The Aztec Diamond Edge-Probability Generating Function

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ABSTRACT

We derive the Aztec diamond edge probability generating function, whose coefficients give the probability that a particular edge is present in a perfect matching of an Aztec diamond graph chosen uniformly at random. This is done via two different approaches: the first is based upon Propp's generalized domino shuffling algorithms, while the second is based upon Speyer's Octahedron Recurrence. After a brief introduction to asymptotic analysis of multivariate generating functions, we derive a more complicated edge probability generating function for weighted Aztec diamonds with a periodic edge weight assignment, which arises from diabolo tilings of fortresses.

1 Background

Charles Dodgson (better known to the public as Lewis Carroll) in 1866 invented a method of computing the determinant of a matrix using a recurrence relating the connected minor a matrix, called Dodgson condensation[1]. Attempting to understand the combinatorics of Dodgson condensation led Mills, Robbins, and Rumsey to their discovery of "alternating sign matrices". As explained by Bressoud and Propp in [BP99], "when Dodgson condensation is applied to an n-by-n matrix and all like monomials are gathered together, the terms in the final expression are associated with the n-by-n matrices of 0's, 1's and -1's in which the nonzero entries in each row and column alternate in sign, beginning and ending with a +1. These are the alternating sign matrices (or ASMs) of order n". The combinatorial question of interest was then naturally the number of ASMs; this question was resolved at first by Doron Zeilberger in 1992, whose proof drew on results and techniques from partition theory, symmetric functions, and constant term identities; its eventual publication required an army of 89 referees (88 people and 1 computer). Then in 1995, Greg Kuperberg provided a much shorter proof that relied on the machinery of statistical mechanics. Kuperberg's work on ASMs began as an outgrowth of his work on enumeration of tilings in collaboration with Noam Elkies, Michael Larsen, and James Propp. As it turned out, the theory of ASMs have strong connections to the problem of counting

[1] See [Dod1866]

domino tilings of certain plane regions known as Aztec diamonds.[2]

The *Aztec diamond of order* n is the union of lattice squares $[a, a+1] \times [b, b+1] \subset \mathbb{R}^2 (a, b \in \mathbb{Z})$ that lie completely inside the tilted square $\{(x, y) : |x| + |y| \leq n + 1\}$. A *domino* is a closed 1×2 or 2×1 rectangle in $\mathbb{R}^2$ with corners in $\mathbb{Z}^2$, and a *tiling of a region* R *by dominoes* is a set of dominoes whose interiors are disjoint and whose union is R. The number of domino tilings of the Aztec diamond of order n is $2^{n(n+1)/2}$, and four different proofs of this formula were given in [EKLP92]: the first proof used the connection between Aztec diamonds and alternating sign matrices - specifically, a bijection between domino tilings of Aztec diamonds and pairs of ASMs that satisfy a particular "compatibility" condition; the second proof established the formula as a special case of a theorem on "monotone triangles", which are also related to alternating sign matrices; the third proof came from the representation theory of the general linear group; finally, the fourth proof used the technique of "domino shuffling", which in the original [EKLP92] paper actually was the name of an entire combinatorial approach to tiling enumeration instead of the specific domino shuffling algorithm that will be discussed later in this section.

If we choose uniformly at random a tiling of a large Aztec diamond, a curious phenomenon is observed: outside of a roughly circular region, the dominoes are "frozen" in a brick-wall pattern while inside the region the domino orientations appear to be random. The boundary of this region seems to be the inscribed circle of the tilted

[2]This history comes from and is recounted in much greater detail in [BP99].

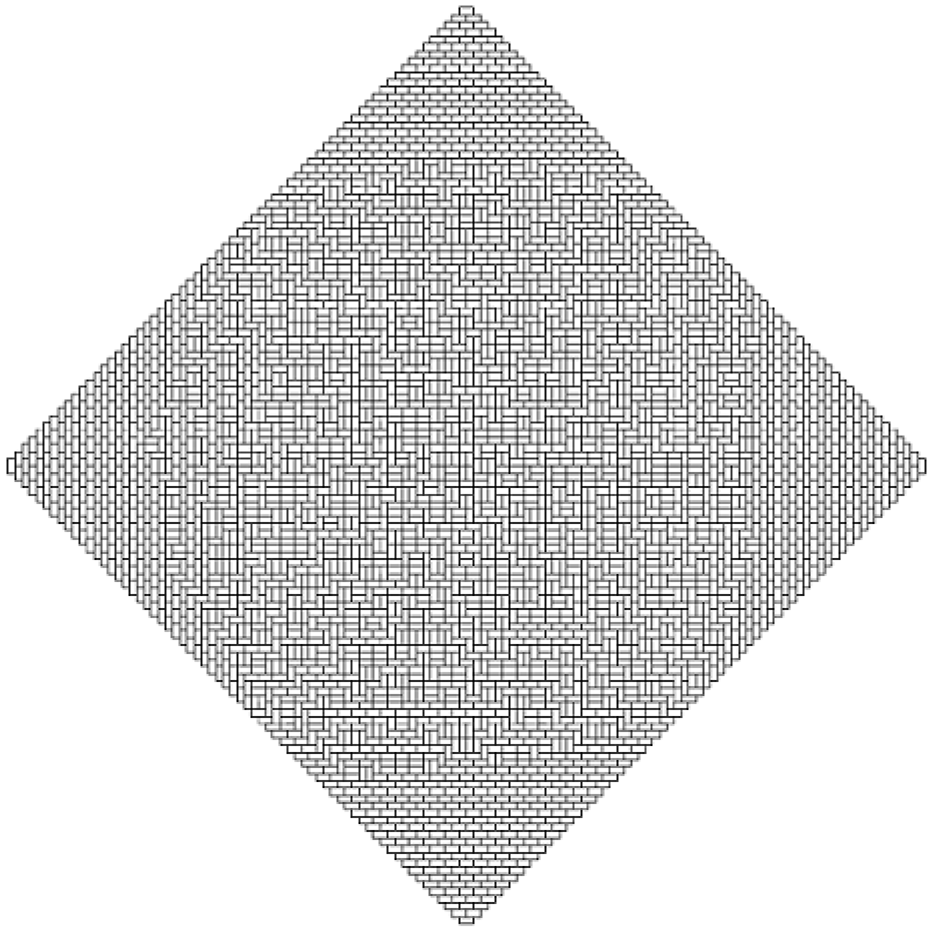

Figure 1: A random tiling of an Aztec diamond of order 64

square. See Figure 1 for a random tiling of an Aztec diamond of order 64. A rigorous analysis of the behavior of the domino shuffling algorithm led to the first proof of the asymptotic circularity of the boundary of the frozen region, the so-called "arctic circle theorem", in [JPS95].

We may "dualize" the Aztec diamond by taking the center of each lattice squre to be a vertex, with an edge adjoining each pair of adjacent lattice squares. Therefore, the problem of enumerating domino tilings of the Aztec diamond is equivalent to the problem of enumerating (perfect) matchings of the dual Aztec diamond graph (ADG). See Figure 2 for an example of a matching of an order-4 ADG and the equivalent tiling of an order-4 Aztec diamond. We may generalize the latter problem by assigning nonnegative real weights to each edge, and make the weight of a particular matching be equal to the product of the weights of the edges in the matching, then summing these weights over all possible matchings. The basic case of simply enumerating the number of matchings thus correspond to calculating the weight-sum where each edge is given weight 1. Around the time that [JPS98] was written, Alexandru Ionescu discovered a recurrence relation related to domino shuffling that allows for efficient computation of the probability that any edge in an unweighted ADG is contained in a random matching of the ADG. This lead Gessel, Ionescu, and Propp (in unpublished work) to derive a multivariate generating function for these edge probabilities. The rediscovery of this "lost" derivation lead to the paper [DGIP], which forms the core of and shares the same title as this thesis. The generating function then allowed Cohn, Elkies, and Propp to give more detailed asymptotic analysis of random tilings of Aztec diamonds in [CEP96] and yield a new proof of the arctic circle theorem.

