Credible Signaling in the “Letter of Recommendation” Game

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Contents

1 Introduction

1.1 What is a signaling problem?

Formal games and game theory are used to model and analyze many interactions. Examples range from familiar play games such as tic-tac-toe, chess, and poker to classic study games like prisoner's dilemma and the Nash bargaining game, to models of complex real life interactions such as the job hunt, price negotiation, and setting a teenager's curfew. In any game, each player has a partial order on the set of possible outcomes, and a real-valued function known as his utility function that quantifies the player's partial order on the outcomes. Each player's goal is to make game decisions that maximize his utility.

A signaling game is a game in which one player controls information that affects both players and the second player controls an outcome that affects both players. The first player and second player attempt to communicate in a mutually beneficial way, i.e. one that comes as close as possible to allowing the players simultaneously maximize their respective utilities. The players' respective utilities depend solely on the information and the action, not on the method of communication. The study of such games, and the conditions under which reasonable equilibrium strategies arise is a field of significant intrinsic and practical interest.

Signaling games may be completely cooperative or only partially cooperative. Internet protocol is optimized when the information exchange occurs completely cooperatively. Since the computers cannot communicate in advance to arrange the information exchange, the goal of the game is to arrange for cooperation in the absence of prior communication. In trade negotiations, both the importer and exporter control private information (the importer knows the maximum he will pay; the exporter knows the minimum he will accept) and attempt to convey information that is accurate enough to make a mutually beneficial deal, but noisy enough to optimize their expected outcomes. We may consider one-way signaling the situation in which a professor has confidential information about the skill of the student and recommends the student with the goal of the student obtaining the best job possible; the employer's goal is of course that the student's skill and the job's requirements are close. This latter problem will the be the focus of this paper.

1.2 Summary Of The Thesis

In the next section, we will lay the formal game-theoretic groundwork required to pursue rigorous analysis of the "letter of recommendation" problem. In Section 3 we will review the work of Crawford and Sobel. In their 1982 paper, *Strategic Information Transmission*, they examine a specific signaling game and classify all of its equilibria. Next we will review the approach of Green and Stokey to the same problem and Chatterjee and Samuelson's analysis of a related but notably different problem. Lastly, in Section 5 we will demonstrate that by expanding Crawford and Sobel's model to allow the sender to recommend two candidates relative to each other but not quantitatively, we achieve new equilibria that did not exist in Crawford and Sobel's original model.

2 Preliminaries and Formal Setup

2.1 Game Theory

We now develop formally the ideas of game theory that we require. Our goal is to create a rigorous framework in which we can view and analyze the decisions made by each player in the context of the game. To analyze a game, we need mathematical ways of representing game states, rules, strategies and players' preferences among the possible outcomes of the game.

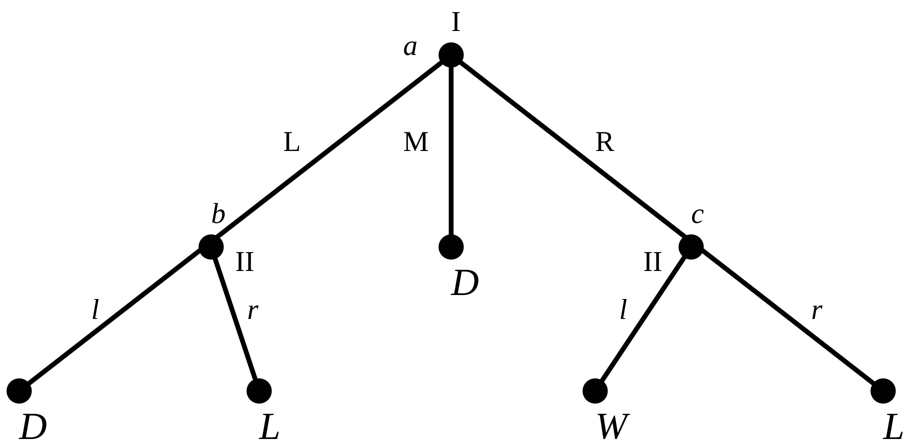
We will introduce the extensive and normal forms of a game, and discuss how they represent games states, rules, and strategies. We will develop utility theory to express player preferences. Lastly, we will show that under certain conditions, games have stable points known as Nash Equilibria.

Definition 1. *The extensive form of a game G (also known as the game tree) is a tree with the following properties:*

1. *The nodes correspond to game states. The root of the tree represents the starting state of the game. Each terminal node is an end of the game.*

2. *Each edge corresponds to an action available to the player whose turn it is at the node from which the edge emanates.*

We say a game is finite if the game tree is finite. We label each non-terminal node with its corresponding game state, and with the name of the player whose turn it is to move at that node. We label each terminal node with its corresponding outcome. Lastly, each edge is labeled with its corresponding game action. Each subtree of the game tree represents a *subgame* of G. Consider the following basic example:



At node a, player I has three choices, L, M, and R. At nodes b and c, player II has two choices, l and r. At the terminal nodes, player I's utility is the number in the upper right corner and player II's utility is number in the lower left corner. At a node, the player whose turn it is chooses an action by specifying for each choice α the probability p_α with which he uses α. may use a pure action, in which he selects one of his choices with probability 1 and the other choices with probability 0, or he may select a random action, in which he selects choice α with probability p_α. In a game of perfect information, both players see the entire game tree when making their decisions. In a game of imperfect information, nodes are grouped into information sets. The set of moves based at two nodes in the same information set look the same.

A *strategy* for a player is a list of which actions he will take at each of his decision points. A *pure strategy* is one in which every action is a pure action. A *mixed strategy* is one in which the player may select either pure or random actions. Each mixed strategy is a convex combination of pure strategies, and the space of mixed strategies is usually represented as the convex hull of pure strategies. Every pure strategy is also a mixed strategy. A *strategy pair* is an ordered pair of strategies (s_I, s_{II}) where s_I is player I's strategy and s_{II} is player II's strategy. The strategies in a strategy pair may be either mixed or pure. If (s_I, s_{II}) is a strategy pair, by $G(s_I, s_{II})$ we mean the outcome of the game when I and II employ s_I and s_{II}; by $U^I(s_I, s_{II})$ we denote I's utility when I and II employ s_I and s_{II}; by $U^{II}(s_I, s_{II})$ we denote II's utility when I and II employ s_I and s_{II}. The *normal* (also known as *strategic*) form of a game is a table of strategies for I and II and the outcomes of the strategy pairs. For the moment, assume that I and II are only allowed to use pure strategies. I's pure strategies are L, M and R. II's pure strategies are ll, lr, rl and rr, where we list first II's action at b and then at c. The normal form of G is

	ll	lr	rl	rr
L	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$
M	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
R	$\mathcal{W}$	$\mathcal{L}$	$\mathcal{W}$	$\mathcal{L}$

2.2 Utility Theory

Next, we develop utility theory, the mathematical language we use to express the players' preferences among the outcomes of the game. Let Ω be the set of outcomes of a game G, and let ω_0, ω_1, ω_2 etc. denote elements of Ω. A *preference relation* is a weak linear partial order $\preceq$ on Ω. Each player has a preference relation.

