ASYMPTOTIC ENUMERATION VIA SINGULARITY ANALYSIS

DISSERTATION

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By

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ABSTRACT

Asymptotic formulae for two-dimensional arrays $(f_{r,s})_{r,s\geq 0}$ where the associated generating function $F(z,w) := \sum_{r,s\geq 0} f_{r,s} z^r w^s$ is meromorphic are provided. Our approach is geometrical. To a big extent it generalizes and completes the asymptotic description of the coefficients $f_{r,s}$ along a compact set of directions specified by smooth points of the singular variety of the denominator of $F(z,w)$. The scheme we develop can lead to a high level of complexity. However, it provides the leading asymptotic order of $f_{r,s}$ if some unusual and pathological behavior is ruled out. It relies on the asymptotic analysis of a certain type of stationary phase integral of the form $\int e^{-s\cdot P(d,\theta)} A(d,\theta) d\theta$, which describes up to an exponential factor the asymptotic behavior of the coefficients $f_{r,s}$ along the direction $d = \frac{r}{s}$ in the (r,s)-lattice. The cases of interest are when either the phase term $P(d,\theta)$ or the amplitude term $A(d,\theta)$ exhibits a change of degree as d approaches a degenerate direction. These are handled by a generalized version of the stationary phase and the coalescing saddle point method which we propose as part of this dissertation. The occurrence of two special functions related to the Airy function is established when two simple saddles of the phase term coalesce. A scheme to study the asymptotic behavior of big powers of generating functions is proposed as an additional application of these generalized methods.

Dedicated to my mother, father and sister.

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Finally, I want to give my appreciation to those unique and valuable friends back in Wisconsin and here in Ohio. In many ways they have compensated for the distance with my family and old friends welcoming me into their lives and cheering up mine. Thank you!

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PUBLICATIONS

1. LLadser, M. and San Martin, J. *Domain of attraction of the quasi-stationary distributions for the Ornstein-Uhlenbeck process.* Journal of Applied Probability, Vol. 37, N2, 511-520, 2000.

FIELDS OF STUDY

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CHAPTER 1

INTRODUCTION

The ultimate aim of the research in this dissertation is to provide asymptotic formulae for two-dimensional arrays $(f_{r,s})_{r,s\geq 0}$ where the generating function $F(z,w) := \sum_{r,s\geq 0} f_{r,s}\, z^r\, w^s$ is known. Such arrays arise most often in combinatorial applications, where classes of objects indexed by one or more positive integers are being counted. When the cardinalities $f_{r,s}$ satisfy various, often recursive relations, the associated generating function $F(z,w)$ will satisfy corresponding equations or functional equations, leading to a complete or partial description of $F(z,w)$ as an analytic function of its arguments. Methods of approximating $f_{r,s}$ given $F(z,w)$ are not well understood, and this is the subject of the present dissertation.

The motivation for this work, as well as some basic techniques for dealing with generating functions, lies in the area of combinatorics. These are summarized in the second and third chapters.

In chapter 2 basic definitions pertaining to generating functions are presented, along with some elementary properties. These are illustrated by several combinatorial examples. Some analytic techniques are introduced as well, specifically those for which an elementary and self-contained exposition is possible.

Chapter 3 summarizes much of what is known about approximating the coefficients $(f_n)_{n\geq 0}$ of a one-variable generating function $F(z) := \sum_{n=0}^{\infty} f_n\, z^n$. The name for this is *singularity analysis in one dimension*, and there are a number of useful techniques and results, some as recent as the 1990's and some dating back to the 1960's and beyond. In one variable, as in the multivariate setting, all analytic techniques start by using Cauchy's integral formula to rewrite f_n as an integral of $\frac{F(z)}{z^{n+1}}$ over a contour in the complex plane. The trick is to make this integral tractable, in the sense that it may be well approximated for large values of n. There is a substantial toolkit available, including the use of well chosen contours passing near the dominant singularity, as well as saddle point methods when the contour must be taken through or arbitrarily close to a stationary phase point.

One of the reasons that this analysis had not previously been extended to the multivariable setting is that the corresponding multivariate complex variable theory is more difficult and the multivariate stationary phase method less systematized. The analytic machinery required to analyze and approximate generating functions in one variable is pretty well known to the combinatorial community. However, the corresponding apparatus required for approximation in the multivariate setting is much less well known to combinatorialists and indeed to most mathematicians outside of the Several Complex Variables and Applied Math communities.

The thrust of the research of Pemantle and Wilson has been to harness the theory of several complex variables in order to derive asymptotic formulae for wide classes of multivariable generating functions. Their approach is geometrical and topological. The multivariate Cauchy integral is represented as a middle-dimensional integral over a cycle in the complement of the singular locus of the generating function, and this is then simplified using tools from multivariate residue theory, stratified Morse theory and multivariate saddle point analysis. The use of powerful theories such as stratified Morse theory depend, to some degree, on having asked the right questions: in this case, asking for asymptotics of $f_{r,s}$ as $(r, s) \to \infty$ with $\frac{r}{s}$ remaining constant, or at any rate not coming near any of a set of *degenerate directions*.

In the present dissertation, we tackle the problem of degenerate directions. Specifically, we investigate asymptotics of $f_{r,s}$ as $(r, s) \to \infty$ and $\frac{r}{s}$ converges to a degenerate direction at a prescribed rate. The goal is to determine the *bandwidth*, that is, the function $g(s)$ such that $f_{r,s}$ exhibits a phase transition when r is of order $g(s)$ and $\frac{r}{s}$ approaches a degenerate direction. Because the powerful topological techniques cannot be immediately applied to this situation, it is necessary to delve deeper into multivariate saddle point methods.

Chapter 4 introduces some basic theory of several complex variables. In particular, the Cauchy's integral formula for several variables is introduced and various canonical forms for multivariate analytic functions are discussed. At the end of this chapter we present a couple of new uniqueness results for canonical representations of analytic functions of several complex variables using one-complex variable methods.

The first of two chapters dealing mainly with new results is chapter 5. This chapter develops some machinery for approximating a certain type of stationary phase integral, namely $\int e^{-s \cdot P(t(s),z)} A(t(s), z)\, dz$ where the phase $P(t, z)$ and/or amplitude $A(t, z)$ near the stationary phase point z_0 exhibit a phase change: the power series for $P(t(s), \cdot)$ has a different leading order at $s = \infty$ versus $s < \infty$. These results draw on existing saddle point methods, though these methods often have been formulated only for special cases and need to be adapted to the present problem. In the cases of interest here, saddle points coalesce, and Airy-type limits are encountered. The end of chapter 5 is devoted to a self-contained discussion on the asymptotic behavior of the coefficient of z^n of a generating function of the form $f(z)^n \cdot g(z)^m \cdot h(z)$. The scheme we develop is applied to rediscover a particular local limit law of the map Airy-type.

The final chapter, chapter 6, applies the results of chapter 5 to the Cauchy integrals arising from bivariate rational generating functions. As an example, some combinatorial problems are analyzed from start to finish: bivariate generating functions are derived, critical directions identified, asymptotics are reduced to computations of certain integrals, and finally asymptotic approximations for these are derived which are uniform throughout the problematic region, i.e., as $(r, s) \to \infty$ with $\frac{r}{s}$ approaching a degenerate direction. Putting these steps together can lead to a high degree of complexity in general, but it is possible to provide a formula for the leading term of $f_{r,s}$ when $F(z, w)$ is a meromorphic function and some unusual and pathological behavior is ruled out.

CHAPTER 2

GENERATING FUNCTION METHODS

2.1 Laplace and the dawn of modern counting

Pierre Simon Laplace (1749-1827) experienced for the good of all humankind a revelation when he discovered the idea of generating functions as a means to count. According to Rota [Rot75]: "Laplace discovered the remarkable correspondence between set theoretic operations and operations on formal power series and put it to great use to solve a variety of combinatorial problems".

The idea of Laplace was to associate to a sequence of numbers of interest a power series. In the modern jargon, the *formal power series* or sometimes also called *generating function* associated to a sequence of (complex) numbers $(a_n)_{n\geq 0}$ is the series

$$A(z) := \sum_{n=0}^{\infty} a_n\, z^n\,. \tag{2.1}$$

The above z is a complex variable. The definition is to be understood in a purely algebraic sense and by no means do we claim that $A(z)$ defines a function of $\mathbf{z}$. After all it is possible that the radius of convergence of $A(z)$ is zero. Thus, a priori, a power series is not meant to be evaluated at any particular value of z. The exception to this is $z = 0$ which is the only point in the complex plane where a priori we can ensure convergence.

We will let $\mathbb{C}[[z]]$ denote the *set of formal power series in the indeterminate* z. For an arbitrary element such as $A(z)$ in (2.1) we will write $[z^n]\,A(z)$ to refer to the coefficient of z^n in the series. Given two elements $F(z) := \sum_{n=0}^{\infty} f_n\, z^n$ and $G(z) := \sum_{n=0}^{\infty} g_n\, z^n$, their *sum* and *product* are respectively defined as

$$F(z) + G(z) \;:=\; \sum_{n=0}^{\infty} u_n\, z^n\,, \tag{2.2}$$

$$F(z) \cdot G(z) \;:=\; \sum_{n=0}^{\infty} v_n\, z^n\,, \tag{2.3}$$

where $u_n := f_n + g_n$ and $v_n := \sum_{k=0}^{n} f_k \cdot g_{n-k}$. These operations are well-defined in $\mathbb{C}[[z]]$ and indeed make of this set an integral domain. Furthermore, if $F(z)$ and $G(z)$ are absolutely convergent for all $|z| < r$, with $r > 0$, then the series in (2.2) and (2.3) are also absolutely convergent for all such z and their limit is precisely to $F(z) + G(z)$ and $F(z) \cdot G(z)$ respectively.

It is convenient to rewrite (2.3) as follows

$$F(z) \cdot G(z) = \sum_{n=0}^{\infty} \left\{ \sum_{p+q=n} f_p \cdot g_q \right\} z^n\,, \tag{2.4}$$

where the indices p and q range over the nonnegative integers.

If $H(z)$ is the the generating function associated to a third sequence $(h_n)_{n\geq 0}$, the reader should not be very surprise to stare at the formula

$$F(z) \cdot G(z) \cdot H(z) = \sum_{n=0}^{\infty} \left\{ \sum_{p+q+r=n} f_p \cdot g_q \cdot h_r \right\} z^n\,, \tag{2.5}$$

where the indices in the middle summation again range over the nonnegative integers.

Observe that parenthesis in left-hand side above are not needed because the right-hand side in (2.5) implies that multiplication in $\mathbb{C}[[z]]$ is associative.

Whether or not a formal power series defines an analytic function, their relevance for counting problems is mainly based upon formulas (2.4) and (2.5) and the generalization of these to a higher number of factors. To see the value of this assertion consider the following problem.

Question: How many ways are there to write the number 2001 as a sum of multiples of 1, 2, 3 and 5? In a more precise way: how many four-tuples (k_1, k_2, k_3, k_5) of nonnegative integers are there such that $k_1 + 2 \cdot k_2 + 3 \cdot k_3 + 5 \cdot k_5 = 2001$?

Define

$$a_n := \text{ the number of } (k_1, k_2, k_3, k_5) \text{ such that } k_1 + 2 \cdot k_2 + 3 \cdot k_3 + 5 \cdot k_5 = n\,.$$

Although our motivating question relates only to the coefficient a_{2001} one could aspire for a formula or procedure to compute explicitly the general coefficient a_n. We may even wonder about an asymptotic formula for the coefficients a_n as $n \to \infty$, after all, one expects these coefficients to be astronomically large if n is large.

We will show that

$$\text{(2.6)} \qquad a_n \;=\; \text{the coefficient of } z^n \text{ of the Taylor series about } z = 0 \text{ of } \frac{1}{1 - z - z^2 + z^4 + z^7 - z^9 - z^{10} + z^{11}}\,.$$

In equivalent words, the generating function associated to the sequence $(a_n)_{n \geq 0}$ is the multiplicative inverse in $\mathbb{C}[[z]]$ of the polynomial

$$(1 - z - z^2 + z^4 + z^7 - z^9 - z^{10} + z^{11}).$$

To show (2.6), consider the power series [1]

$$
\begin{aligned}
F(z) &:= \sum_{k_1=0}^{\infty} z^{k_1} \cdot \sum_{k_2=0}^{\infty} z^{2k_2} \cdot \sum_{k_3=0}^{\infty} z^{3k_3} \cdot \sum_{k_5=0}^{\infty} z^{5k_5}\,, \\
&= (1-z)^{-1} \cdot (1-z^2)^{-1} \cdot (1-z^3)^{-1} \cdot (1-z^5)^{-1} \\
&= (1-z-z^2+z^4+z^7-z^9-z^{10}+z^{11})^{-1}\,.
\end{aligned}
$$

But, on the other hand, using the generalization of formulas (2.4) and (2.5) you may agree that

$$
\begin{aligned}
F(z) &= \sum_{n=0}^{\infty} \#\Big\{(k_1,k_2,k_3,k_5) : k_1+2k_2+3k_3+5k_5 = n\Big\}\, z^n\,, \\
&= \sum_{n=0}^{\infty} a_n\, z^n\,.
\end{aligned}
$$

Since two formal power series are equal if and only if they are associated to the same sequence, (2.6) follows from the above two representations for $F(z)$.

In what remains in this section we will use (2.6) to obtain more tractable expressions for a_n. For example, using that $F(z)$ is analytic near $z=0$ we have that

$$
a_n = \frac{1}{n!} \cdot \frac{d^n}{dz^n}\Big[F(z)\Big]\,. \tag{2.7}
$$

The above formula is very useful to compute a_n for small values of n. To mention some few we obtain that

$$
\begin{array}{llll}
a_0 = 1 \;, & a_1 = 1 \;, & a_2 = 2 \;, & a_3 = 3 \;, \\
a_4 = 4 \;, & a_5 = 6 \;, & a_6 = 8 \;, & a_7 = 10 \;, \\
a_8 = 13 \;, & a_9 = 16 \;, & a_{10} = 20 \;, & a_{11} = 24 \;.
\end{array} \tag{2.8}
$$

[1] The series $\sum_{k=0}^{\infty} z^{n\cdot k}$, with $n>0$ an arbitrary integer, simply represents the power series associated to the sequence $(b_k)_{k\geq 0}$ where $b_k := 1$ if n divides k, however, $b_k := 0$ otherwise. Furthermore, using the definition in (2.4) it follows that $\sum_{k=0}^{\infty} z^{n\cdot k} = \frac{1}{1-z^n}$.

However, (2.7) is not an efficient formula for large values of n. A more suitable approach to deal with the case in which n is large is to determine a linear recursion satisfied by the coefficients $(a_n)_{n\geq 0}$. Indeed, a well-known fact is that a one-dimensional power series is rational if and only if its coefficients satisfy a linear recursion.

To reveal this recursion in our particular case notice that

$$\begin{aligned} 1 &= (1 - z - z^2 + z^4 + z^7 - z^9 - z^{10} + z^{11}) \cdot F(z)\,, \\ &= \sum_{k=0}^{\infty} a_k\,(1 - z - z^2 + z^4 + z^7 - z^9 - z^{10} + z^{11})\, z^k\,. \end{aligned}$$

As a result, by recognizing the coefficient of z^n on both sides of the above identity, we obtain the formula

$$a_n = a_{n-1} + a_{n-2} - a_{n-4} - a_{n-7} + a_{n-9} + a_{n-10} - a_{n-11}\,. \tag{2.9}$$

This linear recursion together with its initial values provided in (2.8) is enough to compute explicitly the numerical value of any a_n. For example, using computer algebra for $n = 2001$ we obtain that there are $44,879,079$ ways to write the number 2001 as sum of multiples of 1, 2, 3 and 5.

An alternative approach to find a rather more explicit formula for a_n proceeds from the partial fraction decomposition of $F(z)$. Indeed if u, v are non-trivial roots of unity such that $u^3 = 1$ and $v^5 = 1$ then one determines that

$$\begin{aligned} F(z) = \frac{1}{(1-z)^4 \cdot (1+z)} &\cdot \frac{1}{(u-z)\cdot(u^2-z)} \\ &\cdot \frac{1}{(v-z)\cdot(v^2-z)\cdot(v^3-z)\cdot(v^4-z)}\,. \end{aligned}$$

The method of partial fractions then assures that there are constants A_j, B_j, C_j and D_j such that

$$(2.10)\quad F(z) = \frac{A_1}{1-z} + \frac{A_2}{(1-z)^2} + \frac{A_3}{(1-z)^3} + \frac{A_4}{(1-z)^4} + \frac{B_1}{z+1} + \frac{C_1}{u-z} + \frac{C_2}{u^2-z} + \frac{D_1}{v-z} + \frac{D_2}{v^2-z} + \frac{D_3}{v^3-z} + \frac{D_4}{v^4-z}\,.$$

Observe that

$$\begin{aligned} A_4 &= \lim_{z\to 1}(1-z)^4 \cdot F(z)\,,\\ &= \frac{1}{2\cdot(2-u-u^2)\cdot(4-v-v^2-v^3-v^4)}\,,\\ &= \frac{1}{2\cdot 3\cdot 5}\,. \end{aligned}$$

The advantage of (2.10) is that the coefficient of z^n of each term in the right-hand side is relatively simpler to determine than for $F(z)$. Indeed, by repeated differentiation of the geometric series one determines for all $a \neq 0$ and nonnegative integer $k > 0$ that

$$(2.11)\quad \begin{aligned} [z^n]\,\frac{1}{(a-z)^k} &= \frac{1}{a^{n+k}}\binom{n+k-1}{k-1}\,,\\ &\sim \frac{n^{k-1}}{a^{n+k}\cdot(k-1)!}\,. \end{aligned}$$

The above asymptotic formula is only valid as $n \to \infty$ and can be derived using Stirling's formula (see (2.52) ahead).

Using the above identity, it is now direct to read-off the the coefficient of z^n on both sides of (2.10). We obtain the explicit formula

$$(2.12)\quad a_n = \sum_{k=1}^{4} A_k \cdot \binom{n}{k-1} + n \cdot \left\{ B_1 \cdot (-1)^n + \sum_{k=1}^{2} \frac{C_k}{u^{k(n+1)}} + \sum_{k=1}^{4} \frac{D_k}{v^{k(n+1)}} \right\}.$$

Furthermore, the asymptotic formula in (2.11) implies that the term $A_4 \cdot \binom{n}{k-1}$ dominates the others terms in the above summation for large values of n. This lets us obtain that

$$a_n \sim \frac{1}{30}\binom{n}{3}, \tag{2.13}$$

as $n \to \infty$.

For example, if we use the above approximation to estimate a_{2001} we obtain that there are approximately $44,511,144$ ways to write the number 2001 as sums of multiples of 1, 2, 3 and 5. This differs from the actual number of ways, which is $44,879,079$, however, the relative error incurred in the approximation is less than 1%. Thus, in any situation where the actual value of a_n is not needed but only an approximation with a small relative error, the estimation of a_n using (2.13) instead of (2.9) or (2.12) should be preferred.

The example we discussed pretty much describes all the known methods to obtain information about rational generating functions in one variable.

It is worth to remark that none of our discussion up to (2.12) required analyticity as a main feature. Indeed, the computation of the first few coefficients a_n could have been done by hand without the need of (2.7). However, analyticity was implicitly used to obtain (2.13) because Stirling's formula is a classical application of the theory of analytic functions of one complex variable.

2.2 Automata and regular languages

An *automaton* is a mathematical model of a machine equipped with memory and capable of performing certain finite number of operations in a finite number of steps.

A first example of an automaton, the so called *Turing machine*, was conceived by Alan Mathison Turing (1912-1954) in his paper *On Computable Numbers* [Tur36]. Since then, several other models have been conceived. In this section we will focus on the so called *finite automata* which have been broadly used in the analysis of *regular languages*.

A *language* is a set of words constructed using a finite alphabet. For example, given the alphabet of two letters $\mathcal{A} := \{a, b\}$ examples of words could be aba, bb or even $\emptyset$ which is the usual notation (in language theory) to refer to the *empty word*. The *length* of a word is defined to be the number of characters it uses. Accordingly, the empty word is defined to have length zero.

Given two words u and v we will write $(u)(v)$ to denote the word obtained by concatenating u and v with u followed by v. Thus, for example, $(aba)(bb) = ababb$ and $(\emptyset)(aba) = aba$.

According to the classification of Chomsky [Cho56] a language is *regular* if it coincides with the set of all words recognized by a deterministic finite-state automaton.

For example, the language $\mathcal{L}_0$ of all the words formed with the alphabet $\{a, b\}$ that contain somewhere the string "*aba*" is a regular language (see figure 2.1). We will refer to the automaton represented in figure 2.1 as "Hal". Its so called *states* are q_0, q_1, q_2 and q_3. q_0 is called the *initial state* whereas q_3 is referred to as the *final state*.

The words recognized by Hal are by definition words formed with the edge labels read along any path in the graph that starts in the state q_0 and ends at q_3. Thus, for example, the word *baaabbababababba* is recognized by Hal for the final state is reached as soon as Hal inspects the ninth character. However, the word *baabba* is not recognized by Hal for the associated path ends in the state q_1.

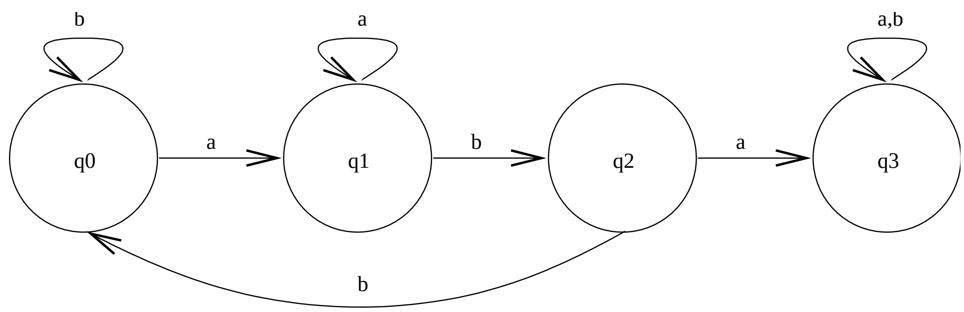

Figure 2.1: Representation of a four-state automaton via a multi-labelled graph. The associated automaton recognizes all words constructed with the alphabet $\{a, b\}$ that contain somewhere the string "*aba*".

The proof that the words recognized by Hal are only words containing the string *aba* is simple and will be omitted. For the converse, suppose that there is a word containing the string *aba* which is not recognizable by Hal. Let x be a shortest of them. x must be of length at least 3. Furthermore, x must start with an "*a*" or otherwise by erasing its first character we would obtain a shorter word still containing the string *aba*. For the same reasons the second letter must be a "*b*". The third letter cannot be an "*a*" for then $x = (aba)(y)$ and any letter of this form is certainly recognized by Hal. But the third letter cannot be a "*b*" either for then by erasing the first character of x we could find a shorter word containing the string *aba*. This is not possible because the third letter in x must be either an "*a*" or a "*b*". We have incurred in a contradiction and therefore such a word x cannot not exist.

The conclusion of the previous paragraph is that Hal recognizes only and all words contained in the (regular) language $\mathcal{L}_0$.

Question: How many words are there in $\mathcal{L}_0$ of length n?

Interestingly, and as discovered by Chomsky and Schützenberger [ChoSch63], Hal may be of great inspiration to answer this question. Before occupying ourselves in any computation we introduce some generic notation. Given an arbitrary integer $n \geq 0$ and a language $\mathcal{L}$ we will define $[\mathcal{L}]_n$ to be the number of words in $\mathcal{L}$ of length n. Accordingly, we will define $[\mathcal{L}](z) := \sum_{n=0}^{\infty} [\mathcal{L}]_n \, z^n$ and will refer to it as the *generating function associated to* $\mathcal{L}$.

Our question is equivalent to the problem determining the coefficients of $[\mathcal{L}_0](z)$. However, motivated by the inner structure of Hal, we will consider the (also regular) languages defined as

$$\begin{aligned}
\mathcal{L}_1 &:= \text{words recognized by the automaton with initial state } q_1\,, \\
\mathcal{L}_2 &:= \text{words recognized by the automaton with initial state } q_2\,, \\
\mathcal{L}_3 &:= \text{words recognized by the automaton with initial state } q_3\,,
\end{aligned}$$

and their associated generating functions $[\mathcal{L}_1](z)$, $[\mathcal{L}_2](z)$ and $[\mathcal{L}_3](z)$.

A simple inspection to figure 2.1 reveals that

$$\mathcal{L}_0 = \{a\} \times \mathcal{L}_1 + \{b\} \times \mathcal{L}_0\,, \tag{2.14}$$

$$\mathcal{L}_1 = \{a\} \times \mathcal{L}_1 + \{b\} \times \mathcal{L}_2\,, \tag{2.15}$$

$$\mathcal{L}_2 = \{a\} \times \mathcal{L}_3 + \{b\} \times \mathcal{L}_0\,. \tag{2.16}$$

The notation used above may require some clarifications. $\{a\}$ and $\{b\}$ are used to represent the languages with the single word "a" and "b" respectively. The symbol

"$\times$" stands for concatenation of languages. Thus, for example, $\mathcal{L}_1 \times \mathcal{L}_0$ denotes the language formed by all words which start with the word in $\mathcal{L}_1$ and are followed by a word in $\mathcal{L}_0$. The symbol "+" replaces the standard union sign, however, it is used to emphasize that the sets in the union are disjoint.

The set theoretical identities in (2.14), (2.15) and (2.16) translate almost directly into the following relations involving generating functions

$$[\mathcal{L}_0](z) = z \cdot [\mathcal{L}_1](z) + z \cdot [\mathcal{L}_0](z)\,, \tag{2.17}$$

$$[\mathcal{L}_1](z) = z \cdot [\mathcal{L}_1](z) + z \cdot [\mathcal{L}_2](z)\,, \tag{2.18}$$

$$[\mathcal{L}_2](z) = z \cdot [\mathcal{L}_3](z) + z \cdot [\mathcal{L}_0](z)\,. \tag{2.19}$$

We will explain only the first of these identities. A similar argument will apply to obtain the other two. First, observe that (2.14) implies for all $n \geq 1$ that

$$\begin{aligned} [\mathcal{L}_0]_n &= [\{a\} \times \mathcal{L}_1 + \{b\} \times \mathcal{L}_0]_n\,, \\ &= [\{a\} \times \mathcal{L}_1]_n + [\{b\} \times \mathcal{L}_0]_n\,, \\ &= [\mathcal{L}_1]_{n-1} + [\mathcal{L}_0]_{n-1}\,, \end{aligned}$$

where for the middle identity we have used that $\{a\} \times \mathcal{L}_1$ and $\{b\} \times \mathcal{L}_0$ are disjoint. Using that $[\mathcal{L}_0]_0 = 0$ we obtain

$$\begin{aligned} [\mathcal{L}_0](z) &= \sum_{n=1}^{\infty} [\mathcal{L}_0]_n\, z^n\,, \\ &= \sum_{n=1}^{\infty} [\mathcal{L}_1]_{n-1}\, z^n + \sum_{n=1}^{\infty} [\mathcal{L}_0]_{n-1}\, z^n\,, \\ &= z \cdot [\mathcal{L}_1](z) + z \cdot [\mathcal{L}_0](z)\,. \end{aligned}$$

This proves (2.17).

Observe that $\mathcal{L}_3$ is simply all words formed with the alphabet $\{a, b\}$. Thus, $[\mathcal{L}_3]_n = 2^n$ and hence, $[\mathcal{L}]_3(z) = \frac{1}{1-2z}$. Because of this, the identities in (2.17), (2.18) and (2.19) can be easily put into a linear form. Namely,

$$\begin{bmatrix} (1-z) & -z & 0 \\ 0 & (1-z) & -z \\ -z & 0 & 1 \end{bmatrix} \begin{bmatrix} [\mathcal{L}_0](z) \\ [\mathcal{L}_1](z) \\ [\mathcal{L}_2](z) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{z}{1-2z} \end{bmatrix}.$$

As a result, by inverting the linear system, we obtain that

$$\begin{aligned} [\mathcal{L}_0](z) &= \frac{z^3}{(z^3 - z^2 + 2z - 1)(2z-1)} \\ &= z^3 + 4z^4 + 11z^5 + 27z^6 + 63z^7 + 142z^8 + 312z^9 + 673z^{10} + \ldots \end{aligned}$$

In particular, we see that there are 11 binary words of length 5 that contain the string *aba*. These are: "*abaaa*", "*abaab*", "*ababa*", "*ababb*", "*aabaa*", "*aabab*", "*babaa*", "*babab*", "*aaaba*", "*baaba*" and "*bbaba*". Also, there are 673 binary words of length 10 containing somewhere the string *aba* although this time is not so easy to list all of them.