By a simple local transformation known as "*urban renewal*", J. Propp gives several examples in [Pro03] whereby a number of other tiling enumeration problems can be reduced to counting weight-sums of matchings of Aztec diamonds with a certain weight-assignment of edges, leading to an embedding of the given graph into an ADG

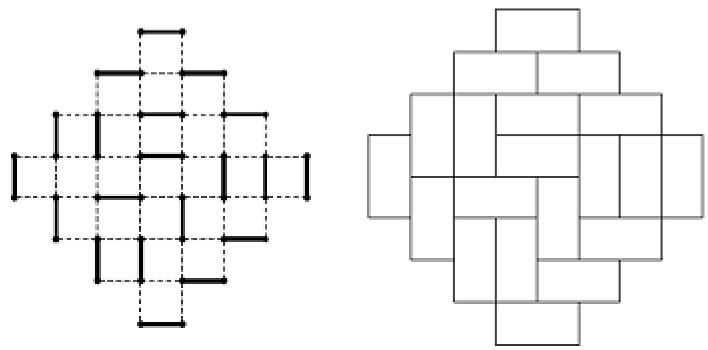

Figure 2: Tiling of an Aztec diamond of order 4

of some order. One of these problems is that of diabolo-tilings of fortresses, which, after transforming to a weighted ADG via urban renewal, has periodic edge weights of 1 and 1/2 in alternating order. The exact details of this weight-assignment scheme will be given in section 4.

1.1 The Weight Sum Algorithm

As its name suggests, the weight sum algorithm allows the computation of the sum of the weights over all possible matchings of a weighted Aztec diamond. J. Propp points out in [Pro03] that it is "essentially a restatement of Mihai Ciucu's cellular graph complementation algorithm". Given an Aztec diamond graph of order n, let a square face centered at (i, j) be called a *cell* if $i+j+n \equiv 1 \pmod 2$. The weight-sum algorithm is reductive, in the sense that one starts with a weighted order-n ADG, apply a edge-weight transformation at the cells, and obtain a (differently) weighted order-$(n-1)$ ADG. In the edge-weight transformation, for a cell with edge weights

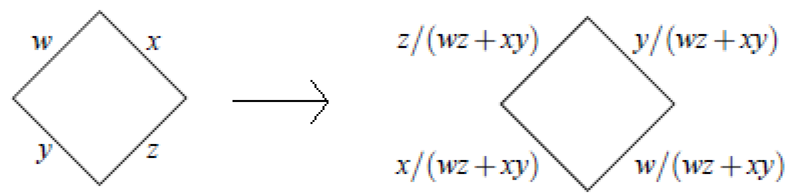

Figure 3: One Step of the Weight Sum Algorithm

w, x, z and y in clockwise order, the weights at those edges become z/c, y/c, w/c and x/c respectively, where c denotes the "cell-factor" $wz + xy$. See Figure 3 for how this sub-step is done. The outer edges of the order-n ADG are then stripped away, leaving an order-$(n-1)$ ADG. At the end, the cell-factors are all multiplied together to give the desired weight-sum. For a more detailed description of this algorithm and its proof, please refer to [Pro03].

1.2 The Edge-Probability Computation Algorithm

With weights assigned to the edges of an ADG, we now have a probability distribution on these edges, where the probability of a particular edge is equal to the sum of the weights of matchings containing that edge divided by the sum of weights of all matchings of the ADG. There is also an iterative algorithm to efficiently compute these edge probabilities, which we give a condensed description below, and the reader is again encouraged to refer to [Pro03] for the full version and the proof.

Given a weighted order-n ADG, we first run through the weight-sum algorithm, recording the weighted order-k ADG we obtain along the way, for $k = n-1, \ldots, 1$. For the purposes of purely computing the edge probabilities, it is not necessary to keep track of the cell factors. In contrast to the weight-sum algorithm, the edge-probability algorithm is constructive in the sense that it "builds up" to the edge probabilities of the given order-n ADG. We start with the weighted order-1 ADG, which is obtained at the final step of the weight-sum algorithm, and compute its edge probabilities according to its weights: let A, B, C, D be the four edges arranged in clockwise order respectively weighted w, x, z, y, then the edge probabilities of A and C are both $wz/(wz + xy)$, while the edge probabilities of B and D are both $xy/(wz + xy)$.

Now the induction step: suppose we have the edge probabilities of the order-$(k-1)$ ADG; we then *embed* the order-$(k-1)$ ADG, with each edge labeled by its probability, concentrically into an order-k ADG. At this point, if an edge has no probability labeled on it (as will be the case for the outside edges), consider its edge probability as 0. Then, in each of the k^2 cells, we *swap* the probabilities across the center of the cell. Now we perform the final *correction* step to obtain the actual edge probabilities of the weighted order-k ADG. Suppose that after the swap, a cell with edges A, B, C, D in clockwise order has probabilities p, q, r, s respectively. Then the *deficit* for that cell is defined as $1 - p - q - r - s$. Suppose A, B, C, D have respective weights w, x, z, y; then the *bias* for edges A and C is $wz/(wz + xy)$, and for edges B and D it is $xy/(wz+xy)$. We now add to the probability of each edge in this cell its bias times the deficit; for example, for edge A, its probability becomes $p+(1-q-r-s)\cdot wz/(wz+xy)$. These are the actual edge probabilities for the order-k weighted ADG that we obtained during the weight-sum algorithm. Note starting with an order-n ADG with all edges given weight 1, the bias is always 1/2; this is because the intermediate weighted ADGs that we obtained in the weight sum algorithm will alternately have uniform edge weights of 1/2 and 1.

1.3 The Domino Shuffling Algorithm

The third algorithm in [Pro03], called *domino shuffling*, is used for generating a random matching of an order-n weighted ADG. We briefly describe it here, as we will invoke it in a later section to explain the significance of the "deficits", also known as "net creation rates". This algorithm works in a similar way as the edge probability computation algorithm, in that we first need to run through the weight-sum algorithm to obtain n weighted ADGs of sizes 1 to n, and then "build up" to the random matching of an order-n weight ADG in n steps, with each step consisting of 3 substeps: *destruction*, *sliding*, and *creation*. The first step, generating a random matching of the order-1 weighted ADG, is equivalent to a single flip of a biased coin, with the bias coming from the weights on the single cell of the order-1 ADG. In the induction step, having already a random matching of an order-$(k-1)$ weighted ADG, we embed it into an order-k ADG, and erase pairs of edges that share a cell

