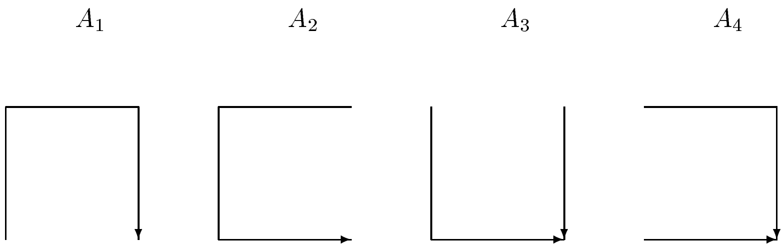


The corresponding event on trees is that the edge upward from the origin be in T, that the path from the origin to infinity be through that edge, that the edge leading right from the origin not be in T and that the path connecting the origin and the point to the right go over the top, rather than around the bottom (speaking homotopically in the plane minus the edge leading right from the origin). We do not know how to compute this probability. There are however some cylinder domino events corresponding to events whose probabilities we do know how to compute. Here is one example.

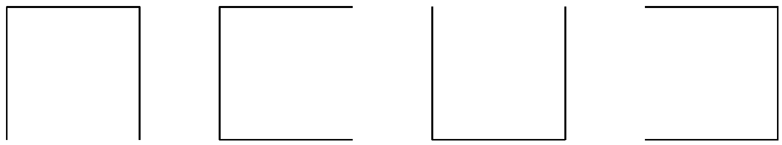
Consider the following contour which can be broken down into dominos in the four ways shown.



We may calulate the probability of finding this contour with the origin at the bottom left. Since $\tilde{\nu}$ is uniform on the interior of any box given the boundary, each of the four configurations inside then has $1/4$ this probability. To carry this out, map by Φ so as to get the following four configurations of directed edges.

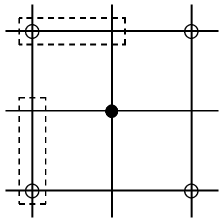


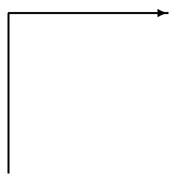
Here, A_1 is the event of there being oriented edges in T leading up out of the origin, right, and back down, while there also being a downwardly directed edge in T^* on the right of the origin. Since this dual edge is implied by the the other three, it is not shown. A similar thing happens with A_2, A_3 and A_4. Now $A_1 \cup A_2 \cup A_3 \cup A_4$ is the event of T containing three out of the four edges of the square with the origin at its lower left, and having the path from this square to infinity exit the square at the lower right. Then $\mathbf{P}(\tilde{A}) = \mathbf{P}(A) = \mathbf{P}(A_1 \cup A_2 \cup A_3 \cup A_4)$ which is by symmetry just $1/4$ times the probability of T containing three of these four edges, with no specified orientation



which is just 1/4 the probability of the vertex in the center of the square being a leaf of T^*, which is $2\pi^{-2} - 4\pi^{-3} \approx .0736$ from Section 5.

The examples which work out this nicely are a small finite class. There is another class of examples of f.d.m.'s we can calculate. We give just one illustration, since the taxonomy is still being worked out. Consider the following pair of dominos and corresponding set of oriented edges of T.





The probability of these two oriented edges is the probability of T containing the two unoriented edges times the conditional probability given that of the path to infinity leaving through the point at the upper right. The first probability is computed by transfer impedances to be

$$\begin{vmatrix} 1/2 & 1/2 - 1/\pi \\ 1/2 - 1/\pi & 1/2 \end{vmatrix} = \pi^{-1} - \pi^{-2}.$$

The conditional probability may be seen to be the probability that a random walk coming in from infinity hits the set of three vertices first at the upper right. The hitting distribution from infinity on a set of vertices $x_1, \ldots, x_k$ is proportional to the k entries of $(1, \ldots, 1)M^{-1}$ where M is the Green's matrix i.e. $M_{ij} = H(x_i, x_j)$ (proof: use a last exit decomposition from $\{x_1, \ldots, x_k\}$ and then invert the linear relations). Then the conditional probability in question is $\pi/(6\pi - 8)$, which gives a total probability of $(1 - \pi^{-1})/(6\pi - 8) \approx .0628$.