The first few coefficients of $[\mathcal{L}_0](z)$ can be computed by repeated differentiation. On the other hand, the method of partial fractions could be used to determine an exact formula for the coefficients of $[\mathcal{L}_0](z)$. Instead, we shall be concerned with the problem of estimating the order of $[\mathcal{L}_0]_n$, as $n \to \infty$.

It is common knowledge among generating functionologists that the closet singularity to the origin of a generating function provides a great deal of information about its coefficients. In our case, $[\mathcal{L}_0](z)$ has as many singularities as zeroes in its denominator.

These last are

$$
\begin{aligned}
z_1 &= 0.5\,,\\
z_2 &= 0.56984\ldots\,,\\
z_3 &= 0.21507\ldots + i\cdot 1.30714\ldots\,,\\
z_4 &= 0.21507\ldots - i\cdot 1.30714\ldots\,.
\end{aligned}
$$

The argument we will present to show that z_1 determines the leading asymptotic order of $[\mathcal{L}_0]_n$ should serve as model of a much more general technique. Since the analyticity of $[\mathcal{L}_0](z)$ will play a fundamental role in our approach it is common among the specialists to describe this approach as an *analytic method.* [2]

Define $\rho_0 := z_1/2$ and $\rho_1 := (z_1 + z_2)/2$ and observe that z_1 is the only singularity of $[\mathcal{L}_0](z)$ within the annulus $[z : \rho_0 \leq |z| \leq \rho_1]$ (see figure 2.2). To obtain an asymptotic formula for $[\mathcal{L}_0]_n$ we first represent this coefficient using Cauchy's formula (see [Rud87]) over the contour $[z : |z| = \rho_0]$, which we parametrize counterclockwise. The resulting integral is then compared with the integral over the contour $[z : |z| = \rho_1]$ by means of the residue theorem (see [Rud87]). In accomplishing this plan we obtain that

$$
\begin{aligned}
[\mathcal{L}_0]_n &= \frac{1}{2\pi i}\int_{|z|=\rho_0}\frac{[\mathcal{L}_0](z)}{z^{n+1}}\,dz\,,\\
&= \operatorname{Res}\left(-\frac{[\mathcal{L}_0](z)}{z^{n+1}}, z = z_1\right) + \frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{[\mathcal{L}_0](z)}{z^{n+1}}\,dz\,,\\
&= 2^n + \frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{[\mathcal{L}_0](z)}{z^{n+1}}\,dz\,.
\end{aligned}
\tag{2.20}
$$

[2] The method of partial fractions is much more efficient to determine the leading asymptotic order of the coefficients of a rational generating function of one variable. However, analytic methods can be used to deal more generally with meromorphic functions even in the realm of several variables.

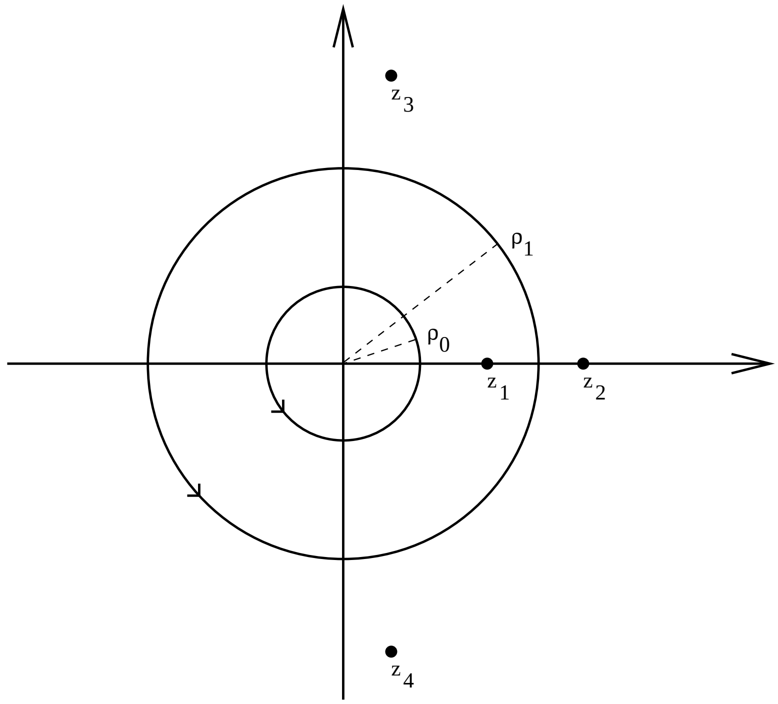

Figure 2.2: $[\mathcal{L}_0](z)$ is analytic everywhere except at $z = z_1$, z_2, z_3, and z_4. Since z_1 is the closet singularity to the origin, the leading asymptotic order of $[\mathcal{L}_0]_n$ coincides with the residue of $\frac{[\mathcal{L}_0](z)}{z^{n+1}}$ at $z = z_1$.

The hope is that the term produced by the residue in (2.20) represents the leading asymptotic order of $[\mathcal{L}_0]_n$, as $n \to \infty$. To verify this, parametrize $[z : |z| = \rho_1]$ letting $z = \rho_1 e^{i\theta}$, with $\theta \in [0, 2\pi]$. One then determines that

$$\left| \frac{1}{2\pi i} \int_{|z|=\rho_1} \frac{[\mathcal{L}_0](z)}{z^{n+1}} \, dz \right| \leq c \cdot \rho_1^{-n} \, ,$$

where c is the maximum value of $|[\mathcal{L}_0](z)|$ as z ranges over the circle of radius ρ_1. This maximum value is finite because $[\mathcal{L}_0](z)$ is a continuous function of z over the circle of radius ρ_1.

Using asymptotic notation, the above inequality implies that

$$\frac{1}{2\pi i}\int_{|z|=\rho_1} \frac{[\mathcal{L}_0](z)}{z^{n+1}}\,dz = O\left(\rho_1^{-n}\right) ,$$

as $n \to \infty$.

As a result, the last formula in (2.20) can be written in the more compact form

$$[\mathcal{L}_0]_n = 2^n \cdot \left[1 + O\left(\left\{\frac{\rho_1}{2}\right\}^{-n}\right)\right] ,$$

as $n \to \infty$. The $O\left(\left\{\frac{\rho_1}{2}\right\}^{-n}\right)$ represents a term which up to a constant factor, which is independent of n, is bounded by $\left\{\frac{\rho_1}{2}\right\}^{-n}$. (See section 3.1 for further reference on the meaning and uses of the big-O symbol.)

Since $\rho_1 < 2$, it follows that $\left\{\frac{\rho_1}{2}\right\}^{-n} \to 0$, as $n \to \infty$, and therefore the

(2.21) $$\lim_{n\to\infty} \frac{[\mathcal{L}_0]_n}{2^n} = 1 .$$

In probabilistic terms the finding in (2.21) is almost obvious. Indeed, the same conclusion could be obtained using Kolmogorov's 0-1 law (see [Dur95]). A more intuitive explanation goes as follows. Since there are a total of 2^n binary words of length n, the probability of picking any one of them at random is $\frac{1}{2^n}$. Therefore, the ratio in the above limit represents the probability of picking up at random a word containing the pattern *aba* among all binary words of size n. Intuitively, if n is big, it is extremely unlikely to not observe the pattern *aba* within a random word of size n. Conversely, if a word of a very big size is chosen at random (among all words of the same size) then it is very likely to observe the pattern *aba* within it. The above limit shows that this intuition is correct.

2.3 Plane trees, algebraic generating functions and the stationary phase method.

A *tree* is an undirected graph which is connected and free of cycles. The *size* of a tree is defined to be its number of nodes. A *plane tree* or also called *ordered tree* is a tree that can be embedded in the plane and has a distinguished node called *root*. More precisely, these are trees with a root where the order given to the subtrees dangling from any node is taken into account. (For a compact reference about the terminology used in graph theory see the Appendix section in [Sta86].)

For example, in figure 2.3, T_3 and T_4 count as different plane trees although, thought of as undirected graphs, they would be equal.

Question: How many plane trees are there of size n?

Let G_n be the number of plane trees of size n and $G(z) := \sum_{n=1}^{\infty} G_n\, z^n$ be the generating function associated to these coefficients. It turns out that the recursive structure of plane trees will translates into a functional equation for $G(z)$. Indeed, we will show that

$$\big\{G(z)\big\}^2 - G(z) + z = 0\,. \tag{2.22}$$

In particular, $G(z)$ is *algebraic* meaning by this it is the solution of a polynomial equation with coefficients in $\mathbb{C}[z]$ i.e. the ring of polynomials in the variable z with complex coefficients.

We will not enter into a deep discussion on algebraic generating functions, however, we want to emphasize that they have been and are nowadays subject of active research. (See section 6 in [Sta99].)

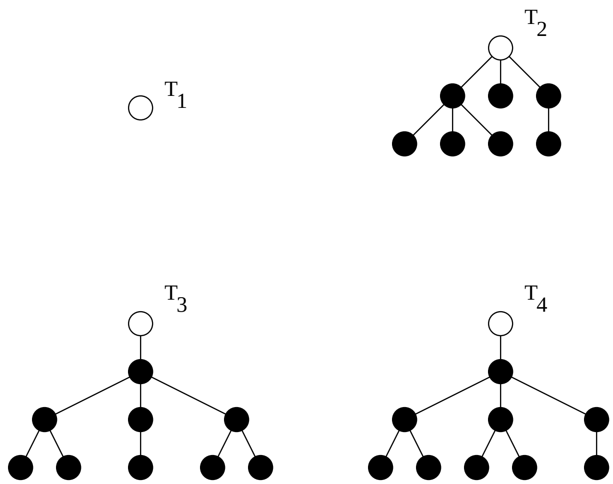

Figure 2.3: Examples of plane trees of various sizes. T_1 is of size 1. T_2 is of size 8. Observe that T_3 and T_4 count as different plane trees of size 10 despite that, as graphs, they are equivalent.

If in (2.22) we complete the square and use that there are no plane trees of size 0, in other words, $G(0) = 0$, we can determine that

$$G(z) = \frac{1 - \sqrt{1-4z}}{2}\,. \tag{2.23}$$

The binomial formula states that

$$[z^n]\,(1 + a \cdot z)^b = a^n \cdot \frac{b(b-1) \cdot \ldots \cdot (b-n+1)}{n!}\,,$$

provided that n is a nonnegative integer and $a, b \in \mathbb{C}$ are nonzero. If in (2.23) we recognize the coefficient of z^n on both sides, the binomial formula lets us conclude, for all $n \geq 1$, that

$$G_n \quad = \quad \frac{(2n-2)!}{n!\,(n-1)!} = \frac{1}{n}\binom{2n-2}{n-1}\,. \tag{2.24}$$

The above coefficients are known as *Catalan numbers*.

To clarify the functional equation in (2.22) we will count the number of plane trees partitioning them according the number of subtrees dangling from the root.

We will let $G_n(k)$ to be the number of trees of size n which have k subtrees dangling from the root. Since the order in which these k subtrees are placed below the root matters, it should be clear, for all $n \geq 1$, that

$$\begin{aligned} G_n(k) &= \sum_{n_1+\ldots+n_k=(n-1)} G_{n_1} \cdot \ldots \cdot G_{n_k} , \\ &= [z^{n-1}] \left\{ \sum_{n_1=1}^{\infty} G_{n_1} z^{n_1} \right\} \cdot \ldots \cdot \left\{ \sum_{n_k=1}^{\infty} G_{n_k} z^{n_k} \right\} , \\ &= [z^{n-1}]\{G(z)\}^k . \end{aligned}$$

Therefore

$$\begin{aligned} [z^n]G(z) &= \sum_{k=1}^{\infty} [z^{n-1}]\{G(z)\}^k , \\ &= [z^n]\, z \cdot \sum_{k=1}^{\infty} \{G(z)\}^k , \\ &= [z^n] \frac{z}{1-G(z)} . \end{aligned}$$

Observe that the above identity not only holds for all $n \geq 1$ but by inspection also for $n = 0$. Consequently, $G(z) = \frac{1}{1-G(z)}$ and this proves (2.22).

The above computations are not as trivial as they may seem at a first glance. The condition $G(0) = 0$ is required to ensure that the $\sum_{k=1}^{\infty}\{G(z)\}^k$ is effectively a power series. Indeed, it implies that $[z^n]\{G(z)\}^k = 0$, for all $k > n$. As a result, the coefficient of z^n in $\sum_{k=1}^{\infty}\{G(z)\}^k$ is well-defined because it is a finite sum of nonzero terms. [3] It is now matter of routine to check that $\big(1 - G(z)\big)$ and $\sum_{k=1}^{\infty}\{G(z)\}^k$ are

[3] Otherwise, if there were infinitely many nonzero terms to consider in the summation the series would diverge for its coefficients are known to be nonnegative integer numbers. A deeper discussion concerning this remark will be given in section 2.5 but in the more general context of *bivariate power series*.

multiplicative inverses in $\mathbb{C}[[z]]$.

The procedure we have just exposed to determine $G(z)$ can be applied to more restricted classes of plane trees. For example, consider the class of *unary-ternary trees*. These are plane trees where each node has either 0, 1 or 3 nodes dangling from it.

Question: How many unary-ternary trees are there of size n?

Let U_n denote this number and let $U(z) := \sum_{n=1}^{\infty} U_n z^n$ be their associated generating function. As you may expect $U(z)$ is algebraic and we would like to determine the functional equation it solves. We will do this by partitioning the set of unary-ternary trees into three sets. Namely those with none, one or three subtrees dangling from the root node.

Define $U(k, n)$ with $k = 0, 1$ or 3 to be number of unary-ternary trees of size n with k subtrees dangling from the root. Accordingly, we set $U_k(z) := \sum_{n=1}^{\infty} U(k, n) \cdot z^n$.

Since there is only one unary-ternary tree (which happens to be of size 1) with no subtrees dangling from its root node then $U_0(z) = z$. On the other hand, it should be clear that there as many unary-ternary trees of size n with only one tree dangling from the root as there are unary-ternary trees of size $(n-1)$. Thus $U(1, n) = U_{n-1}$, for all $n \geq 2$, and therefore $U_1(z) = z \cdot U(z)$. Finally, following the same outline used in the previous discussion on plane trees, we obtain that

$$\begin{aligned}
[z^n]\, U_3(z) &= [z^{n-1}] \left\{ \sum_{i=1}^{\infty} U_i\, z^i \right\} \cdot \left\{ \sum_{j=1}^{\infty} U_j\, z^j \right\} \cdot \left\{ \sum_{k=1}^{\infty} U_k\, z^k \right\}, \\
&= [z^{n-1}] \, \{U(z)\}^3, \\
&= [z^n] \, z\{U(z)\}^3.
\end{aligned}$$

Therefore, $U_3(z) = z \cdot \{U(z)\}^3$.

Summarizing, $U_0(z) = z$, $U_1(z) = z \cdot U(z)$ and $U_3(z) = z \cdot \{U(z)\}^3$. But, due to the disjointness of the three classes of unary-ternary trees considered, it should be clear that $U(z) = U_0(z) + U_1(z) + U_3(z)$. This implies that $U(z)$ is a solution of the functional equation

$$U(z) = z \cdot \Phi(U(z))\,, \tag{2.25}$$

with $\Phi(u) := 1 + u + u^3$. This functional equation cannot be solved as directly as the one in (2.22). One attempt to find $U(z)$ would be to solve (2.25) selecting the proper branch which produces $U(z) = z + z^2 + z^3 + ...$ because there is only one unary-ternary tree of size 1, 2 and 3 respectively. However, it is will prove more fruitful to explore a rather more analytical approach.

For a functional equation of the form in (2.25) there is a standard way to relate the coefficients of $U(z)$ to the coefficients of $\Phi(u)$, this last regarded as a power series in u. The *Lagrange inversion formula* (see section 5.4 in [Sta99]) asserts that

$$U_n = \frac{1}{n}\,[u^{n-1}]\,\{\Phi(u)\}^n\,. \tag{2.26}$$

For relatively small values of n one may easily determine an exact numerical value for U_n using (2.26). However, a manageable formula for U_n is not readily from (2.26) and this identity only constitutes an intermediate step to find more explicit information about the coefficients U_n.

The demonstration of (2.26) proceeds as follows. Observe that $U(z)$ must be analytic in some neighborhood of the origin after all $0 \le U_n \le G_n$ and $G(z)$ itself is analytic near the origin. On the other hand, $U_n = \frac{1}{n}[z^{n-1}]\,U'(z)$. (This last identity is actually true for an arbitrary power series.) Representing $[z^{n-1}]\,U'(z)$ as an integral using Cauchy's formula (see [Rud87]) and then using (2.25) we find that

$$
\begin{aligned}
U_n &= \frac{1}{n} \cdot \frac{1}{2\pi i} \int_\gamma \frac{U'(z)}{z^n}\, dz\,, \\
&= \frac{1}{n} \cdot \frac{1}{2\pi i} \int_\gamma \left\{ \frac{\Phi(U(z))}{U(z)} \right\}^n \cdot U'(z)\, dz\,,
\end{aligned}
$$

where γ is any circle of a sufficiently small radius so that it is contained in the domain of convergence $U(z)$. If we then substitute: $u = U(z)$, then it follows that

$$
U_n = \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{U(\gamma)} \left\{ \frac{\phi(u)}{u} \right\}^n du\,.
$$

The Lagrange inversion formula follows from the fact that $U(\gamma)$ above is a closed contour having winding number equal to one about $u = 0$. This proves (2.26).

As a result, for all $R > 0$ we find that

$$
U_n = \frac{1}{n} \cdot \frac{1}{2\pi i} \int_{|u|=R} \{g(u)\}^n\, du\,, \tag{2.27}
$$

where we have defined

$$
\begin{aligned}
g(u) &:= \frac{\Phi(u)}{u}\,, \\
&= \frac{1}{u} + 1 + u^2\,.
\end{aligned}
$$

The problem of determining the asymptotic behavior of U_n is therefore equivalent to the understanding of the asymptotic behavior of the integral in (2.27). But, it should be expected that for large values of n the main contribution to the integral in (2.27) comes from integration near those u's which maximize the modulus of $|g(u)|$ along the circle $[u : |u| = R]$. To characterize these points observe that

$$
g(u\, e^{i\theta}) = g(u) \cdot \exp\left\{ (\mathcal{L}g)(u) \cdot \theta + \dots \right\},
$$

where we have defined

$$\mathcal{L}g(u) := i\,\frac{u\cdot g'(u)}{g(u)}\,,$$

provided that $g(u)\neq 0$. Thus, if u is a local maximum of $|g(u)|$, with $|u|=R$, then the $\Re\{\mathcal{L}g(u)\}=0$. This is a "first order condition" satisfied by all local maximum of $|g(u)|$ along the circle $[u:|u|=R]$.

This condition is certainly satisfied wherever $g'(u)=0$ and $g(u)\neq 0$. In our context, a point of these characteristics is $r := \frac{1}{\sqrt[3]{2}}$ and, as can be seen from figure 2.4, it turns out that $u=r$ maximizes the $|g(u)|$ along the circle $[u:|u|=r]$. This motivates to select $R=r$ in (2.27). Furthermore, if in (2.27) we normalize the integrand by $\{g(r)\}^n$ and parametrize the contour of integration as $u=re^{i\theta}$, with $\theta\in[-\pi,\pi]$, then we obtain that

$$U_n = \frac{r\cdot\{g(r)\}^n}{2\pi n}\cdot\int_{-\pi}^{\pi}\left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n e^{i\theta}\,d\theta\,. \tag{2.28}$$

To estimate the above integral we do not need to integrate all the way from $-\pi$ to π. Indeed, if we define $m(\epsilon) := \min\limits_{\theta:\epsilon\leq|\theta|\leq\pi}\ln\left|\frac{g(r)}{g(re^{i\theta})}\right|$ then

$$\int_{-\pi}^{\pi}\left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n e^{i\theta}\,d\theta = \int_{-\epsilon}^{\epsilon}\left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n e^{i\theta}\,d\theta + O\left(e^{-n\cdot m(\epsilon)}\right)\,, \tag{2.29}$$

for all $n\geq 0$. Observe that $m(\epsilon)>0$ because, for all nonzero $\theta\in[-\pi,\pi]$, $\left|\frac{g(r)}{g(re^{i\theta})}\right|>1$.

The identity in (2.29) will be of practical use only if the integral on the left-hand side is comparable in size to the integral on the right-hand side; that is, if the part of the original integral we have neglected tends to be of a much smaller size (for large values of n) than the part of the integral we have kept.

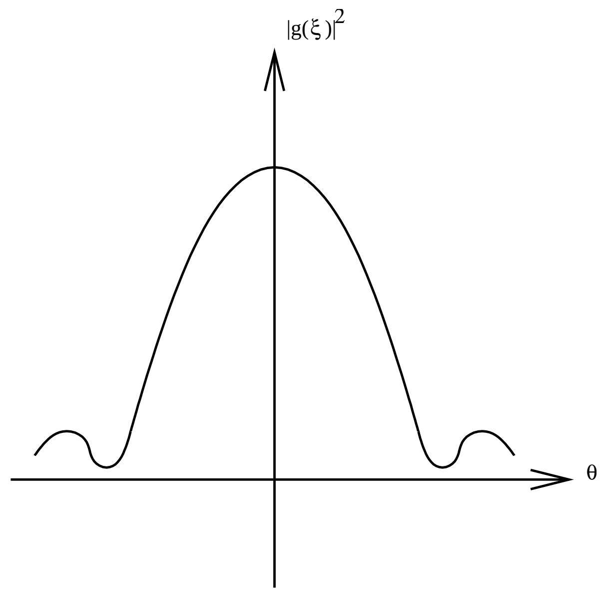

Figure 2.4: Plot of $|g(\xi)|^2$ with $\xi := r\,e^{i\theta}$ and $\theta \in [-\pi, \pi]$. The graph shows that $u = r$ maximizes the $|g(u)|$ along the circle $[u : |u| = r]$.

The estimation of the integral on the right-hand side in (2.29) will be outlined using the *stationary phase method* (see theorem 5.2 and corollary 5.3 in [PemWil01]). We first rewrite

$$\begin{aligned} \left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n \cdot e^{i\theta} &= \exp\left(-n \cdot \ln\left\{\frac{g(r)}{g(re^{i\theta})}\right\}\right) \cdot e^{i\theta}\,, \\ &= \exp\left(-n \cdot \left\{\frac{3}{3+2r}\,\theta^2 + \ldots\right\}\right) \cdot (1 + \ldots)\,, \end{aligned}$$

where, on the second identity, we have made explicit the first nontrivial terms of the Taylor series of $\ln\left\{\frac{g(r)}{g(re^{i\theta})}\right\}$ and $e^{i\theta}$ about $\theta = 0$.

While the Taylor series of $e^{i\theta}$ is known to converge for all values of θ, the series for $\ln\left\{\frac{g(r)}{g(re^{i\theta})}\right\}$ a priori is only known to converge if θ is sufficiently small. The advantage of (2.29) is precisely that we may chose $\epsilon > 0$ small enough to ensure the convergence of this last series for all θ in the disk $[\theta : |\theta| \leq \epsilon]$. The stationary phase method then asserts that to obtain a good approximation of the integral between $[-\epsilon, \epsilon]$ it is enough to consider the leading term in each of the determined series. More precisely, (see section 3.1 for clarifications on the notation) it lets us conclude that

$$\begin{aligned}\int_{-\epsilon}^{\epsilon} \left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n e^{i\theta}\, d\theta \quad &\sim \quad \int_{-\epsilon}^{\epsilon} \exp\left(-\frac{3n}{3+2r}\,\theta^2\right)\, d\theta\,, \\ &= \quad \sqrt{\frac{3+2r}{6n}} \cdot \int_{-\epsilon(n)}^{+\epsilon(n)} e^{-\tau^2/2}\, d\theta\,, \\ &\sim \quad \sqrt{\frac{\pi(3+2r)}{3n}}\,.\end{aligned}$$

Above, for the last identity, we have used that $\epsilon(n) := \epsilon \cdot \sqrt{\frac{6n}{3+2r}} \to \infty$, as $n \to \infty$, and the well-known fact that the $\int_{-\infty}^{\infty} e^{-\tau^2/2} d\tau = \sqrt{2\pi}$.

Thus, the $\int_{-\epsilon}^{\epsilon} \left\{\frac{g(re^{i\theta})}{g(r)}\right\}^n e^{i\theta}\, d\theta$ is of order $n^{-1/2}$. In particular, it is the leading order on the right-hand side in (2.29). As a result, using (2.28) we obtain the asymptotic formula

$$U_n \sim \frac{r}{2} \cdot \sqrt{\frac{3+2r}{3\pi}} \cdot n^{-3/2} \cdot \{g(r)\}^n\,, \tag{2.30}$$

as $n \to \infty$.

The above formula estimates that there are 356 unary-ternary trees of size 10. However, the Lagrange inversion formula implies that there are exactly 349 of these trees. The relative error is less than 2%. On the other hand, there are exactly 86,236913,825615,976816 unary-ternary trees of size 50. If instead we use (2.30) to aproximate this number we obtain 86,660613,708482,684088 and the relative error in the estimation is this time less that 0.5%. Formula (2.30) states that the relative error will approach zero as $n \to \infty$. However, even for a small number like $n = 50$ this relative error is surprisingly small.

2.4 Symbolic combinatorics

The success of Generating function methods in studying combinatorial quantities is mainly due to the way summation and multiplication of power series has been defined. In this section we will support this claim from the more abstract but very general point of view of the so called *symbolic combinatorics*.

Symbolic combinatorics deals with *combinatorial classes* where a notion of *size* is associated to each member in the class. For example, a combinatorial class could be the set of unary-ternary trees where the size of a tree is defined as its number of vertices.

In a more abstract setting consider two arbitrary combinatorial classes $\mathcal{F}$ and $\mathcal{G}$. We model the notion of size in each class prescribing the existence nonnegative integer-valued functions $|\cdot|_{\mathcal{F}}$ and $|\cdot|_{\mathcal{G}}$ defined on $\mathcal{F}$ and $\mathcal{G}$ respectively. Letting the symbol $\#\mathcal{A}$ to stand for the cardinality of an arbitrary set $\mathcal{A}$, consider the quantities

$$\begin{aligned} F_n &:= \#\Big\{f \in \mathcal{F} : |f|_{\mathcal{F}} = n\Big\}, \\ G_n &:= \#\Big\{g \in \mathcal{G} : |g|_{\mathcal{G}} = n\Big\}. \end{aligned}$$

Thus, F_n and G_n count the number of elements of size n in $\mathcal{F}$ and $\mathcal{G}$ respectively. We will assume that these coefficients are finite for all $n \geq 0$. This lets us define the power series

$$\begin{aligned} F(z) &:= \sum_{n=0}^{\infty} F_n\, z^n, \\ G(z) &:= \sum_{n=0}^{\infty} G_n\, z^n, \end{aligned}$$

to which we will refer to as the *generating function associated to (the combinatorial class)* $\mathcal{F}$ *and* $\mathcal{G}$ respectively. The process of assigning a generating function to a combinatorial class in terms of a size function is a standard procedure in symbolic combinatorics. The spirit of symbolic combinatorics as a method for enumeration is that often the generating function associated to an elaborated combinatorial class can be related in a very explicit way to the generating functions associated to simpler classes (usually, subclasses). Indeed, very natural set theoretical operations using combinatorial classes translate into algebraic manipulations of their associated generating functions.

