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Biased Random Walks on Galton-Watson Trees

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Abstract. We consider random walks with a bias toward the root on the family tree T of a supercritical Galton-Watson branching process and show that the speed is positive whenever the walk is transient. The corresponding harmonic measures are carried by subsets of the boundary of dimension smaller than that of the whole boundary. When the bias is directed away from the root and the extinction probability is positive, the speed may be zero even though the walk is transient; the critical bias for positive speed is determined.

§1. Introduction.

Consider a supercritical Galton-Watson branching process with generating function $f(s) = \sum_{k=0}^{\infty} p_k s^k$, i.e., each individual has k offspring with probability p_k, and $m := f'(1) \in (1, \infty)$. Started with a single progenitor, this process yields a random infinite family tree T, called a **Galton-Watson tree**, on the event of nonextinction. We assume throughout that no p_k is equal to 1.

Simple random walk gives some information on the structure of a tree; to explore this structure further, random walks with a bias toward the root have been used (e.g., Berretti and Sokal (1985), Lawler and Sokal (1988), Lyons (1990)). The rate of escape (speed) of a random walk indicates how much of the tree a single path explores, while the dimension of harmonic measure indicates how much of the tree is explored by the ensemble of almost all paths.

For $\lambda \geq 0$, the **λ-biased random walk** on a locally-finite rooted tree T, denoted RW_λ, is the time-homogeneous Markov chain $\langle X_n\, ;\, n \geq 0\rangle$ on the vertices of T such that if u is a vertex with $k \geq 1$ children $v_1, \ldots, v_k$ and parent u_*, then $\mathbf{P}[X_{n+1} = v_i \mid X_n = u] = 1/(k+\lambda)$ for $i = 1, \ldots, k$ and $\mathbf{P}[X_{n+1} = u_* \mid X_n = u] = \lambda/(k+\lambda)$; from the root all

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Figure 1.1. The walk is at the highlighted vertex and will take one of its incident edges with probabilities proportional to the weights indicated.

transitions to its children are equally likely. In case $k = \lambda = 0$, then $\mathbf{P}[X_{n+1} = u_* \mid X_n = u] = 1$. Normally, we fix the initial state X_0 to be the root, ρ. See Figure 1.1.

For almost every Galton-Watson tree T on the event of nonextinction, RW_λ is transient for $1 \le \lambda < m$ (Lyons 1990). Here we show that for $1 < \lambda < m$, the random walk escapes at a positive *speed* and the corresponding harmonic measure has Hausdorff dimension less than that of the whole boundary. For $\lambda = 1$, i.e., the case of simple random walk, this was shown in Lyons, Pemantle and Peres (1995) by using an explicit stationary measure on the space of trees. We know of no such direct construction when $\lambda > 1$; instead, the proof is based on some *a priori* bounds on the Green function and a regeneration argument. The speed of the random walk is the almost sure limit (if it exists) of $|X_n|/n$, where $|x|$ denotes the distance from the root to the vertex x. In Section 5, we use positivity of the speed (and, in particular, the finiteness of the mean time between regenerations) to establish the existence of a finite measure on the space of trees which is absolutely continuous with respect to Galton-Watson measure and is stationary for the λ-harmonic flow. This is the key to the “dimension drop” of harmonic measure. In Corollary 5.3, we deduce that there exists a.s. a subtree $T^{(\epsilon)}$ of T with smaller exponential growth such that RW_λ on T is confined to $T^{(\epsilon)}$ with overwhelming probability.

When the bias is away from the root, i.e., $0 < \lambda < 1$, the walk is obviously transient on any infinite tree, but the walk may have zero speed when too much time is spent at leaves. In Theorem 4.1, we show that for Galton-Watson trees, the speed is positive iff $\lambda > f'(q)$, where q is the extinction probability.

§2. Linear Growth of the Range.

For the speed of RW_λ to be positive, certainly the range of RW_λ must grow linearly in the number of steps taken. In this section, we establish that when $\lambda > 1$, the range grows linearly for *any* tree on which RW_λ is transient; this is false for $\lambda = 1$. We begin with an *a priori* bound on the Green function.

Let $G(x, y) := \sum_{i=0}^{\infty} \mathbf{P}_x[X_i = y]$ be the Green function of RW_λ on T, i.e., the expected number of visits to y when the walk starts at x. Let $d(x)$ denote the number of children of a vertex x.

PROPOSITION 2.1. *Let $\lambda > 1$ and let T be any tree on which RW_λ is transient. Then for every vertex $x \in T$, we have*

$$G(x,x) \le \frac{d(x)+\lambda}{\lambda-1} G(\rho,\rho)\,. \tag{2.1}$$

Proof. Let $\widehat{G}(x,x)$ denote the expected number of visits to x before visiting ρ when starting from x. Let $f(x,y) := \mathbf{P}_x[\exists n > 0 \quad X_n = y]$ denote the probability of visiting y when starting at x (ignoring the initial visit if $x = y$), and let $\widehat{f}(x,y)$ denote the probability of visiting y before visiting ρ when starting at x.

By considering separately the path before and after the first visit to ρ, we see that

$$G(x,x) \;\le\; \widehat{G}(x,x) + f(x,\rho) f(\rho,x) G(x,x) \;\le\; \widehat{G}(x,x) + f(\rho,\rho) G(x,x)$$

and therefore

$$G(x,x) \le \frac{\widehat{G}(x,x)}{1-f(\rho,\rho)} = \widehat{G}(x,x) G(\rho,\rho)\,. \tag{2.2}$$

(This is valid for any transient Markov chain.) Denote by x_* the **parent** of the vertex x (i.e., the neighbor of x that is closer to the root), and observe that

$$1 - \widehat{f}(x,x) \ge \frac{\lambda}{d(x)+\lambda}\big(1 - \widehat{f}(x_*,x)\big)\,.$$

By comparing the steps of RW_λ on the path connecting ρ and x to a simple asymmetric random walk on the integers, and using a standard result on gambler's ruin, we find that $\widehat{f}(x_*,x) \le 1/\lambda$. Therefore

$$1 - \widehat{f}(x,x) \ge \frac{\lambda}{d(x)+\lambda}\Big(1 - \frac{1}{\lambda}\Big) = \frac{\lambda-1}{d(x)+\lambda}\,. \tag{2.3}$$

Since $\widehat{G}(x,x) = 1/(1-\widehat{f}(x,x))$, combining (2.3) and (2.2) yields (2.1). ∎

Let R_n be the number of distinct vertices visited by time n. Our next proposition is interesting in itself.