8 Appendix

8.1 The classical Green's function

Let $H(x,y)$ be the classical Green's function for G defined by

$$H(x,y) = \sum_{n=0}^{\infty} \mathbf{P}(SRW_x(n) = y) \tag{15}$$

when $d = 3$, and

$$H(x,y) = \sum_{n=0}^{\infty} [\mathbf{P}(SRW_x(n) = y) - \mathbf{P}(SRW_x(n) = x)] \tag{16}$$

when $d = 1$ or 2.

Theorem 8.1 *The sums in (15) and (16) converge. Furthermore, H has the following properties:*

(i) (H) is symmetric;
(ii) $H(x,\cdot)$ is harmonic except at x, where its excess is 1;
(iii) H is bounded if $d \geq 3$;
(iv) $H(x,\cdot) - H(y,\cdot)$ is bounded for fixed x,y if $d \leq 2$.

Remark: Theorem 4.1 follows immediately from this and Corollary 3.3.

Proof: Begin with the observation that SRW^G is transient if $d \geq 3$ and recurrent if $d \leq 2$; there are many ways, to see this, one being to watch SRW^G only at the times when it hits $\mathbb{Z}^d \times \{1\}$ which is then a symmetric random walk on $\mathbb{Z}^d$ with $\mathbf{P}(x,y)$ having exponential tails. When $d \geq 3$, the theorem is now easy to prove. The sum converges by definition of transience. Writing $\mathbf{P}(SRW_x(n) = y)$ as the sum of D^{-n} over paths of length n from x to y shows by path reversal that this is equal to $\mathbf{P}(SRW_y(n) = x)$ for each n, hence H is symmetric. Boundedness follows from the fact that $H(x,y) = \mathbf{P}(SRW_x \text{ hits } y)H(y,y) \leq H(y,y)$, and from the fact that $H(y,y)$ takes on only k different values.

Assume now that $d \leq 2$. Since SRW^G is recurrent there is a σ-finite stationary distribution μ, unique up to constant multiple [IM]. It is easy to see that this is uniform. Furthermore, it is well-known [IM] that for any $x,y,z \in G$, the ratio of Cesaro averages converges:

$$\frac{1}{N}\sum_{n=1}^{N} \mathbf{P}(SRW_z(n) = x) / \frac{1}{N}\sum_{n=1}^{N} \mathbf{P}(SRW_z(n) = y) \to \mu(x)/\mu(y) = 1 \tag{17}$$

as $N \to \infty$. Now fix x, y, z and consider the Markov chain $\{Z(n) : n \geq 1\}$ on the space $\{x, y\}$ gotten by looking at SRW_z only when it at x or y. In other words, $Z(n) = x$ if the n^{th} visit of SRW_z to $\{x, y\}$ is at x and $Z(n) = y$ otherwise. The transition matrix for Z is $\begin{pmatrix} a & 1-a \\ 1-b & b \end{pmatrix}$, where $a = \mathbf{P}(SRW_x \text{ hits } x \text{ before } y)$ and $b = \mathbf{P}(SRW_y \text{ hits } y \text{ before } x)$. It follows easily from (17) that the stationary distribution for Z must be half at x and half at y, from which it follows that $a = b$ in the transition matrix.

It is easy to calculate

$$\sum_{n=N}^{\infty} [\mathbf{P}(Z(n) = x) - \mathbf{P}(Z(n) = y)] = [\mathbf{P}(Z(N) = x) - \mathbf{P}(Z(N) = y)]/(2 - 2a). \tag{18}$$

Now for any positive integers $L < M$, we have

$$\begin{aligned} & \sum_{n=1}^{M} [\mathbf{P}(Z(n) = x) - \mathbf{P}(Z(n) = y)] \\ = & \sum_{n=0}^{L} [\mathbf{P}(SRW_z(n) = x) - \mathbf{P}(SRW_z(n) = y)] \\ & + \mathbf{E} \left[\mathbf{E}(\sum_{n=L+1}^{\tau} I(SRW_z(n) = x) - I(SRW_z(n) = y) \,|\, SRW_z(L+1)) \right], \end{aligned}$$

where τ is the time of the M^{th} visit to $\{x, y\}$, and letting $M \to \infty$ while using (18) gives