As a simple example suppose that the combinatorial classes $\mathcal{F}$ and $\mathcal{G}$ are disjoint. The *Union class of* $\mathcal{F}$ *and* $\mathcal{G}$ is defined to be the set $\mathcal{F} \oplus \mathcal{G}$, where the symbol $\oplus$ is used instead of the more standard union sign to emphasize that the sets participating in the union are disjoint. A natural size function on $\mathcal{F} \oplus \mathcal{G}$ is

$$|h| := \begin{cases} |h|_{\mathcal{F}} \quad , \quad h \in \mathcal{F} \\ |h|_{\mathcal{G}} \quad , \quad h \in \mathcal{G} \end{cases} .$$

If $(F \oplus G)(z)$ denotes the generating function associated to $\mathcal{F} \oplus \mathcal{G}$, then it should be clear that

$$(F \oplus G)(z) = F(z) + G(z) . \tag{2.31}$$

As another example consider the *Product class of* $\mathcal{F}$ *and* $\mathcal{G}$ defined to be cartesian product $\mathcal{F} \times \mathcal{G}$. Observe that the former hypothesis of disjointness is no longer needed. The size of a generic element (f, g) in $\mathcal{F} \times \mathcal{G}$ will be defined naturally as

$$|(f, g)| := |f|_{\mathcal{F}} + |g|_{\mathcal{G}} .$$

If $(F \times G)(z)$ denotes the generating function associated to $\mathcal{F} \times \mathcal{G}$ then a simple calculation reveals that

$$(F \times G)(z) = F(z) \cdot G(z) . \tag{2.32}$$

The above formula generalizes to an arbitrary number of combinatorial classes. Thus, for example, given an integer $k \geq 1$, the generating function associated to the class $\mathcal{F}^k$ (where the size of a k^{th}-tuple is defined to be the summation of the sizes of each coordinate) is $\{F(z)\}^k$.

The remarkable fact that the set theoretical constructions $\mathcal{F} \oplus \mathcal{G}$ and $\mathcal{F} \times \mathcal{G}$ translate into the setting of generating functions to the relations in (2.31) and (2.32) is the foundation of symbolic combinatorics. These two relations set the bases to consider quite more complicated set theoretical constructions.

For example, consider a combinatorial class $\mathcal{F}$ with no elements of size 0. (The class of planar trees would satisfy this requirement.) The *Sequence class associated to* $\mathcal{F}$ will be defined to be the set

$$\mathcal{S}(\mathcal{F}) := \{\star\} \oplus \bigoplus_{k=1}^{\infty} \mathcal{F}^k .$$

Above, $\{\star\}$ is used to represent a combinatorial class containing an element "$\star$" which by definition is of size 0 and is disjoint from the class $\mathcal{F}$ itself and all its cartesian powers.

Question: What is the generating function associated to $\mathcal{S}(\mathcal{F})$?

We will denote this generating function as $\mathcal{S}(F)(z)$.

Observe that "$\star$" is the only element of size 0 in $\mathcal{S}(\mathcal{F})$. On the other hand, the number of k-tuples of size n in $\mathcal{S}(F)$ is the coefficient of z^n in $\{F(z)\}^k$. As a result, the number of elements of size n in $\mathcal{S}(\mathcal{F})$ is the $\sum_{k=0}^{\infty} [z^n]\{F(z)\}^k$. This summation is indeed finite for all the terms with $k > n$ vanish. This is because all k-tuples are of size at least k for there are no elements of size 0 in $\mathcal{F}$. We deduce that

$$\mathcal{S}(F)(z) = \frac{1}{1 - F(z)} .$$

The above identity trivializes most of our discussion on plane trees in section 2.3. If $\mathcal{T}$ is used to denote the combinatorial class of plane trees then our findings in there imply that

$$T(z) = \frac{1 - \sqrt{1-4z}}{2}\,. \tag{2.33}$$

From the point of view of symbolic combinatorics a plane tree with k subtrees dangling from its root node can be seen as an ordered pair formed with the root node followed by a k-tuple of trees. If $\{\circ\}$ is used to denote the combinatorial class containing root node then

$$\mathcal{G} = \{\circ\} \times \mathcal{S}(\mathcal{G})\,. \tag{2.34}$$

This equality is not in the standard set theoretical sense but rather as combinatorial classes. It means that the left and right hand side sets have exactly the same number of elements of size n. This is good enough to conclude that the power series associated to the combinatorial class on the left is the same as the one associated to the class on the right. Since the generating function associated to $\{\circ\}$ is z we deduce that

$$T(z) = z \cdot \frac{1}{1 - T(z)}\,.$$

This shows that $T(z)$ is algebraic and (2.33) is the only solution to this equation satisfying $T(0) = 0$.

The identity in (2.34) is a form of specification of the class of plane trees. Indeed, in the jargon of symbolic combinatorics, it is described as a *combinatorial specification of the class* $\mathcal{T}$. Not surprisingly, many generating functions associated to a class with a similar kind of combinatorial specification are algebraic.

To finalize our incursion into symbolic combinatorics we will consider one more construction. Given a combinatorial class $\mathcal{F}$ with no elements of size 0, the *Multiset class associated to* $\mathcal{F}$ is the combinatorial class formed by all multi-subsets of $\mathcal{F}$. [4] We will denote this class with the script $\mathcal{M}(\mathcal{F})$. The size of a multiset is defined to be the summation of the sizes of its elements.

For example, consider $\mathcal{M}(\{\bullet, \blacktriangle\})$ and suppose that the size of "$\bullet$" is defined to be 1 whereas "$\blacktriangle$" has size 3. Examples of multi-subsets are $[\blacktriangle]$, $[\blacktriangle, \bullet, \blacktriangle]$ or even $[\bullet, \bullet, \bullet, \bullet]$. They are respectively of sizes 3, 7 and 4. Since the order in which the elements of a multi-subset are listed does not matter we can think of a multi-subset of $\{\bullet, \blacktriangle\}$ as an element in the product class $\mathcal{S}(\{\bullet\}) \times \mathcal{S}(\{\blacktriangle\})$. After all, if "$\star$" is used to denote the added element of size 0 in $\mathcal{S}(\{\bullet\})$ and $\mathcal{S}(\{\blacktriangle\})$ then we have the correspondences

$$\begin{aligned} [\blacktriangle] &\longleftrightarrow (\star, (\blacktriangle))\,, \\ [\blacktriangle, \bullet, \blacktriangle] &\longleftrightarrow ((\bullet), (\blacktriangle, \blacktriangle))\,, \\ [\bullet, \bullet, \bullet, \bullet] &\longleftrightarrow ((\bullet, \bullet, \bullet, \bullet), \star)\,. \end{aligned}$$

This implies the combinatorial specification

$$\mathcal{M}(\{\bullet, \blacktriangle\}) = \mathcal{S}(\{\bullet\}) \times \mathcal{S}(\{\blacktriangle\})\,.$$

In particular, since the generating function associated to $\mathcal{S}(\{\bullet\})$ and $\mathcal{S}(\{\blacktriangle\})$ are respectively $\frac{1}{1-z}$ and $\frac{1}{1-z^3}$, then $\frac{1}{(1-z)\cdot(1-z^3)}$ is the generating function associated to $\mathcal{M}(\{\bullet, \blacktriangle\})$.

[4] A *multi-subset* is like a finite subset in the sense that the order in which its elements are listed does not matter, however, repetition of elements is allowed.

The above computation generalizes trivially to consider a more abstract case. Suppose that $\mathcal{F}$ is a finite combinatorial class and let $|f|$ denote the size of a generic element $f \in \mathcal{F}$. Let $F(z)$ and $\mathcal{M}(F)(z)$ respectively denote the generating function associated to $\mathcal{F}$ and $\mathcal{M}(\mathcal{F})$. Then, the combinatorial specification: $\mathcal{M}(\mathcal{F}) = \prod_{f\in\mathcal{F}} \mathcal{S}(\{f\})$ implies that

$$\mathcal{M}(F)(z) = \prod_{f\in\mathcal{F}} \frac{1}{1-z^{|f|}}\,.$$

The above formula does not establish yet a clear relation between $\mathcal{M}(F)(z)$ and $F(z)$. To make this relation explicit observe that in the above product the factor $(1-z^n)$ appears as many times as there are elements of size n in $\mathcal{F}$. If this number is accordingly denoted as F_n then

$$\begin{aligned}\mathcal{M}(F)(z) &= \prod_{n=0}^{\infty}\left(\frac{1}{1-z^n}\right)^{F_n}, \\ &= \exp\left\{\sum_{n=0}^{\infty} F_n\cdot\log\left(\frac{1}{1-z^n}\right)\right\}.\end{aligned}$$

Since $\mathcal{F}$ is finite, the summation within the exponential above involves only a finite number of terms. Finally, since $\log\left(\frac{1}{1-w}\right) = w + \frac{w^2}{2} + \frac{w^3}{3} + \ldots$, we obtain that

$$\sum_{n=0}^{\infty} F_n\cdot\log\left(\frac{1}{1-z^n}\right) = \sum_{k=1}^{\infty}\frac{1}{k}\cdot\left\{\sum_{n=0}^{\infty} F_n\cdot z^{n\cdot k}\right\}.$$

The sought formula is thus obtained to be

$$\mathcal{M}(F)(z) = \exp\left\{\sum_{k=1}^{\infty}\frac{F(z^k)}{k}\right\}.$$

The revealed relation between $\mathcal{M}(F)(z)$ and $F(z)$ is very different from what one could have expected. However, it is a fascinating relation. It is the first non-trivial relation we have encountered relating the generating function of a given combinatorial class and one of its derived classes.

We anticipated that many combinatorial classes will have generating function that are algebraic. However, our discussion on multi-sets shows that there is a lot more to be found.

As an application, we will consider the class of *unordered plane trees*. These are rooted-trees where the order of the subtrees dangling from a node does not matter. The notion of size is the same as for plane trees. Hence, for a given size, there are much fewer unordered plane trees than plane trees itself. For example, the unordered plane trees displayed in figure 2.5 are all equal.

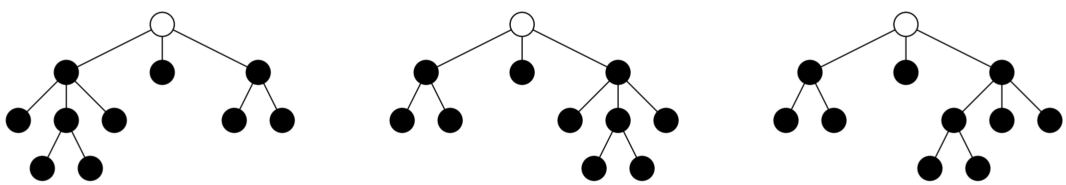

Figure 2.5: Three different representations of the same unordered plane tree.

We will denote the class of unordered plane trees with the script $\mathcal{U}$. Accordingly, $U(z)$ will denote the associated generating function. In contrast with (2.34), $\mathcal{U}$ has the combinatorial specification $\mathcal{U} = \{\circ\} \times \mathcal{M}(\mathcal{U})$, where $\{\circ\}$ represents again the combinatorial class containing the root node. As a result, we obtain that

$$U(z) = z \cdot \exp\left\{U(z) + \frac{U(z^2)}{2} + \frac{U(z^3)}{3} + \dots\right\}. \tag{2.35}$$

To determine a formula in closed form for $U(z)$ from the above relation seems not possible. However, it may be used to establish a recurrence for its coefficients. Letting $U_n := [z^n]\, U(z)$ it should be clear that $U_0 = 0$ and $U_1 = 1$. Thus, $U(z) = z + \ldots$ Consider the auxiliary power series

$$\begin{aligned} W(z) &:= \frac{U(z) - z}{z}\,, \\ &= \sum_{j=1}^{\infty} U_{j+1}\, z^j\,. \end{aligned}$$

A simple algebra manipulation let us rewrite (2.35) in the more appropriate form

$$\log\Big\{1 + W(z)\Big\} = \sum_{k=1}^{\infty} \frac{U(z^k)}{k}\,.$$

Using that: $\log(1+w) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{w^k}{k}$, we obtain that

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \cdot \{W(z)\}^k = \sum_{k=1}^{\infty} \frac{1}{k} \cdot U(z^k)\,. \tag{2.36}$$

In the above form it is simple to recognize the coefficient of z^n on both sides. First observe that we do not need to consider all terms on the summations: since $W(z) = z + \ldots$ and $U(z) = z + \ldots$ then z^k is the smallest power of z appearing in the series of $\{W(z)\}^k$ and $U(z^k)$. Since the coefficient of z^n in $U(z^k)$ is $U_{\frac{n}{k}}$ provided that k divides n (we will write $k|n$ to mean that k divides n) the above identity implies, for all $n \geq 0$, that

$$U_{n+1} - \frac{1}{2} \cdot \left\{ \sum_{j_1+j_2=(n+2)} U_{j_1} \cdot U_{j_2} \right\} + \ldots + \frac{(-1)^{n+1}}{n} \cdot \left\{ \sum_{j_1+\ldots+j_n=2n} U_{j_1} \cdot \ldots \cdot U_{j_n} \right\} = \sum_{k:k|n} \frac{1}{k} \cdot U_{\frac{n}{k}}\,.$$

The subscripts j_l all satisfy the inequality $j_l \geq 2$; in particular, $j_l \leq n$. As a result, U_{n+1} can be defined recursively in terms of $U_1, \ldots, U_n$ allowing, at least theoretically, the exact computation of these coefficients for arbitrarily large values of n. This does not resolve the problem of determining the asymptotic behavior of U_n for large values of n. However, as remarked by Flajolet and Sedgewick (see [FlaSed93], section 1.6), Polya discovered (using singularity analysis) that

$$U_n \sim C \cdot \frac{A^n}{n^{3/2}} ,$$

for some appropriate constants C and $A > 0$.

2.5 Bivariate power series

The natural generalization of power series are the so called bivariate series. Our interest in them is not just theoretical. Indeed, like regular one variable power series, bivariate power series can be, for example, of great use in enumeration problems. A *formal bivariate power series* is of the form

$$F(z, w) := \sum_{r,s \geq 0} f_{r,s} \, z^r \, w^s .$$

Above the indeterminate z and w are complex variables and each coefficient $f_{r,s}$ is a complex number. Conversely, given an array of complex numbers $(f_{r,s})_{r,s\geq 0}$ we will refer to $F(z, w)$ as the *power series* or *generating function associated to the array* $(f_{r,s})_{r,s\geq 0}$.

The set of bivariate power series in the indeterminates z and w will be denoted $\mathbb{C}[[z, w]]$. Two bivariate series will be said to be equal if and only if they have the same coefficients. This lets us define $[z^r \, w^s] \, F(z, w)$ or more briefly $[z^r \, w^s] \, F$ to be the coefficient of $z^r \, w^s$ in the series $F(z, w)$. The special notation $F(0, 0)$ will be used to refer to the constant term in the series $F(z, w)$.

Given two generic bivariate power series

$$F(z,w) = \sum_{r,s\geq 0} f_{r,s}\, z^r\, w^s\,,$$
$$G(z,w) = \sum_{r,s\geq 0} g_{r,s}\, z^r\, w^s\,,$$

we will define their *sum* and *product* respectively as

$$F(z,w) + G(z,w) := \sum_{r,s\geq 0} u_{r,s}\, z^r\, w^s\,, \tag{2.37}$$

$$F(z,w) \cdot G(z,w) := \sum_{r,s\geq 0} v_{r,s}\, z^r\, w^s\,, \tag{2.38}$$

where $u_{r,s} := f_{r,s} + g_{r,s}$ and $v_{r,s} := \sum_{p=0}^{r}\sum_{q=0}^{s} f_{p,q} \cdot g_{r-p,s-q}$. These operations are well-defined in $\mathbb{C}[[z,w]]$ and make of this set an integral domain. Furthermore, $\mathbb{C}[[z]]$ and $\mathbb{C}[[w]]$ can be regarded as sub-rings of $\mathbb{C}[[z,w]]$.

The last observation is of great use in obtaining information about bivariate power series using well-known properties of regular power series. A typical example of this process is the characterization of the units in $\mathbb{C}[[z,w]]$ which we discuss next.

An element $H \in \mathbb{C}[[z,w]]$ is called a *unit* if it has a multiplicative inverse. If it exists it is denoted $H^{-1}(z,w)$ or sometimes $\frac{1}{H(z,w)}$. As might be expected, an element $H(z,w) \in \mathbb{C}[[z,w]]$ is a unit if and only if $H(0,0)$ is nonzero. To prove this assertion we will embed $\mathbb{C}[[z,w]]$ with a topology under which the ring operations are continuous.

A topology of these characteristics is the so called *formal topology*. It is induced by the metric $d(F,G) := 2^{-m(F,G)}$, where

$$m(F,G) := \min\left\{k \geq 0 : [z^r\, w^s](F-G) = 0, \text{ for all } 0 \leq r,s \leq k\right\}.$$

A necessary and sufficient condition in order for a sequence $(F_k)_{k\geq 0}$ of bivariate power series to be convergent (in the formal topology) is the sequence of complex numbers $([z^r\, w^s]\, F_k)_{k\geq 0}$ be eventually constant, for all pairs of nonnegative integers r and s. Furthermore, the $\lim\limits_{k\to\infty} F_k = F$ if and only if for all nonnegative integer n there is m such that $[z^r\, w^s]\,(F_k - F) = 0$, provided that $0 \leq r, s \leq n$ and $k \geq m$.

It follows that

$$\begin{aligned} \lim_{k\to\infty} (F_k + G_k) &= \left(\lim_{k\to\infty} F_k\right) + \left(\lim_{k\to\infty} G_k\right) , \\ \lim_{k\to\infty} (F_k \cdot G_k) &= \left(\lim_{k\to\infty} F_k\right) \cdot \left(\lim_{k\to\infty} G_k\right) , \end{aligned}$$

provided of course that the $\lim\limits_{k\to\infty} F_k$ and $\lim\limits_{k\to\infty} G_k$ exist.

Furthermore, if $H(z,w) = \sum\limits_{r,s\geq 0} h_{r,s}\, z^r\, w^s$ and for each $n \geq 0$ we define $H_n(z) := \sum\limits_{r=0}^{\infty} h_{r,n}\, z^r$ then

$$H(z,w) = \sum_{n=0}^{\infty} H_n(z) \cdot w^n \,, \tag{2.39}$$

in the sense that the partial sums of the above summation converge in the discrete topology to $H(z,w)$.

The characterization of the unit elements of $\mathbb{C}[[z,w]]$ proceeds as follows. It is immediate to verify that a necessary condition in order for $H(z,w)$ to have a multiplicative inverse is to have $H(0,0) \neq 0$. To show the converse suppose that

$$H = \sum_{r,s\geq 0} h_{r,s}\, z^r\, w^s$$

with $h_{0,0} \neq 0$. We need to show that there is $G \in \mathbb{C}[[z,w]]$ such that $H(z,w) \cdot G(z,w) = 1$. Consider the decomposition $H(z,w) = \sum\limits_{s=0}^{\infty} H_s(z) \cdot w^s$, with $H_s(z) := \sum\limits_{r=0}^{\infty} h_{r,s}\, z^r$. The condition $h_{0,0} \neq 0$ implies that $H_0(z)$ is a unit element in $\mathbb{C}[[z]]$ and thus there exists $G_0(z) \in \mathbb{C}[[z]]$ such that $H_0(z) \cdot G_0(z) = 1$. Define $G_s(z)$, with $s \geq 0$, recursively to satisfy the relation $\sum\limits_{r=0}^{s} G_r(z) \cdot H_{s-r}(z) = 0$.

The series $\sum_{n=0}^{\infty} G_n(z) \cdot w^n$ is convergent in $\mathbb{C}[[z,w]]$ because the coefficient of $z^r\, w^s$ in the partial sums $\sum_{n=0}^{k} G_n(z) \cdot w^n$ remains constant for all $k \geq s$. Define $G(z,w) := \sum_{n=0}^{\infty} G_n(z) \cdot w^n$. It follows that

$$\begin{aligned} \sum_{s=0}^{n} G_s(z)\, w^s \cdot \sum_{s=0}^{n} H_s(z)\, w^s &= \sum_{s=0}^{n} \left\{ \sum_{r=0}^{s} G_r(z) \cdot H_{s-r}(z) \right\} w^s + w^{n+1} \cdot P_n(z,w)\,, \\ &= 1 + w^{n+1} \cdot P_n(z,w)\,. \end{aligned}$$

$P_n(z,w)$ is certain power series whose coefficients are of no interest. The left-hand side above converges to $G(z,w) \cdot H(z,w)$ due to the continuity of the product. On the other hand, the $\lim_{n\to\infty} w^{n+1} \cdot P_n(z,w) = 0$ because the coefficient of $z^r\, w^s$ in $w^{n+1} \cdot P_n(z,w)$ is zero for all $n \geq s$. Taking limits both sides we conclude that $G(z,w) \cdot H(z,w) = 1$. This shows that $H(z,w)$ is a unit and completes the proof of our claim.

The preceding discussion provides the main ingredients we will require in relation to the formal theory of bivariate power series. In our context, the term *formal* is to be used to refer to any property of bivariate power series that can be deduced only using the ring structure of $\mathbb{C}[[z,w]]$.

Next we study (as we did in the case of regular power series) the problem of whether a bivariate series can be thought of as a function. As in the one-dimensional case a bivariate generating function does not necessarily define a function of the two-complex variables (z,w). Convergence of bivariate power series is understood in terms of absolute convergence. If the $\sum_{r,s\geq 0} |f_{r,s}\, z_0^r\, w_0^s| < \infty$ then the terms in the series must be bounded and hence there is a constant $c > 0$ such that

$$|f_{r,s}| \leq c \cdot |z_0|^{-r} \cdot |w_0|^{-s}\,,$$

for all $r, s \geq 0$. This implies that the partial sums $\sum_{r,s=0}^{n} f_{r,s}\, z^r\, w^s$ are absolutely convergent for all (z, w) in the set $[|z| \leq |z_0|] \times [|w| \leq |w_0|]$. Moreover, the convergence of the partial sums can be shown to be uniform as long as (z, w) remains in a compact subset of $[|z| < |z_0|] \times [|w| < |w_0|]$. Thus, we may think of $F(z, w)$ not just as a formal power series but indeed as continuous function of z and w for $|z| \leq |z_0|$ and $|w| \leq |w_0|$. Moreover, since the series $\sum_{r,s \geq 0} f_{r,s}\, z^r\, w^s$ is absolutely convergent, we may reorganize the summation in any possible way without altering the convergence nor the limiting value. In particular, we may rewrite

$$F(z, w) = \sum_{s \geq 0} \left\{ \sum_{r \geq 0} f_{r,s}\, z^r \right\} w^s \,.$$

Thus, for each z in the disk $[|z| < |z_0|]$, $F(z, w)$ is an analytic function of w on the disk $[|w| < |w_0|]$. A similar conclusion can be obtained in regards to z in the disk $[|z| < |z_0|]$. As a result, $F(z, w)$ is analytic in each variable separately in the set $[|z| < |z_0|] \times [|w| < |w_0|]$.

We remark that the set where a bivariate series is convergent is not necessarily the product of two disks. For example, the series $\sum_{n=0}^{\infty} z^n\, w^n$ is convergent for all (z, w) such that $|z \cdot w| < 1$, however, it must diverge whenever $|z \cdot w| > 1$. The set $\{(z, w) : |z \cdot w| < 1\}$ is certainly not a product of disks because it contains points of the form $(t, \frac{1}{2t})$ and $(\frac{1}{2t}, t)$ with $t > 0$ arbitrarily small.

Bivariate power series like their one-dimensional counterpart are of use in counting or combinatorial problems. However, the problem of determining asymptotics for the coefficients of bivariate generating functions is far from being a direct generalization of the techniques we have learned to study regular power series.

We will remark further in the complications involved on bivariate generating functions in the coming sections. However, to end our discussion, it will be instructive to consider an example where the toolkit of tricks for regular power series is very suitable to analyze the coefficients of a bivariate power series.

Consider

$$F(z,w) := \frac{1}{2-z-w}\,.$$

This is the multiplicative inverse of the unit element $(2-z-w)$. In particular, there are coefficients $f_{r,s}$ such that $F(z,w) = \sum_{r,s\geq 0} f_{r,s}\, z^r\, w^s$. Our goal is to determine as explicitly as possible the coefficients of $F(z,w)$. A first step in this direction is consequence of the identity $(2-z-w)\cdot F(z,w) = 1$. Multiplying out, it follows that

$$2\cdot f_{0,0} + \sum_{s=1}^{\infty}\Big(2\cdot f_{0,s} - f_{0,s-1}\Big)\, w^s + \sum_{r=1}^{\infty}\Big(2\cdot f_{r,0} - f_{r-1,0}\Big)\, z^r \\ + \sum_{r,s\geq 1}\Big(2\cdot f_{r,s} - f_{r-1,s} - f_{r,s-1}\Big)\, z^r\, w^s = 1$$

This implies that

$$f_{r,s} = \begin{cases} \frac{1}{2^{r+s+1}} & ,\quad r=0 \text{ or } s=0\,, \\ \frac{f_{r-1,s}+f_{r,s-1}}{2} & ,\quad \text{otherwise}\,. \end{cases}$$

If we think of the coefficient $f_{r,s}$ as displayed over a grid on the position (r,s) the above recursion shows that $f_{r,s}$ is the average of its two most immediate neighbors to the South and to the West. With this visualization in mind it should be clear that $0 \leq f_{r,s} \leq \frac{1}{2}$. Moreover, an inductive argument shows that

$$\frac{1}{2^{r+s+1}} \leq f_{r,s} \leq \frac{1}{2}\,, \tag{2.40}$$

for all $r,s\geq 0$. But these inequalities are far from being accurate. To justify that $f_{r,s}$ could be much bigger than $\frac{1}{2^{r+s+1}}$ but much smaller than $\frac{1}{2}$ consider the power series

$$\begin{aligned} L(z,w) &:= \sum_{r,s\geq 0}\frac{1}{2^{r+s+1}}\, z^r\, w^s = \frac{2}{(2-z)\cdot(2-w)}\,, \\ U(z,w) &:= \sum_{r,s\geq 0}\frac{1}{2}\, z^r\, w^s = \frac{1}{2\cdot(1-z)\cdot(1-w)}\,. \end{aligned}$$

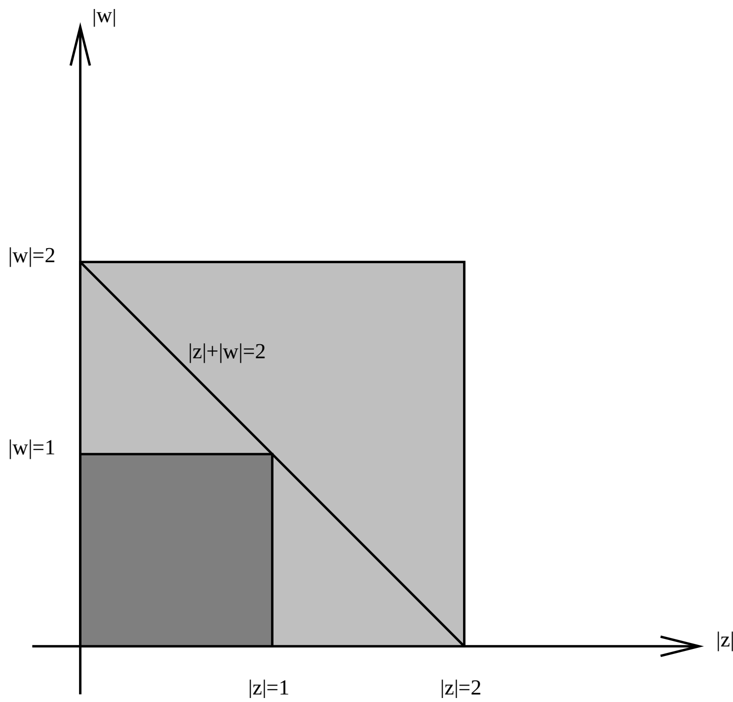

Figure 2.6: Representation of the domain of convergence of $L(z,w)$, $U(z,w)$ and $F(z,w)$.