PROPOSITION 2.2. *Let $\lambda > 1$ and let T be any tree on which RW_λ is transient. Then for all $n \ge 1$,*

$$\frac{\mathbf{E}[R_n]}{n} \ge \frac{1}{n} + \frac{\lambda-1}{2\lambda G(\rho,\rho) + (\lambda-1)}\,.$$

Proof. For every $k \le n$, we have

$$\mathbf{P}[\forall j \in (k,n] \; X_j \ne X_k \mid X_k] \ge G(X_k,X_k)^{-1}\,.$$

Since R_n is the number of epochs at which a vertex is visited for the last time, it follows that

$$\mathbf{1}_{\{\forall j\in(k,n]\ X_j\neq X_k\}}\Bigg] \geq 1+\mathbf{E}\left[\sum_{k=0}^{n-1} G(X_k,X_k)^{-1}\right]$$

$$G(\rho,\rho)^{-1}\mathbf{E}\left[\sum_{k=0}^{n-1}\frac{1}{d(X_k)+\lambda}\right] \tag{2.4}$$

by Proposition 2.1. This bound is effective when the typical degrees are small. To handle large degrees, note that for $x\neq\rho$, the drift at x is

$$\mathbf{E}\big[|X_{k+1}|-|X_k|\bigm| X_k=x\big]=\frac{d(x)-\lambda}{d(x)+\lambda}\,.$$

Therefore,

$$\mathbf{E}[R_n]\geq 1+\mathbf{E}\big[|X_n|\big]\geq 1+\mathbf{E}\left[\sum_{k=0}^{n-1}\frac{d(X_k)-\lambda}{d(X_k)+\lambda}\right]\,. \tag{2.5}$$

Now multiply (2.4) by $2\lambda G(\rho,\rho)/(\lambda-1)$ and add to (2.5). After a small amount of algebra, we obtain the proposition. ∎

REMARK. The expected range can grow linearly even when RW_λ is recurrent, as can be checked for the case $\lambda=2$ on the binary tree.

§3. Speed.

Our aim in this section is to prove the following theorem.

THEOREM 3.1. *For $1<\lambda<m$ and for a.e. Galton-Watson tree T upon nonextinction, the limit $\lim_{n\to\infty}|X_n|/n$ exists a.s. and is a positive constant depending only on λ and the offspring distribution. A lower bound is*

$$\frac{(1-\lambda^{-1})^3}{12}(1-q_\lambda)^2\,, \tag{3.1}$$

where q_λ is the smallest nonnegative number satisfying $f(1-\lambda^{-1}(1-q_\lambda))=q_\lambda$.

Our proof relies on the existence of infinitely many regeneration epochs, where, given a path $\langle X_0,X_1,\ldots\rangle$, we call $n>0$ a **fresh epoch** if $X_n\neq X_k$ for all $k<n$ and a

regeneration epoch if, in addition, $X_{n-1} \neq X_k$ for all $k > n$. Define $\gamma(T)$ to be the probability that, for the tree T' gotten by adjoining a new vertex to the root of T and designating it the root of T', the walk RW_λ on T' never returns to its root. This is the same as the effective conductance from the root of T' to infinity when edges at distance n from the root of T' have conductance λ^{-n}. To establish that there are infinitely many regeneration epochs, we work on the space of trees, not, as in Section 2, on only one tree. At first reading, we recommend that the reader consider only the case $p_0 = 0$. For this and other proofs, let $\mathbf{P}_{\text{non}}$ and $\mathbf{E}_{\text{non}}$ denote probability and expectation conditional on nonextinction.

Lemma 3.2. *Let A be a measurable set of infinite trees and $\mathcal{F}_n$ be the σ-field generated by the events $\{X_i \neq X_j\}$ for $0 \leq i < j \leq n$. Let α be a stopping time with respect to $\langle \mathcal{F}_n \rangle$ such that α is a fresh epoch and let T^α denote the descendant subtree of X_α. Then*

$$\mathbf{P}_{\text{non}}[T^\alpha \in A \mid \mathcal{F}_\alpha] = \mathbf{P}_{\text{non}}[T \in A]\,.$$

Proof. This lemma expresses a strong Markov property, which is evident without the conditioning on nonextinction. Since each of the events $T^\alpha \in A$ and $T \in A$ implies nonextinction of T, we have

$$\mathbf{P}_{\text{non}}[T^\alpha \in A \mid \mathcal{F}_\alpha] = \frac{\mathbf{P}[T^\alpha \in A \mid \mathcal{F}_\alpha]}{1-q} = \frac{\mathbf{P}[T \in A]}{1-q} = \mathbf{P}_{\text{non}}[T \in A]\,. \qquad \blacksquare$$

Lemma 3.3. *Let $1 < \lambda < m$. For a.e. Galton-Watson tree T upon nonextinction and a.e. sample path of* RW_λ*, there are infinitely many regeneration epochs.*

Proof. Condition throughout on nonextinction. It suffices to show that for any N, there is a.s. a regeneration epoch $n \geq N$. Since T is infinite, there is a.s. a fresh epoch $n \geq N$; let α be the first such. From Lemma 3.2, with the same notation, we have

$$\text{ch } \geq N \mid \mathcal{F}_N] \geq \mathbf{P}_{\text{non}}[\alpha \text{ is a regeneration epoch} \mid \mathcal{F}_N]$$
$$= \mathbf{E}_{\text{non}}[\gamma(T)]\,.$$

Denote by $\mathcal{F}_\infty$ the join of all the σ-fields $\mathcal{F}_n$. By martingale convergence, the conditional probability of a regeneration epoch after N given $\mathcal{F}_\infty$ is almost surely

$$
\begin{aligned}
\mid \mathcal{F}_{N+k}] &\ge \liminf_k \mathbf{P}_{\mathrm{non}}[\exists \text{ regeneration } \ge N+k \mid \mathcal{F}_{N+k}] \\
&\ge \mathbf{E}_{\mathrm{non}}[\gamma(T)] \,.
\end{aligned}
$$

Since the regeneration epochs are $\mathcal{F}_\infty$-measurable, there is a.s. a regeneration epoch after each N. ∎

Let the regeneration epochs be $0 < \tau_1 < \tau_2 < \cdots$. These are defined only on the event of nonextinction.