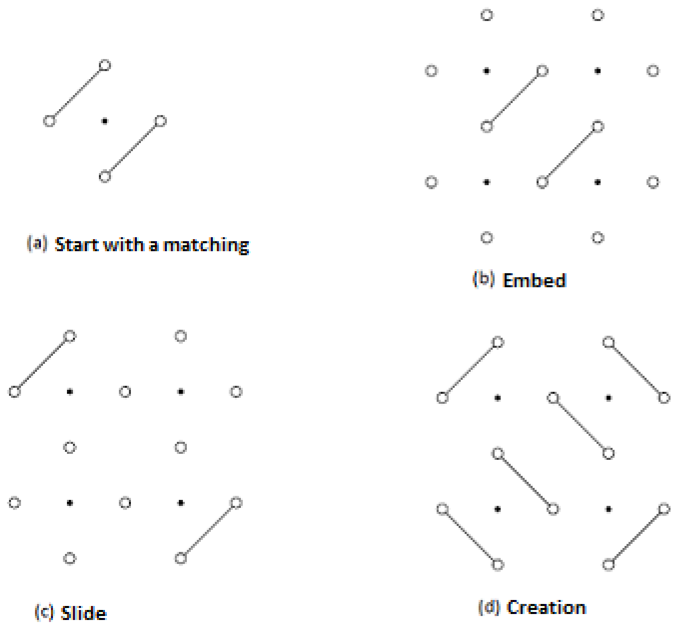

Figure 4: One Step of the Domino Shuffling Algorithm

(destruction); note that these are the edges which share the same "cocell" in the order-$(k-1)$ ADG. Then, in the cells that only contain a single edge, that edge is reflected across the center of the cell (sliding). Finally, the remaining cells will be filled with pairs of edges via coin flips, biased according to the weights on each cell in the order-k weighted ADG. At the end, we get a random matching of the order-n weighted ADG, with the probability distribution given by its edge weights. See Figure 4 for an example of performing one step of this algorithm, to generate a random matching of an order-2 ADG.

2 The Aztec Diamond Edge Probability Generating Function

Let us call a weighted ADG *standard* if it has uniform edge weights. We will now derive the 3-variate generating function that encodes the edge probabilities of standard ADGs, based on the edge probability algorithm described in the previous section. The main reason that such a generating function is possible is the existence of recurrence relations between cell deficits and edge probabilities, due to the fact that edge-weight transformation in the weight-sum algorithm reduces an order-k standard ADG to another order-$(k-1)$ standard ADG.

Color the lattice squares of the Aztec diamond alternately black and white, chessboard-style, with the north-west "side" always consisting of white squares. Call a horizontal pair of squares "north-going" if the left square is white and the right square is black, so that the northernmost two squares are always north-going. In the dual Aztec diamond graph, each square face corresponds to a 2×2 block of squares in the original Aztec diamond, and the northgoing edges are the top edges of the cells. Given a standard ADG, we can by symmetry only consider the northgoing edge probabilities, as the probability of a particular edge is equal to those of the three edges obtained by rotating the ADG by 90, 180, and 270 degrees about its center.

The main result of this section is the following theorem:

Theorem 2.1. *In an Aztec diamond of order n, let $p(i,j,n)$ denote the northgoing edge probability for the cell centered at (i,j). Then we have the following generating function for the quantities $p(i,j,n)$:*

$$\begin{aligned} F(x,y,z) &:= \sum p(i,j,n)x^i y^j z^n \\ &= \frac{z/2}{(1-yz)(1-(x+x^{-1}+y+y^{-1})z/2+z^2)} \end{aligned}$$

The sum is taken over $n \geq 1$ and $-n \leq i,j \leq n$ with $|i-1/2|+|j-1/2| \leq n$ and $i+j+n \equiv 1 \mod 2$.

2.1 Proof via Direct Computation

Proof. Let $F(x,y,z)$ be defined as above, and let $f(x,y,z) = \sum d(i,j,n)x^i y^j z^n$ denote the closely related generating function for the *deficits*, also known as the *net creation rates*. Please refer to the edge probability computation algorithm described in the earlier section for the definition of deficits. We seek to prove the following two identities, from which the formula for $F(x,y,z)$ follows by a simple algebraic manipulation. The proof of the first identity will be a tedious but elementary manipulation, while the second equation is a direct consequence of the edge probability computation algorithm.

$$f = z + \frac{1}{2}z(x+1/x+y+1/y)f - z^2 f \tag{2.1}$$

$$F = yzF + \frac{1}{2}f \tag{2.2}$$

Proof of (2.1): First, we need to interpret this algebraic equation for f as a recurrence relation between its coefficients $d(i,j,n)$. The first z term on the right hand side corresponds to the initial condition $d(0,0,1) = 1$, that is, right after embedding the order-0 (empty) ADG into the order-1 ADG, the four edges each have probability 0, yielding a deficit of 1. So what we need to show is the following recurrence relation:

$$\begin{aligned} d(i,j,n) =& \frac{1}{2}[d(i-1,j,n-1) + d(i+1,j,n-1) + d(i,j-1,n-1) \\ &+ d(i,j+1,n-1)] - d(i,j,n-2) \end{aligned} \tag{2.3}$$

Let $t(i,j,n), b(i,j,n), l(i,j,n), r(i,j,n)$ denote the respective edge probabilities of the top, bottom, left and right edges of the cell centered at (i,j), immediately after the embedding of the order-$(n-1)$ ADG into the order-n ADG, before any swap or correction. So these are “approximate” probabilities, and the exact probability for the top (north-going) edge, for example, will be $p(i,j,n) = b(i,j,n) + d(i,j,n)/2$, the sum of the probability on the bottom edge with the deficit $d(i,j,n)$ times the bias 1/2. We have the following equations, directly derived from the edge probability computation algorithm:

$$d(i,j,n-2) = 1 - [t(i,j,n-2) + b(i,j,n-2) + l(i,j,n-2) + r(i,j,n-2)] \tag{2.4}$$

$$\begin{aligned} d(i-1,j,n-1) = 1 - [&t(i-1,j,n-1) + b(i-1,j,n-1) \\ &+ l(i-1,j,n-1) + r(i,j,n-2) + d(i,j,n-2)/2] \end{aligned} \tag{2.5a}$$

$$d(i+1,j,n-1) = 1 - [t(i+1,j,n-1) + b(i+1,j,n-1) \tag{2.5b}$$
$$+ r(i+1,j,n-1) + l(i,j,n-2) + d(i,j,n-2)/2]$$

$$d(i,j-1,n-1) = 1 - [b(i,j-1,n-1) + l(i,j-1,n-1) \tag{2.5c}$$
$$+ r(i,j-1,n-1) + t(i,j,n-2) + d(i,j,n-2)/2]$$

$$d(i,j+1,n-1) = 1 - [t(i,j+1,n-1) + l(i,j+1,n-1) \tag{2.5d}$$
$$+ r(i,j+1,n-1) + b(i,j,n-2) + d(i,j,n-2)/2]$$

$$d(i,j,n) = 1 - [t(i,j+1,n-1) + b(i,j-1,n-1) \tag{2.6}$$
$$+ l(i-1,j,n-1) + r(i+1,j,n-1)]$$
$$- [d(i,j+1,n-1) + d(i,j-1,n-1)$$
$$+ d(i-1,j,n-1) + d(i+1,j,n-1)]/2$$

(2.4) is the definition of the deficit at the (i,j)-cell in the order-$(n-2)$ ADG. Moving ahead from the $(n-2)$-step to the $(n-1)$-step in the algorithm, the four equations in (2.5) represent the deficits in the four cells surrounding the (i,j)-cell in the order-$(n-1)$ ADG (note that in the order $(n-1)$ ADG, the square at location (i,j) is no longer a cell. Finally, moving from $(n-1)$-step to the n-step in the algorithm yields (2.6).