$L(z,w)$ is absolutely convergent only for all $|z| < 2$ and $|w| < 2$. However, $U(z,w)$ is absolutely convergent if and only if $|z| < 1$ and $|w| < 1$. On the other hand, $F(z,w)$ is absolutely convergent if and only if $|z| + |w| < 2$. (See figure 2.6).

The disparity between these domains of convergence is an indication that the coefficients of $L(z,w)$ go to zero at a much faster rate than the coefficients of $F(z,w)$. This allows convergence for more points in the series of $L(z,w)$ than in the series for $F(z,w)$. By the same reasoning, it is expected that the coefficients of $F(z,w)$ decrease toward zero at a much faster rate than the coefficients of $U(z,w)$. Thus, by all accounts the bounds in (2.40) seem to be very inaccurate for most coefficients $f_{r,s}$.

To reveal more accurate information about the coefficients $(f_{r,s})_{r,s\geq 0}$ we will fit one-variable methods in our analysis.

Since $F(z,w)$ is absolutely convergent for all (z,w) such that $|z|+|w|<2$ we can organize the terms in the series in any particular way. One suitable way is to rearrange them in the form

$$F(z,w) = \sum_{s=0}^{\infty} \left\{ \sum_{r=0}^{\infty} f_{r,s}\, z^r \right\} w^s \,.$$

On the other hand, for a given $0<z<2$, we may think of $F(z,w)$ as a power series in w which is absolutely convergent for $|w|<(2-z)$. For all such w we obtain that

$$F(z,w) \;=\; \frac{1}{2-z}\cdot\frac{1}{1-\frac{w}{(2-z)}} = \sum_{s=0}^{\infty}\frac{1}{(2-z)^{s+1}}\, w^s \,.$$

Using the last two representations for $F(z,w)$ we can conclude that $\sum\limits_{r=0}^{\infty} f_{r,s}\, z^r = \frac{1}{(2-z)^{s+1}}$, for all $0<z<2$. But, this equality must be satisfied for all $|z|<2$. Thus, using (2.11), we obtain the exact formula

$$\begin{aligned} f_{r,s} &= [z^r]\,\frac{1}{(2-z)^{s+1}}\,, \\ &= \frac{1}{2^{r+s+1}}\binom{r+s}{r}. \end{aligned}$$

2.6 Gaussian approximation and the Stirling's formula

One widely popular method to study bivariate generating function is inspired by the techniques used in probability theory to obtain central or local limit theorems. Although the applicability of these ideas to bivariate generating functions is limited many bivariate power series of interest can be studied or at least partially studied via this approach. The term *gaussian approximation* is used to emphasize the appearance of the *standard normal density distribution*

$$p(x) := \frac{1}{\sqrt{2\pi}} \cdot e^{-x^2/2} \,.$$

We motivate the setting of the gaussian approximation and the type of asymptotics which are expected by its use considering a very general probabilistic example. Suppose that $x, x_1, x_2, \ldots$ are independent identically distributed $\mathbb{N}$-valued random variables (see [Dur95]). We will define

$$\mu \;:=\; \sum_{j=0}^{\infty} j \cdot P[x=j] \,, \tag{2.41}$$

$$\sigma^2 \;:=\; \sum_{j=0}^{\infty} (j-\mu)^2 \cdot P[x=j] \,, \tag{2.42}$$

provided that μ is finite. μ is the so called *expected value of* x and is usually denoted $E(x)$. On the other hand, σ^2 is referred to as the *variance of* x and is usually denoted $V(x)$.

If the expected value and variance of x are finite, the well-known *strong law of large numbers* (see [Dur95]) states that the

$$\lim_{s\to\infty} \frac{1}{s} \sum_{j=1}^{s} x_j = \mu \,,$$

almost surely. In particular, if we define $y_s := \sum_{j=1}^{s} x_j$ then, for big values of s, it is likely that y_s will be close to $s \cdot \mu$. It is then an interesting problem to estimate the probabilities $p_{r,s} := P[y_s = r]$ when the ratio $\frac{r}{s}$ is close to μ.

The probabilities $p_{r,s}$ have been well studied in probability theory. The result we will quote in here is the so called *Local Central limit theorem for lattice distributions* (see [Dur95], section 2.5). For simplicity we will assume that the $P[x=n] > 0$, for all nonnegative integer n. Under these circumstances the Local Central limit theorem states that

$$p_{r,s} = \frac{1}{\sqrt{s\,\sigma^2}} \cdot \left\{ p\left(\frac{r - s\cdot\mu}{\sqrt{s\,\sigma^2}}\right) + o(1) \right\}, \tag{2.43}$$

uniformly for all $r \geq 0$, as $s \to \infty$. This can be used to estimate the probabilities $p_{r,s}$ when $\frac{r}{s}$ is close to μ. Indeed, since $p(x)$ zero-free then $p\left(\frac{r-s\cdot\mu}{\sqrt{s\,\sigma^2}}\right)$ remains bounded away from zero as long as $\frac{r-s\cdot\mu}{\sqrt{s\,\sigma^2}}$ remains in a compact subset of the real line. As a result, for all $\delta > 0$, we can conclude that

$$p_{r,s} \sim \frac{1}{\sqrt{s\sigma^2}} \cdot p\left(\frac{r - s\cdot\mu}{\sqrt{s\,\sigma^2}}\right), \tag{2.44}$$

uniformly for all $|r - s\cdot\mu| \leq \delta \cdot s^{3/2}$, as $s \to \infty$. (See figure 2.7.)

The classic proof of the Local Central limit theorem involves the so called *characteristic function of x*. It is defined to be the function $\varphi_x(t) := E(e^{itx})$, with $t \in \mathbb{R}$. Since X is an $\mathbb{N}$-valued random variable, it follows that

$$\varphi_x(t) = \sum_{n=0}^{\infty} P[x = n] \cdot e^{itn},$$

and the series is convergent because the $\sum_{n=0}^{\infty} P[x = n] = 1$.

In a spark of ingenuity one may come to realize that $\varphi_x(t)$ is very close to be the power series associated to the sequence $(P[x = n])_{n\geq 0}$. Indeed, since this last is defined to be

$$X(z) := \sum_{n=0}^{\infty} P[x = n] \cdot z^n, \tag{2.45}$$

it follows that

$$\varphi_x(t) = X(e^{it}). \tag{2.46}$$

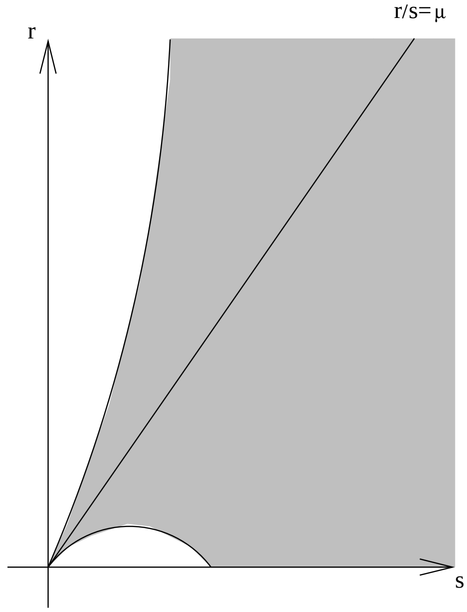

Figure 2.7: Bandwidth about the line $r = s \cdot \mu$ in the (r, s)-lattice where a gaussian approximation of the probabilities $p_{r,s}$ remains valid.

The identity in (2.46) is nowadays a well-known bridge between the theory of asymptotic expansions and probability theory. Expectedly, asymptotic formulas of certain classes of two-dimensional arrangements will involve the standard normal density distribution in a way that resembles (2.44). The following result parallels, at a very basic level, the discussion of Bender in [Ben73]. It is a typical example of the appearance of gaussian approximations in relation to the asymptotic behavior of the coefficients of a bivariate generating function.

Proposition 2.1. (Gaussian approximation.) *Suppose that $A(z, w)$ is of the form*

$$A(z, w) := U(w \cdot X(z)) \,, \tag{2.47}$$

for certain power series

$$U(z) \; := \; \sum_{n=0}^{\infty} u_n \, z^n \,, \tag{2.48}$$

$$X(z) \; := \; \sum_{n=0}^{\infty} p_n \, z^n \,, \tag{2.49}$$

where $U(z)$ has a positive radius of convergence, $p_n > 0$, for all n, and the $\sum_{n=0}^{\infty} p_n = 1$, $\sum_{n=0}^{\infty} n \cdot p_n < \infty$ and $\sum_{n=0}^{\infty} n^2 \cdot p_n < \infty$. Then, for all $\delta > 0$,

$$[z^r \, w^s] \, A(z,w) \sim \frac{u_s}{\sqrt{s \cdot \sigma^2}} \cdot p\left(\frac{r - s \cdot \mu}{\sqrt{s \cdot \sigma^2}}\right) \,, \tag{2.50}$$

uniformly for all (r,s) such that $|r - s \cdot \mu| \leq \delta \cdot s^{3/2}$, as $s \to \infty$.

The proof of proposition 2.1 goes as follows. The first two conditions on the p_n's imply the existence of random variables $x, x_1, x_2, \ldots$ independent and identically distributed such that $P[x = n] = p_n$, for all $n \geq 0$. (See Kolmogorov's extension theorem in [Dur95].) The other two conditions are just another way to say that $\mu < \infty$ and $0 < \sigma^2 < \infty$, with μ and σ as defined in (2.41) and (2.42) respectively.

Define $y_s := \sum_{n=1}^{s} x_n$. For a fixed s, the characteristic function associated to the random variable y_s is found to satisfy

$$\begin{aligned} \varphi_{y_s}(t) := E\left(\prod_{n=1}^{s} e^{itx_n}\right) &= \prod_{n=1}^{s} E(e^{itx_n}) \,, \\ &= \{E(e^{itx})\}^s \,, \\ &= \{X(e^{it})\}^s \,. \end{aligned}$$

The second equality is justified by the independence of $x_1, \ldots, x_n$. The third equality uses that these random variables are equally distributed as x. For the last identity we have used (2.46).

The previous computation implies that the $\sum_{r\geq 0} P[y_s = r]\, z^r = \{X(z)\}^s$, for all $|z| \leq 1$. As a result, we obtain that

$$\begin{aligned}\sum_{r,s\geq 0} u_s \cdot P[y_s = r]\, z^r\, w^s &= \sum_{s=0}^{\infty} u_s \cdot \{X(z)\cdot w\}^s \\ &= U(X(z)\cdot w) \\ &= A(z,w).\end{aligned}$$

Thus, $[z^r\, w^s]\, A(z,w) = u_s \cdot P[y_s = r]$ and (2.50) now follows from (2.44). This completes the proof of proposition 2.1.

We will apply the proposition to deduce the well-known *Stirling's formula* which provides an asymptotic equivalent to $n!$ for big values of n. Our starting point may seem somehow unconnected with the previous discussion, however, it is key for the eventual use of (2.50).

Observe that $n! \leq n^n$. This crude estimate can be improved using Cauchy's formula (see [Rud87]). For all $R > 0$, it implies that

$$\frac{1}{n!} = \frac{1}{2\pi} \int_{|z|=R} \frac{e^z}{z^n} \frac{dz}{iz}\,.$$

As a result,

$$1 \leq \frac{1}{n!} \leq \frac{e^R}{R^n}\,.$$

The optimal upper bound is reached at $R = n$. This is because the function $R \to \frac{e^R}{R^n}$ is strictly decreasing for $R < n$, however, it is strictly increasing for $R > n$. Thus, for all $n > 0$, we obtain that

$$1 \leq \frac{n^n}{n!} \leq e^n\,. \tag{2.51}$$

The above inequalities motivate us to consider the arrangement $a_{r,s} := \frac{s^r}{r!}$.

The generating function associated to the coefficients $\left(a_{r,s}\right)_{r,s\geq 0}$ will be denoted as $A(z,w)$. The discussion in section 2.5 implies that the $\sum_{r,s\geq 0} a_{r,s}\, z^r\, w^s$ is absolutely convergent for all (z,w) such that $|w|\cdot e^{|z|}<1$.[5] Furthermore, observe that

$$\begin{aligned} A(z,w) &= \sum_{s=0}^{\infty} w^s \cdot \sum_{r=0}^{\infty} \frac{(sz)^r}{r!}\,, \\ &= \sum_{s=0}^{\infty} w^s \cdot e^{sz}\,, \\ &= \frac{1}{1-w\cdot e^z}\,. \end{aligned}$$

This identity resembles (2.47), with $U(z)=\frac{1}{1-z}$ and $X(z)=e^z$. The only difficulty to use proposition 2.1 is that $X(1)\neq 1$. To overcome this problem consider, for each $t>0$, the power series

$$B_t(z,w) \;:=\; A\left(t\cdot z, \frac{w}{X(t)}\right) = U\left(w\cdot \frac{X(t\cdot z)}{X(t)}\right)\,.$$

This series is precisely in the setting of proposition 2.1. Furthermore, $\frac{X(t\cdot z)}{X(t)}$, thought of a power series in z, is induced by a random variable x_t such that $P[x_t=n]=\frac{t^n\,e^{-t}}{n!}$. x_t is consequently a Poisson random variable with parameter t (see [Dur95]). Its mean and variance are easily determined to be $\mu_t=\sigma_t^2=t$.

Observe that

$$B_t(z,w)=\sum_{r,s\geq 0} \frac{t^r\cdot a_{r,s}}{\{X(t)\}^s}\, z^r\, w^s\,.$$

To obtain an asymptotic formula for

$$a_{n,n}=\frac{n^n}{n!}\,,$$

it is enough to determine an asymptotic formula for $[z^r, w^s]\, B_t(z,w)$, as $(r,s)\to\infty$ and (r,s) stays "nearby" the line $r=s$. We plan to do this using proposition 2.1.

[5]Observe that the series is not absolutely convergent for all (z,w) such that $|w\cdot e^z|<1$. As a counterexample consider $(2i,1/2)$.

Select $t = 1$ to have $\mu_t = 1$. With this selection of t, $\sigma_t^2 = 1$. As a result, for all $\delta > 0$, proposition 2.1 implies that

$$\begin{aligned} e^{-s} \cdot a_{r,s} &= \frac{1}{\sqrt{s}}\, p\left(\frac{r-s}{\sqrt{s}}\right), \\ &\sim \frac{1}{\sqrt{2\pi s}}\, e^{-(r-s)^2/(2s)}, \end{aligned}$$

uniformly for all (r, s) such that $|r - s| \le \delta \cdot s^{3/2}$, as $s \to \infty$. In particular, letting $r = s = n$, we obtain the Stirling's formula. Namely,

$$n! \sim \sqrt{2\pi} \cdot n^{n+1/2} \cdot e^{-n}, \tag{2.52}$$

as $n \to \infty$.

The procedure described to obtain (2.52) is very generic. Typically, asymptotics for the coefficients $a_{r,s}$ of a power series $A(z, w)$ can be computed on a bandwidth along a particular direction in the (r, s)-lattice which depends on certain parameter.

In the case of Stirling's formula this direction was the line $r = s$ and the parameter was the nonnegative real number t. To obtain asymptotics for the coefficients along the diagonal line the parameter t was adjusted to have $\mu_t = 1$.

This two-step procedure is very standard in the asymptotic study of bivariate power series even in situations that are not related to a gaussian approximation. At a first exposure, however, it may seem strange the sudden appearance of a parameter like, for example, t was in relation to Stirling's formula. To amplify on this, we reconsider the bivariate generating function

$$A(z, w) := \frac{1}{1 - w \cdot e^z}.$$

Its domain of absolute convergence is the set

$$\left\{(z, w) : |w| \cdot e^{|z|} < 1\right\}.$$

In particular, each point of the form (t, e^{-t}), with $t > 0$, is a boundary point of this domain. However, a more remarkable fact is that (t, e^{-t}) is the only singularity of $A(z, w)$ on the closed set

$$[|z| \leq t] \times [|w| \leq e^{-t}] .$$

In the terminology introduced by Pemantle and Wilson in [PemWil01], this means that each point of the form (t, e^{-t}), with $t > 0$, is a *strictly minimal singularity* of $A(z, w)$. Moreover, in this particular example, all points of this form are also simple poles of $A(z, w)$. (See figure 2.8.)

It turns out that each such singularity of $A(z, w)$ is informative of the asymptotic behavior of its coefficients but only along a direction in the (r, s)-lattice specified by the singularity itself. From this new perspective, the seemingly artificial introduction and later adjustment of the parameter t is equivalent to the problem of identifying a singularity of $A(z, w)$ which produces asymptotics for the coefficients $[z^r\, w^s]\, A(z, w)$ along a direction of interest. A discussion on this approach will be treated in great detail in the introduction of chapter 6. There, we will summarize the scheme developed by Pemantle and Wilson in [PemWil01] to obtain asymptotics for the coefficients of bivariate meromorphic functions along directions specified by simple poles.

We remark that the situation in the presence of poles of higher order is radically different. In that case, the pole usually determines the asymptotic behavior of the coefficients not along a single direction by rather in a cone of directions. Further, along these directions, the coefficients behave polynomially in (r, s) up to an error which is rapidly decreasing as $(r, s) \to \infty$. For further details on these findings refer to [PemWil02].

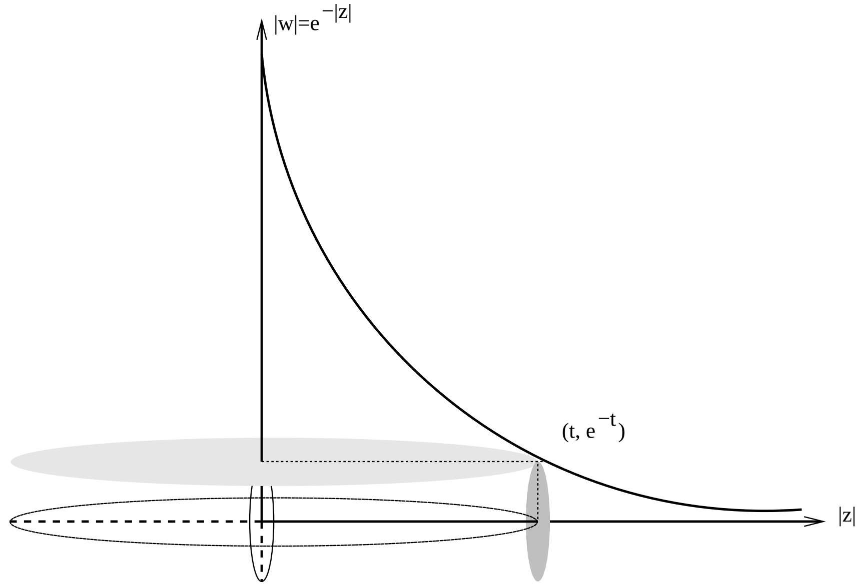

Figure 2.8: Representation of a strictly minimal singularity. For all $t > 0$, (t, e^{-t}) is the only pole of $A(z, w)$ in the set $[z : |z| \le t] \times [w : |w| \le e^{-t}]$.

CHAPTER 3

SINGULARITY ANALYSIS IN ONE DIMENSION

3.1 Asymptotic expansions

To set the terminology we will briefly review the basic aspects of the theory of asymptotic expansions. For a compact reference on the subject of asymptotic expansions we recommend [Bru81] and [BleHan86].

R will represent a subset of $\mathbb{C}$ or of the Riemann sphere embedded with the induced topology (see [Rud87]). Typically, $R = \{0, 1, 2, \ldots\} \cup \{\infty\}$, $R = [0, \infty]$ or it is an open subset of the Riemann sphere. Given $x_0 \in R$, suppose that $f(x)$ and $g(x)$ are complex-valued functions defined for all x in some punctured neighborhood of x_0. We say that $f(x)$ *is a big-O of* $g(x)$ *as* $x \to x_0$, and write: "$f(x) = O(g(x))$, as $x \to x_0$" provided that there is a neighborhood N of x_0 and a constant $c > 0$ such that $|f(x)| \leq c \cdot |g(x)|$, for all $x \in N \setminus \{x_0\}$.

We say that $f(x)$ *is a little-o of* $g(x)$ *as* $x \to x_0$, and write: "$f(x) = o(g(x))$, as $x \to x_0$" provided that for all $\epsilon > 0$ there is a neighborhood N_ϵ of x_0 such that $|f(x)| \leq \epsilon \cdot |g(x)|$, for all $x \in N_\epsilon \setminus \{x_0\}$.

Suppose that $g(x)$ is zero-free in some punctured neighborhood of x_0. We will say that $f(x)$ *is of the same order as* $g(x)$ *as* x *approaches* x_0, and write: "$f(x) \sim g(x)$, as $x \to x_0$" provided that the $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$. Equivalently, $f(x) - g(x) = o(g(x))$, as $x \to x_0$.

Suppose that $(g_n(x))_{n \geq 0}$ is a sequence functions defined in a punctured neighborhood of x_0. We say that $(g_n(x))_{n \geq 0}$ *is an asymptotic sequence as x approaches* x_0 provided that for any nonnegative integer N, $g_{N+1}(x) = o(g_N(x))$, as $x \to x_0$. Given a sequence $(\alpha_n)_{n \geq 0}$ of complex numbers we will write $f(x) \approx \sum_{n=0}^{\infty} \alpha_n \cdot g_n(x)$, as $x \to x_0$, provided that for all N,

$$f(x) - \sum_{n=0}^{N} \alpha_n \cdot g_n(x) = O(g_{N+1}(x)) \,, \text{ as } x \to x_0 \,.$$

The series $\sum_{n=0}^{\infty} \alpha_n \cdot g_n(x)$ is then said to be an *asymptotic expansion* (or asymptotic development) of $f(x)$ with respect to the asymptotic sequence $(g_n(x))_{n \geq 0}$. We observe that if $f(x) \approx \sum_{n=0}^{\infty} \beta_n \cdot g_n(x)$, as $x \to x_0$, then necessarily $\alpha_n = \beta_n$.

For example, for all $x_0 \in \mathbb{C}$, the functions $g_n(x) := (x - x_0)^n$ form an asymptotic sequence in the complex plane as $x \to x_0$. Furthermore, if $f(x)$ is analytic near the $x = x_0$ then $f(x) \approx \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} \cdot (x - x_0)^n$, as $x \to x_0$. A bit more intriguing example is the asymptotic expansion

$$e^{-1/|x - x_0|} \approx \sum_{n=0}^{\infty} 0 \cdot (x - x_0)^n \,, \text{ as } x \to x_0 \,.$$

The series on the right-hand side is convergent for all x, however, the left and right-hand side are nowhere equal. This shows that a convergent asymptotic expansion does not necessarily converge to the function it expands.

As another example, consider the functions $g_n(x) := \frac{1}{x^n}$, which form an asymptotic sequence as $x \to \infty$ in the Riemann sphere. For example, by repeated integrations by parts, it is simple to verify that

$$e^x \cdot \int_x^\infty \frac{e^{-t}}{t}\, dt \approx \sum_{n=0}^{\infty} \frac{(-1)^n \cdot n!}{x^{n+1}}\,, \text{ as } x \to \infty\,.$$

Observe that the series on the right-hand side is nowhere convergent on the complex plane. Thus, the example shows that an asymptotic development does not necessarily converge.

Another notion we will repeatedly use in the coming discussion is the following. For the special case in which $R = (0, \infty)$ or R is a cone in the complex plane we will say that a function $f(x)$ *is rapidly decreasing at* ∞ provided that, for all $N \geq 0$, $f(x) = O(|x|^{-N})$, as $x \to \infty$. Thus, rapidly decreasing functions are smaller at infinity than any functions with asymptotic developments in powers of x^{-1}. Smaller yet are the functions of *exponential decay at* ∞, namely those satisfying $f(x) = O(e^{-c\cdot|x|})$, as $x \to \infty$, for some positive constant c.

The preceding discussion generalizes to consider the so called *uniform big-O* and *uniform little-o* when functions are indexed by a parameter. Let T be also a subset of $\mathbb{C}$ or of the Riemann sphere embedded with the induced topology. Given $x_0 \in R$, suppose that $f(t, x)$ and $g(t, x)$ are complex-valued functions defined for all x in some punctured neighborhood of x_0 and for all $t \in T$ (except, possibly $t = \infty$, if T is a neighborhood of infinity in the Riemann sphere). We will think of $f(t, x)$ and $g(t, x)$ as a functions of x indexed by the parameter t.

We will write: "$f(t, x) = O(g(t, x))$ uniformly for all $t \in T$, as $x \to x_0$" provided that there is a neighborhood N of x_0 and a constant $c > 0$ such that $|f(t, x)| \leq c \cdot |g(t, x)|$, for all $x \in N \setminus \{x_0\}$ and $t \in T$. Accordingly, we will write: "$f(t, x) = o(g(t, x))$ uniformly for all $t \in T$, as $x \to x_0$" provided that for all $\epsilon > 0$ there is a neighborhood N_ϵ of x_0 such that $|f(t,x)| \leq \epsilon \cdot |g(t,x)|$, for all $x \in N_\epsilon \setminus \{x_0\}$ and $t \in T$.

The definitions of

(a) "$f(t,x) \sim g(t,x)$ uniformly for all $t \in T$, as $x \to x_0$",

(b) "$f(t,x)$ is rapidly decreasing uniformly for all $t \in T$, as $x \to \infty$", and

(c) "$f(t,x)$ is exponentially decreasing uniformly for all $t \in T$, as $x \to \infty$",

are the trivial adaptation of the former definitions but using the notion of uniform big-O and uniform little-o.

In regards to *uniform asymptotic expansions*, we will mostly deal with expansions of the form: "$f(t,x) \approx \sum_{n=0}^{\infty} \alpha_n(t) \cdot g_n(t,x)$ uniformly for all $t \in T$, as $x \to x_0$." The notation again assumes that $(g_n(t,x))_{n\geq 0}$ is an asymptotic sequence as $x \to x_0$; that is to say, for all nonnegative integer N, $g_{N+1}(t,x) = o(g_N(t,x))$ uniformly for all $t \in T$, as $x \to x_0$. The natural condition is now to require that for all nonnegative integer N, the difference

$$f(x) - \sum_{n=0}^{N} \alpha_n(t) \cdot g_n(t,x) = O(g_{N+1}(t,x))\,, \text{ uniformly for all } t \in T, \text{ as } x \to x_0\,.$$

3.2 The method of partial fractions

The method of partial fraction is very suitable to determine explicitly the coefficients of a rational function of one complex variable. However, this method is also suitable to obtain, with very little effort, the leading asymptotic order of its coefficients. The lesson to learn from the discussion that will follow is that (with the exception of rare cases) the closest singularity to the origin of a rational function contains a great deal of information about the leading asymptotic order of its coefficients.

Let $p(z)$ and $q(z)$ be polynomials in the variable z of degree d and e respectively and assume that $q(0) \neq 0$. Our interest is in determining the coefficient of z^n in the power series representation of $F(z) := \frac{p(z)}{q(z)}$ about $z = 0$. Without loss of generality we may assume that $p(z)$ and $q(z)$ do not have common zeroes.

The linearity in the numerator motivates to study first the rational function $\frac{1}{q(z)}$. We first consider the case in which all the roots of $q(z)$ are distinct. If these are r_1, $\ldots$,r_e then $q(z) = q(0) \cdot \prod_{j=1}^{e} (1 - r_j^{-1} \cdot z)$. The method of partial fractions implies that there are nonzero constants c_j such that $\frac{1}{q(z)} = \sum_{j=1}^{e} \frac{c_j}{1-r_j^{-1} \cdot z}$. [1] As a result, we obtain the exact formula

$$\begin{aligned} [z^n]\frac{1}{q(z)} &= \sum_{j=1}^{e} c_j \cdot [z^n]\, \frac{1}{1 - r_j^{-1} \cdot z}\,, \\ &= \sum_{j=1}^{e} c_j \cdot r_j^{-n}\,. \end{aligned}$$

More generally, if $p(z) = \sum_{k=0}^{d} p_k \cdot z^k$ we obtain the exact formula

$$\begin{aligned} [z^n]\tfrac{p(z)}{q(z)} &= \sum_{j=1}^{e} \sum_{k=0}^{d} c_j \cdot p_k \cdot r_j^{k-n}\,, \\ &= \sum_{j=1}^{e} \tfrac{-p(r_j)}{r_j \cdot q'(r_j)} \cdot r_j^{-n}\,. \end{aligned} \tag{3.1}$$

Since $p(z)$ and $q(z)$ do not have common roots it follows that $p(r_j) \neq 0$, for all j. As a result, the leading order(s) within the summation is of course r^{-n} where $r := \min\big\{|r_j| : j = 1, \ldots, n\big\}$. Hence, if cancellation is ruled out among the terms of

[1] Although it is not relevant for our discussion, we remark that $c_j = \frac{-1}{r_j \cdot q'(r_j)}$.

order r^{-n} then the following asymptotic formula applies

$$[z^n]\frac{p(z)}{q(z)} \sim \sum_{j:|r_j|=r} \frac{-p(r_j)}{r_j \cdot q'(r_j)} \cdot r_j^{-n} \,,$$

as $n \to \infty$.