$$\begin{aligned} & \sum_{n=0}^{L} [\mathbf{P}(SRW_z(n) = x) - \mathbf{P}(SRW_z(n) = y)] \\ = & [2\mathbf{P}(SRW_z \text{ hits } x \text{ before } y) - 1)/(2 - 2a) \\ & - [\mathbf{P}(SRW_z(\tau_L) = x) - \mathbf{P}(SRW_z(\tau_L) = y)]/(2 - 2a), \end{aligned}$$

where τ_L is the first time after L that SRW_z hits $\{x, y\}$. The last term is converging to zero as $L \to \infty$, hence letting $z = x$, the sum in (16) converges. Moreover, when $z = x$ the sum converges to $1/(2 - 2a)$, and having shown that $a = b$ in the transition matrix, we see that this is symmetric in x and y, proving (i). Along with the relation $\mathbf{P}(SRW_x(n) = y) = \mathbf{P}(SRW_y(n) = x)$, this also establishes that $\sum_{n=0}^{\infty} [\mathbf{P}(SRW_x(n) = x) - \mathbf{P}(SRW_y(n) = y)] = 0$. From the fact that $\mathbf{P}(SRW_x(n) = w) \to 0$ for any w and from the relation

$$\sum_{n=0}^{N} [\mathbf{P}(SRW_x(n) = y) - \mathbf{P}(SRW_x(n) = x)] = \delta_x(y) + D^{-1} \sum_{z \sim y} \sum_{n=0}^{N-1} [\mathbf{P}(SRW_x(n) = z) - \mathbf{P}(SRW_x(n) = x)]$$

it now follows that $H(x,\cdot)$ is harmonic except at x and has excess 1 at x.

Finally, to check that $H(x,\cdot)-H(y,\cdot)$ is bounded for fixed x and y, use $\sum_{n=0}^{\infty}[\mathbf{P}(SRW_x(n)=x)-\mathbf{P}(SRW_y(n)=y)]=0$ to conclude that $H(x,z)-H(y,z)=\sum_{n=0}^{\infty}[\mathbf{P}(SRW_z(n)=x)-\mathbf{P}(SRW_z(n)=y)]$. This is just $(2\mathbf{P}(SRW_z$ hits x before $y)-1)/(2-2a)$, and the numerator is bounded between -1 and 1, which proves that $H(x,\cdot)-H(y,\cdot)$ is bounded. $\square$

8.2 Harnack lemmas

Lemma 3.1 is developed in [La1] through a series of theorems beginning with a local central limit theorem for SRW on $\mathbb{Z}^d$. We first remark that [La1, Theorem 1.2.1] actually holds for the following more general random walk. Let $\{X_n : n \geq 0\}$ be an irreducible aperiodic random walk on $\mathbb{Z}^d$ with symmetric transition probabilities (i.e. $\mathbf{P}(x,x+a)=\mathbf{P}(x,x-a)$) that decay exponentially (i.e. $\mathbf{P}(x,x+a)=O(e^{-c|a|})$). Then the characteristic function for X_n is still given by $\phi(\theta)^n$ where $\phi(\theta)$ is real and equal to $1-<\theta,\theta>+O(<\theta,\theta>^2)$ near zero for some positive definite form $<,>$. Then the proof of [La1, Theorem 1.2.1] gives

Theorem 8.2 (Local CLT) *Under the above assumptions on X_n, there exists a $C>0$ and a positive definite form $<,>$ for which $\mathbf{P}_n(x,y)=Cn^{-d/2}e^{-<x-y,x-y>/2n}(1+O(\min(n^{-1},<x-y,x-y>)))$.* $\square$

Now let $\{Y_n : n \geq 0\}$ be a SRW^G started at the point $(0,1)$. Write $Y_n=(X_n,Z_n)$, where $X_n \in \mathbb{Z}^d$ and Z_n is the projected RW on S. It can be shown that X_n and Z_n are exponentially asymptotically independent in the sense that the joint distribution of X_n and Z_n is within e^{-cn} in total variation of the product distribution with the correct marginals. Applying Theorem 8.2 to a time change of X_n, it can be shown that X_n obeys the same local central limit theorem, the correction for the time change being smaller than the error bounds in the CLT. This gives

Theorem 8.3 (Local CLT for G) *Let Y_n be a SRW^G. Then there exists a $C>0$ and a positive definite form $<,>$ for which*

$$\mathbf{P}_n((x,i),(y,j))=Cn^{-d/2}e^{-<x-y,x-y>/2n}(1+O(\min(n^{-1},<x-y,x-y>))).$$

$\square$

This is sufficient to establish part (i) of Lemma 3.1 along the following lines, as pointed out to us by Maury Bramson (personal communication). For $x \in B_n$ and $z \in \partial B_m$, $\nu_x^{B_m}(z)$ is the sum of probabilities of paths starting from x and hitting ∂B_m for the first time at z. Reversing the paths, shows that this is the expected occupation of x by a SRW starting from z and killed when it hits ∂B_m again.