Substituting (2.6) into (2.3) and rearranging, we see that proving the recurrence relation (2.3) reduces to showing the following equality:

$$d(i,j+1,n-1)+d(i,j-1,n-1)+d(i-1,j,n-1)+d(i+1,j,n-1) \tag{2.7}$$
$$= 1-[t(i,j+1,n-1)+b(i,j-1,n-1)$$
$$+ l(i-1,j,n-1)+r(i+1,j,n-1)]+d(i,j,n-2)$$

Summing up the four equations in (2.5), and then substituting (2.4) into the right hand side yields the following unwieldly equation:

$$d(i,j+1,n-1)+d(i,j-1,n-1)+d(i-1,j,n-1)+d(i+1,j,n-1) \tag{2.8}$$
$$= 4-[t(i-1,j,n-1)+b(i-1,j,n-1)+l(i-1,j,n-1)$$
$$+ t(i+1,j,n-1)+b(i+1,j,n-1)+r(i+1,j,n-1)$$
$$+ b(i,j-1,n-1)+l(i,j-1,n-1)+r(i,j-1,n-1)$$
$$+ t(i,j+1,n-1)+l(i,j+1,n-1)+r(i,j+1,n-1)]$$
$$- [1+d(i,j,n-2)]$$

To simplify the right hand side of (2.8) into that of (2.7), we need to recall the fact that in the edge probability computation algorithm, at the end of step n, after the “swap” and “correct” sub-steps, the resulting edge probabilities are the actual edge probabilities for the ADG of order n. Therefore, the edge probabilities around each vertex needs to sum up to 1. Applying this to the four vertices of the (i,j)-square in the order-$(n-1)$ ADG, we have the following equations:

$$l(i,j+1,n-1)+t(i-1,j,n-1) \tag{2.9a}$$
$$=1-[r(i,j,n-2)+b(i,j,n-2)+d(i,j,n-2)]$$

$$r(i,j+1,n-1)+t(i+1,j,n-1) \tag{2.9b}$$
$$=1-[l(i,j,n-2)+b(i,j,n-2)+d(i,j,n-2)]$$

$$l(i,j-1,n-1)+b(i-1,j,n-1) \tag{2.9c}$$
$$=1-[r(i,j,n-2)+t(i,j,n-2)+d(i,j,n-2)]$$

$$r(i,j-1,n-1)+b(i+1,j,n-1) \tag{2.9d}$$
$$=1-[l(i,j,n-2)+t(i,j,n-2)+d(i,j,n-2)]$$

Summing up these four equations, and plugging in (2.4) yields

$$l(i,j+1,n-1)+r(i,j+1,n-1)+t(i-1,j,n-1)+b(i-1,j,n-1)+ \tag{2.10}$$
$$l(i,j-1,n-1)+r(i,j-1,n-1)+t(i+1,j,n-1)+b(i+1,j,n-1)$$
$$=2[l(i,j,n-2)+r(i,j,n-2)+t(i,j,n-2)+b(i,j,n-2)]$$
$$=2-2d(i,j,n-2)$$

Note that the eight terms on the left hand side of (2.10) appear in the right side of (2.8), so we make the substitution and simplify (2.8) to obtain (2.7), the equation we needed to prove:

$$d(i,j+1,n-1)+d(i,j-1,n-1)+d(i-1,j,n-1)+d(i+1,j,n-1) \quad (2.11)$$

$$= 4-[2-2d(i,j,n-2)+t(i,j+1,n-1)+b(i,j-1,n-1)$$

$$+ l(i-1,j,n-1)+r(i+1,j,n-1)]-[1+d(i,j,n-2)]$$

$$= 1-[t(i,j+1,n-1)+b(i,j-1,n-1)$$

$$+ l(i-1,j,n-1)+r(i+1,j,n-1)]+d(i,j,n-2)$$

□

Proof of (2.2): Again, we first interpret this algebraic relation between $F(x,y,z)$ and $f(x,y,z)$ as a relation between their coefficients, so what we need to prove is the following:

$$[x^iy^jz^n]F = [x^iy^jz^n](yzF+\frac{1}{2}f) \Longrightarrow p(i,j,n) = p(i,j-1,n-1)+d(i,j,n)/2 \quad (2.12)$$

To see that this relation holds, we simply recall that $p(i,j,n)$ represent only the probabilities of the North-going edges, which are the top edges in the cells of the order-n ADG. So, we trace through one whole step of the edge probabilitiy computation algorithm: in the "embed" sub-step, the exact top-edge probability at the $(i,j-1)$-cell in the order-$(n-1)$ ADG becomes the approximate bottom-edge probability at the (i,j)-cell in the order-n ADG, so in the "swap" sub-step it moves upwards to the top edge of that same cell. Finally, in the "correct" sub-step, we add $d(i,j,n)$ (deficit) times 1/2 (bias) to this top-edge probability to get $p(i,j,n)$, the exact top-edge probability at the (i,j)-cell in the order-n ADG.

From (2.1), we can solve for f to obtain

$$f(x,y,z) = \frac{z}{1 - \frac{z}{2}(x + x^{-1} + y + y^{-1}) + z^2}$$

From the above formula and (2.2), we solve for F to be

$$F(x,y,z) = \frac{z/2}{(1-yz)(1-(x+x^{-1}+y+y^{-1})z/2+z^2)}$$

□

2.2 Alternative Derivation Via the Octahedron Recurrence

The recurrence relation (2.3), which is the key step in obtaining the ADG edge probability generating function, can also be derived using the octahedron recurrence, which is described in detail in [Spe06]. Just like alternating sign matrices and Aztec diamonds, the octahedron recurrence is also a product of research on Dodgson condensation. Speyer notes that "the study of algebraic relations between determinants and how they are effected by various vanishing conditions is essentially the study of the flag manifold and Schubert varieties." This study led Fomin and Zelevinsky to their invention of cluster algebras.[3]

In the octahedron recurrence framework, the recurrence relation satisfied by edge probabilities of Aztec diamonds can be seen as just one example from a whole family of possible recurrence relations that arise from matchings of certain bipartite planar graphs, all of which can be encoded in the octahedron recurrence. This alternative

[3] [Spe06], section 1.1

derivation is sketched out in [PS05], but as the main focus of that paper is on groves, their derivation for the Aztec diamond edge probability generating function is missing some details and contains some small errors. Here we present a more complete version of the derivation, along with an interpretation of deficits as "net creation rates".

The octahedron recurrence is a recurrence for quantities $g_{i,j,n}$ where i, j, n are indexed by the 3-dimensional lattice and each $g_{i,j,n}$ is a Laurent polynomial in the initial conditions. For the uniformly-weighted Aztec diamond, the octahedron recurrence is

$$g_{i,j,n}g_{i,j,n-2} = g_{i-1,j,n-1}g_{i+1,j,n-1} + g_{i,j-1,n-1}g_{i,j+1,n-1} \tag{2.13}$$

and initial conditions are $g_{i,j,n} = x_{i,j,n}$ for $n = 0, -1$. These are the "face variables" that we can use to encode all matchings of the order-n ADG (arbitrarily centered at $(0, 0)$) by $g_{0,0,n}$ as follows:

$$g_{0,0,n} = \sum_T m(T)$$

Where T ranges over all matchings of the order-n ADG, and each matching corresponds to a Laurent monomial $m(T)$ in the variables $x_{i,j,\delta}$, where $\delta = 0$ or -1 according to whether $i+j+n$ is respectively even or odd. $m(T)$ is defined as follows:

$$m(T) = \prod_{|i|+|j|\leq n} x_{i,j,\delta}^{1-\alpha(i,j)}$$

$\alpha(i, j)$ is defined as the number of edges surrounding the face centered at (i, j). Note that since the centers of the faces satisfy $|i| + |j| \leq n$, these faces consist of not only the unit squares contained in the order-n ADG, but also the unit squares outside of the order-n ADG that share at least one edge with it. Also note that in contrast to the previous approach based on the edge-probability computation algorithm, we no longer restrict our attention to the cells, but also all the squares adjacent to the cells.