The case in which $q(z)$ has roots of repeated multiplicity is slightly more complicate yet a similar conclusion can be obtained. Suppose that $q(z)$ has m-distinct roots, $r_1, \ldots, r_m$, and that r_k is of multiplicity m_k. We let r to denote the modulus of the root(s) closest to the origin.

The method of partial fractions this time implies that there are coefficients $c_{k,l}$ such that $\frac{1}{q(z)} = \sum_{k=1}^{m}\sum_{l=1}^{m_k} \frac{c_{k,l}}{(1-r_k^{-1}\cdot z)^l}$. Moreover, we must have $c_{k,m_k} \neq 0$, for all k. Furthermore, since $\frac{1}{(1-r^{-1}\cdot z)^l} = \sum_{n=0}^{\infty} \binom{n+l-1}{l-1} \cdot r^{-n} \cdot z^n$ and $\binom{n+l-1}{l-1} \sim \frac{n^{l-1}}{(l-1)!}$, as $n \to \infty$, we deduce that

$$\begin{aligned} [z^n]\frac{p(z)}{q(z)} &= \sum_{j=1}^{d} p_j \cdot \sum_{k=1}^{m}\sum_{l=1}^{m_k} c_{k,l} \cdot \binom{n-j+l-1}{l-1} \cdot r_k^{-n} \,, \\ &= \sum_{k=1}^{m} \frac{c_{k,m_k} \cdot p(r_k)}{(m_k-1)!} \cdot n^{m_k-1} \cdot r_k^{-n} \cdot \left\{1 + O\left(\frac{1}{n}\right)\right\} \,, \end{aligned}$$

as $n \to \infty$. Since for all k, $p(r_k) \neq 0$, it follows that r^{-n} is the leading order among the terms in the summation.

3.3 An analytic approach

The discussion in the previous section shows that the leading asymptotic behavior of the coefficients of a rational generating function $F(z) = \frac{p(z)}{q(z)}$ is determined by the closest zero of $q(z)$ to the origin. The coefficient of z^n in $F(z)$ will be denoted as f_n.

In a more analytical prospect, the assumption we made that $q(0) \neq 0$ implies that the disk of convergence of $F(z)$ is $[z : |z| < r]$ where r refers to the modulus of the root(s) closest to the origin. As a result, the

$$\limsup_{n\to\infty} \frac{\ln|f_n|}{n} = -\ln|r| \,.$$

Thus we get the correct exponential rate, at least for the lim sup, for the coefficients f_n with no work at all. To determine more explicitly the leading order of f_n we use Cauchy's integral formula (see [Rud87]) to write

$$f_n = \frac{1}{2\pi i}\int_{|z|=\rho_0} \frac{F(z)}{z^{n+1}}\,dz\,,$$

where $0 < \rho_0 < r$. Let $\rho_1 > r$ and assume that there is only one root of q of minimum modulus and that the moduli of other roots is greater than r. Denote this root by r_M. We then have, by the residue theorem (see [Rud87]), that

$$\frac{1}{2\pi i}\int_{|z|=\rho_0} \frac{F(z)}{z^{n+1}}\,dz - \frac{1}{2\pi i}\int_{|z|=\rho_1} \frac{F(z)}{z^{n+1}}\,dz = -\mathrm{Res}\left(\frac{F(z)}{z^{n+1}}; z = r_M\right)\,.$$

If we assume that r_M is a zero of multiplicity one then $F(z) = \frac{p(z)}{q(z)}$ has a simple pole at $z = r_M$; in particular, the residue term above is then just $\frac{p(r_M)}{r_M \cdot q'(r_M)} \cdot r_M^{-n}$. On the other hand, the integral over the circle $[z : |z| = \rho_1]$ is bounded from above by: $\sup\limits_{|z|=R} |F(z)| \cdot \rho_1^{-n}$, and is therefore exponentially smaller than the residue term (which is of order r^{-n}). Thus the leading term asymptotic for f_n is

$$f_n = -\frac{p(r_M)}{r_M \cdot q'(r_M)} \cdot r_M^{-n} + O(\rho_1^{-n})\,.$$

In fact, if we let ρ_1 to tend to infinity, each zero of $q(z)$ contributes by a residue term. Thus, for example, if all the zeroes of $q(z)$ have multiplicity one then all the poles of $F(z)$ are simple and one recovers (3.1).

If there is more than one root of minimum modulus, simply sum the contributions. If the root r_j appears with multiplicity $m_j > 1$ and we define $q_j(z) := \frac{q(z)}{(z-r_j)^m}$ then the residue at $z = r_j$ is found to be instead

$$\begin{aligned}\operatorname{Res}\left(\frac{F(z)}{z^{n+1}}; z = r_j\right) &= \frac{1}{(m_j-1)!}\frac{\partial^{m_j-1}}{\partial z^{m_j-1}}\left[\frac{p(z)}{z^{n+1}\cdot q_j(z)}\right](r_j)\,, \\ &= (-1)^{m_j-1}\cdot\binom{n+m_j-1}{n}\cdot\frac{p(r_j)}{r_j^{m_j}\cdot q_j(r_j)}\cdot r_j^{-n} + O\binom{n+m_j-2}{m_j-2}\,, \\ &= \frac{(-n)^{m_j-1}}{(m_j-1)!}\cdot\frac{p(r_j)}{r_j^{m_j}\cdot q_j(r_j)}\cdot r_j^{-n} + O(n^{m_j-2})\,,\end{aligned}$$

as $n \to \infty$. This gives a "polynomial correction" for the case of multiple roots.

The advantage of the analytic solution is that it vastly more general. The method of partial fractions requires $F(z) = \frac{p(z)}{q(z)}$ to be a quotient of polynomials. Instead, suppose that $p(z)$ and $q(z)$ are only required to be analytic in some disk $B(0, R)$, and that $q(z)$ has a zero, say z_0, inside the disk. The same computation then gives

$$\begin{aligned}[z^n]\,F(z) &= \frac{1}{2\pi i}\int_{|z|=\rho_1}\frac{F(z)}{z^{n+1}}\,dz\,, \\ &= \frac{1}{2\pi i}\int_{|z|=\rho_2}\frac{F(z)}{z^{n+1}}\,dz - \operatorname{Res}\left(\frac{F(z)}{z^{n+1}}; z = z_0\right)\,,\end{aligned} \tag{3.2}$$

provided that $0 < \rho_1 < |z_0| < \rho_2 < R$ and no other pole of $F(z)$ is in $[z : |z| \leq \rho_2]$. Then, $\frac{1}{2\pi i}\int_{|z|=\rho_2}\frac{F(z)}{z^{n+1}}\,dz = O(\rho_2^{-n})$ and the residue is easily computed as before. For instance, in the case of a simple root of $q(z)$ at $z = z_0$, the residue is still $\frac{p(z_0)}{z_0\cdot q'(z_0)}\cdot z_0^{-n}$.

Question: What does the analytic approach give us for a general function $F(z)$?

If the power series $F(z)$ is purely formal, that is to say, nowhere convergent, then we learn nothing. If $F(z)$ is entire, we learn very little, though more can be said by means beyond the scope of this discussion. Assume then that the radius of convergence of $F(z)$ is positive and finite. If the minimal modulus singularity of $F(z)$ is a pole (or poles) then the preceding analysis applies. If it is a branch-point, there are standard modifications of the method which we will discuss shortly. If it is an isolated singularity, there are good prospects for a successful modification of the method, though it is more of an art than a science; this will be discussed too.

Logically, the worst case is if the domain of convergence of $F(z)$ has an entire circle as its natural boundary (this means that $F(z)$ can not be continued analytically locally across any point in the boundary of the circle). This can and does happen. For example, the generating function $\sqrt{\frac{1+z}{1-z}} \cdot \prod_{n=1}^{\infty} \cos\left(\frac{z^{2n}}{2n}\right)$, appearing in the problem of counting permutations of $\{1, \ldots, n\}$ which admit a square root, has every $4m^{th}$-root of unity (with $m \geq 1$) as a singularity and therefore, has the unit circle as its natural boundary.

A point of philosophy. We have motivated the problem of understanding the asymptotic behavior of the coefficients of generating functions due to their application in enumerative combinatorics. A classical theorem of Pólya-Carlson [2] states that:

> *"If $F(z)$ has integer coefficients and radius of convergence 1, then either $F(z)$ is rational or $F(z)$ has the unit circle for its natural boundary."*

In regards to the class of generating functions with the boundary of their disk of convergence as their natural boundary, there is a recent contribution due to Flajolet et al. [FGPP03] which provides a hybridization of the Darboux' and singularity analysis methods to tackle the asymptotic analysis of the coefficients of a variety of generating functions in this class.

In terms of rational functions and although the asymptotic analysis of their coefficients is trivial in one-dimension this commodity just disappears when considering

[2] With the help of Philippe Flajolet we could find more details on the history of this theorem. It was first conjectured by Pólya and finally proved by Carlson (see [Car21]). In [Pól23], Pólya wrote: "En poursuivant les remarques de MM. Borel et Fatou sur les séries à coefficients entiers, je suis arrivé à quelques résultats, mais je n'ai pu ni démontrer ni réfuter la supposition suivante: *Lorsqu'une série entière à coefficients entiers converge dans un cercle de rayon un, ou bien la fonction représentée est rationnelle ou bien son domaine d'existence est limité par le cercle de convergence.* J'ai dû me contenter d'énoncer ce théorème et de proposer aux mathématiciens de décider s'il est vrai ou non. [...] quelques années plus tard, M. Carlson le résolut par l'affirmative."

more than one variable even in the simplest case of bivariate power series. Our findings in this regard will be presented in chapter 6.

3.4 Dealing with other types of singularities

The remaining discussion of one-variable asymptotics will be as follows. A discussion of an example with an isolated essential singularity will shed some light on the role of oscillating integrals and the method of stationary phase. We will then look at the classical formal power series derivation of asymptotics for branch-points induced by non-integral powers following the discussion of Henrici (see theorem 11.10 in [Hen77]). These will be compared to the so called *transfer theorems* following the discussion of Flajolet and Odlyzko (see [FlaOdl90]) which use analytic methods to obtain the leading term asymptotics for a very wide class of power series.

Since contour integration is an art, it is wise to ask first for the guiding principles behind choosing the contour. To answer this, we recall the Cauchy integral formula (see [Rud87])

$$[z^n]\, F(z) = \frac{1}{2\pi i} \int_\gamma \frac{F(z)}{z^{n+1}}\, dz\,, \tag{3.3}$$

where γ is any closed contour encircling the origin and no singularity of $F(z)$. If $[z^n]\, F(z)$ is of order r^{-n} then the radius of convergence of the power series associated to $F(z)$ is precisely r. However, if we use the above formula to represent $[z^n]\, F(z)$ and select γ to be a circle of radius $\rho \in (0, r)$ then the integrand has order of magnitude ρ^{-n}. Since $[z^n]\, F(z)$ is known to be of order r^{-n} we conclude that in the integration there is a lot of cancellation taking place. Indeed, the cancellation reduces the integral by an exponential factor of $\left(\frac{\rho}{r}\right)^n$. As ρ expands toward r the oscillation kills at a lower exponential rate, until, right near r, it is not really killing at all. This leads to the *stationary phase principle*: find a stretch of the contour where the integrand is not oscillating; this will be the leading contribution to the integral. A related technique is the *saddle point method*: if there is no point of stationary phase on the starting contour, move the contour until you go through one, and make sure you go through at the right angle (that minimizes the oscillation of the integrand).

We do not plan to have a systematic discussion of the stationary phase method or the saddle point method. These are part of a whole toolkit to deal with oscillatory integrals. However, their principles will be used and illustrated in great detail in chapter 5 where we present with a generalized version of both methods. We turn now to some examples.

Example 3.1. (An isolated essential singularity.)

Let $F(z) = \exp\left(\frac{z}{1-z}\right)$. This function is analytic for all $z \in \mathbb{C}$ except for the point $z = 1$ which is an (isolated) essential singularity. By this we mean that there is no $n \geq 0$ such that the $\lim\limits_{z\to 1} |(1-z)^n \cdot F(z)| < \infty$. This follows from the identity

$$\begin{aligned} F(z) &= \tfrac{1}{e} \cdot \exp\left(\tfrac{1}{1-z}\right) , \\ &= \tfrac{1}{e} \sum_{n=0}^{\infty} \tfrac{1}{n!(1-z)^n} . \end{aligned}$$

It is possible to compute the coefficients of $F(z)$ directly from its combinatorial interpretation: F is the exponential generating function for the number of unordered partitions of an n element set into ordered sequences. The analytic approach, however, will allow us to compute these coefficients asymptotically.

Although the disk of convergence of the power series associated to $F(z)$ is the disk $[z : |z| < 1]$, the fact that $F(z)$ is analytic in $\mathbb{C} \setminus \{1\}$ let us choose a variety of contours (not necessarily contained in this disk) to use Cauchy's integral formula as in (3.3). Indeed, since $F(z)$ is analytic at infinity (meaning by this that its is bounded at infinity) we may choose γ to be any vertical line contained in the strip $[z : 0 < \Re\{z\} < 1]$. (See figure 3.1.)

Next, we seek for stationary points of the integrand in (3.3). These are solutions of the equation: $\frac{d}{dz}\left[\frac{F(z)}{z^{n+1}}\right] = 0$. There is only one stationary point within the strip $[z : 0 < \Re\{z\} < 1]$, namely

$$\begin{aligned} z_n &:= 1 - \sqrt{\tfrac{1}{n+1} + \tfrac{1}{4(n+1)^2}} + \tfrac{1}{2(n+1)}\,, \\ &= 1 - n^{-1/2} + O(n^{-1})\,. \end{aligned}$$

The motivation one has to look for stationary points is that it is always possible to chose a contour going through them such that the integrand is locally maximized (along the contour) precisely at the stationary point. The hope is that, for big values of n, most of the contribution to the integral in (3.3) is from integration near this point. The existence of a contour of these characteristics is warranted by the basic principles of the method of steepest descents (see chapter 7 in [BleHan86]). However, it is usually intricate to describe such a contour in a precise manner. Luckily, in most applications, it is not necessary to determine the contour of steepest descents but rather to chose a contour that passes through the stationary point with the right angle. We will clarify what we mean by "the right angle" in the coming computations.

A convenient choice of γ in (3.3) is $\gamma := \{z_n + i \cdot t : t \in \mathbb{R}\}$. Furthermore, if in (3.3) we normalize the integrand by $\frac{F(z_n)}{z_n^{n+1}}$ we obtain that

$$[z^n]\, F(z) = \frac{F(z_n)}{z_n^{n+1}} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} G_n(t)\, dt\,, \tag{3.4}$$

where accordingly it has been defined $G_n(t) := \frac{F(z_n+it)}{F(z_n)} \left(\frac{z_n}{z_n+it}\right)^{n+1}$.

The factor $\frac{F(z_n)}{z_n^{n+1}}$ in (3.4) is easily shown to be asymptotically equivalent to $e^{2\sqrt{n}-\frac{1}{2}}$. Therefore, to determine the leading order of $[z^n]\,F(z)$ all reduces to find the leading order of the integral in (3.4).

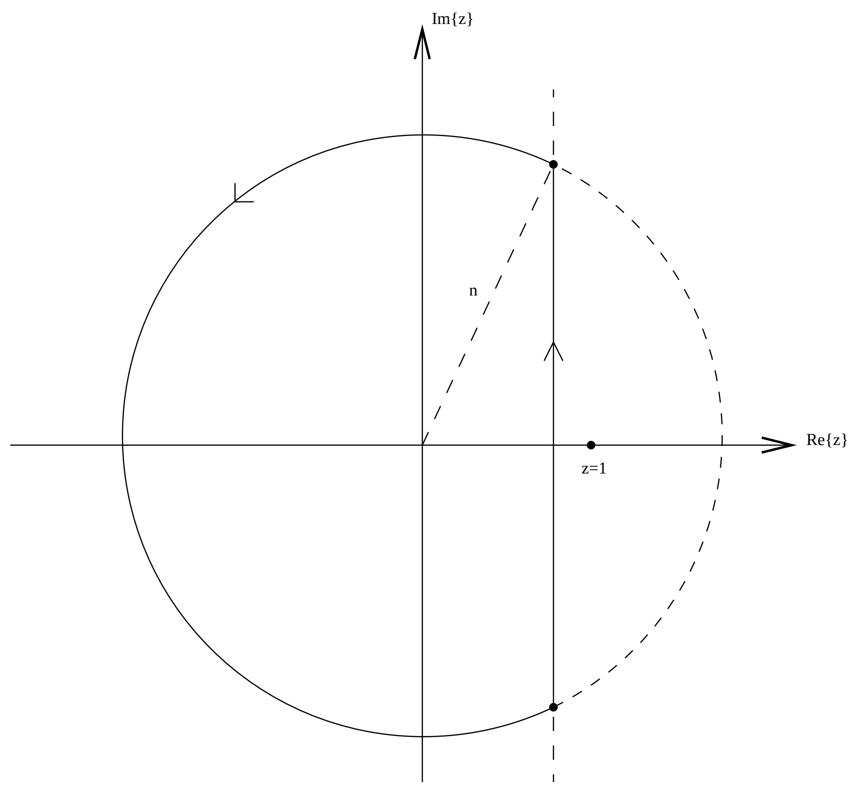

Figure 3.1: Since $F(z)$ is analytic at infinity, the integration of $\frac{F(z)}{z^{n+1}}$ over the circular arc tends to zero, as $n \to \infty$. As a result, $[z^n]\,F(z) = \frac{1}{2\pi i}\int_\gamma \frac{F(z)}{z^{n+1}}dz$, where γ can be chosen to be any vertical line contained in the strip $[z : 0 < \Re\{z\} < 1]$.

To justify the appropriateness of the contour γ, it is convenient to rewrite $G_n(t)$ in the exponential form $e^{-H_n(t)}$. Indeed, using Taylor's formula (see [Rud87]) it follows that there is $\epsilon > 0$ such that

$$(3.5) \qquad \begin{cases} H_n(t) &= H_n''(0) \cdot t^2 + O(n^2 t^3)\,, \\ H_n''(0) &= n^{3/2} \cdot (1 + o(1))\,, \end{cases}$$

uniformly for all $t \in [-\epsilon, \epsilon]$, as $n \to \infty$. This implies that, in an interval of the form $[-t_n, t_n]$ with $n^{1/2} \cdot t_n = o(1)$, the function $H_n(t)$ is minimized solely at $t = 0$. The hope now is that

$$(3.6) \qquad \begin{aligned} \frac{1}{2\pi} \int_{-\infty}^{\infty} G_n(t)\, dt &\sim \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-H_n''(0) \cdot t^2}\, dt \\ &= \frac{1}{\sqrt{4\pi H_n''(0)}}\,, \\ &\sim \frac{1}{\sqrt{4\pi n^{3/2}}}\,, \end{aligned}$$

as $n \to \infty$.

Only the first asymptotic formula in (3.6) deserves a justification. This is verified as follows. (3.5) implies that the main contribution to the integral on the right-hand side in (3.6) comes from a region where $|t|$ is not much bigger than $n^{-3/4}$. We hope the main contribution to the left-hand side in (3.6) comes from this region as well and is nearly the same. Accordingly, we pick a cutoff a little greater, say

$$L_n = n^{-3/4} \cdot \sqrt{\ln n}\,,$$

and break the left and right-hand integral in (3.6) into two parts, $|t| \le L_n$ and $|t| \ge L_n$.

We will show that up to the cutoff the two integrals are close, however, past the cutoff they are both small.

To deal with the case $|t| \leq L_n$, observe that

$$(3.7) \quad \left| \frac{1}{2\pi} \int_{-L_n}^{L_n} G_n(t) - e^{-H_n''(0)\cdot t^2}\, dt \right| \leq \frac{1}{2\pi} \int_{-L_n}^{L_n} \left| e^{H_n''(0)\cdot t^2 - H_n(t)} - 1 \right| \cdot e^{-H_n''(0)\cdot t^2}\, dt\,, \\ = o(1) \cdot \int_{-L_n}^{L_n} e^{-H_n''(0)\cdot t^2}\, dt\,.$$

But, it is simple to derive that

$$(3.8) \quad \frac{1}{2\pi} \int_{-L_n}^{L_n} e^{-H_n''(0)\cdot t^2}\, dt = \frac{1}{2\pi\sqrt{2\,H_n''(0)}} \int_{-\sqrt{2\,H_n''(0)}\,L_n}^{\sqrt{2\,H_n''(0)}\,L_n} e^{-\tau^2/2}\, d\tau\,, \\ \sim \frac{1}{\sqrt{4\pi n^{3/2}}}\,.$$

(3.7) and (3.8) combined imply that

$$(3.9) \quad \frac{1}{2\pi} \int_{-L_n}^{L_n} G_n(t)\, dt \sim \frac{1}{\sqrt{4\pi n^{3/2}}}\,,$$

as $n \to \infty$.

On the other hand, observe that

$$|G_n(t)| = \left| \frac{z_n}{z_n + it} \right|^{n+1} \cdot \exp\left(\Re \left\{ \frac{1}{1 - z_n - it} - \frac{1}{1 - z_n} \right\} \right)\,, \\ \leq (1 + t^2)^{-(n+1)/2} \cdot \exp\left(- \frac{(1 - z_n)^{-3} \cdot t_n^2}{1 + (1 - z_n)^{-2} \cdot t_n^2} \right)\,,$$

for all $|t| \geq L_n$. Since $\frac{(1-z_n)^{-3}\cdot t_n^2}{1+(1-z_n)^{-2}\cdot t_n^2} \sim \ln n$, as $n \to \infty$, it follows for all $\delta > 0$ sufficiently small that

$$(3.10) \quad \frac{1}{2\pi} \int_{|t| \geq L_n} G_n(t)\, dt = O\left(\frac{1}{n^{1-\delta}} \right)\,,$$

as $n \to \infty$. Selecting $0 < \delta < \frac{1}{4}$, (3.9) and (3.10) imply the asymptotic formula in (3.6). As a result, back in (3.4) we obtain the desired asymptotic formula

$$[z^n]\, F(z) = \frac{e^{2\sqrt{n}}}{\sqrt{4\pi e n^{3/2}}} \cdot \big(1 + o(1)\big)\,.$$

Example 3.2. (A branch-point.)

Among singularities that are not isolated, the nicest are branch-points. Many though not all branch-points can be written as a non-integral power times an analytic function. Thus we consider the class of functions of the form $F(z) = \left(1 - \frac{z}{a}\right)^{-b} \cdot \psi(z)$, where a is a nonzero complex-number, b is a complex number that is not an integer, and $\psi(z)$ is an analytic function in a disk of the form $[z : |z| < r]$, with $r > |a|$. In particular, $F(z)$ is analytic on the disk slit from a to the boundary. Following Henrici's exposition (see [Hen77]), the following can be said about the coefficients of $F(z)$.

Theorem 3.3. (Darboux's theorem.) *Let $F(z)$, $\psi(z)$, etc. be as defined before and suppose that $\psi(z) = \sum_{j=0}^{\infty} \alpha_j\,(z-a)^j$ is the power series representation of $\psi(z)$ near $z = a$. Then, for any nonnegative integer $k \geq \Re\{b\}$,*

$$[z^n]\,F(z) = (-a)^{-n} \left\{ \sum_{j=0}^{k} \alpha_j \cdot (-a)^j \cdot \binom{j-b}{n} + o\left(|a|^k \cdot \binom{k-b}{n} \right) \right\}, \tag{3.11}$$

as $n \to \infty$, where $\binom{x}{n} := \frac{1}{n!} \prod_{j=1}^{n} (x - j + 1)$.

Proof. Recall that $\left(1 - \frac{z}{a}\right)^{-b} = \sum_{n=0}^{\infty} (-a)^{-n} \binom{-b}{n} z^n$. Thus, formally speaking, (3.11) results from the calculation

$$\begin{aligned} F(z) &= \sum_{j=0}^{\infty} \alpha_j \cdot (-a)^j \cdot \left(1 - \frac{z}{a}\right)^{j-b}, \\ &= \sum_{n=0}^{\infty} (-a)^{-n} \cdot \left\{ \sum_{j=0}^{\infty} \alpha_j \cdot (-a)^j \cdot \binom{j-b}{n} \right\} z^n. \end{aligned}$$

To justify all this, fix k. The hypothesis on $\psi(z)$ implies that

$$\psi(z) = \sum_{j=0}^{k} \alpha_j \cdot (z-a)^j + R_k(z) \cdot (z-a)^{k+1},$$

where $R_k(z)$ is certain analytic function in the disk $[z : |z| < r]$. In particular, for all $|z| < |a|$, it applies that

$$F(z) - \sum_{j=0}^{k} \alpha_j \cdot (-a)^j \cdot \left(1 - \frac{z}{a}\right)^{j-b} = (-a)^{k+1} \cdot \left(1 - \frac{z}{a}\right)^{k+1-b} \cdot R_k(z)\,.$$

The coefficient of z^n in the left-hand side above is recognized to be: $[z^n]\, F(z) - (-a)^{-n} \cdot \sum_{j=0}^{k} \alpha_j \cdot (-a)^j \binom{j-b}{n}$. Accordingly, if we define

$$G_k(z) := \left(1 - \frac{z}{a}\right)^{k+1-b} \cdot R_k(z)\,,$$

to conclude (3.11) all reduces to show that

$$[z^n]\, G_k(z) = o\left(|a|^{-n} \cdot \binom{k-b}{n}\right)\,. \tag{3.12}$$

It is for this that the assumption $k \geq \Re\{b\}$ will be used. Indeed, it implies that $G_k(z)$ is analytic for $|z| < |a|$ but continuous for $|z| \leq |a|$. Furthermore, $G_k(z)$ is $(l+1)$-times continuously differentiable, with l a nonnegative integer maximal with the property $l \leq \Re\{k-b\}$. This lets us use Cauchy's formula and to integrate by parts $(l+1)$-times to obtain

$$\begin{aligned} [z^n]\, G_k(z) &= \frac{1}{2\pi i} \int_{|z|=|a|} \frac{G_k(z)}{z^{n+1}}\, dz\,, \\ &\vdots \\ &= \frac{1}{n(n-1)\cdot\ldots\cdot(n-l)} \cdot \frac{1}{2\pi i} \int_{|z|=|a|} \frac{G_k^{(l+1)}(z)}{z^{n-l}}\, dz\,. \end{aligned}$$

The Riemann-Lebesgue lemma (see [Rud87]) implies that the $\int_{|z|=|a|} \frac{G_k^{(l+1)}(z)}{z^{n-l}}\, dz$ is a $o(a^{-n})$. As a result, $[z^n]\, G_k(z)$ is a $o(n^{-(l+1)}\, |a|^{-n})$. But, using Stirling's formula (2.52) it follows that $\binom{k-b}{n}$ is of order $n^{-\Re\{k-b\}}$. (3.12) then follows by noticing that $\Re\{k-b\} < (l+1)$. This completes the proof of the theorem. □

Example 3.4. (Transfer theorems.)

The previous example was in some sense a very special case: we knew the expansion of $\left(1-\frac{z}{a}\right)^{-b}$ explicitly, and were able to show how the series behaved under a perturbation that multiplied by an analytic factor. The next example also deals with a special class of functions, but a very wide and hence useful class.