A *lottery* is a discrete distribution on Ω. Let the set Ω^* be the set of all lotteries. Since the distribution in which outcome ω occurs with probability 1 and all other outcomes occur with probability 0 is itself a lottery, $\Omega \subseteq \Omega^*$. If λ_1 and λ_2 are lotteries and $0 \leq p \leq 1$, then $p\lambda_1 + (1-p)\lambda_2$ is the lottery in which $P(\omega|p\lambda_1 + (1-p)\lambda_2) = p \cdot P(\omega|\lambda_1) + (1-p) \cdot P(\omega|\lambda_2)$.

A *utility function* is function $U : \Omega \to \mathbb{R}$ that respects a player's preference relation, i.e. if $\omega_0 \succeq \omega_1$ then $U(\omega_0) \geq U(\omega_1)$. We can extend a utility function to the set Ω^* by setting $U(\lambda) = E(U(\lambda))$, i.e. if λ is the lottery that assigns probability p_k to the outcome ω_k, $1 \leq k \leq n$, then

$$U(\lambda) = \sum_{k=1}^{n} p_k U(\omega_k). \tag{1}$$

It may appear that we have made a strong assumption that the player is risk-neutral. In fact, choosing an appropriate utility function on Ω allows us to model the behavior of players who are risk-loving, risk-neutral, and risk averse. This extension of U gives a preference relation on Ω^* wherein

$$\lambda_1 \succeq \lambda_2 \text{ if and only if } U(\lambda_1) \geq U(\lambda_2). \tag{2}$$

Alternatively, given a preference relation on Ω^*, there exists a utility function U on Ω that gives rise to such an extension if

1. If λ_1, λ_2, and μ are in Ω^* and $0 \leq p \leq 1$, then $\lambda_1 \preceq \lambda_2$ if and only if $p\lambda_1 + (1-p)\mu \preceq p\lambda_2 + (1-p)\mu$.

2. For all λ_1, λ_2, and μ in Ω^*, $\lambda_1 \prec \lambda_2$ implies there exists $p > 0$ such that $p_1 \prec p\mu + (1-p)\lambda_2$.

3. For all λ_1, λ_2, and μ in Ω^*, $\lambda_1 \prec \lambda_2$ implies there exists $p > 0$ such that $p\mu + (1-p)\lambda_1 \prec \lambda_2$.

Given a preference relation on Ω^*, a von Neumann Morgenstern utility function is a utility function that is an extension of a utility function on Ω.

We now return to our original example of a game. Assume that both players can see the game tree. If the game progresses to node b, II (being rational) will play r, since $U^{II}(\mathcal{L}) > U^{II}(\mathcal{D})$. Similarly, if the game progresses to node c, II will select r, since $U^{II}(\mathcal{L}) > U^{II}(\mathcal{W})$. If I plays L or R the game will progress to b or c and the outcome will be $\mathcal{L}$. However, if I plays M, the game terminates immediately with the outcome $\mathcal{D}$. Since $U^I(\mathcal{D}) > U^I(\mathcal{L})$, I (being rational) will play M at a, guaranteeing an outcome of $\mathcal{D}$. In essence, I has a choice of outcomes, $\mathcal{D}$ or $\mathcal{L}$.

2.3 Zero Sum Games

There are several interesting things to observe about the game, the method of analysis and the results thereof. First, note our assumption that both I and II see the game tree. Not all games have this feature; those that do are games of *perfect information.* Second, note that we could have chosen utility functions so that $U^I = -U^{II}$. Such games are called *zero-sum.* These are the simplest of games and we will discuss them in some detail as a basis for discussing signaling games. Last, observe our method of analysis, known as Zermelo's Algorithm or backwards induction. We began by determining rational behavior in the smallest subgames of G, and then determined rationality at a based on the known outcomes of the following subgames at b and c. Since I can move in a way that renders II's strategy choice irrelevant, we say that I *forces* $\mathcal{D}$. It is not a coincidence that player I could force a result.

Lemma 1. *Let G be a finite two player zero-sum game of perfect information with a set of outcomes Ω. Suppose T is a non-empty subset of Ω. Either I can force an outcome in T or II can force an outcome in $\Omega \backslash T$.*

Proof. We proceed using induction on the total number of turns. In a game of one turn (without loss of generality we assume it is player I's turn) I's action simply selects the outcome of the game. If I can select an outcome in T, then I has just forced an outcome in T; else by default II forces an outcome in $\Omega \backslash T$. Now assume that the lemma holds for games of n or fewer turns. In a game of $n + 1$ turns player I selects on the first turn from games of n or fewer turns. If I can select a game in which he can subsequently force an outcome in T then on his first turn he has forced an outcome in T; else II can force an outcome in $\Omega \backslash T$. $\square$

Corollary 1. *Let G be a finite two player zero-sum game of perfect information, with a set of outcomes Ω. There exists an outcome $v \in \Omega$ and a strategy pair (s_I, s_{II}) such that*

1. *If $\tilde{s_{II}}$ is any other strategy for player II, then $U^{II}(s_I, \tilde{s_{II}}) \leq U^{II}(s_I, s_2)$.*

2. *If $\tilde{s_I}$ is any other strategy for player I, then $U^I(\tilde{s_I}, s_{II}) \leq U^I(s_I, s_2)$.*

Proof. Suppose that $\Omega = \{\omega_0, \omega_1, \cdots \omega_n\}$. Suppose further that

$$\omega_0 \prec_I \omega_1 \prec_I \cdots \prec_I \omega_n$$

and

$$\omega_n \prec_{II} \omega_{n-1} \prec_{II} \cdots \prec_{II} \omega_0.$$

Let $W_{\omega_k} = \{\omega \,|\, \omega \succeq_I \omega_k\}$ and let $L_{\omega_k} = \{\omega \,|\, \omega \succeq_{II} \omega_k\}$. There certainly exists a smallest set W_{ω_k} in which I can force an outcome. Suppose $v = \omega_k$. Then I cannot force an outcome in $W_{\omega_{k+1}}$, ergo II can force an outcome in L_{ω_k}. Therefore, I and II have pure strategies that can force v and no better, ergo G has a value v. $\square$

The last important thing to note about our analysis of G is the strategy pair (M, rr) that Zermelo's Algorithm selected. Neither player, knowing his opponent's strategy in advance, can improve on the result $\mathcal{D} = G(M, rr)$ by changing his strategy. Such a strategy pair is a Nash Equilibrium, and classifying Nash Equilibria in signaling games is the focus of this paper.