First suppose $d \geq 3$ and fix $\epsilon > 0$. Then the local CLT for G allows us to pick $L > n$ large enough so that for $w \in \partial B_L$, the occupation measures at x and y for SRW_w will be within a factor of $1+\epsilon$ of each other for any $x, y \in B_n$. If m is then chosen large enough, the occupation measure for SRW_w at any point in B_n will be at most $1+\epsilon$ times the occupation measure for SRW_w killed upon hitting ∂B_m. Now use the Markov property to write the occupation measure at x for SRW_z killed upon hitting ∂B_m as a linear combination over w of the occupation measure at x of SRW_w killed upon hitting ∂B_m. This shows the measures at x and y to be within a factor of $(1+\epsilon)^2$, and since ϵ was arbitrary this establishes the Harnack principle (i).

On the other hand, if $d \leq 2$ then SRW_G is recurrent, then for any $\epsilon > 0$ and n there is an m large enough so that $\mathbf{P}(SRW_x \text{ hits } y \text{ before } \partial B_m) \geq 1 - \epsilon$ for all $x, y \in B_n$. Then $\nu_x^{B_m} \geq (1-\epsilon)\nu_y^{B_m}$ for all $x, y \in B_n$, establishing (i).

The remaining parts of the theorem are derived as follows. To get (ii) from (i), pick $x \in \partial B_n$ and write

$$\nu_x^{B_m} = \mathbf{P}(SRW_x \text{ does not return to } B_n)\rho_x^{B_m B_n} + \mathbf{P}(SRW_x \text{ returns to } B_n)\nu'$$

where ν' is a mixture over $y \in \partial B_n$ of $\nu_y^{B_m}$, the mixing measure being given by the return hitting distribution of SRW_x on ∂B_n. Since B_n is held fixed, $\nu'(\{z\})/\nu_x^{B_m}(\{z\})$ is converging to 1 uniformly in z as $m \to \infty$. Since $\mathbf{P}(SRW_x \text{ returns to } B_n)$ is bounded away from one (for fixed B_n), solving for $\rho_x^{B_m}$ gives that $\sup_z \rho_x^{B_m}/\nu_x^{B_m} \to 1$ as $m \to \infty$. To get (iv) from (ii), restate (ii) as saying that the sum, call it $\pi(x, z, n, m)$, of $D^{-|\gamma|}$ over paths γ from $x \in B_n$ to $z \in B_m$ that avoid ∂B_n and ∂B_m except at the endpoints is equal to $(1 + o(1))f(x)g(z, m)$ and functions f, g as $m \to \infty$. The restatement of (iv) is easily seen to be identical by time-reversal. To get (iii) from (iv), just note that $\nu_x^{B_n}$ is a mixture over $y \in \partial(B_m^c)$ of $\rho_y^{B_n B_m}$.

Finally, to get (v) choose $L > n$ so that B_L contains all the contracted edges. For (i) and (ii), write $\nu_x^{B_m}$ and $\rho_x^{B_m B_n}$ as a mixture over $y \in \partial B_L$ of $\rho_y^{B_m B_L}$ and observe that $\rho_y^{B_m B_L}$ for the contracted graph is equal to $\rho_y^{B_m B_L}$ for the uncontracted graph, so that all the measures being mixed are identical up to a factor of $(1 + o(1))$. For (iii) and (iv), write $\nu_x^{B_n}$ and $\rho_x^{B_n B_m}$ as a mixture over $y \in \partial B_L$ of $\nu_y^{B_n}$ and observe that this time it is the mixing measures, which are just $\rho_x^{B_L B_m}$ that are all within a factor of

$(1 + o(1))$ as x varies with $m \to \infty$.