We will let $\mathcal{C}$ to denote the class of functions of the form

$$G(z) := (1-z)^{\alpha} \cdot g\left(\frac{1}{1-z}\right),$$

where $g(z) = (\log z)^{\gamma} \cdot (\log\log z)^{\delta}$ for arbitrary real numbers α, γ and δ. Instead of requiring $F(z) = (1-z)^{\alpha} \cdot \psi(z)$, for $\psi(z)$ analytic in an open disk centered at the origin and containing $z = 1$, we derive information under the assumption only that $F(z) = O(G(z))$ or $F(z) = o(G(z))$, as $z \to 1$. (Naturally, we can normalize so that the dominant singularity appears somewhere else other than $z = 1$.) The price we pay is that we require $F(z)$ to be analytic in a sector of the form (see figure 3.2)

$$\Delta := \left\{z : |z| \le (1+\epsilon)\,,\, |\arg(z-1)| \ge \delta\right\},$$

for certain sufficiently small $\epsilon > 0$ and a fixed $\delta \in \left(0, \frac{\pi}{2}\right)$.

The transfer method of Flajolet and Odlyzko [FlaOdl90] consists of three results. The first is an explicit asymptotic description of the coefficients of all functions in $\mathcal{C}$. The second and third are the following theorems.

Theorem 3.5. (big-O theorem.) *Suppose that $F(z) = \sum_{n=0}^{\infty} f_n\, z^n$ is analytic in Δ and that $G(z) = \sum_{n=0}^{\infty} g_n\, z^n$ is in the class $\mathcal{C}$. If $F(z) = O(G(z))$, as $z \to 1$ along Δ, then*

$$f_n = O(g_n)\,, \ \text{as } n \to \infty\,.$$

Theorem 3.6. (little-o theorem.) *Suppose that* $F(z) = \sum_{n=0}^{\infty} f_n z^n$ *is analytic in* Δ *and that* $G(z) = \sum_{n=0}^{\infty} g_n z^n$ *is in the class* $\mathcal{C}$*. If* $F(z) = o(G(z))$*, as* $z \to 1$ *along* Δ*, then*

$$f_n = o(g_n)\,,\ \ as\ n \to \infty\,.$$

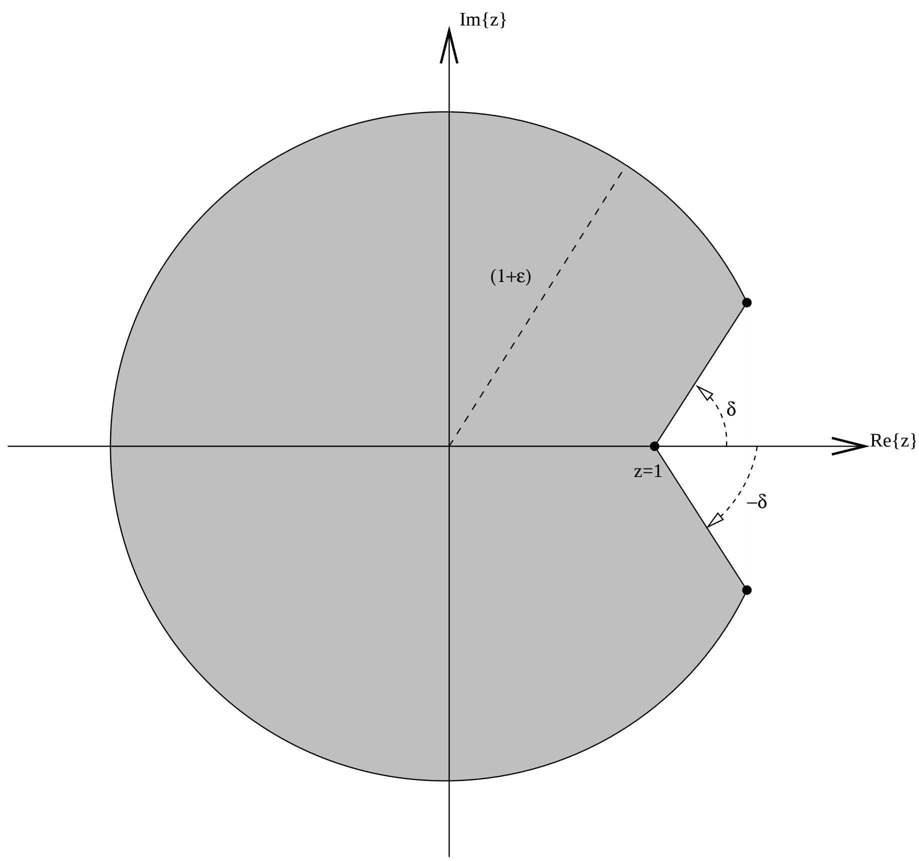

Figure 3.2: Representation of a domain Δ where analyticity is required to use the big-O and little-o theorems.

To see an application of the transfer theorems in the context of asymptotic expansions, consider the class $\mathcal{C}'$ containing only functions of the form $G(z) = (1-z)^\alpha \cdot g\left(\frac{1}{1-z}\right)$, with α an arbitrary real number, and $g(z) = (\log z)^\gamma$, with γ a nonnegative integer.

Corollary 3.7. (Σ-transfer.) *Suppose that $F(z)$ is analytic in Δ and $F(z) \approx \sum_{j=0}^{\infty} G_j(z)$, as $z \in \Delta \to 1$, where, for all $j \geq 0$, $G_j \in \mathcal{C}'$ and $G_{j+1}(z) = o(G_j(z))$, as $z \in \Delta \to 1$. If $F(z) = \sum_{n=0}^{\infty} f_n\, z^n$ and $G_j(z) = \sum_{n=0}^{\infty} g_n^{(j)}\, z^n$ then*

$$f_n \approx \sum_{j=0}^{\infty} g_n^{(j)}\,, \text{ as } n \to \infty\,.$$

Proof. Observe that elements in the class $\mathcal{C}'$ are analytic in Δ. As a result, since $G_{j+1}(z) = o(G_j(z))$, as $z \to 1$ along Δ, the little-o theorem let us conclude that $g_n^{(j+1)} = o(g_n^{(j)})$, as $n \to \infty$. Therefore, $(g_n^{(j)})_{j\geq 0}$ is effectively an asymptotic sequence as $n \to \infty$. Define

$$h_k(z) := F(z) - \sum_{j=0}^{k} G_j(z)\,.$$

By hypothesis, for each k, $h_k(z) = o(G_k)$, as $z \to 1$ along Δ. Another application of the little-o theorem gives that $[z^n]\, h_k(z) = o(g_n^{(k)})$, as $n \to \infty$. The corollary then follows by noticing that $[z^n]\, h_k(z) = f_n - \sum_{j=0}^{k} g_n^{(j)}$. □

The counterpart to corollary 3.7 is the asymptotic determination of the coefficients of functions in the class $\mathcal{C}'$. In general, there are several cases to consider, depending on whether α is or not an integer. For example, corollary 5 in [FlaOdl90] implies that

$$[z^n]\,(1-z)^\alpha \cdot \left(\log \frac{1}{1-z}\right)^\gamma \sim \frac{n^{-\alpha-1}}{\Gamma(-\alpha)} \cdot (\log n)^\gamma\,,$$

this provided that $\alpha, \gamma \notin \{0, 1, \ldots\}$.

To finalize our discussion we will prove the big-O theorem. We remark that the little-o theorem follows from the big-O theorem if one can keep track of the constants. That is, if $F(z) = o(G(z))$, as $z \to 1$ along Δ, then for all $\epsilon > 0$, $|F(z)| \leq \epsilon \cdot |G(z)|$, for all z in some neighborhood of $z = 1$ in Δ. Thus if the constant in the conclusion of the big-O theorem can be made to go to zero as the constant in the hypothesis goes to zero, the little-o theorem is proved.

To not get too far afield, we will only prove the big-O theorem for the restricted class $\mathcal{C}''$ in place of $\mathcal{C}$, where $\mathcal{C}''$ contains all functions of the form $G(z) = (1-z)^{\alpha}$, with α a real number. Our exposition parallels the original discussion of Flajolet and Odlyzko in [FlaOdl90].

First note that for $G(z) = (1-z)^{\alpha} \in \mathcal{C}''$, $[z^n]\, G(z)$ is of order $n^{-(\alpha+1)}$. Thus, to prove the big-O theorem, it will be enough to show that $f_n = O(n^{-(\alpha+1)})$, as $n \to \infty$. Next, note that the assumption that $F(z) = O(|1-z|^{\alpha})$ near $z = 1$ implies (using only continuity, not analyticity) that for some constant $c > 0$, $|F(z)| \leq c \cdot |1-z|^{\alpha}$, for all $z \in \Delta \setminus \{1\}$.

Cauchy's formula implies that

$$f_n = \frac{1}{2\pi i} \int_{\gamma_n} \frac{F(z)}{z^{n+1}}\, dz\,, \tag{3.13}$$

where γ_n is a contour carefully selected as the union of four elementary contours (see figure 3.3). Since we will bound the integral in (3.13) we need not to worry about the orientations.

Let $\gamma_n^{(1)}$ be the circular arc parameterized by $\left(1 + \frac{e^{it}}{n}\right)$, for $\delta \leq t \leq (2\pi - \delta)$.

Let $\gamma_n^{(2)}$ be the line segment between $\left(1 + \frac{e^{i\delta}}{n}\right)$ and the number β of modulus

$(1+\epsilon)$ such that $\arg(\beta - 1) = \delta$.

Let $\gamma^{(3)}$ be the arc on the circle of radius $(1+\epsilon)$ running between β and $\overline{\beta}$ the long way, and let $\gamma_n^{(4)}$ be the ray joining $\bar{\beta}$ with $\left(1 + \frac{e^{-i\delta}}{n}\right)$.

Observe that

$$\left|\frac{1}{2\pi i}\int_{\gamma_n^{(1)}} \frac{F(z)}{z^{n+1}}\,dz\right| \le \frac{c}{2\pi}\cdot\left(1-\frac{1}{n}\right)^{n+1}\cdot n^{-(\alpha+1)}\cdot\int_{\gamma_n^{(1)}}|dz|\,. \tag{3.14}$$

Since the sequence $\left(1-\frac{1}{n}\right)^n$ is bounded and the $\int_{\gamma_n^{(1)}}|dz|$ is at most $\frac{2\pi}{n}$, we conclude that

$$\left|\frac{1}{2\pi i}\int_{\gamma_n^{(1)}} \frac{F(z)}{z^{n+1}}\,dz\right| = O(n^{-(\alpha+1)})\,,$$

as $n \to \infty$.

On the other hand,

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_{\gamma^{(3)}} \frac{F(z)}{z^{n+1}}\,dz\right| &\le \sup_{z\in\gamma^{(3)}}|F(z)|\cdot(1+\epsilon)^{-n}\,,\\ &= o(n^{-(\alpha+1)})\,,\end{aligned}$$

as $n \to \infty$.

The symmetry between $\gamma_n^{(2)}$ and $\gamma_n^{(4)}$ lets us reduce the computation only for $\gamma_n^{(2)}$. We parametrize the integral as $z = 1 + \frac{t\cdot e^{i\delta}}{n}$, for $t = 1$ to $t = n\cdot|\beta - 1|$. Since, $|F(z)| \le c\cdot|z-1|^{\alpha} \le c\left(\frac{t}{n}\right)^{\alpha}$, for all $z \in \gamma_n^{(2)}$, we obtain that

$$\left|\frac{1}{2\pi i}\int_{\gamma_n^{(2)}} \frac{F(z)}{z^{n+1}}\,dz\right| \le \frac{c}{2\pi}\cdot n^{-(\alpha+1)}\cdot\int_1^{\infty} t^{\alpha}\cdot\left(1+\frac{t\cdot\cos(\delta)}{n}\right)^{-n}dt\,.$$

The integral on the right-hand side is convergent for all $n \ge (\alpha+1)$. Since the integrand is a decreasing function of n, the above inequality implies that

$$\left|\frac{1}{2\pi i}\int_{\gamma_n^{(2)}} \frac{F(z)}{z^{n+1}}\,dz\right| = O(n^{-(\alpha+1)})\,,$$

as $n \to \infty$. Finally, since we have bounded all four integrals by multiples of $n^{-(\alpha+1)}$, the big-O theorem for the class $\mathcal{C}''$ follows from (3.13).

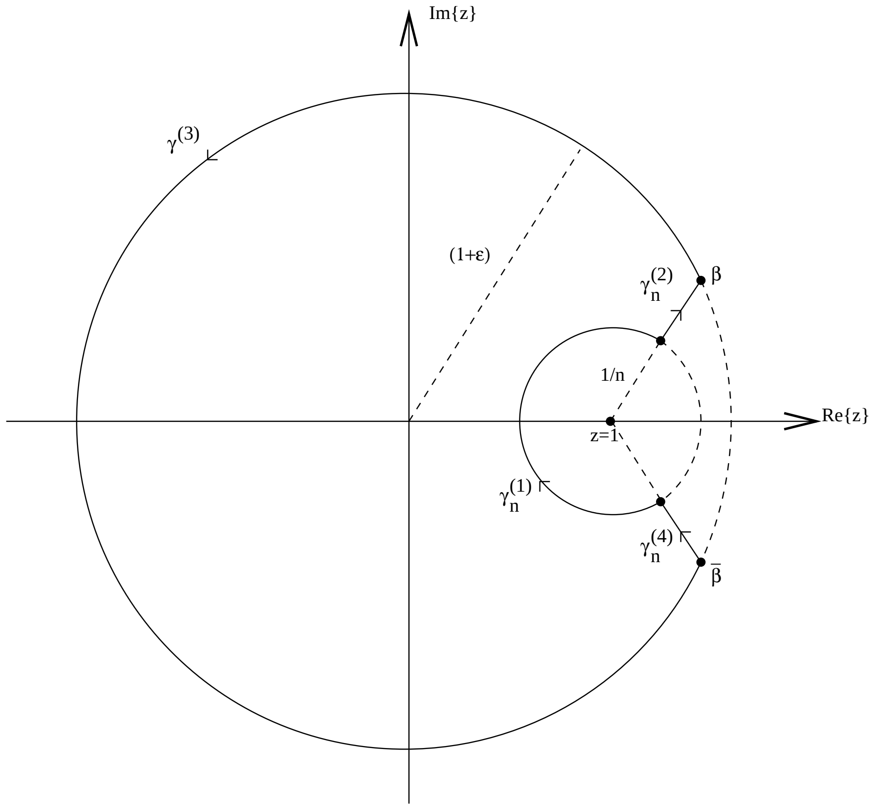

Figure 3.3: Plot of the contour $\gamma_n := \gamma_n^{(1)} + \gamma_n^{(2)} + \gamma^{(3)} + \gamma_n^{(4)}$ used to prove the big-O theorem of Flajolet and Odlyzko on the class $\mathcal{C}''$.

CHAPTER 4

MULTI-VARIABLE GENERATING FUNCTIONS

4.1 Basic notation.

Throughout this chapter $d \geq 2$ will be a fixed integer and $i := \sqrt{-1}$. We will use boldface to denote vectors in $\mathbb{C}^d$ and hence also in $\mathbb{R}^d$ or $\mathbb{N}^d$. As otherwise stated we will assume that all vectors are d-dimensional.

The coordinates of a vector will be denoted consistently using the letter used to name it; thus, for example, the coordinates of a $\mathbf{z} \in \mathbb{C}^d$ will be $(z_1, \ldots, z_d)$ whereas the ones for $\mathbf{n}$ will be $(n_1, \ldots, n_d)$. We will reserve the notation $\mathbf{0}$ to refer to the zero vector. On the other hand, $\mathbf{1}$ will represent the vector with all entries equal to 1.

The scripts $\mathbf{n}$ and $\mathbf{k}$ will be always assumed to be vectors with nonnegative integer coordinates. We will define $\langle \mathbf{n} \rangle := \sum_{j=1}^{d} n_j$. The notation $\mathbf{n} \leq \mathbf{k}$ will be used to mean that $n_j \leq k_j$ for all $j = 1, \ldots, d$. Similarly, we will write $\mathbf{n} < \mathbf{k}$ to mean that $n_j < k_j$ for all $j = 1, \ldots, d$.

The scripts $\mathbf{z}$ and $\mathbf{w}$ will mostly denote vectors with complex-valued entries. A specific notation relating their real and imaginary part will be used. We will define $x_j + i\, y_j := z_j$ and $u_j + i\, v_j := w_j$ where x_j, y_j, u_j and v_j are real numbers. The *dot product* of $\mathbf{z}$ and $\mathbf{w}$ is defined as $\mathbf{z} \cdot \mathbf{w} := \sum_{j=0}^{\infty} z_j \cdot \bar{w}_j$. The Euclidean norm of $\mathbf{z}$ will be denoted $\|z\|_2 := \sqrt{\mathbf{z}\cdot\mathbf{z}}$.

For convenience we will define $\mathbf{n}! := n_1! \cdot \ldots \cdot n_d!$ and $\mathbf{z}^{\mathbf{n}} := z_1^{n_1} \cdot \ldots \cdot z_d^{n_d}$.

We will also use the convention that a function, ostensibly of one-complex variable, applied to an element of $\mathbb{C}^d$ acts on each coordinate separately — thus, for example, we will define

$$e^{\mathbf{z}} := (e^{z_1}, \ldots, e^{z_d}) .$$

The only exceptions to this rule will be $\frac{1}{\mathbf{z}} := \frac{1}{z_1 \cdot \ldots \cdot z_d}$, provided that the denominator does not vanish, and $|\mathbf{z}| := |z_1| \cdot \ldots \cdot |z_d|$.

4.2 Formal power series.

A *multidimensional array indexed by d-tuples* is a function $f : \mathbb{N}^d \to \mathbb{C}$. We will use the alternative notation $(f_{\mathbf{n}})$ to refer to the multidimensional array f such that $f(\mathbf{n}) = f_{\mathbf{n}}$, for all $\mathbf{n}$. The *formal power series* or *generating function associated to multidimensional array* $(f_{\mathbf{n}})$ is defined to be the

$$F := \sum_{\mathbf{n}} f_{\mathbf{n}}\, \mathbf{z}^{\mathbf{n}} .$$

The *set of formal power series in the variables* $z_1, \ldots, z_d$ will be denoted as $\mathbb{C}[[\mathbf{z}]]$.

For $F \in \mathbb{C}[[\mathbf{z}]]$ we will write $[\mathbf{z}^{\mathbf{n}}]\, F$ to refer to the coefficient of $\mathbf{z}^{\mathbf{n}}$ in F. The special notation $F(\mathbf{0})$ will be adopted to denote the constant term in the series. Furthermore, as we did in the one-dimensional case, we will attach to a naming notation relating generating functions and their coefficients. Thus, for example, the coefficients of the formal power series F, G and H will be consistently denoted and without further explanation as $f_{\mathbf{n}}$, $g_{\mathbf{n}}$ and $h_{\mathbf{n}}$ respectively.

Given two formal power series $F(\mathbf{z})$ and $G(\mathbf{z})$ we will define their *sum* and *product* respectively as

$$F + G := \sum_{\mathbf{n}} u_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}} \,, \tag{4.1}$$

$$F \cdot G := \sum_{\mathbf{n}} v_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}} \,, \tag{4.2}$$

where $u_{\mathbf{n}} := f_{\mathbf{n}} + g_{\mathbf{n}}$ and $v_{\mathbf{n}} := \sum_{\mathbf{k}:\mathbf{0}\leq\mathbf{k}\leq\mathbf{n}} f_{\mathbf{k}} \cdot g_{\mathbf{n}-\mathbf{k}}$. With these operations $\mathbb{C}[[\mathbf{z}]]$ is a *commutative ring*. Moreover, it is also an *integral domain* because if F, $G \in \mathbb{C}[[\mathbf{z}]]$ are such that $F \cdot G = 0$ then either $F = 0$ or $G = 0$.

Following an argument similar to the one given for bivariate power series in section 2.5, one can show that a formal power series F has a multiplicative inverse if and only if $F(\mathbf{0}) \neq 0$. In other words, the *units* of $\mathbb{C}[[\mathbf{z}]]$ are series with a nonzero constant term. We will mostly use the notation $\frac{1}{F}$ to refer to the multiplicative inverse of a unit element F instead of the more conventional notation F^{-1}.

We will introduce in $\mathbb{C}[[\mathbf{z}]]$ *pseudo-differential operators*. We define the *partial derivative of F with respect to z_j* as the formal power series defined as

$$\frac{\partial F}{\partial z_j} := \sum_{\mathbf{n}:n_j\geq 1} n_j \, f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}-\mathbf{e}_j} \,, \tag{4.3}$$

where $\mathbf{e}_j$ is the vector that has all its coordinates identically zero, however, its jth coordinate equals 1. If we regard $\mathbb{C}[[\mathbf{z}]]$ as a vector space over $\mathbb{C}$ then $\frac{\partial}{\partial z_j}$ is a linear operator. Since the operators $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial z_k}$ commute we can unambiguously define, for all $\mathbf{k}$, the pseudo-derivative

$$\frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}} := \frac{\partial^{k_1}}{\partial z_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial z_d^{k_d}} F = \sum_{\mathbf{n}:\mathbf{n}\geq\mathbf{k}} \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!} \, f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}-\mathbf{k}} \,. \tag{4.4}$$

In particular, for all $\mathbf{k}$, we have the identity

$$[\mathbf{z}^{\mathbf{k}}]F = \frac{1}{\mathbf{k}!} \frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}}(\mathbf{0}) \,.$$

4.3 Domain of convergence.

Hereafter, we will reserve the term *polyradius* to refer to vectors $\mathbf{r}$ such that $\mathbf{r} > \mathbf{0}$. The *open polydisk centered at $\boldsymbol{z}$ and with polyradius $\boldsymbol{r}$* is set defined as $\Delta(\mathbf{z}, \mathbf{r}) := \{\mathbf{w} : |w_j - z_j| < r_j, \text{ for all } j\}$. Accordingly, the *closed polydisk centered at* $\mathbf{z}$ *with polyradius* $\mathbf{r}$ it is defined to be the set $\Delta[\mathbf{z}, \mathbf{r}] := \{\mathbf{w} : |w_j - z_j| \leq r_j, \text{ for all } j\}$. The special notation $\Delta(\mathbf{z})$ and $\Delta[\mathbf{z}]$ will be used to refer to the open and closed polydisk centered at $\mathbf{0}$ with polyradius $(|z_1|, \ldots, |z_d|)$ respectively. Moreover, we will define $\mathcal{T}[\mathbf{z}] := \{\mathbf{w} : |w_j| = |z_j|, \text{ for all } j\}$. Observe that $\mathcal{T}[\mathbf{z}]$ is a manifold with real dimension d and therefore it is strictly contained in the boundary of $\Delta(\mathbf{z})$ which has real dimension $(2d - 1)$.

In one-complex variable the domain of convergence of a power series is either empty, an open disk or the entire complex plane (see section 10.5 in [Rud87]). In several complex variables the situation is quite more intriguing. For example, the bivariate power series $\sum_{n=0}^{\infty} x^n y^n$ is absolutely convergent only for those $(x, y) \in \mathbb{C}^2$ such that $|x| \cdot |y| < 1$. This domain is not even convex hence it cannot be a disk (using the euclidian norm in $\mathbb{C}^2$) nor a polydisk in $\mathbb{C}^2$.

Our discussion on the domain of convergence of formal power series begins with the following result.

Theorem 4.1. *Let $\boldsymbol{w}$ be such that $\boldsymbol{w}_j \neq 0$ for all j and suppose that a formal power series F is absolutely convergent at $\boldsymbol{w}$; that is to say, the $\sum_{\boldsymbol{n}} |f_{\boldsymbol{n}} \boldsymbol{w}^{\boldsymbol{n}}| < \infty$. Then, for all $\boldsymbol{z} \in \Delta(\boldsymbol{w})$ and $\boldsymbol{k}$ the series $\frac{\partial^{\boldsymbol{k}} F}{\partial \boldsymbol{z}^{\boldsymbol{k}}}$ is uniformly absolutely convergent on $\Delta[\boldsymbol{z}]$.* [1]

[1]Given a set $\mathcal{D} \subset \mathbb{C}^d$ and functions $g_{\mathbf{n}}(\mathbf{z})$ defined for all $\mathbf{z} \in \mathcal{D}$, the series $\sum_{\mathbf{n}} g_{\mathbf{n}}(\mathbf{z})$ is said to be *uniformly absolutely convergent (in $\mathcal{D}$)* if there is a finite constant $c > 0$ such that the $\sum_{\mathbf{n}} |g_{\mathbf{n}}(\mathbf{z})| \leq c$, for all $z \in \mathcal{D}$.

Proof. Define $c := \sum_{\mathbf{n}} |f_{\mathbf{n}} \mathbf{w}^{\mathbf{n}}|$; in particular, $|f_{\mathbf{n}}| \leq \frac{c}{|\mathbf{w}^{\mathbf{n}}|}$, for all $\mathbf{n}$.

Given $\mathbf{z} \in \Delta(\mathbf{w})$ it follows for all $\mathbf{x} \in \Delta[\mathbf{z}]$ and $\mathbf{k}$ that

$$\begin{aligned} \sum_{\mathbf{n}:\mathbf{n}\geq\mathbf{k}} \left| \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!} f_{\mathbf{n}} \, \mathbf{x}^{\mathbf{n}-\mathbf{k}} \right| &\leq \frac{c \cdot \mathbf{k}!}{|\mathbf{w}^{\mathbf{k}}|} \sum_{\mathbf{n}:\mathbf{n}\geq\mathbf{k}} \frac{\mathbf{n}!}{\mathbf{k}!\,(\mathbf{n}-\mathbf{k})!} \left| \frac{\mathbf{z}^{\mathbf{n}-\mathbf{k}}}{\mathbf{w}^{\mathbf{n}-\mathbf{k}}} \right| , \\ &= \frac{c \cdot \mathbf{k}!}{|\mathbf{w}^{\mathbf{k}}|} \prod_{j=1}^{d} \frac{1}{(1-|z_j/w_j|)^{k_j+1}} , \end{aligned}$$

where the last identity proceeds from (2.11).

The above inequality implies that there is a finite constant $c' > 0$ such that

$$\sum_{\mathbf{n}:\mathbf{n}\geq\mathbf{k}} \left| \frac{\mathbf{n}!}{(\mathbf{n}-\mathbf{k})!} f_{\mathbf{n}} \, \mathbf{x}^{\mathbf{n}-\mathbf{k}} \right| \leq c' , \text{ for all } \mathbf{x} \in \Delta[\mathbf{z}] .$$

This shows that the series $\frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}}$ as defined in (4.4) is uniformly absolutely convergent over $\Delta[\mathbf{z}]$. This completes the proof of the theorem. ☐

We may rephrase theorem 4.1 as follows.

Corollary 4.2. *A sufficient condition in order for a formal power series F and its pseudo-derivatives of any order to be uniformly absolutely convergent on any compact subset of a polydisk $\Delta(\mathbf{0}, \mathbf{r})$ is that there is a finite constant $c > 0$ such that*

$$|f_{\boldsymbol{n}}| \leq c \cdot \boldsymbol{r}^{-\boldsymbol{n}} , \text{ for all } \boldsymbol{n} . \tag{4.5}$$

Proof. Given $\lambda \in (0,1)$ the above inequality implies that

$$\begin{aligned} \sum_{\mathbf{n}} |f_{\mathbf{n}} \, (\lambda \, \mathbf{r})^{\mathbf{n}}| &\leq c \cdot \sum_{\mathbf{n}} \lambda^{\langle \mathbf{n} \rangle} , \\ &= \frac{c}{(1-\lambda)^d} . \end{aligned}$$

Therefore F is absolutely convergent at each point of the form $\lambda \, \mathbf{r}$. Using theorem 4.1 we can conclude that for all $\mathbf{k}$ the formal power series $\frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}}$ is uniformly absolutely convergent over the polydisk $\Delta[\lambda\,\mathbf{r}]$. The corollary follows by noticing that any compact set of $\Delta(\mathbf{0},\mathbf{r})$ is contained in a polydisk of the form $\Delta[\lambda\,\mathbf{r}]$, for some $\lambda \in (0,1)$. □

The condition in (4.5) indeed ensures that F defines a $\mathcal{C}^\infty$-function. Before we state our next result it will be useful to introduce some notation.