Definition 2. *A Nash Equilibrium is a strategy pair* (s_I, s_{II}) *such that*

1. *If* $\tilde{s_{II}}$ *is any other strategy for player* II*, then* $U^{II}(s_I, \tilde{s_{II}}) \leq U^{II}(s_I, s_2)$*.*

2. *If* $\tilde{s_I}$ *is any other strategy for player* I*, then* $U^I(\tilde{s_I}, s_{II}) \leq U^I(s_I, s_2)$*.*

It is now fairly easy to view the Nash Equilibria from the strategic form, repeated here for convenience:

	ll	lr	rl	rr
L	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{L}$	$\mathcal{L}$
M	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
R	$\mathcal{W}$	$\mathcal{L}$	$\mathcal{W}$	$\mathcal{L}$

The Nash Equlibria are the strategy pairs with an outcome that is the best for I in the column of II's strategy choice and the best outcome for II in the row of I's strategy choice. As we have seen, the pair (M, rr) is a Nash Equilibrium, since $\mathcal{D}$ is the best outcome for I in column rr and the best outcome for II in row M. The pair (M, rr) is also the strategy pair chosen by Zermelo's algorithm. There is another Nash Equilibrium, (M, lr). However, Zermelo's algorithm does not select (M, lr) since it involves II making the irrational choice of l at b, i.e. the subgame whose initial state is b is not at equilibrium. An equilibrium in which all the subgames are also at equilibrium is called a subgame-perfect equilibrium. In general, Zermelo's algorithm selects subgame-perfect equlibria. The extensive form of the game is generally more useful to understanding how the game progresses; the normal form is generally better for finding the value of the game and Nash Equilibria.

Theorem 1. *Let G be finite zero-sum, two-player game of perfect information. G has a pure strategy pair (s_I, s_{II}) that is a Nash Equilibrium.*

Proof. This result follows immediately from Corollary 1. □

2.4 Non Zero Sum Games

Now we will lift the assumption that the game is zero-sum, and simply assume that U^I and U^{II} are smooth von Neumann Morgenstern utility functions. We also lift the assumption about perfect information; we assume that only I knows ν_0 and that each player knows only his own utility function. We the concepts of value of a game and the players' associated utilities with considering players' respective expected utilities. Since neither player has perfect information, neither can adopt a strategy that he knows will force a specific outcome. Instead, rational play now dictates that players select strategies that will maximize their respective expected utilities. We now have two results that parallel Lemma (1) and Theorem (1).

Theorem 2. *In a finite game G in which U^I and U^{II} are smooth von Neumann Morgenstern utility functions, neither I nor II know the other's utility function, and only I knows the initial state ν_0, both players have expected utilities.*

Proof. We will prove this lemma using induction on the number of turns. Without loss of generality, assume that player I goes first. In a game G of one turn, I's expected utility is

$$E_I(G) = \sum_{\alpha \in A} p_\alpha U^I(\alpha(\nu_0)). \tag{3}$$

II's expected utility is

$$E_{II}(G) = \sum_{\alpha \in A} p_\alpha U^{II}(\alpha(\nu_0)). \tag{4}$$

Since von Neumann Morgenstern utility functions preserve preferences over compound lotteries, we may assume that in all games of n or fewer turns both players have expected utilities consistent with compounding expected utility over all the turns. In a game G of $n+1$ turns, at the first turn player I's is a choice of games $G_1, G_2, \cdots G_m$ of n or fewer turns, so the respective expected utilities are

$$E_I(G) = \sum_{i=1}^{m} p_i E_I(G_i) \tag{5}$$

and

$$E_{II}(G) = \sum_{i=1}^{m} p_i E_{II}(G_i) \tag{6}$$

□

Theorem 3. *A finite game G in which U^I and U^{II} are smooth von Neumann Morgenstern utility functions, neither I nor II know the other's utility function, and only I knows the initial state ν_0, has a Nash Equilibrium.*

Proof. See [Fer] A-9. □

2.5 Signaling Games

Definition 3. *A signaling game is a game with are three players, ν, S (sender) and R (receiver) with the following rules and payoffs:*

The action spaces for each player are:

1. *The set Θ of ν's moves, the space in which a random variable m, density $f(m)$, takes a value.*

2. *The set A of S's moves, measurable functions $\alpha : \Theta \to \Phi$.*

3. *The set B of R's moves, measurable functions $\beta : \Phi \to \Theta$.*

ν moves first, then S, then R. Players S and R have utility functions $U^S(\beta, m)$ and $U^R(\beta, m)$ respectively, which are independent of α, S's move.

We will assume that Θ is a topological space in order to analyze equilibria arising in specific signaling games.

S's and R's goals are to maximize their respective expected utilities. Nash Equilibrium occurs when, given his opponent's strategy, neither player can select a strategy that increases his expected utility. Formally, a Nash Equilibrium is a pair of strategies $q(\alpha|m)$ for S and $\beta(\alpha(m))$ for R so that

1. For each $m \in \Theta$,

$$\int_A q(\alpha|m) d\alpha = 1 \tag{7}$$

   and if $\alpha^* \in \text{Supp } q(\cdot|m)$, then α^* solves $\max_{\alpha \in A} U^S(\beta(\alpha(m)), m)$.

2. For each $\alpha(m)$, $\beta(\alpha(m))$ solves

$$\max_{\beta} \int_{\Theta} U^R(\beta(\alpha(m)), m) p(m|\alpha) dm \tag{8}$$

where $p(\cdot|\alpha) = q(\alpha|m)f(m)/\int_{\Theta} q(\alpha|t)f(t)dt$.

We have allowed S to adopt mixed strategies but have restricted R to pure strategies. In equilibrium, R's strategy is necessarily to maximize his expected utility, which he achieves by adopting the pure strategy we have described. It is therefore unnecessary to consider the possibility that R will select a mixed strategy.