8.3 Convergence of probability measures on trees via moments

Propositions 8.4 - 8.7 are adaptations of classical tightness criteria to the setting of tree-valued random variables. The development is brief and essentially copied from [Du]. Theorem 8.8 is less trivial and the proof is given in full detail.

Fix a positive integer r until further notice and restrict attention to trees of height at most r. Recall the definitions of $|t|$, $t \wedge r$ and $N(u;t)$ from Section 5. We say a family of probability distributions $\{\mathbf{P}_n\}$ on such trees is *tree-tight* if for all $\epsilon > 0$ there is a K for which $\limsup_n \mathbf{P}_n(|\mathbf{T}| > K) < \epsilon$.

Proposition 8.4 *If $\{\mathbf{P}_n\}$ is tree-tight then every sequence of measures from $\{\mathbf{P}_n\}$ has a subsequence that converges in distribution to a probability measure.*

Proof: Since the $\mathbf{P}_n$ laws of $|\mathbf{T}|$ are tight in the usual sense, every sequence has a subsequence $\mathbf{P}_{n_j}$ for which the laws of $|\mathbf{T}|$ converge in distribution to some probability measure. For each k, the conditional distribution $(\mathbf{P}_{n_j} \mid |\mathbf{T}| = k)$ is finitely supported, hence has a subsequence converging to a probability measure, and diagonalizing over k gives the desired subsequence. $\square$

Proposition 8.5 *Let g, h be functions from trees of height at most r to the reals. Suppose that $g > 0$ and that $h(t)/g(t) \to 0$ as $|t| \to \infty$. Let $\mathbf{P}_n$ be probability measures on these trees with $\mathbf{P}_n \stackrel{\mathcal{D}}{\to} \mathbf{P}_\infty$ and $\limsup_n \mathbf{E}_n g < \infty$, where $\mathbf{E}_n$ is expectation with respect to $\mathbf{P}_n$. Then $\mathbf{E}_n h \to \mathbf{E}_\infty h$.*

Proof: $|\mathbf{E}_n h - E_\infty h| \leq |\mathbf{E}_n h I(|\mathbf{T}| > K) - \mathbf{E}_\infty h I(|\mathbf{T}| > K)| + |\mathbf{E}_n h I(|\mathbf{T}| \leq K)| + |\mathbf{E}_\infty h I(|\mathbf{T}| \leq K)|$. The last two of these can be made small uniformly in n by using $|h|/g \to 0$ and choosing K large enough. The first goes to zero for any fixed K. $\square$

Proposition 8.6 *Suppose $\mathbf{P}_n \stackrel{\mathcal{D}}{\to} \mathbf{P}_\infty$ and for each tree t, $\mathbf{E}_n N(\mathbf{T};t)$ converges to a finite limit $m(t)$. Then $\mathbf{E}_\infty N(\mathbf{T};t) = m(t)$ for all t.*

Proof: For any tree t of height at most r, let $\mathcal{C}(t)$ be the set of all t' of height at most r that extend t by adding to some vertex in t a single finite chain of descendants. Let

$$g_t(u) = \sum_{t' \in \mathcal{C}(t)} N(u; t').$$

Notice that $g_t(u) \geq (|u| - |t|)N(u;t)$ since each tree-map $\phi : t \to u$ can be extended to a tree-map of some $t' \in \mathcal{C}(t)$ into u in at least as many ways as there are vertices in $u \setminus$ Image (ϕ). Then

$$\limsup_n \mathbf{E}_n g_t(\mathbf{T}) \leq \sum_{t' \in \mathcal{C}(t)} \limsup_n \mathbf{E}_n N(\mathbf{T}; t) \leq \sum_{t' \in \mathcal{C}(t)} m(t) < \infty.$$

Applying the previous proposition with $g = g_t$ and $h = N(\cdot; t)$ finishes the proof. □

Proposition 8.7 *Suppose $\mathbf{E}_n N(\mathbf{T}; t) \to \mathbf{E}_\infty N(\mathbf{T}; t) < \infty$ for all t and that $\mathbf{P}_\infty$ is uniquely determined by the values of $\mathbf{E}_\infty N(\mathbf{T}; t)$. Then $\mathbf{P}_n \stackrel{\mathcal{D}}{\to} \mathbf{P}_\infty$.*