PROPOSITION 3.4. *For $1 < \lambda < m$, on the event of nonextinction, the differences between successive regeneration epochs $\{\tau_{n+1} - \tau_n\}_{n \ge 1}$ are i.i.d. as are the increments $\big\{|X_{\tau_{n+1}}| - |X_{\tau_n}|\big\}_{n \ge 1}$.*

Proof. The proof of this intuitively clear assertion requires more formal notation. Label the edges from each vertex x to its children by the integers $1, \ldots, d(x)$ so that each vertex is identified with the sequence of labels leading to it from the root. This identifies the tree T with a set $[T]$ of finite sequences of positive integers. For every vertex x, let $T(x)$ denote the tree of descendants of x, rooted at x; we identify $T(x)$ with the set $[T(x)]$ of sequences which, when appended to the sequence identifying x, correspond to vertices in T. A (finite or infinite) path $\mathbf{X} := \langle X_k \,;\ k \ge 0 \rangle$ is described by the sequence of non-negative integers $\dot{\mathbf{X}} := \langle \dot{X}_k \,;\ k \ge 1 \rangle$, where $\dot{X}_k$ is 0 if X_k is the parent of X_{k-1} and is otherwise the label on the edge from X_{k-1} to X_k. Here, as in the sequel, we use angle brackets $\langle \cdots \rangle$ to denote a sequence (rather than a set).

Conditional on the event of nonextinction, the sequence of fresh trees $T(X_{\tau_n})$ seen at regeneration epochs is clearly stationary, but not i.i.d. However, as we establish below, the part of a tree between regeneration epochs, together with the path taken through this part of the tree, is independent of the rest of the tree and of the rest of the walk. We call this part a **slab**: (see Figure 3.1)

$$\mathsf{Slab}_n := \left(\left[T(X_{\tau_n}) \setminus T(X_{\tau_{n+1}}) \cup X_{\tau_{n+1}}\right] \;;\; \langle \dot{X}_{\tau_n+1}, \dot{X}_{\tau_n+2}, \ldots, \dot{X}_{\tau_{n+1}} \rangle\right). \tag{3.2}$$

(These are defined only on the event of nonextinction. Note that Slab_n is rooted at X_{τ_n}.) The stationarity of the sequence of fresh trees seen at regeneration epochs implies that the random variables Slab_n are identically distributed.

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Figure 3.1. The slab shown is a portion of the whole tree. The path taken is highlighted. The tree that is not part of the slab is joined only through the first and last vertices of the path.

Now we demonstrate that the slabs are mutually independent given nonextinction, which implies the proposition.

Note that for $k \le n$, the variables τ_k are measurable with respect to $\langle X_k \;;\; k < \tau_n \rangle$; in particular, τ_n is just the length of this sequence. Thus it suffices to show that for $n \ge 1$, the fresh tree $[T(X_{\tau_n})]$ and the remaining walk $\langle \dot{X}_{\tau_n+k} \;;\; k \ge 1 \rangle$ are independent of $[T \setminus T(X_{\tau_n}) \cup \{X_{\tau_n}\}]$ and $\langle \dot{X}_k \;;\; k \le \tau_n \rangle$ given nonextinction. Define the maps ϕ_t and ψ_t by

$$\phi_t([T] \;;\; \dot{\mathbf{X}}) := ([T \setminus T(X_t) \cup X_t] \;;\; \langle \dot{X}_k \;;\; 1 \le k \le t \rangle)$$

and

$$\psi_t([T] \;;\; \dot{\mathbf{X}}) := ([T(X_t)] \;;\; \langle \dot{X}_{t+k} \;;\; k \ge 1 \rangle)\,.$$

Let $\mathbf{GW}$ be the measure on trees given by the Galton-Watson process and let $\mathbf{P} = \mathsf{RW}_\lambda \times \mathbf{GW}$ be the associated probability measure defined on a space Ω of paths in trees. Let T be a Galton-Watson tree and let $\mathbf{X}'$ be a sample from RW_λ on the enlarged tree T' started, however, at the root of T. Let $\mathbf{Q}$ be the distribution of the pair $(T\ ;\ \mathbf{X}')$ [*not* $(T'\ ;\ \mathbf{X}')$], so that $\mathbf{Q}$ is a probability measure on a space Ω' which contains Ω, in the sense that the set of pairs $(T\ ;\ \mathbf{X}') \in \Omega'$ such that $\mathbf{X}'$ remains in T may be identified with Ω. Note that $\mathbf{Q}(\Omega) = \mathbf{E}[\gamma(T)]$.

Likewise, for any time t, we have $\mathbf{P}[\psi_t \in \Omega \mid t \text{ fresh}, \phi_t] = \mathbf{Q}(\Omega)$. More generally, for any event $B \subseteq \Omega$, we have

$$\mathbf{P}[\psi_t \in B \mid t \text{ fresh}, \phi_t] = \mathbf{Q}(B)\,. \tag{3.3}$$

For $1 \le k < t$, denote by C_k^t the event that t is a fresh epoch and that there are exactly k regeneration epochs before time t when the walk is killed at time t. Let Ω_{non} be the intersection of Ω and the event of nonextinction. Then for any time t, any positive integer n, and any events $B \subseteq \Omega_{\text{non}}$ and F, we have by (3.3) that

$$\mathbf{P}[\psi_t \in B\ ,\ \phi_t \in F\ ,\ \tau_n = t] = \mathbf{P}[\psi_t \in B\ ,\ \phi_t \in F\ ,\ C_{n-1}^t] = \mathbf{Q}(B)\mathbf{P}[C_{n-1}^t\ ,\ \phi_t \in F]\,.$$