We will call a Nash Equilibrium a partition equilibrium if it satisfies the following conditions:

1. S's strategy is
   (a) Partition Θ into path-connected sets $L_1, L_2, \cdots L_N$ satisfying
      i. $\bigcup_{i=1}^{N} L_i = \Theta$
      ii. int $L_i \cap$ int $L_j = \emptyset$ if $i \neq j$.
   (b) Uses the signaling rule

$$\alpha(m) = l_i \text{ if } m \in L_i \tag{9}$$

2. R's strategy is

$$\beta(l_i) = \max_{y} \int_{L_i} U^R(y, m) f(m) dL_i \tag{10}$$

3. If $m \in L_i$, then for all $i \neq j$,

$$U^S(\beta(l_i), m) \geq U^S(\beta(l_j), m). \tag{11}$$

In particular, if $m \in L_i \cap L_j$ then

$$U^S(\beta(l_i), m) = U^S(\beta(l_j), m). \tag{12}$$

Intuitively, a partition equilibrium is one in which S sends the same signal for similar values of m, the partition determining "how similar" two values of m must be in order to be represented by the same signal. R's strategy is very naturally the one in which R maximizes his expected utility on L_i. Condition (3) requires that, given Conditions (1) and (2), S can never improve his expected utility by violating the strategy set forth in Condition (1). Alternatively, we can interpret Condition (3) to be the requirement that boundary between regions L_i and L_j be the indifference curve between the two regions.

We are primarily interested in the partition equilibria due to their relative simplicity. We will explore Crawford and Sobel's work, in which they prove that all the equilibria in a specific signaling game are partition equilibria. We will then explore an expanded version of Crawford and Sobel's game in which we demonstrate that again all the equilibria are partition equilibria.

3 Crawford and Sobel

3.1 The Model

Crawford and Sobel classify equilibria in the signaling game where $\Theta = [0,1]$, and A and B are measurable functions $[0,1] \to [0,1]$. S observes the value ν chooses in $[0,1]$ and takes action $q(\alpha|m)$ in A where q is a probability density function. S signals $n = \alpha(m)$. R observes n and selects action $y = \beta(n)$, where y takes a value in $[0,1]$. Further, the von Neumann Morgenstern utility functions $U^S(y,m,b)$ and $U^R(y,m)$ satisfy:

1. For all $m \in \Theta$, there exists $y \in \Theta$ such that $\partial U^i/\partial y = 0$ for $i = S, R$.

2. $\partial^2 U^i/\partial y^2 < 0$.

3. $\partial^2 U^i/\partial y \partial m > 0$.

Conditions (1) and (2) guarantee that for each m, S's and R's payoffs have maxima in y. Condition (3) says the S's and R's respective preferred outcomes are increasing functions of m. Additionally, Crawford and Sobel stipulate that U^S also depends on a parameter b that measures how closely the interests of S and R coincide.

3.2 The Equilibria

Before we can proceed, we need to introduce some notation. Since $\Theta = [0,1]$, there exist $0 = a_0 < a_1 < \cdots < a_N = 1$ so that $L_i = [a_{i-1}, a_i]$. We will say that m induces

y if there exist strategies β and α such that $\beta(\alpha(m)) = y$ and the probability of S using α is positive. Also, for $m \in [0, 1]$ we define

1. $y^S(m, b) = \arg\max_y U^S(y, m, b)$

2. $y^R(m) = \arg\max_y U^R(y, m)$.

Crawford and Sobel show that all equilibria are partition equilibria and then determine what those equilibria are. First, let us see why all equilibria are partition equilibria.

Lemma 2. *Suppose that for all m, $y^S(m, b) \neq y^R(m)$. Suppose also that there exist m_1 and m_2 that induce u and v respectively in equlibrium. Then u and v cannot be arbitrarily close. Also, there are a finite number of assignments induced in equilibrium.*

Proof. Suppose that $u < v$ are induced in equilibrium. Then if m induces u in equilibrium, $U^S(u, m, b) \geq U^S(v, m, b)$. Since U^S is continuous, there exists $\overline{m}$ such that $U^S(u, \overline{m}, b) \geq U^S(v, \overline{m}, b)$. $U^S(y, \overline{m}, b)$ has a local maximum in y, therefore

$$u < y^S(\overline{m}, b) < v. \tag{13}$$

Further, since $\partial^2 U^S / \partial y \partial m > 0$,

1. There is no $m > \overline{m}$ that induces u in equilibrium

2. There is no $m < \overline{m}$ that induces v in equilibrium

Conditions (1) and (2) coupled with the fact that $\partial^2 U^R/\partial y \partial m > 0$ tells us that

$$u \leq y^R(\overline{m}) \leq v. \tag{14}$$

By hypothesis, $y^S(m,b) \neq y^R(m)$ for all m, ergo there exists $\epsilon > 0$ such that $|y^S(m,b) - y^R(m)| \geq \epsilon$ for all $m \in [0,1]$. Equations (13) and (14) guarantee that $|u - v| \geq |y^S(m,b) - y^R(m)| \geq \epsilon$ ergo u and v are not arbitrarily close.

Since any two assignments induced in equilibrium are at a distance of at least ϵ from each other, there can be at most $1 + \lceil 1/\epsilon \rceil$ possible assignments. □

Theorem 4. *Suppose that for all $m \in [0,1]$, $y^S(m,b) \neq y^R(m)$. Then there exists an integer $N(b)$ such that for each integer $1 \leq N \leq N(b)$*

1. *There exists a partition of $[0,1]$ into sets $L_i = [a_i, a_{i+1}]$ for $0 \leq i \leq N-1$ and an S action $\alpha(m) = l_i$ if $m \in L_i$.*

2. $\beta(l_i) = \max_{y \in [0,1]} \int_{a_i}^{a_{i+1}} U^R(y,m) f(m) dm$

3. *For all $1 \leq i \leq N-1$,*

$$U^S(\beta(l_i), a_i, b) = U^S(\beta(l_{i-1}), a_i, b) \tag{15}$$

Further, these are all of the equilibria.

Proof. It is immediately clear that such a strategy pair satisfies the definition of a partition equilibrium. We need only verify three things:

1. The existence of $N(b)$.

2. The existence of such an equilibrium for all $1 \leq N \leq N(b)$.

3. That any equilibrium is one of these equilibria.

It follows from Lemma (2) that $N(b)$ exists. Suppose there is no upper bound on N. Then for all $\epsilon > 0$, we can choose N large enough so that in any partition of $[0,1]$ into N intervals we can find i so that $|\beta(l_i) - \beta(l_j)| \leq \epsilon$, contradicting Lemma (2).