We will define for $j = 1,\dots,d$ the differential operator

$$\frac{\partial}{\partial z_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right). \tag{4.6}$$

The above operators commute over any class of $\mathcal{C}^\infty$-functions defined on an open subset of $\mathbb{C}^d$. This motivates to define for each $\mathbf{k}$ the higher order differential operator

$$\frac{\partial^{\mathbf{k}}}{\partial \mathbf{z}^{\mathbf{k}}} := \frac{\partial^{k_1}}{\partial z_1^{k_1}} \cdots \frac{\partial^{k_d}}{\partial z_d^{k_d}}.$$

To distinguish between pseudo-differential operators (whose domain is the ring of formal power series) and the ones just defined (whose domain are $\mathcal{C}^\infty$-functions defined over an open set in $\mathbb{C}^d$) we will use the notation $\frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}}(\mathbf{z})$ whenever F is a $\mathbb{C}^\infty$-function defined in some open neighborhood of a point $\mathbf{z}$. The following result is now an almost direct consequence of corollary 4.2.

Corollary 4.3. *The condition in (4.5) implies that the formal power series F converges to a $\mathcal{C}^\infty$-function on the polydisk $\Delta(\boldsymbol{0},\boldsymbol{r})$. Moreover, for all $\boldsymbol{z}$ in this polydisk and for all $\boldsymbol{k}$*

$$\frac{\partial^{\boldsymbol{k}} F}{\partial \boldsymbol{z}^{\boldsymbol{k}}}(\boldsymbol{z}) = \sum_{\boldsymbol{n}:\boldsymbol{n}\geq\boldsymbol{k}} \frac{\boldsymbol{n}!}{(\boldsymbol{n}-\boldsymbol{k})!}\, f_{\boldsymbol{n}}\, \boldsymbol{z}^{\boldsymbol{n}-\boldsymbol{k}},$$

and the series is uniformly absolutely convergent on any compact subset of $\Delta(\boldsymbol{0},\boldsymbol{r})$; in particular, for all $\boldsymbol{n}$ one finds that

$$f_{\boldsymbol{n}} = \frac{1}{\boldsymbol{n}!}\frac{\partial^{\boldsymbol{n}} F}{\partial \boldsymbol{z}^{\boldsymbol{n}}}(\boldsymbol{0}). \tag{4.7}$$

Proof. It is almost direct to see that condition (4.5) implies that for all $\mathbf{k}$ there is a constant $c_{\mathbf{k}} > 0$ such that

$$\left| [\mathbf{z}^{\mathbf{n}}] \, \frac{\partial^{\mathbf{k}} F}{\partial \mathbf{z}^{\mathbf{k}}} \right| \leq c_{\mathbf{k}} \cdot \mathbf{r}^{-\mathbf{n}} \, ,$$

for all $\mathbf{n}$.

Therefore, using an inductive argument on the order of the partial derivatives, to prove the corollary, it will be enough to show that F is $\mathcal{C}^1$ over $\Delta(\mathbf{0}, \mathbf{r})$. Without loss of generality, all reduces to prove that F has a continuous partial derivative with respect to z_d. We prove this by induction on the number of variables

Corollary 4.2 implies that the $\sum_{\mathbf{n}} f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}}$ converges uniformly over compact subsets of $\Delta(\mathbf{0}, \mathbf{r})$. This lets us think of F as a continuous function such that $\sum_{\mathbf{n}} f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}} = F(\mathbf{z})$, for all $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$. Furthermore, for the same reasons, for $j = d$, the formal series in (4.3) converges uniformly over compact subsets of $\Delta(\mathbf{0}, \mathbf{r})$ to certain continuous function $F_j(\mathbf{z})$. The uniform convergence implies that

$$F(\mathbf{z}) = \int_0^{z_d} F_d(z_1, \ldots, z_{d-1}, \xi) \, d\xi + F(z_1, \ldots, z_{d-1}, 0) \, ,$$

for all $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$. This implies immediately that F is continuously differentiable with respect to z_d. Moreover, it follows that

$$\frac{\partial F}{\partial z_d}(\mathbf{z}) = F_d(\mathbf{z}) = \sum_{\mathbf{n}:\mathbf{n} \geq \mathbf{e}_d} n_d \, f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}-\mathbf{e}_d} \, ,$$

for all $\mathbf{z}$ in this polydisk. This completes the proof of the corollary. □

The *domain of convergence* of a formal power series $F \in \mathbb{C}[[\mathbf{z}]]$ is defined to be the set of points $\mathbf{z}$ such that the series $\sum_{\mathbf{n}} f_{\mathbf{n}} \, \mathbf{z}^{\mathbf{n}}$ is uniformly absolutely convergent for all points in an open polydisk containing $\mathbf{z}$. We will use the notation $\mathcal{D}(F)$ to refer to the domain of convergence of F.

Clearly, $\mathcal{D}(F)$ is an open subset of $\mathbb{C}^d$. Moreover, due to corollary 4.2, if $\mathbf{z} \in \mathcal{D}(F)$ then $\Delta[\mathbf{z}] \subset \mathcal{D}(F)$. This last fact is expressed in the literature of several complex variables by saying that $\mathcal{D}(F)$ is a *complete Reinhardt domain centered at* $\boldsymbol{0}$. As a result, we obtain that

$$\mathcal{D}(F) = \bigcup_{\mathbf{z} \in \mathcal{D}(F)} \Delta(\mathbf{z}) ,$$

and hence $\mathcal{D}(F)$ is a union of open polydisk centered at $\mathbf{0}$.

Another set of interest related to a formal power series F is

$$\mathcal{B}(F) := \Big\{ \mathbf{z} : \text{ there is a finite constant } c > 0 \text{ such that : } |f_{\mathbf{n}}\, \mathbf{z}^{\mathbf{n}}| \leq c, \text{ for all } \mathbf{n} \Big\} .$$

An often useful relation between $\mathcal{D}(F)$ and $\mathcal{B}(F)$ is that

$$\mathcal{D}(F) = \text{Interior}\big(\mathcal{B}(F)\big) . \tag{4.8}$$

The proof of the above identity proceeds as follows. The fact, $\text{Interior}\big(\mathcal{B}(F)\big) \subset \mathcal{D}(F)$ is almost an immediate consequence of corollary 4.2. On the other hand, if $\mathbf{z} \in \mathcal{D}(F)$ then the series $\sum_{\mathbf{n}} |f_{\mathbf{n}}\, \mathbf{z}^{\mathbf{n}}|$ is convergent and hence there is a finite constant $c > 0$ such that $|f_{\mathbf{n}}\, \mathbf{z}^{\mathbf{n}}| \leq c$, for all $\mathbf{n}$. Thus $\mathbf{z} \in \mathcal{B}(F)$. This shows that $\mathcal{D}(F) \subset \mathcal{B}(F)$. But, being $\mathcal{D}(F)$ open, we can conclude that $\mathcal{D}(F) \subset \text{Interior}\big(\mathcal{B}(F)\big)$. This proves the identity.

Observe that $\mathcal{B}(F)$ like $\mathcal{D}(F)$ is also a complete Reinhardt domain centered at $\mathbf{0}$. Thus both sets are characterized by its intersection with $\mathbb{R}^d_+$. This motivates to define the sets

$$\begin{aligned} \log \mathcal{D}(F) &:= \{\mathbf{x} \in \mathbb{R}^d : e^{\mathbf{x}} \in \mathcal{D}(F)\} , \\ \log \mathcal{B}(F) &:= \{\mathbf{x} \in \mathbb{R}^d : e^{\mathbf{x}} \in \mathcal{B}(F)\} . \end{aligned}$$

Corollary 4.4. *The set* $\log \mathcal{D}(F)$ *is open and convex.*

Proof. To show that $\log \mathcal{D}(F)$ is open first observe that the set $\mathcal{D}(F) \cap \mathbb{R}_+^d$ is open in $\mathbb{R}^d$. On the other hand, an argument similar to the one used to prove (4.8) shows that: $\mathcal{D}(F) \cap \mathbb{R}_+^d = \text{Interior}\Big(\mathcal{B}(F) \cap \mathbb{R}_+^d\Big)$. But the transformation $e^{\mathbf{x}} : \mathbb{R}^d \to \mathbb{R}_+^d$ is an homeomorphism; in particular, since the pre-image of $\mathcal{D}(F) \cap \mathbb{R}_+^d$ is $\log \mathcal{D}(F)$ whereas the pre-image of $\mathcal{B}(F) \cap \mathbb{R}_+^d$ is $\log \mathcal{B}(F)$ we obtain that

$$\log \mathcal{D}(F) = \text{Interior}\Big(\log \mathcal{B}(F)\Big)\,. \tag{4.9}$$

This implies that $\log \mathcal{D}(F)$ is open.

Recall that the interior of a convex set in $\mathbb{R}^d$ is also convex. Thus, to show the convexity of $\log \mathcal{D}(F)$ it will be enough to show that $\log \mathcal{B}(F)$ is convex. Hence, suppose that $\mathbf{x}, \mathbf{y} \in \log \mathcal{B}(F)$; in particular, there is $c > 0$ such that $|f_{\mathbf{k}}|\, e^{\mathbf{k}\cdot\mathbf{x}} \leq c$ and $|f_{\mathbf{k}}|\, e^{\mathbf{k}\cdot\mathbf{y}} \leq c$, for all $\mathbf{k}$. Then for all $\lambda \in (0,1)$ we have that

$$|f_{\mathbf{k}}|\, e^{\mathbf{k}\cdot(\lambda\mathbf{x}+(1-\lambda)\mathbf{y})} = \Big\{|f_k| e^{\mathbf{k}\cdot\mathbf{x}}\Big\}^{\lambda} \cdot \Big\{|f_{\mathbf{k}}| e^{\mathbf{k}\cdot\mathbf{y}}\Big\}^{1-\lambda} \leq c\,.$$

This shows that $\{\lambda\mathbf{x} + (1-\lambda)\mathbf{y}\} \in \log \mathcal{B}(F)$ and hence that $\log \mathcal{B}(F)$ is convex. This completes the proof of the corollary. □

Because of corollary 4.4, $\mathcal{D}(F)$ is described as being *logarithmically convex*. This can be of great use to determine in many situations an upper bound for the *exponential growth* of the coefficients of F in a predetermined direction $\mathbf{k}$; that is, an upper bound for the $\log |f_{s\cdot\mathbf{k}}|$, as $s \to \infty$. Indeed, suppose that the quantity

$$\gamma(\mathbf{k}) := -\sup_{\mathbf{x}\in\partial \log \mathcal{D}(F)} \mathbf{k}\cdot\mathbf{x}$$

is attained at certain $\mathbf{x} \in \partial \log \mathcal{D}(F)$ and that $\log \mathcal{B}(F)$ is closed. Then, due to (4.9), it follows that $e^{\mathbf{x}} \in \mathcal{B}(F)$ and hence there is a finite constant $c > 0$ such that

$$\log |f_{s\cdot\mathbf{k}}| \leq s \cdot \gamma(\mathbf{k}) + \log(c)\,. \tag{4.10}$$

The cases where most is known about the asymptotic behavior of the coefficients $f_{s\cdot\mathbf{k}}$ is when their exponential growth is exactly $s \cdot \gamma(\mathbf{k})$; that is, when there is a constant $c > 0$ such that $|f_{s\cdot\mathbf{k}}| \sim c \cdot e^{s\cdot\gamma(\mathbf{k})}$, as $s \to \infty$. In this context typically a local analysis of the function F near the point $e^{\mathbf{x}}$ which makes $\gamma(\mathbf{k}) = -\mathbf{k} \cdot \mathbf{x}$ is enough to get a great deal of asymptotic information of the coefficients of F along the direction specified by $\mathbf{k}$.

To finish our discussion we quote a result (see [Pem02], chapter 5) which answers negatively to the question of whether the exponential rate of growth of $f_{s\cdot\mathbf{k}}$ is $s \cdot \gamma(\mathbf{k})$.

Lemma 4.5. *Let F be a power series and suppose that $\boldsymbol{x} \in \partial \log \mathcal{D}(F)$ is such that $\gamma(\boldsymbol{k}) = -\boldsymbol{k} \cdot \boldsymbol{x}$. If the hyperplane through $\boldsymbol{x}$ normal to $\boldsymbol{k}$ is not a support hyperplane of $\log \mathcal{D}(F)$ then the quantity $|f_{s\cdot\boldsymbol{k}}| \cdot e^{-s\cdot\gamma(\boldsymbol{k})}$ decreases exponentially, as $s \to \infty$.*

Proof. The hypotheses of the lemma imply that there is a $\mathbf{y} \in \log \mathcal{D}(F)$ such that

$$(\mathbf{y} - \mathbf{x}) \cdot \mathbf{k} > 0\,.$$

In addition to this, (4.9) implies that $\mathbf{y} \in \log \mathcal{B}(D)$ and hence, there is a constant $c > 0$ such that $|f_{s\cdot\mathbf{k}}| \cdot e^{s(\mathbf{k}\cdot\mathbf{y})} \leq c$, for all s. As a result, we obtain that

$$|f_{s\cdot\mathbf{k}}| \cdot e^{-s\cdot\gamma(\mathbf{k})} \leq c \cdot e^{-s(\mathbf{y}-\mathbf{x})\cdot\mathbf{k}}\,,$$

and this proves the lemma. □

4.4 Analytic functions

In this section we explore in more depth properties that characterize a function which can be represented as a convergent power series near the origin. For the sake of generality it will be convenient to consider power series centered at other points besides the origin. Such a point will be generically denoted by $\mathbf{a}$. A *formal power series centered at* $\boldsymbol{a}$ is by definition any series of the form $\sum_{\mathbf{n}} f_{\mathbf{n}} (\mathbf{z}-\mathbf{a})^{\mathbf{n}}$.

Consider for $j = 1, \ldots, d$ the differential operators

$$\frac{\partial}{\partial \bar{z}_j} := \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right), \tag{4.11}$$

whose domain are $\mathcal{C}^\infty$-functions defined on an open subset of $\mathbb{C}^d$.

An observation of major relevance is that

$$\frac{\partial}{\partial \bar{z}_j}\left[(\mathbf{z}-\mathbf{a})^{\mathbf{k}}\right] = 0\,,$$

for all $j = 1, \ldots, d$. This implies that any finite linear combination of terms of the form $(\mathbf{z}-\mathbf{a})^{\mathbf{k}}$ for various $\mathbf{k}$'s is in the kernel of the differential operators $\frac{\partial}{\partial \bar{z}_j}$, $j = 1, \ldots, d$. Moreover, if $F \in \mathbb{C}[[\mathbf{z}]]$ has a nonempty domain of convergence $\mathcal{D}(F)$, corollary 4.3 promise that

$$\frac{\partial F}{\partial \bar{z}_j}(\mathbf{z}) = 0\,, \tag{4.12}$$

for all $j = 1, \ldots, d$ and for all $\mathbf{z} \in \mathcal{D}(F)$.

Given an open set $\mathcal{D} \subset \mathbb{C}^d$, a $\mathcal{C}^1(\mathcal{D})$-function $F(\mathbf{z})$ that holds the above equations for all $j = 1, \ldots, d$ and all $\mathbf{z} \in \mathcal{D}$ is said to satisfy the *Cauchy-Riemann equations* over $\mathcal{D}$. In terms of the theory of one-complex variable this simply means that F is analytic in each coordinate provided that the other $(d-1)$ coordinates remain fixed.

Because of this, $F(\mathbf{z})$ is said to be *analytic* or sometimes *holomorphic* on $\mathcal{D}$. We will denote the set of analytic functions over an open set $\mathcal{D}$ with the symbol $\mathcal{H}(\mathcal{D})$.

With the introduced terminology we see that if the domain of absolute convergence of a power series F is not empty then $F \in \mathcal{H}(\mathcal{D}(F))$. A form of converse to this property is the following result whose proof will be momentarily postponed.

Theorem 4.6. *Suppose that $F(\boldsymbol{z})$ is a continuous function in an open set $\mathcal{D}$ which is a complete Reinhardt domain centered at $\boldsymbol{0}$; that is, for all $\boldsymbol{z} \in \mathcal{D}$ the closed polydisk $\mathcal{D}[\boldsymbol{z}] \subset \mathcal{D}$. If $F(\boldsymbol{z})$ satisfies the Cauchy-Riemann equations over $\mathcal{D}$ then*

$$F(\boldsymbol{z}) = \sum_{\boldsymbol{n}} \frac{F^{(\boldsymbol{n})}(\boldsymbol{0})}{\boldsymbol{n}!} \, \boldsymbol{z}^{\boldsymbol{n}} \,, \tag{4.13}$$

and the series is uniformly absolutely convergent on any compact subset of $\mathcal{D}$. Furthermore, the coefficients of F admit the alternative representation

$$[\boldsymbol{z}^{\boldsymbol{n}}]F = \frac{1}{(2\pi i)^d} \int_{\mathcal{T}[\boldsymbol{w}]} \frac{F(\boldsymbol{u})}{\boldsymbol{u}^{\boldsymbol{n}+\boldsymbol{1}}} \, d\boldsymbol{u} \,, \tag{4.14}$$

valid for all $\boldsymbol{n}$ and all $\boldsymbol{w} \in \mathcal{D}$.

The representation of an analytic function as a power series such as in (4.13) is primarily the consequence of the so called *Cauchy's integral formula (in several complex variables).* This is a generalization of the very same well-known formula in the theory of one-complex variable (see [Rud87]). Under the same conditions imposed to $F(\mathbf{z})$ in theorem 4.6, Cauchy's integral formula states that

$$F(\mathbf{z}) = \frac{1}{(2\pi i)^d} \int_{T[\mathbf{w}]} \frac{F(\mathbf{u})}{\mathbf{u} - \mathbf{z}} \, d\mathbf{u} \,, \tag{4.15}$$

for all $\mathbf{z} \in \mathcal{D}(\mathbf{w})$ and $\mathbf{w} \in \mathcal{D}$.

Since the above integral is defined through a convolution we may differentiate both sides to obtain the more generally the formula

$$\text{(4.16)} \qquad \frac{F^{(\mathbf{n})}(\mathbf{z})}{\mathbf{n}!} = \frac{1}{(2\pi i)^d} \int_{T[\mathbf{w}]} \frac{F(\mathbf{u})}{(\mathbf{u}-\mathbf{z})^{\mathbf{n}+\mathbf{1}}}\, d\mathbf{u}\,.$$

To prove Cauchy's formula consider $\mathbf{z}$ and $\mathbf{w}$ as in (4.15). Since F is analytic over $\mathcal{D}$, the function $u_1 \to F(u_1, z_2, \ldots, z_d)$ is analytic for $|u_1| < |w_1|$. Thus we may use the one-complex variable Cauchy's formula to obtain that

$$F(\mathbf{z}) = \frac{1}{2\pi i} \int_{|u_1|=|w_1|} \frac{F(u_1, z_2, \ldots, z_d)}{u_1 - z_1}\, du_1\,.$$

We may repeat the preceding argument this time considering the function $u_2 \to F(u_1, u_2, \ldots, z_d)$ which is analytic for $|u_2| < |w_2|$. Another use of the one-complex variable Cauchy's integral formula produces the new identity

$$F(\mathbf{z}) = \frac{1}{(2\pi i)^2} \int_{|u_1|=|w_1|} \left[\int_{|u_2|=|w_2|} \frac{F(u_1, u_2, \ldots, z_d)}{(u_1 - z_1)\cdot(u_2 - z_2)}\, du_2 \right] du_1\,.$$

Repeated applications of the same argument finally leads to the iterated integral

$$F(\mathbf{z}) = \frac{1}{(2\pi i)^d} \int_{|u_1|=|w_1|} \cdots \int_{|u_d|=|w_d|} \frac{F(\mathbf{u})}{(u_1 - z_1)\cdot \ldots \cdot (u_d - z_d)}\, du_d \ldots du_1\,.$$

Cauchy's formula in (4.15) follows from the above identity by means of Fubini's theorem (see [Rud87]) due to the fact that, for each fixed $\mathbf{z} \in \mathcal{D}(\mathbf{w})$, the integrand above is a continuous function of $\mathbf{u}$ over $\mathcal{T}[\mathbf{w}]$.

We have now all the elements we required to prove theorem 4.6. Thus, let us consider $\mathbf{w} \in \mathcal{D}$. Since $\mathcal{D}$ is a Reinhardt domain centered at $\mathbf{0}$ we may use Cauchy's formula to represent $F(\mathbf{z})$ for $\mathbf{z} \in \Delta(\mathbf{w})$ as an integral like in (4.15).

On the other hand, it is well-known that the $\sum_{n=0}^{\infty} t^n$ converges uniformly over compact subsets of $[t : |t| < 1]$ toward $\frac{1}{1-t}$. This implies that the $\sum_n \left(\frac{z_j}{u_j}\right)^n$ is uniformly convergent toward $\frac{1}{1-z_j/u_j}$, for all $\mathbf{u} \in \mathcal{T}(\mathbf{w})$ and $\mathbf{z}$ restricted to a compact subset of $\Delta(\mathbf{w})$. In particular, since

$$\frac{1}{\mathbf{u}-\mathbf{z}} = \frac{1}{u_1\left(1-\frac{z_1}{u_1}\right)\cdot \ldots \cdot u_d\left(1-\frac{z_d}{u_d}\right)}\,,$$

then it follows that

$$\begin{aligned} \frac{1}{\mathbf{u}-\mathbf{z}} &= \frac{1}{\mathbf{u}}\sum_{n_1}\left(\frac{z_1}{u_1}\right)^{n_1}\cdot \ldots \cdot \sum_{n_d}\left(\frac{z_d}{u_d}\right)^{n_d}\,, \\ &= \sum_{\mathbf{n}} \frac{\mathbf{z}^{\mathbf{n}}}{\mathbf{u}^{\mathbf{n}+\mathbf{1}}}\,, \end{aligned}$$

and this last series is uniformly convergent for all $\mathbf{u}$ and $\mathbf{z}$ as before. Thus, back in (4.15), we may exchange the order of the integral with the above series to obtain that

$$F(\mathbf{z}) = \sum_{\mathbf{n}} \left\{ \frac{1}{(2\pi i)^d} \int_{\mathcal{T}[w]} \frac{F(\mathbf{u})}{\mathbf{u}^{\mathbf{n}+\mathbf{1}}}\, d\mathbf{u} \right\} \mathbf{z}^{\mathbf{n}}\,.$$

The compactness of $\mathcal{T}[\mathbf{w}]$ ensures that the above series is uniformly absolutely convergent for all $\mathbf{z}$ restricted to a compact subset of $\mathcal{D}(\mathbf{w})$. Theorem 4.6 is now a direct consequence of (4.16).

A remarkable fact is that the Cauchy-Riemann equations is all what it is needed to ensure that a function can be represented as a power series. It was Hartogs (see theorem 1.7 on [Nis01]) who showed that if F is analytic over $\mathcal{D}$ — and notice that this only requires the partial derivatives in (4.12) to exist and vanish everywhere over $\mathcal{D}$ — then F must be continuous over $\mathcal{D}$ and hence the representation in (4.13) also applies. The same conclusion can be obtained under much weaker conditions. For example, Ohsawa (see [Ohs98], theorem 1.10) shows that if a function F is locally square integrable and satisfies the Cauchy-Riemann equations in the distributional sense — that is the $\int_{\mathcal{D}} F(\mathbf{z})\, \frac{\partial g}{\partial \bar{z}_j}(\mathbf{z})\, dx_1\, dy_1 \ldots dx_d\, dy_d = 0$, for all $g \in \mathcal{C}_0^\infty(\mathcal{D})$ and for all $j = 1, \ldots, d$ — then F must be analytic.

In what remains on this section we will state some basic yet relevant facts about analytic functions of several complex-variables. In the remaining discussion and without further mention it will be assumed that $\mathcal{D}$ is an open subset of $\mathbb{C}^d$.

Theorem 4.7. (Liouville's theorem.) *Suppose that $F : \mathbb{C}^d \to \mathbb{C}$ is analytic and that there is an integer $l \geq 1$ and a polyradius $\boldsymbol{r}$ such that the: $\max_{\boldsymbol{z}\in\mathcal{T}[\rho\,\boldsymbol{r}]} |F(\boldsymbol{z})| = O(\rho^l)$, as $\rho \to \infty$. Then F must be polynomial of degree less or equal to l. In particular, if $|F|$ is bounded then F must be a constant function.*

Proof. The hypotheses of the theorem imply that there is a constant $c > 0$ such that $|F(\mathbf{z})| \leq c \cdot \rho^l$, for all $\mathbf{z} \in \mathcal{T}[\rho\,\mathbf{r}]$ and all ρ sufficiently large. As a result, using (4.14) we obtain that

$$\left|\frac{F^{(\mathbf{n})}(\mathbf{0})}{\mathbf{n}!}\right| \leq \frac{1}{(2\pi)^d}\int_{\mathcal{T}[\rho\,\mathbf{r}]} \frac{|F(\mathbf{u})|}{|u^{\mathbf{n+1}}|}\,|d\mathbf{u}| \leq \frac{c\,\rho^{l-\langle\mathbf{n}\rangle}}{\mathbf{r}^{\mathbf{n}}}\,.$$

Letting $\rho \to \infty$ we obtain that $F^{(\mathbf{n})}(\mathbf{0}) = 0$, for all $\langle\mathbf{n}\rangle > l$. In particular, back in (4.13) we obtain that $F(\mathbf{z}) = \sum_{\mathbf{n}:\langle\mathbf{n}\rangle\leq l} \frac{F^{(\mathbf{n})}(\mathbf{0})}{\mathbf{n}!}\mathbf{z}^{\mathbf{n}}$ and hence $F(\mathbf{z})$ is a polynomial of degree at most l. This shows the theorem. □

Theorem 4.8. (Weierstrass' theorem.) *If sequence of holomorphic functions $(F_j)_{j\geq 0}$ converges uniformly to a function F on each compact subset of $\mathcal{D}$ then F is analytic over $\mathcal{D}$.*

Proof. First observe that F must be continuous over $\mathcal{D}$ for it is the uniform limit of continuous functions. Let $\mathbf{a} \in \mathcal{D}$ and $\mathbf{r}$ be a polyradius such that $\Delta[\mathbf{a},\mathbf{r}] \subset \mathcal{D}$. To prove the theorem it is enough to show that the Cauchy-Riemann equations are satisfied over $\Delta(\mathbf{a},\mathbf{r})$. For each j we may use Cauchy's integral formula to rewrite

$$F_j(\mathbf{z}) = \int_{\mathbf{a}+\mathcal{T}[\mathbf{r}]} \frac{F_j(\mathbf{u})}{\mathbf{u}-\mathbf{z}}\,d\mathbf{u}\,,$$

for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{r})$. The Bounded Convergence theorem lets us then assert that

$$F(\mathbf{z}) = \int_{\mathbf{a}+\mathcal{T}[\mathbf{r}]} \frac{F(\mathbf{u})}{\mathbf{u}-\mathbf{z}}\, d\mathbf{u}\,,$$

for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{r})$. In particular, differentiating both sides we obtain that

$$\frac{\partial}{\partial \bar{z}_j}\Big[F(\mathbf{z})\Big] = \int_{\mathbf{a}+\mathcal{T}[\mathbf{r}]} F(\mathbf{u})\, \frac{\partial}{\partial \bar{z}_j}\left[\frac{1}{\mathbf{u}-\mathbf{z}}\right] d\mathbf{u} = 0\,.$$

This shows that F satisfies the Cauchy-Riemann equations over $\mathcal{D}$ and thus it is analytic in there. This proves the theorem. □

Theorem 4.9. (Identity theorem.) *Suppose that $\mathcal{D}$ is connected and that $F, G \in \mathcal{H}(\mathcal{D})$. If the set $\{\boldsymbol{z} \in \mathcal{D} :\ F(\boldsymbol{z}) = G(\boldsymbol{z})\}$ has non-empty interior then $F(\boldsymbol{z}) = G(\boldsymbol{z})$, for all $\boldsymbol{z} \in \mathcal{D}$.*

Proof. Define $H := (F - G) \in \mathcal{H}(\mathcal{D})$ and consider the set $\mathcal{Z} := \{\mathbf{z} \in \mathcal{D} : H^{(\mathbf{k})}(\mathbf{z}) = 0,\ \text{for all } \mathbf{k}\}$. The hypotheses of the theorem imply that $\mathcal{Z}$ is non-empty. Moreover, observe that $\mathcal{Z}$ is closed in $\mathcal{D}$ and, in addition, from theorem 4.6 it also follows that $\mathcal{Z}$ is open. Since $\mathcal{D}$ is connected we can conclude that $\mathcal{D} = \mathcal{Z}$. In particular, for all $\mathbf{z} \in \mathcal{D}$, $H(\mathbf{z}) = 0$, and this proves the theorem. □

In one-complex variable the weaker condition that the set $\{\mathbf{z} \in \mathcal{D} :\ F(\mathbf{z}) = G(\mathbf{z})\}$ has an accumulation point in $\mathcal{D}$ is enough to ensure that $F = G$ over $\mathcal{D}$. However, in several complex variables this assertion is not true. For example, consider the entire functions $F(z, w) := z$ and $G(z, w) := w$. Observe that the origin is an accumulation point of the zero set of $(F - G)$, however, there is no open set in $\mathbb{C}^2$ where $(F - G)$ is identically zero. Theorem 4.9 constitutes for us a first indication that in the theory of several complex variables new phenomena occur which were not seen in one-complex variable.