Therefore,

$$\sum_{t\ge n} \mathbf{Q}(B)\mathbf{P}[C_{n-1}^t\ ,\ \phi_t \in F]$$

$$,\ \phi_t \in F]\mathbf{Q}(\Omega_{\text{non}}) = \frac{\mathbf{Q}(B)}{\mathbf{Q}(\Omega_{\text{non}})}\sum_{t\ge n} \mathbf{P}[\phi_t \in F\,,\ \tau_n = t]$$

$$. \qquad (3.4)$$

In the case that F is the whole universe, $\{\phi_{\tau_n} \in F\}$ is the event of nonextinction and we get $\mathbf{P}[\psi_{\tau_n} \in B] = (1-q)\mathbf{Q}(B)/\mathbf{Q}(\Omega_{\rm non})$. Substitution into (3.4) yields

$$\frac{\mathbf{P}[\psi_{\tau_n} \in B\ ,\ \phi_{\tau_n} \in F]}{1-q} = \frac{\mathbf{P}[\psi_{\tau_n} \in B]}{1-q}\frac{\mathbf{P}[\phi_{\tau_n} \in F]}{1-q}\,,$$

which establishes the desired independence. ∎

Corollary 3.5. *For $1 < \lambda < m$, the differences between successive regeneration epochs, $\{\tau_{n+1} - \tau_n\}_{n \geq 1}$, have finite means conditional on the event of nonextinction. An upper bound on their mean is the reciprocal of (3.1).*

Proof. The expected number of regeneration epochs in $[1, n]$ is the sum over $k \in [1, n]$ of the probability that k is a regeneration epoch. For each k, this is $\mathbf{E}[\gamma(T)]$ times the probability that k is a fresh epoch. The sum over $[1, n]$ of the probabilities that k is a fresh epoch equals $\mathbf{E}[R_n]$. Therefore, by Proposition 2.2, the expected number of regeneration epochs grows linearly in time with a lower bound of

$$\lim_{n\to\infty} \mathbf{E}[\gamma(T)]\mathbf{E}[\frac{R_n}{n}] \geq \mathbf{E}[\gamma(T)]\mathbf{E}\left[\frac{\lambda-1}{3\lambda G(\rho,\rho)}\right] = \frac{\lambda-1}{3\lambda}\mathbf{E}[\gamma(T)]^2\,. \tag{3.5}$$

Since the times between regeneration epochs are i.i.d. given nonextinction, it follows by the strong law of large numbers that $\mathbf{E}_{\rm non}[\tau_2 - \tau_1] < \infty$. Moreover, according to (3.5), an upper bound for their mean is $3\lambda/[(\lambda-1)\mathbf{E}[\gamma(T)]^2]$. In order to make this bound more explicit, we use the connection between random walks and percolation of Lyons (1992). Define $\gamma'(T)$ to be the effective conductance from the root of T' to infinity when the edge from the root of T' to the root of T has unit conductance, while edges at distance $n \geq 1$ from the root of T' have conductance $\lambda^{1-n}/(\lambda-1)$. Also, let $p(T)$ be the probability that the component of the root of T is infinite when the edges of T are removed independently with probability $1-\lambda^{-1}$ each. Then the inequality at the bottom of p. 2047 of Lyons (1992) says that

$$\gamma'(T) \leq p(T) \leq 2\gamma'(T).$$

It is easy to calculate that $\gamma(T) \geq (\lambda-1)\gamma'(T)/\lambda$, whence

$$\mathbf{E}[\gamma(T)] \geq \frac{\lambda-1}{2\lambda}\mathbf{E}[p(T)] = \frac{\lambda-1}{2\lambda}(1-q_\lambda),$$

since $\mathbf{E}[p(T)]$ is the probability of nonextinction of a Galton-Watson branching process with probability generating function $s \mapsto f(1-\lambda^{-1}+\lambda^{-1}s)$. ∎

Proof of Theorem 3.1. Condition on nonextinction. By the strong law of large numbers, $\tau_n/n \to \mathbf{E}_{\mathrm{non}}[\tau_2-\tau_1]$ a.s. and $|X_{\tau_n}|/n \to \mathbf{E}_{\mathrm{non}}\big[|X_{\tau_2}|-|X_{\tau_1}|\big]$ a.s. Therefore,

$$\frac{|X_{\tau_n}|}{\tau_n} \to \frac{\mathbf{E}_{\mathrm{non}}\big[|X_{\tau_2}|-|X_{\tau_1}|\big]}{\mathbf{E}_{\mathrm{non}}[\tau_2-\tau_1]} \quad \text{a.s.} \tag{3.6}$$

Since $\lim \tau_n/n$ exists and is finite by Corollary 3.5, we have $\tau_{n+1}/\tau_n \to 1$ and the theorem follows. The lower bound arises from the upper bound in Corollary 3.5 and the observation that the numerator of (3.6) is at least 1. ∎

§4. Outward-biased Random Walks.

If $\lambda < 1$ and $p_0 = 0$, the argument of the preceding section works to give the existence and positivity of the speed of RW_λ, provided we substitute the easy (2.5) for Proposition 2.2. Thus, when $\lambda < 1$, the most interesting possibility occurs when $p_0 > 0$: the walk may have zero speed by spending too much time at leaves. Recall that q is the extinction probability of the Galton-Watson process.

THEOREM 4.1. *Suppose that $p_0 > 0$. Let T be a Galton-Watson tree conditioned on nonextinction. The speed of RW_λ exists and is constant a.s. It is positive if $f'(q) < \lambda < 1$ and zero if $0 \le \lambda \le f'(q)$.*

Proof. Since the case $\lambda = 0$ is obvious, we assume that $\lambda > 0$. Let $g(s) := [f(s) - f(qs)]/(1-q)$ and $h(s) := f(qs)/q$. Then an f-Galton-Watson tree T_f conditioned on nonextinction may be generated by first generating a g-Galton-Watson tree T_g and then appending to each vertex x of T_g a random number N_x of h-Galton-Watson **shrubs**, where N_x has a distribution dependent on $d_{T_g}(x)$ only and, given T_g and the numbers N_x, the shrubs are i.i.d. We shall not need the explicit form of the distribution of N_x (see Lyons (1992)). Call the union of the N_x shrubs at x a **bush**. See Figure 4.1.

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Figure 4.1. Part of the tree T_f decomposed as the tree T_g (solid lines) together with bushes (dashed lines).