It also follows from Lemma (2) that any equilibrium is a partition equilibrium.

It only remains to show that for all integers $1 \leq N \leq N(b)$ there exists an equilibrium of size N. We will follow Crawford and Sobel's proof exactly. They proceed first by constructing determining the value of $N(b)$ and then demonstrate that we can construct partitions of all sizes $1 \leq N \leq N(b)$.

First, we need a definition:

$$\overline{y}(\underline{a}, \overline{a}) = \begin{cases} \int_{\underline{a}}^{\overline{a}} U^R(y,m) f(m) dm \text{ if } \underline{a} \leq \overline{a} \\ y^R(\underline{a}) \text{ if } \underline{a} = \overline{a} \end{cases}$$

Our assumption that U^R is increasing in y and m implies that $\overline{y}$ is also increasing in both of its arguments. We now show that based on a partial partition of $[0,1]$ that satisfies Condition (3) we can construct exactly one more element of the partition. Let a^i denote a partial partition $0 = a_0 < a_1 < \cdots < a_1$ that satisfies Condition(3). Since U^S is convex in y, there exists uniquely $\tilde{y}$ so that

$$U^S(\tilde{y}, a_i, b) = U^S(\beta(l_i), a_i, b) \tag{16}$$

hence there also exists uniquely a_{i+1} so that $\tilde{y} = \overline{y}(a_i, a_{i+1})$. Therefore, any partial partition determines only the next term $a_{i+1} > a_i$.

Now Crawford and Sobel define

$$K(a) = \max\{i \,|\, \exists\, 0 < a < a_2 < \cdots < a_i \leq 1 \text{ satisfying Condition (3)}\}$$

By hypothesis, $y^S(m,b) \neq y^R(m)$, so by Lemma (2), $\overline{y}(a_i, a_{i+1}) - \overline{y}(a_{i-1}, a_i) \geq \epsilon$ for some $\epsilon \geq 0$. If $a_{i+2} - a_i$ can be arbitrarily small, then $\overline{y}(a_{i+1}, a_{i+2}) - \overline{y}(a_i, a_{i+1})$ can be arbitrarily small, contradicting Lemma (2), therefore $a_{i+2} - a_i$ is has a positive greatest lower bound, hence the construction of the partition terminates in a finite number of steps. Therefore, $K(a)$ is finite, well-defined and uniformly bounded, so there exists $\overline{a} \in [0,1]$ so that $\sup_{0<a\leq 1} K(a)$ achieved for some a. Define $N(b) = K(\overline{a})$.

Now that we have constructed $N(b)$, we are prepared to show the existence of an equilibrium of size N for all integers $1 \leq N \leq N(b)$.

Let $a^{K(a)}$ denote a partial partition of $[0,1]$ in which $a_1 = a$. Since the values of $a_2, a_3, \cdots, a_{K(a)}$ vary continuously with a., if $a_{K(a)} < 1$, K is locally continuous at a. Since $K(1) = 1$ and K changes by at most 1 at a discontinuity, K achieves all integer values $1 \leq N \leq N(b)$. If $K(a_1) = N$ and K is discontinuous at a, then $a_0 = 1$, $a_N = 1$ and the partition satisfies Condition (3). □

3.3 An Example

Their article computes all the equilibria in the case where m is uniformly distributed on $[0,1]$, $U^S(y,m,b) = -(y-(m+b))^2$ and $U^R(y,m) = -(y-m)^2$ where $b > 0$. Since $b > 0$, S's and R's interests never coincide. It is easily verified that the partials of U^S and U^R satisfy the requirements set out in Crawford and Sobel's model.

The first step is to determine R's best strategy on L_i and $N(b)$. R maximizes his utility on L_i by playing

$$\beta(l_i) = \max_y \int_{a_{i-1}}^{a_i} -(y-m)^2 dm. \tag{17}$$

The right hand side of Equation (17) is maximal when $y = (a_{i-1} + a_i)/2$, i.e. R plays the center of mass of L_i.

Second, we want to determine the partitions which give rise to equilbrium. Since

$$U^S(\beta(l_i), a_i, b) = U^S(\beta(l_{i+1}), a_i, b) \tag{18}$$

then

$$-\left(\frac{a_{i-1} + a_i}{2} - a_i - b\right)^2 = -\left(\frac{a_i + a_{i+1}}{2} - a_i - b\right)^2. \tag{19}$$

Since $a_{i+1} > a_{i-1}$, it follows that $a_{i+1} = 4b + 2a_i - a_{i-1}$. Given that $a_0 = 0$, we can express all the a_i in terms of a_1. Inductively, we see that

$$a_i = ia_1 + 2i(i-1)b. \tag{20}$$

$N(b)$ is the largest integer i such that $2i(i-1)b < 1$. Completing the square,

$$N(b) = \left\lfloor \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} \right\rfloor. \tag{21}$$

As we expect, as $b \to \infty$, $N(b) \to 1$. In other words, as the players' interests diverge, the only equilibrium is the trivial non-informative one. All equilibria are now determined by b and Equation (20). We work out a few specific cases:

Example 1. *If* $b > 1/4$, *then the only equilibrium is the uninformative one. If* $b > 1/4$, *then* $1 < \frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{2}{b}} < 2$, *so* $N(b) = 1$.

Example 2. *If* $b = 1/10$, *then* $N(b) = 2$. *Equation (20) tells us that if* $N = 2$, $a_1 = 3/20$, *and* $a_2 = 1$.

4 Other approaches

Many other authors have taken up this problem, modeling related situations and achieving similar results. We will examine the work of Jerry Green and Nancy Stokey, and Kalyan Chatterjee and William Samuelson.

4.1 Green and Stokey

Green and Stokey examine the problem of group decision making – one may imagine a hiring committee consisting of an interview team and a hiring manager. Green and Stokey use a principal/agent model, with the goal of understanding how more informative equilibria improve expected utility.

4.1.1 The Model

The initial states of the game (the states of nature) are the elements of $\Theta = \{\theta_1, \cdots \theta_M\}$ and the possible outcomes are $\Phi = \{\phi_1, \cdots \phi_K\}$. The set of signals is $Y = \{y_1, \cdots y_N\}$. The agent receives on observation $y_n \in Y$ of $\theta_m \in \Theta$ and then sends a possibly noisy signal (again an element of Y) to the principal, who then selects an action $\phi_k \in \Phi$.