Clearly, each matching T gives rise to a Laurent monomial $m(T)$. What is less obvious is that this mapping is actually a bijection; that is, the matching T can be uniquely recovered from $m(T)$, so that each coefficient in $g_{0,0,n}$ is 1. This is Proposition 17 in [Spe07]. Its proof involves the idea that for any edge e, the indicator function $\mathbb{1}_{e \in T}$ is equal to the negative sum of the exponents of a subset of face variables in $m(T)$. This subset of faces is roughly one of the four regions that we obtain when we divide the order-n ADG by two lines of slope 1 and -1 intersecting at the center of e. Similar formulas for $\mathbb{1}_{e \in T}$ can be found by computing the sum of exponents of face variables from the other three regions. These formulas can be verified by adjoining a formal variable t onto the terms of the octahedron recurrence which contain face variables from the given region in the order-n ADG, and doing some careful bookkeeping.

We now use the octahedron recurrence to prove (2.3):

Proof. Consider a probabilistic version of the ADG octahedron recurrence, by assigning a probability of $2^{-\binom{n+1}{2}}$ to each matching of the order-n ADG, according to the uniform distribution over all matchings. We thus define

$$G_{0,0,n} = \sum_{T} 2^{-\binom{n+1}{2}} m(T)$$

with T ranging over all matchings of the order-n ADG. Looking back at (2.13), we see that now there is a factor of $2^{-\binom{n+1}{2}} \cdot 2^{-\binom{n-1}{2}}$ on the left hand side, and a factor of $(2^{-\binom{n}{2}})^2$ on the right hand side. Since $\binom{n+1}{2} + \binom{n-1}{2} = 2 \cdot \binom{n}{2} + 1$, we have the following probabilistic octahedron recurrence for $G_{0,0,n}$:

$$G_{i,j,n}G_{i,j,n-2} = \frac{1}{2}(G_{i-1,j,n-1}G_{i+1,j,n-1} + G_{i,j-1,n-1}G_{i,j+1,n-1}) \tag{2.14}$$

If we look at the expected exponent $E_n(i,j)$ of $x_{i,j,\delta}$ in $G_{0,0,n}$, we see that it is calculuated by

$$\begin{aligned} E_n(i,j) = & -1 \cdot \text{pr}(\text{2 edges at face (i,j)}) + 0 \cdot \text{pr}(\text{1 edge at face (i,j)}) \\ & + 1 \cdot \text{pr}(\text{0 edge at face (i,j)}) \end{aligned}$$

where pr($\star$) denotes the probability of $\star$, i.e. the number of matchings satisfying $\star$ divided by $2^{\binom{n+1}{2}}$.

In the language of domino shuffling, this corresponds to the probability of “creation” (having no edges) subtracted by the probability of “destruction” (having two edges) at the face (i,j), thereby we call $E_n(i,j)$ the “net creation rate”. Recall that (2.2) is equivalent to the expression $p(i,j,n) = p(i,j-1,n-1) + d(i,j,n)/2$; we may now interpret this equation as follows: the probability of a north-going edge at the (i,j)-cell in the order-n ADG is equal to the probability that it comes from sliding, plus the probability that it comes from creation, minus the probability that it undergoes destruction. Note that the factor of 1/2 corresponds to the fact that in the standard ADG, there is equal probability for the pair of top-bottom edges as the pair of left-right edges to be created or destroyed. Since (2.2) is derived directly from domino shuffling, the net creation rates are exactly the deficits that are used in computing edge probabilities - that is, $E_n(i,j) = d(i,j,n)$.

To get the expectation of the exponent of the face variable $x_{i_0,j_0,\delta}$, we differentiate the probability generating function $G_{0,0,n}$ with respect to this variable and then set all variables equal to 1:

$$E_n(i_0, j_0) = \frac{\partial}{\partial x_{i_0,j_0,\delta}}(G_{0,0,n})|_{x_{i,j,\delta}=1}$$

So we perform this operation on (2.14), applying the product rule for derivatives, and obtain the following linear recurrence for $E_n(i,j)$:

$$\begin{aligned} E_n(i,j) + E_{n-2}(i,j) = \frac{1}{2}(&E_{n-1}(i-1,j) + E_{n-1}(i+1,j) \\ &+ E_{n-1}(i,j-1) + E_{n-1}(i,j+1)) \end{aligned} \tag{2.15}$$

This is exactly the recurrence relation (2.3) for deficits in the earlier proof of theorem (2.1), with slightly different notation. □

3 Deriving asymptotics using generating functions

Having the generating function for a sequence of combinatorial interest often enables us to derive asymptotics for the sequence. We now give a short introduction to the (relatively new) research realm of analytic combinatorics, with a focus on deriving asymptotics from multivariate generating functions, and how it has been applied to the Aztec diamond edge probability generating function derived in the previous section.

3.1 The univariate case

Given a sequence $\{a_n : n \geq 0\}$ of complex numbers, let $f(z) := \sum_n a_n z^n$ be its associated generating function. Certain recurrences for a_n lead to functional equations that can be solved for f; for example, linear recurrences with constant coefficients always lead to rational generating functions. Deriving asymptotics for a_n, i.e. making estimate for how a_n grows as n tends to infinity, is fairly well understood and somewhat mechanized; at its basis is Cauchy's integral formula

$$a_n = \frac{1}{2\pi i} \int z^{-n-1} f(z) dz \tag{3.1}$$

to which complex analytic methods can be applied to obtain good estimates for a_n.

Depending on the form of the generating function f, one of several known methods may be use to derive its asymptotics. We begin with a basic estimate: if f has radius of convergence R, then taking the contour of integration in (3.1) to be a circle of radius $R - \epsilon$ gives $a_n = O(R - \epsilon)^{-n}$ for any $0 < \epsilon < R$. If f is continuous on the closed disk of radius R then $a_n = O(R^{-n})$. We now mention three methods, which all refine this basic estimate by pushing the contour out far enough to make the resulting upper bound asymptotically sharp.

When f is entire, we can use saddle point methods, where the contour of integration in the Cauchy integral is moved so that it passes through a point (the "saddle point") where the integrand is not rapidly oscillating, and then a two-term Taylor approximation is used for the integrand. Note that this method also works in the case where f is not entire, but the saddle point lies within the domain of convergence of f.

In the case where f is algebraic but has a positive finite radius of convergence R, if there is no saddle point of $z^{-n-1}f(z)$ in the open disk of radius R, we employ the circle method (also known as Darboux's method[4]). Here we push the contour of integration near or onto the circle of radius R, with the refinement that if f extends to the circle of radius R and is k times continuously differentiable there, then integration by parts yields

$$\int z^{-n}f(z)dz = O(n^{-k}R^{-n}).$$

The transfer theorems of Flajolet and Odylyzko[5] are refinements of Darboux's theorem. They involve using a contour consisting of an arc of a circle of radius $1/n$ around r, an arc of a circle of radius $r-\epsilon$ centered at the origin, and two line segments connecting the corresponding end points of the two circular arcs. This contour is contained in a so-called "Camembert-shaped region", the region of analyticity of an $f(z)$ singular at r, and allows for asymptotic extractions for the class of functions named **alg-log**, which are a product of a power of $r - z$, a power of $\log(1/(r - z))$, and a power of $\log\log(1/(r - z))$.

The above short overview is mostly based on sections 1 and 2 of [PW08]. For detailed treatises on univariate asymptotic methods, please refer to Chapter 5 of

[4] See [Hen91]

[5] See [FO90]

[Wil06] or Part B of [FS09].