If we observe RW_λ on T_f only at the times σ_n that it makes a transition along an edge of T_g, then we see a sample $Y_n := X_{\sigma_n}$ of RW_λ on T_g. Between these observations, there are excursions of random lengths, possibly zero. To determine the lengths of these excursions, we consider a single bush. The expected length of time that RW_λ takes to return to the root on a fixed finite tree Γ is equal to the reciprocal of the stationary probability of the root of Γ. Since RW_λ is reversible, this is easily calculated to be $2\sum_{n\geq 1}\Gamma_n\lambda^{1-n}/\Gamma_1$, where Γ_n is the number of vertices in generation n. In particular, for h-Galton-Watson bushes, this sum has expectation

$$2\sum_{n\geq 1} h'(1)^{n-1}\lambda^{1-n} = \begin{cases} 2/(1-f'(q)\lambda^{-1}) & \text{if } \lambda > f'(q), \\ \infty & \text{otherwise.} \end{cases} \tag{4.1}$$

When $0 < \lambda \leq f'(q)$, it follows that the expected time between regeneration epochs on T_f is infinite, whence by the strong law of large numbers, the speed is a.s. zero. (Note that the expected distance between successive regeneration loci on T_f is the same as on T_g, hence is finite.)

Now assume that $f'(q) < \lambda < 1$. Between times σ_n and σ_{n+1}, the walk $\langle X_k\rangle$ makes a random number of excursions into the bush at Y_n. The number of excursions has a geometric distribution minus 1 with mean $(d_{T_f}(Y_n) - d_{T_g}(Y_n))/(\lambda + d_{T_g}(Y_n))$. In conjunction with (4.1), this implies that

$$\mathbf{E}_{\mathrm{non}}[\sigma_{n+1} - \sigma_n \mid Y_n] \leq c d_{T_f}(Y_n) \tag{4.2}$$

for some constant c depending only on λ and f. Let $Z_1, \ldots, Z_{K_n}$ be the distinct vertices among $Y_1, \ldots, Y_n$. Let $U_i = \sum_{j=1}^{\infty} \mathbf{1}_{\{Y_j = Z_i\}}$. Then

$$\sum_{i=1}^{n} d_{T_f}(Y_i) \le \sum_{k=1}^{K_n} U_k d_{T_f}(Z_k),$$

so that

$$\mathbf{E}_{\mathrm{non}}\left[\sum_{i=1}^{n} d_{T_f}(Y_i)\right] \le \mathbf{E}_{\mathrm{non}}\left[\sum_{k=1}^{K_n} U_k d_{T_f}(Z_k)\right].$$

For each k, comparison to asymmetric simple random walk and use of Lemma 3.2 gives

$$\begin{aligned})] &= \mathbf{E}_{\mathrm{non}}\left[d_{T_f}(Z_k)\mathbf{E}_{\mathrm{non}}[U_k \mid d_{T_f}(Z_k)]\right] \\ &\le \mathbf{E}_{\mathrm{non}}\left[d_{T_f}(Z_k)\frac{1+\lambda}{1-\lambda}\right] = \frac{m}{1-q}\,\frac{1+\lambda}{1-\lambda}. \end{aligned}$$

Therefore,

$$\mathbf{E}_{\mathrm{non}}\left[\sum_{i=1}^{n} d_{T_f}(Y_i)\right] \le n\frac{m}{1-q}\,\frac{1+\lambda}{1-\lambda}.$$

In conjunction with (4.2), this yields

$$\mathbf{E}_{\mathrm{non}}[\sigma_n/n] \le \frac{cm(1+\lambda)}{(1-q)(1-\lambda)},$$

whence by Fatou's lemma, $\liminf_{n\to\infty} \sigma_n/n < \infty$ a.s. Because regenerations occur with positive frequency on T_g, it follows that $\liminf_{k\to\infty} \tau_k/k < \infty$ a.s., where τ_k are the regeneration epochs of $\mathbf{X}$. By the strong law of large numbers, it follows that $\mathbf{E}[\tau_{k+1} - \tau_k] < \infty$, and the above lim inf is a limit a.s. with constant value $\mathbf{E}[\tau_2 - \tau_1]$. Now for

$\tau_k \le n < \tau_{k+1}$, we have $|X_{\tau_k}| \le |X_n| < |X_{\tau_{k+1}}|$. Since $\lim \tau_{k+1}/\tau_k = 1$, it follows that

$$\lim_{n\to\infty} |X_n|/n = \lim_{k\to\infty} |X_{\tau_k}|/\tau_k \ge \lim_{k\to\infty} k/\tau_k > 0\,. \qquad \blacksquare$$

§5. Dimension of Harmonic Measure.

Recall that the Hausdorff dimension of a Borel measure ν on a metric space is defined as the infimum of Hausdorff dimensions of Borel sets with full ν-measure.

Given a rooted tree T, let ∂T denote the set of infinite self-avoiding paths from the root of T. This becomes a compact metric space when equipped with the standard metric that assigns distance e^{-n} to any pair of self-avoiding paths with exactly n edges in common. The Hausdorff dimension of ∂T is $\log m$ for a.e. Galton-Watson tree T (Hawkes 1981). Let UNIF_T denote the measure on ∂T which is the weak limit of measures uniform on the vertices in the n-th generation of T; this limit exists on a.e. Galton-Watson tree T: see, e.g., Equation (6.2) in Lyons, Pemantle and Peres (1995). When the random walk RW_λ is transient and cycles are erased from the path, the path converges almost surely to an element of ∂T whose law is denoted HARM_T^λ. Let HARM^λ be the function which assigns to every tree T the probability measure on its first generation corresponding to HARM_T^λ, i.e.,

$$\mathsf{HARM}^\lambda(T)(x) = \mathsf{HARM}_T^\lambda\{\text{paths passing through } x\}$$

for a vertex x in the first generation of T. This gives transition probabilities for a Markov chain on the space of trees if we let $\mathsf{HARM}^\lambda(T)(x)$ be the transition probability from T to the descendant tree $T(x)$.

Call t an **exit epoch** for the path $\langle X_k\,;\ k \ge 0\rangle$ if X_{t-1} is the parent of X_t and $X_k \ne X_{t-1}$ for all $k > t$. Let $\langle t_k\rangle$ be the successive exit epochs. Then $\langle X_{t_k}\rangle$ forms a random ray of T with distribution HARM_T^λ by definition. Therefore,

$$\text{The subtrees } T(X_{t_k}) \text{ form a } \mathsf{HARM}^\lambda\text{-Markov chain.} \tag{5.1}$$

For a fixed offspring distribution, let $\mathbf{GW}$ denote the resulting Galton-Watson measure on the space of trees.