Proof: First notice that $\{\mathbf{P}_n\}$ is tree-tight: letting t_j be a single chain of $j+1$ vertices, $|\mathbf{T}| = \sum_{j=0}^r N(\mathbf{T}; t_j)$, so $\limsup_n \mathbf{E}_n |\mathbf{T}| = \limsup_n \sum_{j=0}^r \mathbf{E} N(\mathbf{T}; t_j) < \infty$ by hypothesis, and using $\mathbf{P}(|\mathbf{T}| \geq k) \leq k^{-1} \mathbf{E}|\mathbf{T}|$ establishes tightness. Now each sequence in $\{\mathbf{P}_n\}$ has a subsequence converging to a probability measure G and the previous proposition shows that $\mathbf{E}_n N(\mathbf{T}; t) \to \mathbf{E}_G N(\mathbf{T}; t)$ for each t. Then $G = \mathbf{P}_\infty$ by the uniqueness assumption. □

For a positive integer-valued random variable X, the moments of X determine its distribution at least under a condition on the rate of growth of these moments. The remainder of the development of the method of moments for trees is to prove the following analogous fact for for tree-valued random variables. Let U be a random variable taking values in the space of locally finite rooted trees of height at most r.

Theorem 8.8 *Suppose $\mathbf{E} N(U; t)$ is bounded by $e^{k|t|}$ for some k. Then the law of U is uniquely determined by the values of $\mathbf{E} N(U; t)$ as t varies over finite rooted trees of height at most r.*

The proof is based on the following version of the integer-valued case. Let $(A)_s = A(A-1)\cdots(A-s+1)$ denote the s^{th} lower factorial of A.

Lemma 8.9 *Suppose $Y \in \mathbb{Z}^+$ and X are random variables and for some fixed $s > 0$ suppose the values $c_j = \mathbf{E} X (Y)_{s+j}$ exist for $j \geq 0$ and are bounded above by e^{kj} for some k. Then $\mathbf{E} X I(Y = s)$ is uniquely determined by the values of the c_j's.*

Proof: Let $Z = X(Y)_s$, so that $c_j = \mathbf{E}Z(Y-s)\cdots(Y-s-j+1)$. Then the values $d_j \overset{def}{=} \mathbf{E}Z(Y-s)^j$ are determined from the values of $c_i, i \le j$ by linear combination. The coefficients are bounded by some exponential e^{kj}, so the d_j are all bounded by some exponential e^{kj}. Let

$$h(t) = \sum_{j \ge 0} (it)^j d_j / j! = \mathbf{E}Z e^{it(Y-s)}.$$

The power series converges for all t, uniformly for $t \in [0, 2\pi]$ and hence

$$\mathbf{E}ZI(Y-s=0) = (1/2\pi)\int_0^{2\pi} h(t)dt,$$

which yields

$$\mathbf{E}XI(Y=s) = (1/2\pi s!)\int_0^{2\pi} h(t)dt.$$

□

Proof of Theorem 8.8: The proof is by induction on r. The induction will in fact show that for any random variable f and any r, if U has height at most r then the values of $\mathbf{E}N(U;t)f$ as t varies over trees of height at most r determine the values of $\mathbf{E}I(U=t)f$, provided that $\mathbf{E}N(U;t)f$ is bounded by $e^{k|t|}$ for some k; using $f \equiv 1$ will then prove the theorem. The initial step $r=0$ is trivial, since then $U \wedge r$ is always a single vertex.

Assume then for induction that the theorem is true for some r. For any tree t let $b(t)$ be the number of children of the root of t. Define a random vector $Z = (Z_1^U, \ldots, Z_{b(U)}^U)$ whose length is always $b(U)$ and whose distribution conditional upon $b(U)$ is uniform over all $b(U)!$ permutations of the $b(U)$ subtrees below the children of the root of U. In other words Z is this multiset of subtrees presented in uniform random order.

For any i and s with $s \ge i \ge 0$ and any $t_1, \ldots, t_s$ of height at most r, let

$$g_i = g_i(t_1, \ldots, t_s) = I(Z_1^U = t_1)\cdots I(Z_i^U = t_i)N(Z_{i+1}^U; t_{i+1})\cdots N(Z_s^U; t_s)I(b(U) = s).$$

We set up a second induction on i to show that for any $t_i, \ldots, t_s$ of height at most r and any random variable f, the value of $\mathbf{E}g_i(t_1, \ldots, t_s)f$ is uniquely determined by the values of $\mathbf{E}N(U;t)f$.