3.2 The multivariate case

For this sub-section we will focus only on results from the Pemantle-Wilson-Baryshnikov vein of research on multivariate asymptotics, as it not only yielded the tools for asymptotic extraction in the widest class of multivariate generating functions, but also led to the asymptotics of the Aztec diamond edge probability generating function. Let d denote the number of variables, so that $\mathbf{z} = (z_1, \ldots, z_d)$. Let $\mathbf{z}^{\mathbf{r}} := \prod_{j=1}^{d} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$, so that we have the multivariate generating function

$$F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$$

for a multivariate array $\{a_{\mathbf{r}} : \mathbf{r} \in \mathbb{N}^d\}$. When d is small, we use (x, y, z) for (z_1, z_2, z_3) and (r, s, t) for (r_1, r_2, r_3). Now we are interested in the asymptotic behavior of $a_{\mathbf{r}}$. Unfortunately, things get a lot more complicated and difficult in the multivariate case. Even for rational functions, multivariate asymptotics is far from trivial and must be classified into a myriad of different cases. Furthermore, simple-looking generating functions can lead to complicated asymptotic analyses and expressions: take for example the binomial coefficients, $a_{rs} = \binom{r+s}{r,s}$. From the recursion $a_{rs} = a_{r-1,s} + a_{r,s-1}$, we have

$$F(x, y) = \sum_{r,s \geq 0} a_{rs} x^r y^s = \frac{1}{1 - x - y}$$

It turns out, after employing the highly nontrivial Theorem 1.3 of [PW08], that

$$a_{rs} \sim \left(\frac{r+s}{r}\right)^r \left(\frac{r+s}{s}\right)^s \sqrt{\frac{r+s}{2\pi rs}},$$

which agrees with Stirling's formula. In contrast, Stirling's formula can be derived using a quick application of the saddle point method for the univariate exponential generating function e^z.[6]

It will be necessary to separate $\mathbf{r}$ into its scale $|\mathbf{r}|$ and direction $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$; we are concerned with asymptotics when $|\mathbf{r}| \to \infty$ with $\hat{\mathbf{r}}$ remaining in some specified set, bounded away from the coordinate planes. In general, for the meromorphic multivariate generating function $F = G/H$, with G and H locally analytic and sharing no common factor, the asymptotic analysis proceeds as follows:

(i) Asymptotics in the direction $\hat{\mathbf{r}}$ are determined by the geometry of the pole variety $V = \{\mathbf{z} : H(\mathbf{z}) = 0\}$ near a finite set, $\mathbf{crit}_{\hat{\mathbf{r}}}$ of *critical points.*

(ii) As in the univariate case, fundamental to all the derivation is the Cauchy integral representation

$$a_{\mathbf{r}} = \frac{1}{(2\pi i)^d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) d\mathbf{z}$$

where T is the product of sufficiently small circles around the origin in each of the coordinates, $\mathbf{1}$ is the d-vector of all ones, and $d\mathbf{z}$ is the holomorphic volume form $dz_1 \wedge \cdots \wedge dz_d$.

(iii) Observe that T may be replaced by any cycle homologous to $[T]$ in $H_d(\mathcal{M})$, where

[6] See [FS09] Example VIII.3.

$\mathcal{M}$ is the domain of holomorphy of the integrand.

(iv) Deform the cycle to lower the modulus of the integrand as much as possible; use Morse theoretic methods to characterize the minimax cycle in terms of *critical points*.

(v) Use algebraic methods to find the critical points; these are points of V that depend on the direction $\hat{\mathbf{r}}$ of the asymptotics, and are saddle points for the magnitude of the integrand.

(vi) Reduce this set of critical points to a set $\mathbf{contrib}_{\hat{\mathbf{r}}} \subset \mathbf{crit}_{\hat{\mathbf{r}}}$ of *contributing critical points* using topological methods. Then replace the integral over T by an integral over *quasi-local* cycles $\mathcal{C}(z_j)$ near each $z_j \in \mathbf{contrib}_{\hat{\mathbf{r}}}$.

(vii) Evaluate the integral over $\mathcal{C}$ by a combination of residue and saddle point techniques.

These steps lead to the meta-formula

$$a_{\mathbf{r}} \sum_{\mathbf{z} \in \mathbf{contrib}_{\overline{\mathbf{r}}}} \mathbf{formula}(\mathbf{z})$$

where $\mathbf{formula}(\mathbf{z})$ is a function of the local geometry for smooth, multiple, and cone points. The main research papers that led to this method of multivariate asymptotic extraction are [PW02], [PW04], and [BP08]. The above short summary is based upon the beginning sections of the survey paper [PW08] and the forthcoming book [PW11].

3.3 Asymptotics of the Aztec diamond edge probability generating function

We first recall the result of Theorem 2.1: the generating function for the edge probabilities of the Aztec diamond is

$$
\begin{aligned}
F(x,y,z) &:= \sum p(i,j,n)x^i y^j z^n \\
&= \frac{z/2}{(1-yz)(1-(x+x^{-1}+y+y^{-1})z/2+z^2)}
\end{aligned}
$$

For the simpler creation rate generating function, $(1-(x+x^{-1}+y+y^{-1})z/2+z^2)^{-1}$, the formula for its coefficients were derived in [CEP96] via a relation to Krawtchouk polynomials. Then these were summed via contour integrals to prove the Arctic Circle Theorem. However, this computation was quite specialized and does not generalize easily to other tiling problems that exhibit arctic circle-type asymptotic behavior. In [BP08], a new powerful machinery was introduced that not only proves the arctic circle theorem for Aztec diamonds, but also generalizes to other generating functions with the same type of contributing points of the singular variety as that of the Aztec diamond generating function - the so-called quadratic points, also known as cone points. The following summary is excerpted from Section 1.3 of [BP08].

The fundamental result of [BP08][7] is that the asymptotics of a generating function with irreducible quadratic denominator are the same as its Fourier transform's, which is the dual quadratic. This is the continuous analogue of the Krawtchouk polynomials

[7] Theorem 3.7 in [BP08]

that appear when the computations are done in the discrete setting. Multiplying the denominator by a smooth factor h corresponds to convolving the Fourier transform with a heavyside function; this is the analogue of summing.[8] In other words, the Fourier transform of a cone is the dual cone, and the Fourier transform of $1/(Qh)$ is the integral of the Fourier transform of $1/Q$. The details of integral computations (and thus the proofs of [BP08] Theorems 3.7 and 3.9) involve the notion of generalized functions, and are presented in Section 6 of [BP08].

As the Aztec diamond edge probability generating function contains a quadratic and a smooth factor in its denominator, Theorem 3.9 of [BP08] can be applied to derive its asymptotics. This computation is carried out in full detail in Section 4.1 of [BP08] and yields a new proof of Theorem 1 of [CEP96], which implies the arctic circle theorem.