THEOREM 5.1. *For $0 \le \lambda < m$, conditional on nonextinction, the Hausdorff dimension of HARM_T^λ is $\mathbf{GW}$-a.s. strictly less than $\log m$. For $0 \le \lambda_1 < \lambda_2 < m$, the measures $\mathsf{HARM}_T^{\lambda_1}$ and $\mathsf{HARM}_T^{\lambda_2}$ are $\mathbf{GW}$-a.s. mutually singular. (We allow $\lambda = 0$ only if $p_0 = 0$.)*

The proof depends on the following lemma.

LEMMA 5.2. *Assume $p_0 = 0$. For $0 \le \lambda < m$, there is a finite stationary measure for the HARM^λ-Markov chain, denoted μ_{HARM}, that is absolutely continuous with respect to $\mathbf{GW}$.*

Proof of Theorem 5.1. Because of the decomposition described in the previous section, the theorem reduces to the case $p_0 = 0$. Theorem 7.1 of Lyons, Pemantle and Peres (1995) shows that the dimension of HARM_T^λ will be a.s. less than $\log m$ as long as HARM^λ has a stationary measure absolutely continuous with respect to **GW**, and as long as HARM_T^λ is not a.s. equal to UNIF_T. The argument of Proposition 8.3 in that paper applies in the present case to show that HARM_T^λ is not a.s. equal to UNIF_T, and Lemma 5.2 of the present work thus shows that $\dim(\mathsf{HARM}_T^\lambda) < \log m$ a.s. Theorem 7.1 of Lyons, Pemantle and Peres (1995) also shows that $\mathsf{HARM}_T^{\lambda_1}$ and $\mathsf{HARM}_T^{\lambda_2}$ are a.s. mutually singular if they are not a.s. equal. To see that they are a.s. unequal, note that a.s. equality would force the vector

$$\left\langle \frac{\gamma_{\lambda_1}(T(x))}{\gamma_{\lambda_2}(T(x))} \right\rangle_{|x|=1} \tag{5.2}$$

to be a multiple of the constant vector **1** since

$$\mathsf{HARM}_T^\lambda(x) = \frac{\gamma_\lambda(T(x))}{\sum_{|y|=1} \gamma_\lambda(T(y))} .$$

For Galton-Watson trees, each component of this vector has the same law as that of $\gamma_{\lambda_1}(T)/\gamma_{\lambda_2}(T)$. Thus, the independence of $T(x)$ and $T(y)$ for two distinct children x and y of the root implies that the random vector (5.2) is, in fact, constant **GW**-a.s. Thus, $\gamma_{\lambda_1}(T)/\gamma_{\lambda_2}(T)$ is a constant **GW**-a.s. This is easily seen to imply that some p_k equals 1, which contradicts our standing assumption. ∎

Proof of Lemma 5.2. The case $\lambda = 1$ was done in Lyons, Pemantle and Peres (1995), so assume that $\lambda \neq 1$. We provide only a sketch due to space restrictions. Let $\Psi_n := \langle T(X_{\tau_n}), T(X_{\tau_n+1}), \ldots, T(X_{\tau_{n+1}-1})\rangle$ be the sequence of forward trees seen by the walk during the nth slab. Then $\langle \Psi_n\,;\ n \geq 1\rangle$ is a stationary Markov chain. There is at least one exit epoch occurring in each slab, namely, τ_n. For each n, let Φ_n be the finite sequence of trees $\langle T(X_t)\,;\ t$ an exit epoch in the nth slab$\rangle$. Thus, $\langle \Phi_n\,;\ n \geq 1\rangle$ is a factor of $\langle \Psi_n\,;\ n \geq 1\rangle$. Let $h(\langle \Phi_n\rangle) \geq 1$ be the length of the sequence Φ_1. The tower over $\langle \Phi_n\rangle$ with height function h yields a shift-invariant distribution for $\langle T(X_{t_k})\rangle$. Examination of the tower construction shows that this last sequence is a HARM^λ-Markov chain. It is necessarily stationary, with some initial distribution μ_{HARM}.

It remains to prove that μ_{HARM} is absolutely continuous with respect to $\mathbf{GW}$. Now for any Borel subset A of trees,

$$\mu_{\mathsf{HARM}}(A) \leq \int \sum_{n=\tau_1}^{\tau_2-1} \mathbf{1}_A(T(X_n))\, d\mathbf{GW} =: \nu(A)\,.$$

Thus, it suffices to show that if $\mathbf{GW}(A) = 0$, then $\nu(A) = 0$. Indeed,

$$\nu(A) \leq \int \sum_{v\in T} \mathbf{1}_A(T(v))\, d\mathbf{GW}\,.$$

For each vertex v in a Galton-Watson tree T, the forward tree $T(v)$ is also a Galton-Watson tree, so the last integral vanishes. ∎

We now demonstrate how the drop in dimension of harmonic measure implies the confinement of RW_λ to a smaller subtree. Given a tree T and positive integer n, let T_n be the vertices of T at distance n from the root and $|T_n|$ be the cardinality of T_n. We remark that the following proof is both easier and more general than the analogous proof of Theorem 9.9 in Lyons, Pemantle and Peres (1995).

COROLLARY 5.3. *Assume that $p_0 = 0$. Fix an offspring distribution and $\lambda \in [0, m)$. For* **GW**-*almost all trees T and for every $\epsilon > 0$, there is a subtree $T^{(\epsilon)} \subseteq T$ such that*

$$\mathsf{RW}_\lambda \Big\{ X_n \in T^{(\epsilon)} \textit{ for all } n \Big\} \geq 1 - \epsilon \tag{5.3}$$

and

$$\frac{1}{n} \log |T_n^{(\epsilon)}| \to \dim(\lambda),$$

where $\dim(\lambda) < \log m$ is the dimension of HARM_T^λ. Furthermore, any subtree $T^{(\epsilon)}$ satisfying (5.3) must have growth

$$\liminf \frac{1}{n} \log |T_n^{(\epsilon)}| \geq \dim(\lambda).$$

Proof. Let $t_k := 1 + \max\{t\,;\ |X_t| = k\}$ be the k-th exit epoch and $D(x,k)$ be the set of descendants y of x with $|y| \leq |x| + k$. We shall use three sample path properties of RW_λ on a fixed tree:

$$\lim_{\to\infty} \frac{|X_n|}{n} = \mathrm{speed}(\lambda) > 0 \qquad \text{a.s.} \tag{5.4}$$

$$\lim_{\to\infty} \frac{1}{k} \log \frac{1}{\mathsf{HARM}_T^{\lambda}(X_{t_k})} = \dim(\lambda) \quad \text{a.s.} \tag{5.5}$$

$$\forall\delta > 0 \qquad \limsup_{n\to\infty} \frac{\log |D(X_n, \delta |X_n|)|}{|X_n|} \le \delta \log m \quad \text{a.s.} \tag{5.6}$$

(In fact, the limit in (5.6) exists and equals the right-hand side, but this is not needed.)