The principal's and agent's respective utilities depend only on θ_m and ϕ_k. Green and Stokey record the players respective von Neumann Morgenstern utilities in $K \times M$ matrices; U^P and U^A will be the principal and agent's respective utility matrices, in which the km-th entry u^i_{km} of U^i is player i's utility ($i = P, A$) if the state of nature is θ_m and the principal chooses ϕ_k.

We represent the strategies available to the principal and agent as Markov matrices. A Markov matrix is a matrix in which every number belongs to the interval $[0, 1]$. A strategy for the agent is an $N \times N$ Markov matrix R, in which the nn'-th entry $r_{nn'}$ is the probability that the agent will signal $y_{n'}$ given that his observation is y_n. A strategy for the principal in an $N \times K$ Markov matrix Z, in which the nk-th entry z_{nk} is the probability that the principal will choose outcome ϕ_k on receiving the signal y_n.

Green and Stokey now define the information structure of the game. The information structure is an $M \times N$ Markov matrix Λ in which the mn-th entry λ_{mn} is the probability that the agent observes y_n given that the state of nature is θ_m. How the players interpret the signals depends on Λ and their respective prior beliefs about the entries of Λ. For $i = P, A$ let π^i_m be i's prior probability that the state of nature is θ_m and also let $\pi^P = (\pi^P_1, \cdots, \pi^P_M)$ and $\pi^A = (\pi^A_1, \cdots, \pi^A_M)$ be vectors of the principal's and agent's respective prior beliefs. Now let Π^i be the $M \times M$ matrix with π^i on the diagonal and zeros everywhere else. Then Bayes's rule tells us that the mn-th entry $p^i(\theta_m|y_n) = \lambda_{mn}\pi^i_m$ of the $M \times N$ matrix $\Pi^i\Lambda$ is player i's posterior. Now given strategy choices Z and R, the principal's and agent's respective expected utilities are

$$E(U^P) = \text{tr } U^P \Pi^P \Lambda R Z \tag{22}$$

$$E(U^A) = \text{tr } U^A \Pi^A \Lambda R Z \tag{23}$$

The pair (Z, R) is a Nash Equilibrium if

1. For all $N \times K$ Markov matrices Z',

$$\text{tr } U^P \Pi^P \Lambda R Z \geq \text{tr } U^P \Pi^P \Lambda R Z'. \tag{24}$$

2. For all $N \times N$ Markov matrices R',

$$\text{tr } U^A \Pi^A \Lambda R Z \geq \text{tr } U^A \Pi^A \Lambda R' Z. \tag{25}$$

4.1.2 The Equilibria

Green and Stokey classify three types of equilibria: partition equilibria, determinate action equilibria, and random action equilibria.

An $M \times N'$ information structure Λ' is a partition of Λ if there exist permutation matrices P and P' and a block diagonal $N \times N'$ Markov matrix D in which each block has rank one such that $\Lambda' = \Lambda P D P'$. The matrices P, D, and P' determine a partition the signal space Y. We look at a quick example to understand how P, D, and P' determine the partition.

Example 3. *Suppose* $\Theta = \{\theta_1, \theta_2\}$, $Y = \{y_1, y_2, y_3\}$, *and*

$$\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}.$$

Then the 2×2 identity matrix Λ' is a partition of Λ since

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

In the initial information structure, the signals y_2 and y_3 are essentially equivalent, since $p(\theta_2 | y_2 \text{ or } y_3) = 1$. Since the $2, 2$-th entry of the new information structure Λ' is 1, then $p(\theta_2 | y_2) = 1$. P, D, and P' partitioned Y into two subsets that are equivalent for signaling purposes: $\{y_1\}$ and $\{y_2, y_3\}$, and identifies y_2 with $\{y_2, y_3\}$. A partition equilibrium is then one in which ΛR is a partition of Λ.

Now the authors define three types of equilibria. A partition equilibrium is a pair (Z, R) such that ΛR is a partition of Λ; a determinate action equilibrium is a pair (Z, R) such that ΛR is not a partition of Λ and each column of Z receiving positive weight under R has only a single positive element; a random action equilibrium is a pair (Z, R) such that ΛR is not a partition of Λ and some column of Z receiving positive weight under R has two or more non-zero entries. The authors subsequently focus solely on partition equilibria, since the other two types are unsuitably unstable for their purposes.

We now examine the example that Green and Stokey employ to illustrate the three types of equilibria.

Example 4. *Suppose that there are two states, two signals and two outcomes, i.e.*

$K = N = M$. Let

$$\Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, U^P = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}, U^A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix}, \pi^P = \pi^A = (1/2, 1/2).$$

A knows the state, since for $i = 1, 2$ he observes $p(y_i|\theta_i) = 1$. If the state is θ_1, P prefers outcome ϕ_1 while A is indifferent between ϕ_1 and ϕ_2. If the state is θ_2, both players prefer outcome ϕ_2. Now consider the following four equilibria:

$Z = I$ and $R = I$, i.e. A's signal is not noisy and P's action is assumes the signal is not noisy. This equilibrium is a partition equilibrium since ΛR is obviously a partition of Λ.

The babbling equilibrium, which we saw in Section 2.5, looks like

$$Z = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \text{ and } R = \begin{bmatrix} \alpha & 1-\alpha \\ \alpha & 1-\alpha \end{bmatrix}, \alpha \in [0, 1].$$

A's transmission strategy is indifferent to the state A observes, and P's assignment strategy is indifferent to the state A transmits. The babbling equilibrium is another example of a partition equilibrium in which Y is partitioned into one set.

The pair

$$Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 1-\epsilon & \epsilon \\ 0 & 1 \end{bmatrix}, 0 < \epsilon \leq 1/2$$

is an example of a determinate action equilibrium. A is indifferent between ϕ_1 and ϕ_2 if the state is θ_1, hence randomizing between y_1 and y_2 when A observes y_1 does not change A's expected utility.

The pair

$$Z = \begin{bmatrix} 1 & 0 \\ \delta & 1-\delta \end{bmatrix}, R = \begin{bmatrix} 1/2 & 1/2 \\ 0 & 1 \end{bmatrix}, \delta \in (0,1)$$

is an example of a random action equilibrium.