For the initial step $i=0$, choose any $t_1, \ldots, t_s$ and f and write t_* for the tree whose root has s children with subtrees $t_1, \ldots, t_s$. Write $t_{*,j}$ for a copy of t_* to which has been added j leaves that are children of the root. Observe that a map from $t_{*,j}$ into U is given by choosing s ordered distinct subtrees

$u_1, \ldots, u_s$ from children of the root of u, mapping each t_i into u_i, and then choosing an ordered j-tuple of vertices from the remaining $b(U) - s$ children of the root of U. Thus

$$N(U; t_{*,j}) = \sum_{u_1,\ldots,u_s} N(u_1; t_1) \cdots N(u_s; t_s)(b(U) - s)_j$$

where the sum is over ordered s-tuples of subtrees from distinct children of the root of U. For each $(u_1, \ldots, u_s)$, $\mathbf{P}(Z_1, \ldots, Z_s) = (u_1, \ldots u_s) = 1/(b(U))_s$. Consequently,

$$\begin{aligned}
& \mathbf{E}N(Z_1; t_1) \cdots N(Z_s, t_s)(b(U))_{s+j} f \\
= \;& \mathbf{E}\frac{1}{(b(U))_s} \sum_{u_1,\ldots,u_s} N(u_1; t_1) \cdots N(u_s; t_s)(b(U) - s)_j \frac{(b(U))_{s+j}}{(b(U) - s)_j} f \\
= \;& \mathbf{E}\frac{1}{(b(U))_s} N(U; t_{*,j}) \frac{(b(U))_{s+j}}{(b(U) - s)_j} f \\
= \;& \mathbf{E}N(U; t_{*,j}) f.
\end{aligned}$$

As j varies, $|t_{*,j}|$ increases linearly with j. By the hypothesis of the theorem this implies that $\mathbf{E}N(U; t_{*,j}) f$ is bounded by e^{kj} for some k. Then we may apply Lemma 8.9 with $Y = b(U)$ and $X = \mathbf{E}N(Z_1; t_1) \cdots N(Z_s, t_s) f$ to get that these expectations uniquely determine $\mathbf{E}g_0 = N(Z_1; t_1) \cdots N(Z_s; t_s) I(b(U) = s) f$.

Now assume for induction that for any $t_1, \ldots, t_s$, $\mathbf{E}g_i f$ is determined by the values of $\mathbf{E}N(U; t) f$ and write

$$\mathbf{E}g_{i+1} = \mathbf{E}I(Z_{i+1} = t_{i+1})[I(Z_1 = t_1) \cdots I(Z_i = t_i) N(Z_{i+2}; t_{i+2}) \cdots N(Z_s; t_s) I(b(U) = s) f].$$

We may apply Lemma 8.9 with $Y = N(Z_{i+1}; t_{i+1})$ and X being the rest of the RHS, provided $\mathbf{E}X(Y)_{s+j}$ is bounded by e^{kj} for some k. But $\mathbf{E}X(Y)_j \leq \mathbf{E}N(U; T) f$ where T is a tree whose root has children with subtrees: j copies of t_{i+2} and one copy of $t_1, \ldots t_i$ and $t_{i+1}, \ldots, t_s$. Since we have assumed that $\mathbf{E}N(U; t) f$ is bounded by some $e^{k|t|}$, $\mathbf{E}N(U, T) f$ must be bounded by some $e^{k' j}$, whence the lemma applies and $\mathbf{E}g_{i+1}$ is indeed determined, completing the induction on i.

Setting $i = s$ and $f \equiv 1$ now showns that for any s and any $t_1, \ldots t_s$, $\mathbf{P}(b(U) = s, Z_1^U = t_1, \ldots, Z_s^U = t_s)$ is determined by the values of $\mathbf{E}N(U; t)$. But this probability is just $C(t_*)/s!$ times $\mathbf{P}(U = t_*)$, where $C(t_*)$ is the number of permutations of $\{1, \ldots s\}$ for which $t_i = t_{\pi(i)}$ for all i. Thus $\mathbf{P}(U = t_*)$ is determined for an arbitrary t_* of height $r + 1$, so the induction on r is completed and the theorem is proved. $\square$

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