The first property (5.4) was proved in Section 2 and the second (5.5) follows from a result of Billingsley and an idea of Furstenberg once the absolute continuity in Lemma 5.2 has been established; see Lyons, Pemantle and Peres (1995), Lemma 4.1 and Section 5. In order to see that (5.6) holds for **GW**-a.e. tree, denote by Y_k the k-th fresh point visited by RW_λ. Then (5.6) can be written as

$$\forall\delta > 0 \qquad \limsup_{k\to\infty} |Y_k|^{-1} \log |D(Y_k, \delta |Y_k|)| \le \delta \log m$$

and since $|Y_k|/k$ has a positive a.s. limit, this is equivalent to

$$\forall\delta^* > 0 \qquad \limsup_{k} k^{-1} \log |D(Y_k, \delta^* k)| \le \delta^* \log m\,. \tag{5.7}$$

Now the random variables $|D(Y_k, \delta^* k)|$ are identically distributed, though not independent. Indeed, the descendant subtree of Y_k has the law of **GW**. Since the expected number of descendants of Y_k at generation $|Y_k|+j$ is m^j for every j, we have

$$\mathbf{P}(|D(Y_k,\delta^* k)| \geq m^{\delta' k}) \leq m^{-\delta' k} \sum_{j=0}^{\delta^* k} m^j \,.$$

If $\delta' > \delta^*$, then the right-hand side decays exponentially in k, so by the Borel-Cantelli lemma, we get (5.7), hence (5.6).

Now (5.5) alone implies the last assertion of Corollary 5.3.

Applying Egorov's theorem to the two almost sure asymptotics (5.4) and (5.5), we see that for each $\epsilon > 0$, there is a set of paths A_ϵ with $\mathsf{RW}_\lambda(A_\epsilon) > 1-\epsilon$ and such that the convergence is uniform on A_ϵ. Thus, we can choose $\langle \delta_n \rangle$ decreasing to 0 such that on A_ϵ, for all k and all n,

$$\mathsf{HARM}_T^\lambda(X_{t_k}) > e^{-k(\dim(\lambda)+\delta_k)} \quad \text{and} \quad \left| \frac{|X_n|}{n \text{ speed } (\lambda)} - 1 \right| < \delta_n \,. \tag{5.8}$$

Now since $\{\delta_n\}$ is eventually less than any fixed δ, (5.6) implies that

$$\limsup_{n\to\infty} |X_n|^{-1} \log |D(X_n, 3\delta_{|X_n|}|X_n|)| = 0 \text{ a.s.},$$

so applying Egorov's theorem again and replacing A_ϵ by a subset thereof (which we continue to denote A_ϵ), we may assume that there exists a sequence $\{\eta_n\}$ decreasing to 0 such that

$$|D(X_n, 3\delta_{|X_n|}|X_n|)| \leq e^{|X_n|\eta_n} \quad \text{for all } n \tag{5.9}$$

on A_ϵ.

Define $F_0^{(\epsilon)}$ to consist of all vertices $v \in T$ such that either $\delta_{|v|} \geq 1/3$ or both

$$\mathsf{HARM}_T^\lambda(v) \geq e^{-|v|(\dim(\lambda)+\delta_{|v|})} \quad \text{and} \quad |D(v, 3\delta_{|v|}|v|)| \leq e^{|v|\eta_{|v|}}\,.$$

Finally, let

$$F^{(\epsilon)} = \bigcup_{v \in F_0^{(\epsilon)}} D(v, 3\delta_{|v|}|v|)$$

and denote by $T^{(\epsilon)}$ the component of the root in $F^{(\epsilon)}$. Since the number of vertices $v \in T_n$ satisfying $\mathsf{HARM}_T^\lambda(v) \geq e^{-|v|(\dim(\lambda)+\delta_{|v|})}$ is at most $e^{n(\dim(\lambda)+\delta_n)}$, the bound on $|D(v, 3\delta_{|v|}|v|)|$ bounds the growth rate from above as asserted in the statement of the corollary. It remains to establish that RW_λ stays inside $F^{(\epsilon)}$ forever on the event A_ϵ, since that will imply that the walk is confined to $T^{(\epsilon)}$ on this event. The points visited at exit epochs t_k are in $F_0^{(\epsilon)}$ by the first part of (5.8) and (5.9). Fix a path $\{X_j\}$ in A_ϵ and a time n, and suppose that the last exit epoch before n is t_k, so that $t_k \leq n < t_{k+1}$. Denote by $N := t_{k+1} - 1$ the time preceding the next exit epoch, and observe that $X_N = X_{t_k}$. If $\delta_n \geq 1/3$, then X_n is in $F_0^{(\epsilon)}$ since $\delta_{|X_n|} \geq \delta_n$, so consider the case that $\delta_n < 1/3$. By the second part of (5.8), we have

$$\frac{|X_n|}{n \text{ speed } (\lambda)} < 1 + \delta_n \qquad \text{and} \qquad \frac{|X_N|}{n \text{ speed } (\lambda)} \geq \frac{|X_N|}{N \text{ speed } (\lambda)} > 1 - \delta_N \geq 1 - \delta_n \,.$$

Dividing, we find that

$$|X_n| \leq \frac{1 + \delta_n}{1 - \delta_n} |X_N| \leq (1 + 3\delta_n)|X_N| \,.$$

It follows that X_n is in $D(X_{t_k}, 3\delta_{|X_{t_k}|}|X_{t_k}|)$ and this completes the proof. ∎

§6. Dependence on the Bias Parameter λ.