Next, Green and Stokey demonstrate that determinate and random action equilibria are not suitably stable for their purposes, since small changes in U^P and U^A may result in drastic changes in the equilibrium strategies. They also prove an analogous result to Theorem 4:

Theorem 5. *Given Λ, U^P and U^A, if Z and R form a partition equilibrium*

1. *If the nth row of R has a positive entry, then the nth row of Z is uniquely determined in the optimal response to R.*

2. *R is the unique best response to Z and R is a partition of Λ.*

4.1.3 Comparison of Results

The primary differences between the two models are (1) Green and Stokey's players have discrete utility functions, whereas Crawford and Sobel's have smooth utilities and (2) Green and Stokey's model has an extra layer of noise. The extra layer of noise does not have a significant effect on the model, it merely introduces another layer of calculations. The discrete utility is the cause of the extra equilibria in Green and Stokey's model. Two results follow.

First, if Λ, U^P, U^A, Z and R are a partition equilibrium, then we can fit smooth functions $U^{P\prime}(\theta, \phi)$ and $U^{A\prime}(\theta, \phi)$ to the entries of U^P and U^A that preserve the equilibrium. These continuous functions correspond to appropriate utility functions for which there exists a partition equilibrium in Crawford and Sobel's model.

Second, if Λ, U^P, U^A, Z and R are a determinate or random action equilibrium, it is impossible to fit smooth functions to the utilities in U^P and U^A that preserve the equilibrium. The fact that such smooth functions do not exist illuminates why determinate and random action equilibria are not stable in the sense of partition equilibria. Determinate and random action equilibria do not vary continuously with continuous changes in utility.

4.2 Chatterjee and Samuelson

Chatterjee and Samuelson explore a much broader game than that of Crawford and Sobel or of Green and Stokey. They model a bargaining problem in which the buyer and seller both have confidential information and each simultaneously transmits an offer to the other. Chatterjee and Samuelson explore equilibria in this bidirectional game of information transmission.

4.2.1 The Model

There are two players, the buyer B and the seller S attempting to agree on a price for a good. The players have reserve prices, v_b and v_s respectively, where v_b is the highest price that B will pay for the good and v_s is the lowest price S will accept. Each player knows his own reserve price but not his opponent's reserve price. B assigns a distribution $F_b : [\underline{v_s}, \overline{v_s}] \to \mathbb{R}$ summarizing his prior beliefs about v_s with $F_b(\underline{v_s}) = 0$ and $F_b(\overline{v_s}) = 1$. S assigns a distribution $F_s[\underline{v_b}, \overline{v_b}] \to \mathbb{R}$ summarizing his prior beliefs about v_b with $F_s(\underline{v_b}) = 0$ and $F_s(\overline{v_b}) = 1$. B proposes a price b and S proposes a price s. If $b < s$, there is no bargain. If $b \geq s$, then a deal is struck at a price $P = bk + (1-k)s$ where $0 \leq k \leq 1$. The players' utilities are $U^S = P - v_s$ and $U^B = v_b - P$. Therefore, the buyer's expected utility is

$$\pi_b(b, v_b) = \begin{cases} \int_{\underline{s}}^{b} (v_b - P) g_b(s) ds \text{ if } b \geq \underline{s} \\ 0 \text{ if } b < \underline{s} \end{cases} \tag{26}$$

where $g_b(s)$ is the density function of the induced distribution $G_b = F_b \circ S^{-1}$. Similarly, the seller's expected utility is

$$\pi_s(s, v_s) = \begin{cases} \int_{s}^{\overline{b}} (P - v_s) g_s(b) db \text{ if } s \leq \overline{b} \\ 0 \text{ if } s > \overline{b} \end{cases} \tag{27}$$

where $g_s(b)$ is the density function of the induced distribution $G_s = F_s \circ B^{-1}$. As before, the strategy pair (s^*, b^*) is a Nash Equilibrium if for all s, $\pi(s^*, v_s) \geq \pi(s, v_s)$ and $\pi(b^*, v_b) \geq \pi(b, v_b)$.

5 A Possible Generalization

We will now pursue a broader example which includes Crawford and Sobel's equilibria, but also introduces an opportunity to find additional equilibria when the U^S and U^R

do not satisfy Crawford and Sobel's hypotheses on the partial derivatives. As a consequence of lifting the hypotheses on the partials of U^S and U^R, we will no longer be able to rely on Lemma 2. To compensate, S observes his type in $[0,1]\times[0,1]$ rather than $[0,1]$. We intuitively consider a type in $[0,1]\times[0,1]$ to represent the case where S has two candidates and recommends them simultaneously, thereby simply ranking the candidates relative to each other rather than quantitatively. If U^S and U^R satisfy Crawford and Sobel's hypotheses, it is still possible for S to signal the coordinates of his type independently and for R to assign outcomes independntly, thereby playing two parallel versions of Crawford and Sobel's game.

5.1 The Model

Our revised model is largely similar to Crawford and Sobel's model. We let $\Theta = [0,1]\times[0,1]$. Then S partitions $[0,1]\times[0,1]$ into regions $L_1, L_2, \cdots, L_N$ as described above, and associates the signal l_i to the region L_i for all $1 \leq i \leq N$. ν chooses an ordered pair $(m, \widehat{m}) \in [0,1]\times[0,1]$. S observes the region L_k in which $(m, \widehat{m})$, henceforth $(m_k, \widehat{m}_k)$, lies and signals l_k. R assigns an outcome $(y_k, \widehat{y}_k)$ and we evaluate the players' respective payoffs. We will classify Nash Equilibria when $U^S((m,\widehat{m}),(y,\widehat{y})) = my + \widehat{m}\widehat{y}$ and $U^R((m,\widehat{m}),(y,\widehat{y})) = -(y-m)^2 - (\widehat{y}-\widehat{m})^2$.

5.2 The Equilibria

A partition together with signaling and assignment rules is a Nash Equilibrium if

1. For all $1 \leq i \leq N$, R's assignment rule is

$$\beta(l_i) \quad = \quad \arg\max_{(y_i,\widehat{y_i})} \int_{L_i} U^R((m,\widehat{m}),(y_i,\widehat{y_i}))dmd\widehat{m} \tag{28}$$

2. For all $1 \leq i \leq n-1$, $L_i \bigcap L_{i+1}$ is the set of points $(t, f_i(t))$ such that

$$U^S((t,f_i(t)),(y_i,\widehat{y_i})) = U^S((t,f_i(t)),(y_{i+1},\widehat{y_{i+1}})). \tag{29}$$

Given that U^R is coordinate-wise quadratic loss, Condition (1) requires for all i, on the region L_i that R select the point minimizing the expected square distance from any point $(m,\widehat{m})$ in L_i to $(y_i,\widehat{y_i})$ in L_i. This point is the centroid of the region L_i. Condition (2) requires that the boundary between adjacent regions be S's indifference curve between R's assignments on those adjacent regions.