4 The Fortress Edge Probability Generating Function

4.1 The Standard-Weight Fortress

In an n-by-n array of unit squares, cut each square by both of its diagonals, forming $4n^2$ identical isoceles right triangles. Color the triangles alternately black and white,

[8]Theorem 3.9 in [BP08]

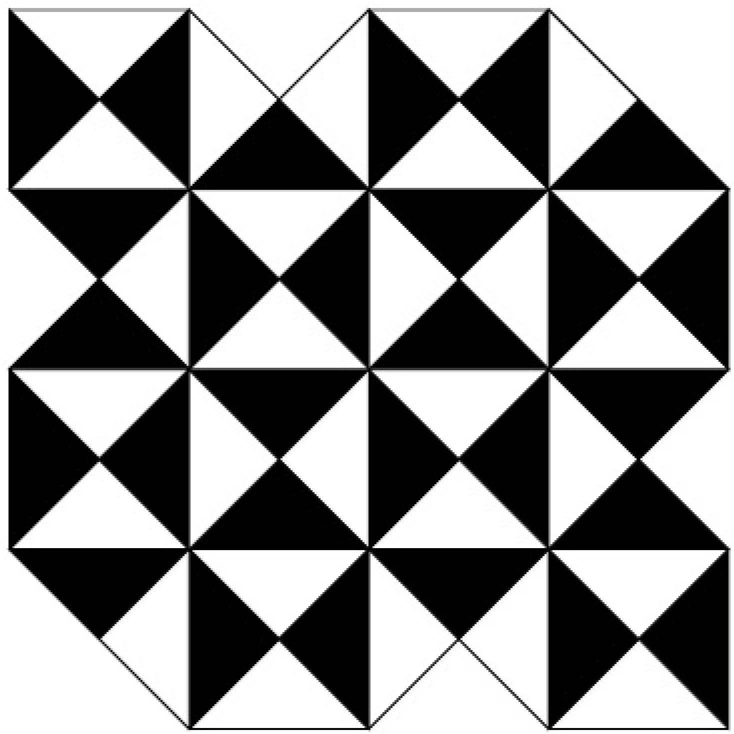

Figure 5: A Fortress of Order 4

so that each black triangle is surrounded by three white triangles, and vice versa. Then remove all the black (respectively white) triangles that have an edge on the top or bottom (respectively left or right) boundaries of the array. The region that remains is called a *fortress of order n*. For even n, the two ways of coloring the triangles lead to mirror-image fortresses, while for odd n, the two ways of coloring the triangles lead to genuinely different fortresses. See Figure 5 for a fortress of order 4.

We call the small isoceles right triangles *monobolos*, and pairs of two adjacent monobolos are called *diabolos*, which are either squares or isoceles triangles. Analogous to domino-tilings, we may define diabolo-tilings of fortresses. The dual graph to the fortress consists of squares and octagons, which, after urban renewal[9], can be turned into an Aztec diamond graph where edges are weighted according to the following scheme:

1. Each edge has weight 1 or weight 1/2.

2. If two horizontal (resp. vertical) edges are related by a unit vertical (resp. horizontal) displacement, they have the same weight, but if they are related by a unit horizontal (resp. vertical) displacement, their weights differ.

3. If n is 1 or 2 (mod 4), the four extreme-most edges have weight 1/2; if n is 0 or 3 (mod 4), these four edges have weight 1.

For the fortress, we can derive its edge probability generating function in the same way as before, but now we need to distinguish the four types of cells: type E, those with all edge weights 1/2, type F, those with all edge weights 1, type G, those with horizontal edge weights 1/2 and vertical edge weights 1, and type H, those with horizontal edge weights 1 and vertical edge weights 1/2. See Figure 6 for an example of classifying the cell types in a fortress ADG of order 2. The derivation of the generating functions for net creation rates (e, f, g, h) and for edge probabilities (E, F, G, H) is essentially the same as before, only now we have to keep track of the relative positions of the four types of cells in the weight reduction process. For example, suppose we have an order-$(k-1)$ fortress-weighted ADG, which is derived

[9]See [Pro03].

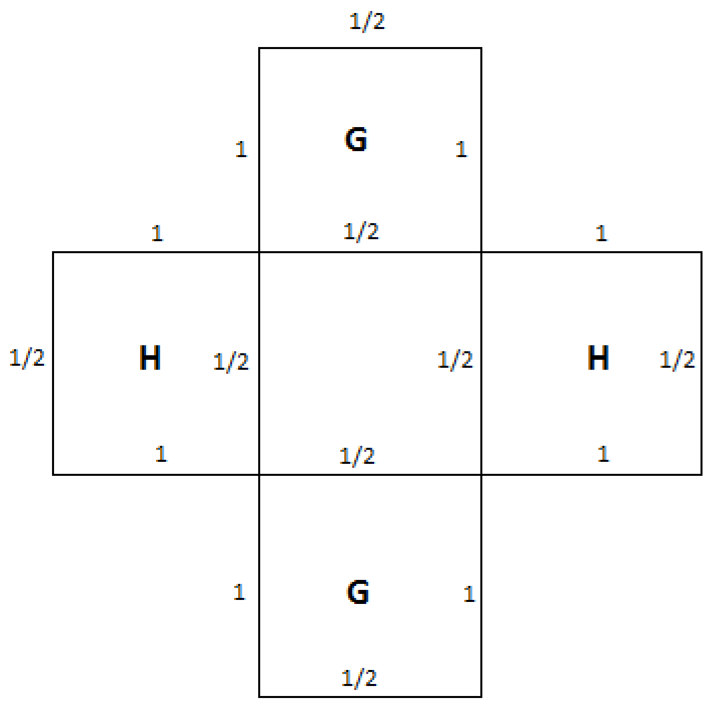

Figure 6: Standard-Weight Fortress ADG of Order 2

from an order-k fortress-weighted ADG by one step of the weight sum computation algorithm. Then in the order-$(k-1)$ ADG, if the cocell at location (i,j) is surrounded by G-cells on the left and right and H-cells above and below, then in the order-k ADG there is a E cell at location (i,j). So we have the following relations:

$$e = z + \frac{1}{5}z((x+x^{-1})g + (y+y^{-1})h) - z^2 f$$

$$f = \frac{4}{5}z((x+x^{-1})h + (y+y^{-1})g) - z^2 e$$

$$g = \frac{1}{2}z((x+x^{-1})f + (y+y^{-1})e) - z^2 h$$

$$h = \frac{1}{2}z((x+x^{-1})e + (y+y^{-1})f) - z^2 g$$

$$E = Hyz + \frac{1}{2}e$$

$$F = Gyz + \frac{1}{2}f$$

$$G = Eyz + \frac{1}{5}g$$

$$H = Fyz + \frac{4}{5}h$$

We can use a computer algebra system such as Maple to solve these relations simultaneously. The first four equations are treated as a system of four linear equations in e, f, g, h with $x, 1/x, y, 1/y$ and z being part of the coefficients, and can be solved as a 4 by 4 matrix. Then once e, f, g, h are found, the last four equations can be solved again as a system of four linear equations in E, F, G, H. The edge probability generating function for all the north-going edges is then $I(x,y,z) = E + F + G + H$, and turns out to be quite a bit more complicated than that of the standard ADG. Just looking at the denominator (as it dictates the asymptotics), we again have a factor of $1 - yz$, and we name what remains as Q. We have the lengthy expression

$$\begin{aligned} Q = {} & 25z^8y^4x^4 - 25z^6x^3y^5 - 25z^6y^5x^5 - 25z^6y^3x^3 \\ & - 25z^6x^5y^3 - z^4y^8x^4 + 25z^4x^4y^6 + 2z^4y^6x^2 \\ & + 2z^4x^6y^6 - z^4y^4 + 25z^4x^6y^4 + 25z^4x^2y^4 - z^4x^8y^4 \\ & + 46z^4y^4x^4 + 25z^4y^2x^4 + 2y^2z^4x^2 + 2z^4x^6y^2 - z^4x^4 \\ & - 25x^3z^2y^5 - 25z^2y^5x^5 - 25z^2y^3x^3 - 25z^2y^3x^5 + 25x^4y^4 \end{aligned} \tag{4.1}$$

There is quite a bit of symmetry in Q, especially after factoring out $x^4y^4z^4$. Making the substitutions $u = (x + 1/x)/2$, $v = (y + 1/y)/2$, $w = (z + 1/z)/2$, we have the following more compact expression for Q:

$$\begin{aligned} Q &= x^4y^4z^4(100u^2 + 200uv + 100v^2 - 400w^2 + 400w^4 \\ &\quad - 400uvw^2 - 16u^4 + 32u^2v^2 - 16v^4) \\ &= x^4y^4z^4[100((u+v)^2 - (2w)^2) + 400w^2(w^2 - uv) - 16(u^2 - v^2)^2] \end{aligned} \tag{4.2}$$

In Maple we can make a 3-D plot (Figure 7) of the set of points (u, v, w) where $Q = 0$, as this is the singular variety that is at the base of asymptotic analysis of the fortress edge probability generating function. It was noted in [BP08] that the techniques developed in that paper can be extended to deal with the isolated quartic singularity of the fortress generating function and derive its asymptotics, though the details were left for another paper.