Fix an offspring distribution, and recall that speed(λ) denotes the a.s. constant speed of RW_λ on Galton-Watson trees upon nonextinction. Similarly, denote by dim(λ) the a.s. constant dimension of the harmonic measure HARM_T^λ. The methods of this paper are not well suited to analyze the dependence of speed(λ) and dim(λ) on the parameter λ. We state explicitly two questions in this direction, and refer to the survey Lyons, Pemantle and Peres (1996) for further questions and relevant examples.

Question 1: *Assume that the offspring distribution satisfies* $p_0 = 0$. *Is* speed(λ) *monotonic nonincreasing for* $\lambda \in [0, m)$ *?*

Though a positive answer is intuitively compelling, the evidence available indicates that if monotonicity holds, it is a special property of Galton-Watson trees. The calculations in Section 4 show that the assumption $p_0 = 0$ cannot be dropped. Even if we restrict attention to trees without leaves, there exist family trees of two-type Galton-Watson processes for which $\mathrm{speed}\,(\lambda)$ is not monotonic in λ (see Lyons, Pemantle and Peres 1996).

Question 2: *Determine the smoothness properties of* $\mathrm{speed}\,(\lambda)$ *and* $\dim\,(\lambda)$ *for* $\lambda \in [0, m)$.

In particular, the methods of the present paper do not yield the intuitively "obvious" inequality

$$\liminf_{\lambda \to 1}\ \mathrm{speed}(\lambda) > 0, \tag{6.1}$$

since the a priori bound for the Green function in Proposition 2.1 blows up as $\lambda \downarrow 1$. Of course, continuity of the speed at $\lambda = 1$ would immediately imply (6.1).

Continuity for $\lambda < 1$ is easier to establish, since comparison with simple asymmetric random walk on the integers is possible.

PROPOSITION 6.1. *If* $p_0 = 0$, *then* $\mathrm{speed}\,(\lambda)$ *is continuous for* $\lambda \in [0, 1)$.

Proof. We construct a richer probability space on which random walks with laws RW_λ are simultaneously defined for all $\lambda \geq 0$. Pick a tree T according to Galton-Watson measure.

Label the edges of T as in the proof of Proposition 3.4. Let $\langle U_n \rangle$ be a sequence of i.i.d. random variables uniformly distributed on [0,1]. For every $\lambda \geq 0$, we define inductively a sequence of vertices $\langle X_n^\lambda \rangle$ as follows. First, let X_0^λ be the root of T. For $n \geq 1$, denote by $d_{n-1}(\lambda)$ the number of children of X_{n-1}^λ. If X_{n-1}^λ is the root, then define $\overset{\bullet}{X}{}_n^\lambda := \lceil d_{n-1}(\lambda) \cdot U_n \rceil$. Otherwise, let

$$\overset{\bullet}{X}{}_n^\lambda := \lceil (\lambda + d_{n-1}(\lambda)) \cdot U_n \rceil \tag{6.2}$$

if the right-hand side is at most $d_{n-1}(\lambda)$, and $\overset{\bullet}{X}{}_n^\lambda := 0$ if the right-hand side of (6.2) is strictly greater than $d_{n-1}(\lambda)$. This defines the path $\langle X_n^\lambda \rangle$ as in the proof of Proposition 3.4.

Given T, the sequence $\langle X_n^\lambda \rangle$ is a clearly a sample from RW_λ. For any fixed $\lambda_0 \geq 0$ and $n \geq 1$, we clearly have pointwise convergence:

$$X_n^\lambda \to X_n^{\lambda_0} \text{ almost surely as } \lambda \to \lambda_0 \,. \tag{6.3}$$

Pick $\lambda_{\max} < 1$. Denote by $\tau_k(\lambda)$ the k-th regeneration epoch of $\langle X_n^\lambda \rangle$. We shall show continuity of speed for $\lambda \in [0, \lambda_{\max}]$ by using the formula

$$\text{speed}(\lambda) = \frac{\mathbf{E}\left[|X_{\tau_2(\lambda)}^\lambda| - |X_{\tau_1(\lambda)}^\lambda| \right]}{\mathbf{E}\big[|\tau_2(\lambda) - \tau_1(\lambda)| \big]} \,. \tag{6.4}$$

Using the random variables U_n, we also define an asymmetric simple random walk $\langle Y_n \rangle$ on the integers. Let $Y_0 := 0$ and for $n \geq 1$, let

$$Y_n := Y_{n-1} + \operatorname{sign}\left(\frac{1}{1+\lambda_{\max}} - U_n\right).$$

Whenever $Y_n > Y_{n-1}$, necessarily $|X_n^\lambda| > |X_{n-1}^\lambda|$ for all $\lambda \in [0, \lambda_{\max}]$. Therefore every regeneration epoch for the process $\langle Y_n \rangle$ is also a regeneration epoch for each of the processes $\langle X_n^\lambda \rangle$ with $\lambda \leq \lambda_{\max}$. Denoting the k-th regeneration epoch for $\langle Y_n \rangle$ by τ_k^Y, we see that $\tau_k(\lambda) \leq \tau_k^Y$ for all $\lambda \leq \lambda_{\max}$, and therefore $\tau_k(\lambda) \to \tau_k(\lambda_0)$ when $\lambda \to \lambda_0 \leq \lambda_{\max}$. Because the speed of $\langle Y_n \rangle$ is positive, τ_k^Y is integrable for each k (indeed, it has an exponentially decaying tail — see, e.g., Lemma 5.1 in Dembo, Peres, and Zeitouni (1995)). Thus, continuity of speed(λ) in the interval $[0, \lambda_{\max}]$ follows from (6.3), (6.4) and Lebesgue's dominated convergence theorem. ∎

REMARK. Similarly, if $p_i = 0$ for $i < N$, then speed(λ) is continuous for $\lambda \in [0, N)$.

REMARK. Very similar methods allow us to deduce Theorem 3.1 for $1 < \lambda < \rho$ for positive-regular nonsingular multitype branching processes such that each particle has at least one child (a stronger condition than a.s. nonextinction, but analogous to $p_0 = 0$), where ρ is the maximal eigenvalue of the mean matrix. We do not know how to prove that the speed of simple random walk ($\lambda = 1$) is positive on multitype trees.

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