Lemma 3. *If $U^S((m,\widehat{m}),(y,\widehat{y})) = my + \widehat{m}\widehat{y}$, then in equilibrium, for $1 \leq i \leq n-1$ the boundary between L_i and L_{i+1} is a line through the origin.*

Proof. By Condition 2, the boundary between L_i and L_{i+1} satisfies

$$ty_i + f_i(t)\widehat{y}_i = ty_{i+1} + f_i(t)\widehat{y}_{i+1} \tag{30}$$

$$f_i(t) = t\frac{y_{i+1} - y_i}{\widehat{y}_i - \widehat{y}_{i+1}} \tag{31}$$

$\square$

Now we are prepared to classify partitions that give rise to equilibria.

Theorem 6. *If $n = 2$, there are three distinct partitions that give rise to equilibrium. If $n = 3$, there is one partition that gives rise to equilibrium. If $n \geq 4$, there are no equilibria.*

Case 1: $n = 2$: Since $f_1(t)$ is a line through the origin, we let $f_1(t) = a_1 t$, $L_1 = \{(t,u) \in [0,1] \times [0,1] | u \leq a_1 t\}$, and $L_2 = \{(t,u) \in [0,1] \times [0,1] | u \geq a_1 t\}$, and determine values of a_1 such that we are in equilibrium.

First, suppose $0 \leq a \leq 1$. Then the centroid of L_1 is $(2/3, a_1/3)$ and the centroid of L_2 is $((3-2a_1)/(6-3a_1), (3-a_1^2)/(6-3a_1))$. By Lemma 3, we are at equilibrium when

$$a_1 = \frac{\frac{3-2a_1}{6-3a_1} - \frac{2}{3}}{\frac{a_1}{3} - \frac{3-a_1^2}{6-3a_1}}$$

Solving for a_1, we are at equilibrium when $a_1 = 1/2$ or $a_1 = 1$. By symmetry, we are also at equilibrium when $a_1 = 2$.

Case 2: $n = 3$: Let $L_1 = \{(t,u) \in [0,1] \times [0,1] | u \leq at\}$, $L_2 = \{(t,u) \in [0,1] \times [0,1] | at \leq u \leq bt\}$, and $L_3 = \{(t,u) \in [0,1] \times [0,1] bt \leq u\}$. Without loss of generality, assume that $a < b$. We now consider two subcases:

Case 2a: $0 < a < b \leq 1$ or $1 \leq a < b$: The centroid of L_1 is $(2/3, a/3)$ and the centroid of L_2 is $(2/3, (b+a)/(3b-3a))$. By Lemma 3, we are at equilibrium when

$$a = \frac{\frac{b+a}{3b-3a} - \frac{a}{3}}{\frac{2}{3} - \frac{2}{3}}$$

which is clearly impossible. By symmetry, there are no equilibria when $1 \leq a < b$.

Case 2b: $0 < a \leq 1 \leq b$: The centroid of L_1 is $(2/3, a/3)$. The centroid of L_2 is $(\frac{3b^2-2ab^2-1}{6b^2-3ab^2-3b}, \frac{3b-a^2b-2}{6b-3ab-3})$. The centroid of L_3 is $(1/3b, 2/3)$. By Lemma 1, we are in equilibrium if

$$\frac{\frac{2}{3} - \frac{3b^2-2ab^2-1}{6b^2-3ab^2-3b}}{\frac{3b-a^2b-2}{6b-3ab-3} - \frac{a}{3}} = a$$

and

$$\frac{\frac{3b^2-2ab^2-1}{6b^2-3ab^2-3b} - \frac{1}{3b}}{\frac{2}{3} - \frac{3b-a^2b-2}{6b-3ab-3}} = b.$$

We solve this system using Maple's Gröbner package:

We are at equilibrium when $a = 2 - \sqrt{3}$ or 1 and $b = 2 + \sqrt{3}$ or 1 respectively. If $a = b = 1$, then L_2 has measure zero. If $a = 2 - \sqrt{3}$, then $b = 1/a = 2 + \sqrt{3}$ and none of the regions L_i has measure zero, so the equilibrium is non-trivial.

Case 3: $n \geq 4$: Let $f_i(t) = a_i t$ be the boundary between L_i and L_{i+1}. Since $n \geq 4$, there exist at least two integers $i < j$ between 1 and N such that either $0 < a_i < a_j \leq 1$ or $1 \leq a_i < a_j$. In either case, by the analysis in Case 2a, it is impossible have equilibrium if one of these statements is true, hence there are no equilibria in this case.

6 Conclusion

This thesis presents an approach to understanding signaling games, reviews Crawford and Sobel's method of approach and work, and examines a new example that expands on Crawford and Sobel's solution. While our example achieves equilibrium in a case that Crawford and Sobel could not, the new equilibria are not as informative as the equilibria in Crawford and Sobel's work. Our results suggest that allowing S to recommend candidates relative to each other allows more opportunities for cooperation, but decreases how much information S can actually communicate.

If we generalize further to the case in which $\Theta = [0,1]^3$, we expect that this trend will continue. The game will have equilibria for an increasingly large collection of payoff functions, but the equilibria will become less informative than in our example. In general, we expect that if S's signal consists of ranking n candidates relative to each other, the number of payoff functions increases as the equilibria become less informative.

As long as we have smooth payoff functions in the 2-candidate game, we expect that the resulting equilibria will still be partition equilibria, since the indifference curves should still be smooth functions. If the functions cease to be smooth or even cease to be continuous, the equilibria will presumably become significantly more complicated.

It would be interesting to pursue further generalization on either of these paths, in particular to determine the equilibria that arise under more general payoff in the 2-candidate game.

References

[Bin] Binmore, Ken. *Fun and Games.* D.C. Heath and Company. Lexington, MA, 1992.

[ChS] Chatterjee, Kalyan and William Samuelson. "Bargaining Under Incomplete Information," *Operations Research,* September 1983, 835-851.

[CrS] Crawford, Vincent and Joel Sobel. "Strategic Information Transmission," *Econometrica,* vol. 50, no. 6, November 1982, 1431-1451.

[Fer] Ferguson, Thomas S. *Game Theory Text.* http://www.math.ucla.edu/~tom/Game_Theory/Contents.html

[GS] Green, Jerry and Nancy Stokey. "A Two-Person Game of Information Transmission," Harvard University, H.I.E.R. Discussion Paper, 751, March 1980.