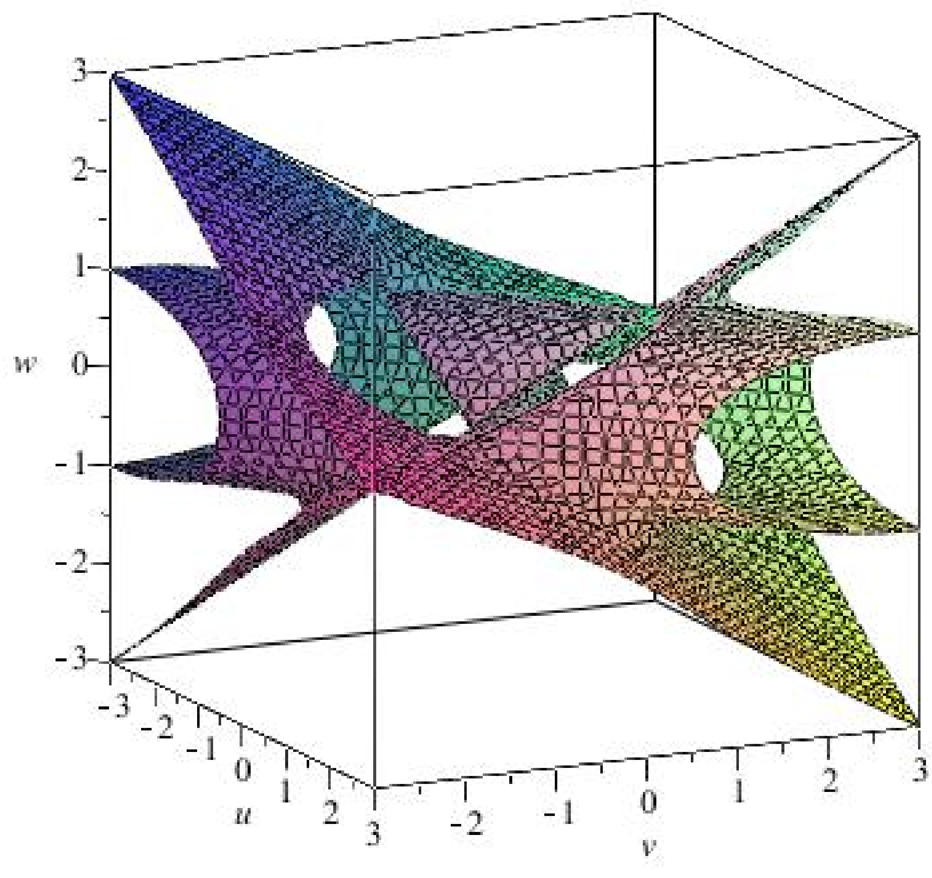

Figure 7: Singular variety of the fortress edge probability generating function

4.2 The General-Weight Fortress

In the more general case where we use the same periodic weight scheme, but with a general parameter t instead of 1/2, the generating functions becomes yet more complicated. The four types of cells E, F, G, H are defined in the same way as before (with t replacing 1/2). The relations between them and their corresponding creation rate generating functions are now

$$e = z + \frac{t^2}{1+t^2}\, z((x + x^{-1})g + (y + y^{-1})h) - z^2 f$$

$$f = \frac{1}{1+t^2}\, z((x + x^{-1})h + (y + y^{-1})g) - z^2 e$$

$$g = \frac{1}{2} z((x + x^{-1})f + (y + y^{-1})e) - z^2 h$$

$$h = \frac{1}{2} z((x + x^{-1})e + (y + y^{-1})f) - z^2 g$$

$$E = Hyz + \frac{1}{2} e$$

$$F = Gyz + \frac{1}{2} f$$

$$G = Eyz + \frac{t}{1+t^2}\, g$$

$$H = Fyz + \frac{1}{1+t^2}\, h$$

Solving these equations, $E + F + G + H$ again has a factor of $1 - yz$ in its denominator, and the remaining factor Q is the following:

$$
\begin{aligned}
Q = {} & 8y^4x^4t^4z^4 + 2y^2x^2t^2z^4 - 8t^2z^2x^3y^3 + 8t^2z^4x^2y^4 + 8t^2z^4y^2x^4 \\
& - 4z^2y^3x^3 + 4z^4y^2x^4 + 4z^4x^2y^4 - 4y^5x^3z^6 - 4y^3x^5z^6 \\
& + 8y^4x^4z^4 - 4y^3x^3z^6 + 4y^4x^4z^8 - 4y^5x^5z^6 + 4y^4x^4t^4 \\
& + 8y^4x^4t^2 - 4y^5x^3z^2 - 4y^3x^5z^2 + 4y^4x^6z^4 + 4y^6x^4z^4 \\
& - 4y^5x^5z^2 - t^2z^4y^4 - t^2x^4z^4 + 4y^4x^4 - 8y^5x^3t^2z^6 \\
& - 8y^3x^5t^2z^6 + 12y^4x^4t^2z^4 - 8y^3x^3t^2z^6 + 4y^4x^4t^4z^8 + 8y^4x^4t^2z^8 \\
& - 8y^5x^5t^2z^6 - 8y^5x^3t^2z^2 - 8y^3x^5t^2z^2 + 8y^4x^6t^2z^4 + 8y^6x^4t^2z^4 \\
& - 8y^5x^5t^2z^2 - 4t^4z^6x^5y^5 - 4t^4x^3z^6y^3 + 2t^2z^4x^6y^2 + 4t^4z^4x^6y^4 \\
& + 2t^2z^4x^6y^6 - t^2z^4x^8y^4 + 4t^4z^4y^4x^2 + 2t^2z^4y^6x^2 - 4t^4x^5z^6y^3 \\
& + 4t^4x^4z^4y^2 - 4t^4z^6y^5x^3 + 4t^4z^4x^4y^6 - t^2z^4x^4y^8 - 4z^2y^3x^3t^4 \\
& - 4t^4z^2x^5y^5 - 4t^4z^2x^5y^3 - 4z^2y^5x^3t^4
\end{aligned}
\tag{4.3}
$$

Now there seems to be symmetry after factoring out $t^2x^4y^4z^4$. Making the substitutions $u = (x + 1/x)/2$, $v = (y + 1/y)/2$, $w = (z + 1/z)/2$, and $s = (t + 1/t)/2$, we have the following (much) more compact expression for Q:

$$
\begin{aligned}
Q &= t^2x^4y^4z^4(64s^2u^2 + 128s^2uv + 64s^2v^2 - 256s^2w^2 \\
&\quad - 16u^4 + 32u^2v^2 - 16v^4 + 256s^2w^4 - 256s^2uvw^2 \\
&= t^2x^4y^4z^4[64s^2((u + v)^2 - (2w)^2) + 256s^2w^2(w^2 - uv) - 16(u^2 - v^2)^2]
\end{aligned}
\tag{4.4}
$$

In the standard-weight fortress, $t = 1/2$ so $s = 5/4$. Substituting this into (4.4) yields (4.2), as expected. It remains to be investigated which other values of t will correspond to something of combinatorial significance.

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