Krantz (see [Kra92], section 0.3) said it well: "at a quick glance, one might be tempted to think of the analysis of several complex variables (or several real variables, for that matter) as being essentially one variable theory with the additional complication of multi-indices. This perception turns out to be incorrect. Deep new phenomena and profound (and yet unsolved) problems present themselves in the theory of several complex variables."

Another striking difference between the theories of analytic functions over $\mathbb{C}$ vs. $\mathbb{C}^d$ is given by the following result. We recall that in $\mathbb{C}$ a function cannot have an accumulation point of zeroes (see [Rud87], theorem 10.18) unless it is identically zero in some neighborhood of the accumulation point.

Theorem 4.10. *In $\mathbb{C}^d$ the zeroes of an analytic function are never isolated.*

To prove the above theorem the following decomposition will be suitable. We will think of $\mathbb{C}^d$ as the product space $\mathbb{C}^{d-1} \times \mathbb{C}$. Given a vector $\mathbf{z}$ we will define $\mathbf{z}' \in \mathbb{C}^{d-1}$ through the relation $\mathbf{z} = (\mathbf{z}', z_d)$. Thus, for example, given a polyradius $\mathbf{r}$ we have that $\Delta(\mathbf{z}, \mathbf{r}) = \Delta(\mathbf{z}', \mathbf{r}') \times \Delta(z_d, r_d)$.

Proof. Let $\mathcal{D}$ be a non-empty open subset of $\mathbb{C}^d$. By contradiction suppose that $F \in \mathcal{H}(\mathcal{D})$ and that there is $\mathbf{a} \in \mathcal{D}$ and a polyradius $\mathbf{r}$ such that $\mathbf{z} = \mathbf{a}$ is the only solution of the equation: $F(\mathbf{z}) = 0$, $\mathbf{z} \in \Delta[\mathbf{a}, \mathbf{r}]$. Consider a sequence $(\mathbf{z}'_n)$ such that $|\mathbf{z}'_n - \mathbf{a}'| > 0$ however $\mathbf{z}'_n \to \mathbf{a}'$ as $n \to \infty$. Then, for each n, the function $\frac{1}{F(\mathbf{z}'_n, z_d)}$ regarded solely as a function of z_d is analytic over $\Delta[a_d, r_d]$. Thus, from the maximum modulus principle in complex-variable (see [Rud87]) we can conclude that

$$\left| \frac{1}{F(\mathbf{z}'_n, a_d)} - \frac{1}{F(\mathbf{z}'_m, a_d)} \right| \leq \sup_{|z_d - a_d| = r_d} \left| \frac{1}{F(\mathbf{z}'_n, z_d)} - \frac{1}{F(\mathbf{z}'_m, z_d)} \right| .$$

Since F must be uniformly continuous over $\Delta[\mathbf{a},\mathbf{r}]$ it follows that $F(\mathbf{z}_n',z_d) \to F(\mathbf{a}',z_d)$ uniformly for $|z_d-a_d|=r_d$, as $n\to\infty$. Being the limiting function zero-free for $|z_d-a_d|=r_d$ we conclude that $\frac{1}{F(\mathbf{z}_n',z_d)} \to \frac{1}{F(\mathbf{a}',z_d)}$ uniformly for $|z_d-a_d|=r_d$, as $n\to\infty$. The above inequality implies that the limit $\lim\limits_{n\to\infty} \frac{1}{F(\mathbf{z}_n',a_d)}$ exists. This is only possibly if $F(\mathbf{a})\neq 0$. This contradicts our original premise and therefore $\mathbf{a}$ cannot be an isolated zero of F. □

The remaining results we present in this section are very much the generalization of established properties of analytic functions in one variable.

Theorem 4.11. (Open Mapping theorem.) *If F is a nonconstant analytic function over $\mathcal{D}$ and $\mathcal{A}$ is an open subset of $\mathcal{D}$ then the set $F(\mathcal{A})$ is also open.*

Proof. It is enough to show that $F(\mathbf{z})$ is in the interior of $F(\mathcal{D})$, for all $\mathbf{z}\in\mathcal{D}$. With $\mathbf{z}$ as before let $\mathbf{r}$ be a polyradius such that $\Delta[\mathbf{z},\mathbf{r}]\subset\mathcal{D}$. Then the function $f(u_1):=F(u_1,z_2,\ldots,z_d)$ is analytic for $|u_1-z_1|<r_1$ and, in particular, the open mapping theorem of complex-variable (see [Rud87]), let us conclude that $f\{u_1 : |u_1-z_1|<r_1\}$ is an open neighborhood of $F(\mathbf{z})$. The statement follows by noticing that $f\{u_1:|u_1-z_1|<r_1\}\subset F(\mathcal{D})$. □

Theorem 4.12. (Maximum modulus principle.) *Suppose that $\mathcal{D}$ is connected. If $F\in\mathcal{H}(\mathcal{D})$ and the $|F|$ attains its maximum value at a point in $\mathcal{D}$ then F is constant over $\mathcal{D}$.*

Proof. Suppose that $\mathbf{z}\in\mathcal{D}$ is such that $|F(\mathbf{z})|\geq|F(\mathbf{w})|$, for all $\mathbf{w}\in\mathcal{D}$. By contradiction, if F were nonconstant over $\mathcal{D}$ then the Open mapping theorem would imply that $F(\mathbf{z})$ is in the interior $F(\mathcal{D})$; in particular, there must be $\mathbf{w}\in\mathcal{D}$ such that $|F(\mathbf{z})|<|F(\mathbf{w})|$. This contradicts our premise and therefore F must be constant. □

4.5 Hartogs' series.

Theorem 4.6 implies that any analytic function on an open set $\mathcal{D}$ can be represented locally as a power series. Our next result provides an alternative type of series to represent an analytic function of several variables. The identification $\mathbb{C}^d = \mathbb{C}^{d-1} \times \mathbb{C}$ used to prove theorem 4.10 will be continued in here and in the remaining sections of this chapter.

Theorem 4.13. *Let* $\boldsymbol{a}$ *be a vector and suppose that* $\mathcal{D} \subset \mathbb{C}^d$, $\mathcal{U}' \subset \mathbb{C}^{d-1}$ *are open sets and* $r_d > 0$ *is such that* $\boldsymbol{a} \in \mathcal{U}' \times \Delta(a_d, r_d) \subset \mathcal{D}$. *If* $F \in \mathcal{H}(\mathcal{D})$ *then there are unique function* $f_n \in \mathcal{H}(\mathcal{U}')$ *such that*

$$F(\boldsymbol{z}) = \sum_{n=0}^{\infty} f_n(\boldsymbol{z}') \cdot (z_d - a_d)^n \,, \tag{4.17}$$

for all $\boldsymbol{z} \in \mathcal{U}' \times \Delta(a_d, r_d)$. *Moreover, the series in (4.17) is uniformly absolutely convergent on any compact subset of* $\mathcal{U}' \times \Delta(a_d, r_d)$.

We will refer to the series in (4.17) as the *Hartogs series of* F *in the variable* z_d *at* $\boldsymbol{a}$.

Proof. We first show the uniqueness of the representation in (4.17). Thus, consider $\mathbf{z} \in \mathcal{U}' \times \Delta(a_d, r_d)$ and let γ be any circle centered at a_d and contained in $\Delta(a_d, r_d)$. Since, by assumption, the series in (4.17) is uniformly convergent toward F over the set $\{\mathbf{z}'\} \times \gamma$, Cauchy's integral formula in one-variable (see [Rud87]) implies that

$$\begin{aligned} \frac{1}{2\pi i} \int_\gamma \frac{F(\mathbf{z}', \xi)}{(\xi - a_d)^{k+1}} \, d\xi &= \frac{1}{2\pi i} \sum_{n=0}^{\infty} f_n(\mathbf{z}') \cdot \int_\gamma \frac{1}{(\xi - a_d)^{k-n+1}} \, d\xi \,, \\ &= f_k(\mathbf{z}') \,. \end{aligned}$$

This proves the uniqueness of the coefficients in (4.17).

To show the existence of a Hartogs series we first consider the simpler case in which $\mathcal{U}' = \Delta(\mathbf{a}', \mathbf{r}')$ for certain polyradius $\mathbf{r} := (\mathbf{r}', r_d)$. First, we use theorem 4.6 to represent

$$F(\mathbf{z}) = \sum_{\mathbf{n}} f_{\mathbf{n}} \cdot (\mathbf{z} - \mathbf{a})^{\mathbf{n}}, \tag{4.18}$$

for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{r})$. Since the above series is uniformly absolutely convergent on any compact subset of $\Delta(\mathbf{a}, \mathbf{r})$, it follows, for each nonnegative integer k, that

$$\sum_{\mathbf{n}:n_d=k} f_{\mathbf{n}} \cdot (\mathbf{z} - \mathbf{a})^{\mathbf{n}} = (z_d - a_d)^k \cdot \sum_{\mathbf{n}:n_d=k} f_{\mathbf{n}} \cdot (\mathbf{z}' - \mathbf{a}')^{\mathbf{n}'}, \tag{4.19}$$

and the series on the right-hand side is uniformly absolutely convergent as long as $\mathbf{z}'$ remains within a compact subset of $\Delta(\mathbf{a}', \mathbf{r}')$. In particular, the Weierstrass theorem 4.8 implies that there is $f_k \in \mathcal{H}\big(\Delta(\mathbf{a}', \mathbf{r}')\big)$ such that $f_k(\mathbf{z}') = \sum_{\mathbf{n}:n_d=k} f_{\mathbf{n}} \cdot (\mathbf{z}' - \mathbf{a}')^{\mathbf{n}'}$. Finally, the uniform absolute convergence of the series in (4.18) allows us to reorganize the terms in the summation in any possible way without destroying the type of convergence of the original series nor affecting its limiting value. This implies that

$$\begin{aligned} F(\mathbf{z}) &= \sum_{k=0}^{\infty} \left\{ \sum_{\mathbf{n}:n_d=k} f_{\mathbf{n}} \cdot (\mathbf{z} - \mathbf{a})^{\mathbf{n}} \right\}, \\ &= \sum_{k=0}^{\infty} f_k(\mathbf{z}') \cdot (z_d - a_d)^k, \end{aligned}$$

and the series is uniformly absolutely convergent on compact subsets of $\mathcal{U}' \times \Delta(a_d, r_d)$. This proves the theorem for the particular case in which $\mathcal{U}' = \Delta(\mathbf{a}', \mathbf{r}')$.

For the general case rewrite $\mathcal{U}' = \bigcup_{j=1}^{\infty} \mathcal{U}'_j$ where each $\mathcal{U}'_j$ is an open polydisk in $\mathbb{C}^{d-1}$. The above discussion implies that

$$F(\mathbf{z}) = \sum_{n=0}^{\infty} f_{j,n}(\mathbf{z}') \cdot (z_d - a_d)^n, \tag{4.20}$$

for all $\mathbf{z} \in \mathcal{U}'_j \times \Delta(z_d, r_d)$. Furthermore, each $f_{j,n}$ in (4.20) is analytic over $\mathcal{U}'_j$. But, the uniqueness of the Hartogs series representation of F forces that $f_{j,n} = f_{l,n}$ over $\mathcal{U}'_j \cap \mathcal{U}'_l$. This let's us define $f_n(\mathbf{z}') := f_{j,n}(\mathbf{z}')$ if $\mathbf{z}' \in \mathcal{U}'_j$. The Identity theorem 4.9 shows that f_n is indeed analytic over $\mathcal{U}'$. Moreover, (4.20) implies that

$$F(\mathbf{z}) = \sum_{n=0}^{\infty} f_n(\mathbf{z}') \cdot (z_d - a_d)^n\,,$$

for all $\mathbf{z} \in \mathcal{U}' \times \Delta(z_d, r_d)$. This proves (4.17).

To finalize the proof of the theorem, it only remains to show that the above series is uniformly absolutely convergent over compact subsets of $\mathcal{U}' \times \Delta(a_d, r_d)$. For this, observe that, without any loss of generality, we could have assumed that for all $j \geq 1$ there is $k > j$ such that $\bar{\mathcal{U}}'_j \subset \mathcal{U}'_k$. Thus, since every compact subset of $\mathcal{U}' \times \Delta(z_d, r_d)$ is finitely covered by sets of the form $\mathcal{U}'_j \times \Delta[z_d, \rho \cdot r_d]$, for some $\rho \in (0,1)$, and the series in (4.20) is uniformly absolutely convergent over $\bar{\mathcal{U}}'_j \times \Delta[z_d, \rho \cdot r_d]$, we conclude that the above series is uniformly absolutely convergent over compact subsets of $\mathcal{U}' \times \Delta(z_d, r_d)$. This completes the proof of the theorem. □

Hartogs series let us think of an analytic function of several complex variables as a one complex-variable function indexed by $(d-1)$-variables. To state our first application of Hartogs series we require a definition. This definition will be also of use to state the main results of this dissertation in chapters 5 and 6.

Suppose that F is analytic in some open neighborhood of a point $\mathbf{a}$. F is said to *vanish to order k in z_d at $\mathbf{a}$* provided that the function $F(\mathbf{a}', z_d)$ vanishes to degree k at $z_d = a_d$. In other words, the Hartogs series of F in the variable z_d at $\mathbf{a}$ is of the form $\sum_{n=0}^{\infty} f_n(\mathbf{z}') \cdot (z_d - a_d)^n$ with $f_n(\mathbf{a}') = 0$ for all $n < k$, however, $f_k(\mathbf{a}') \neq 0$.

Example 4.14. Consider the bivariate rational function

$$F(z,w) := \frac{w^2 + w^2 z - w^2 z^2 - w^2 z^3 + w^3 + w^3 z + w^4}{1 + z + w}\,.$$

The denominator of F vanishes at $(0,-1)$, however, the numerator does not vanish in there. This implies that the numerator of F is not in the ideal generated by the polynomial $(1+z+w)$ and, as a result, we can conclude that the domain of convergence of F is the set $\{(z,w) \in \mathbb{C}^2 : |z| + |w| < 1\}$.

For example, we see that $(1,0)$ is in the boundary of the domain of convergence of $F(z,w)$. But, since the denominator of F does not vanish in there, $F(z,w)$ is analytic in a neighborhood of this point. A simple calculation then reveals that $F(1,w) = w^3$. Thus, F vanishes to order 3 in w about $(1,0)$. Indeed, the Hartogs series of F in the variable w about $(1,0)$ is of the form

$$F(z,w) = (1-z^2)\,w^2 + z\,w^3 + \frac{1-z}{1+z}\,w^4 + ...$$

□

It is possible that a function $F(\mathbf{z})$ vanishes to an infinite degree in z_d at a point $\mathbf{a}$. However, as our next result shows, in a new local system of coordinates, it is always possible to assume that this degree is finite.

Lemma 4.15. *Suppose that F is analytic near a point $\boldsymbol{a}$ and it is not identically zero in any neighborhood of it. Then, there are open neighborhoods $\mathcal{U}$ and $\mathcal{V}$ of $\boldsymbol{a}$ and a biholomorphic function $\Phi : \mathcal{U} \to \mathcal{V}$ such that $F(\Phi(\boldsymbol{z}))$ vanishes to a finite degree in z_d about $\boldsymbol{a}$.* [2]

[2]For $j = 1$ or 2, suppose that $d_j \geq 1$ and that $\mathcal{D}_j \subset \mathbb{C}^{d_j}$ is an open set. A function $G : \mathcal{D}_1 \to \mathcal{D}_2$ is said to be *analytic* provided that each $G_j : \mathcal{D}_1 \to \mathbb{C}$ is analytic, where $G =: (G_1, \ldots, G_{d_2})$ are the coordinate functions of G. If $d_1 = d_2$, G is said to be *biholomorphic* provided that G and its inverse function are analytic. The Cauchy-Riemann equations imply that the composition of analytic functions is analytic. In particular, the composition of biholomorphic functions is biholomorphic.

Proof. Without loss of generality we may assume that $\mathbf{a} = \mathbf{0}$ and that F is analytic over certain polydisk $\Delta(\mathbf{0}, \mathbf{r})$. Given $\mathbf{h}' \in \mathbb{C}^{d-1}$ where all coordinates satisfy $|h_j| < 1$ consider the 1-to-1 transformation $\Phi_{\mathbf{h}'}(\mathbf{z}) := (z_1 + h_1 \cdot z_d, \ldots, z_{d-1} + h_{d-1} \cdot z_d, z_d)$ defined for $\mathbf{z} \in \Delta(\mathbf{0}, (\mathbf{r}', \delta)/2)$ with $\delta := \min_{j=1,\ldots,d} r_j$. Observe that δ is defined so that the range of $\Phi_{\mathbf{h}'}$ is contained in $\Delta(\mathbf{0}, \mathbf{r})$.

We claim that there is $\mathbf{h}'$ such that $F(\Phi_{\mathbf{h}'}(\mathbf{z}))$ vanishes to finite order in z_d at $\mathbf{0}$. By contradiction if we suppose otherwise then $F(h_1 \, z_d, \ldots, h_{d-1} \, z_d, z_d) = 0$, for all $\mathbf{h}'$ and $|z_d| < \delta$. But the range of the transformation $(\mathbf{h}', z_d) \to (h_1 \, z_d, \ldots, h_{d-1} \, z_d, z_d)$ defined for $|h_j| < 1$ and $|z_d| < \delta$ contains the open set

$$\left\{ \mathbf{w} : |w_j| < \frac{\delta}{2} \text{ for all } j = 1, \ldots, (d-1) \text{ and } \frac{\delta}{2} < |w_d| < \delta \right\} .$$

Thus, F must vanish identically on the above set. The Identity theorem 4.9 lets us conclude that F is identically zero over $\Delta(\mathbf{0}, \mathbf{r})$. This contradicts our original premise and the claim follows. □

The relevance of the previous result can be appreciated in the following lemma. In some sense, this is a refinement of theorem 4.10.

Lemma 4.16. *Suppose that F is analytic near a point $\boldsymbol{a}$ where it vanishes to a finite degree k in z_d. Then for any sufficiently small polyradius $\boldsymbol{s}$ there is a polyradius $\boldsymbol{r} < \boldsymbol{s}$ such that, for each $\boldsymbol{z}' \in \Delta(\boldsymbol{a}', \boldsymbol{r}')$,*

$$(4.21) \qquad \begin{cases} F(\boldsymbol{z}', z_d) \text{ has exactly } k \text{ zeroes, counting multiplicity,} \\ \text{in the disk } \Delta(a_d, r_d) \text{ and no other zeroes on } \Delta[a_d, r_d]. \end{cases}$$

Proof. Without loss of generality we may assume that $\mathbf{a} = \mathbf{0}$ and that $\mathbf{s} > \mathbf{0}$ is such that $F \in \mathcal{H}(\Delta(\mathbf{0}, \mathbf{s}))$. The case $k = 0$ is trivial. Thus, assume that F vanishes to degree $k \geq 1$ in z_d at $\mathbf{0}$; in particular, $F(\mathbf{0}', 0) = 0$. Furthermore, since k is finite, the zeroes of the one-variable analytic function $z_d \longrightarrow F(\mathbf{0}', z_d)$ must be isolated and hence, we may find $0 < r_d < s_d$ such that $z_d = 0$ is the only zero of $F(\mathbf{0}', z_d)$ in $\Delta[0, r_d]$.

Define $m := \min\{|F(\mathbf{0}', z_d)| : |z_d| = r_d\}$. Since F is uniformly continuous on any compact subset of its domain we may find $0 < \mathbf{r}' < \mathbf{s}'$ such that $|F(\mathbf{z}', z_d) - F(\mathbf{0}', z_d)| < m$, for all $\mathbf{z}' \in \Delta[\mathbf{0}', \mathbf{r}']$ and $|z_d| = r_d$. Define $\mathbf{r} := (\mathbf{r}', r_d)$. Observe that $F(\mathbf{z}', z_d) \neq 0$, for all $\mathbf{z}' \in \Delta[\mathbf{0}', \mathbf{r}']$ and $|z_d| = r_d$.

To obtain (4.21) it will be enough to show that, for each $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$, $F(\mathbf{z}', z_d)$, regarded as a function of z_d, has k zeroes repeated according their multiplicity within the disk $\Delta(0, r_d)$. But, given any such $\mathbf{z}'$, observe that

$$|F(\mathbf{z}', z_d) - F(\mathbf{0}', z_d)| < |F(\mathbf{0}', z_d)|\,,$$

for all $|z_d| = r_d$. Rouche's theorem (see [Rud87]) then lets us conclude that $F(\mathbf{z}', z_d)$ has the same number of zeroes (repeated according to their multiplicity) as $F(\mathbf{0}', z_d)$ does in the disk $|z_d| < r_d$. The lemma follows by noticing that $z_d = 0$ is a zero of order k of the function $F(\mathbf{0}', z_d)$. $\square$

An interesting application of Hartogs series is the following result which will be a key ingredient in the coming section. To state the theorem we require a definition.

Given an open set $\mathcal{D}$ in $\mathbb{C}$ or $\mathbb{C}^d$, a set $\mathcal{Z} \subset \mathcal{D}$ will be said to be *thin* if for each $\mathbf{z} \in \mathcal{D}$ there exists a neighborhood $\mathcal{U}$ of $\mathbf{z}$ and a function $F \in \mathcal{H}(\mathcal{U})$ such that $F = 0$ over $(\mathcal{Z} \cap \mathcal{U})$, however, F does not vanish identically on any neighborhood of $\mathbf{z}$. Observe that, if $\mathcal{Z}$ is a thin subset of an open set $\mathcal{D}$ then $\big(\mathcal{D} - \mathcal{Z}\big)$ cannot have an empty interior.

For example, any finite set $\mathcal{Z} \subset \mathbb{C}$ contained in a disk of the form $\Delta(z_0, r)$ is a thin subset of the disk: just consider the polynomial $F(z)$ which vanishes at each point in $\mathcal{Z}$.

A well-known fact in one-complex variable which will be of use to prove our next result is the following.

Lemma 4.17. *In $\mathbb{C}$, suppose that $\mathcal{Z}$ is a finite subset of $\Delta(z_0, r)$. Then, every function $f(z)$ analytic and bounded in $\Delta(z_0, r) \setminus \mathcal{Z}$ admits an analytic extension to the whole disk $\Delta(z_0, r)$.*

Proof. Without loss of generality we may assume that $\mathcal{Z} = z_0$. We need to show that any function $f(z)$ which is analytic and bounded in the punctured disk $\Delta(z_0, r) - \{z_0\}$ extends to an analytic function in the whole disk. We will show that

$$g(z) := \frac{1}{2\pi i} \int_{|\xi - z_0| = r/2} \frac{f(\xi)}{\xi - z} \, d\xi \tag{4.22}$$

extends f analytically near z_0. It should be clear that g is analytic over $\Delta(z_0, \frac{r}{2})$. Moreover, if $|z - z_0| < \frac{r}{2}$ and $0 < \epsilon < |z - z_0|$ then the residue theorem (see [Rud87]) implies that

$$g(z) - f(z) = \frac{1}{2\pi i} \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \, d\xi \, ,$$

where γ_ϵ is the circle $|\xi - z_0| = \epsilon$. Thus, to conclude that $g(z) = f(z)$, it will be enough to show that the $\lim\limits_{\epsilon \to 0^+} \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \, d\xi = 0$. But, because f is bounded over $\Delta(z_0, r) - \{z_0\}$, we can conclude that there is a constant $c > 0$ such that $\left| \frac{f(\xi)}{\xi - z} \right| \leq c$, for all ξ sufficiently close to z_0. This implies that for all sufficiently small ϵ the $\left| \int_{\gamma_\epsilon} \frac{f(\xi)}{\xi - z} \, d\xi \right| \leq 2\pi c \cdot \epsilon$. Letting $\epsilon \to 0$, it follows that $g = f$ over $\Delta(z_0, \frac{r}{2}) - \{z_0\}$. This shows the lemma because $g(z)$ is analytic over $\Delta(z_0, \frac{r}{2})$. □

The following result generalizes the previous discussion to an arbitrary number of complex variables.

Theorem 4.18. (Removable singularity theorem.) *Let $\mathcal{D}$ be an open set and suppose that F is bounded and analytic over $(\mathcal{D} \setminus \mathcal{Z})$, where $\mathcal{Z}$ is a thin subset of $\mathcal{D}$. Then F has a unique analytic extension to $\mathcal{D}$.*

Proof. To prove the theorem it will be enough to show that for all $\mathbf{a} \in \mathcal{D}$ there is a polyradius $\mathbf{r}$ such that $\Delta(\mathbf{a}, \mathbf{r}) \subset \mathcal{D}$ and F admits an analytic extension to this whole polydisk. For this, notice that the thinness of $\mathcal{Z}$ implies that there is a polyradius $\mathbf{r}$ and a function $H \in \mathcal{H}(\Delta(\mathbf{a}, \mathbf{r}))$ such that $H = 0$ identically over $\Delta(\mathbf{a}, \mathbf{r}) \cap \mathcal{Z}$, however, H is not identically zero over $\Delta(\mathbf{a}, \mathbf{r})$.

Since the notion of thinness is invariant under biholomorphic transformations, lemma 4.15 lets us assume without loss of any generality that H vanishes to degree $k \geq 1$ in the variable z_d at $\mathbf{a}$. Lemma 4.16 can be now quoted to conclude that there is a polyradius $\mathbf{s} < \mathbf{r}$ such that, for each $\mathbf{z}' \in \Delta(\mathbf{a}', \mathbf{s}')$, $H(\mathbf{z}', z_d)$ vanishes at k points (repeated according to their multiplicity) within the disk $\Delta(a_d, s_d)$, however, there are no other zeroes on $\Delta[a_d, s_d]$. This lets us define

$$G(\mathbf{z}) := \frac{1}{2\pi i} \int_{|\xi - a_d| = s_d} \frac{F(\mathbf{z}', \xi)}{\xi - z_d} \, d\xi \,,$$

for each $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{s})$. The function G is certainly analytic in $\mathbf{z}$. Moreover, since F is bounded over $(\mathcal{D} \setminus \mathcal{Z})$, an argument similar to the one used to prove lemma 4.17 implies, for each $\mathbf{z}' \in \Delta(\mathbf{a}', \mathbf{s}')$, that $G(\mathbf{z}', z_d) = F(\mathbf{z}', z_d)$ wherever $H(\mathbf{z}) \neq 0$. In particular, $G = F$ over $\Delta(\mathbf{a}, \mathbf{s}) \setminus \mathcal{Z}$ and hence, G is an analytic extension of F. The identity theorem 4.9 implies that this extension is unique because $\Delta(\mathbf{a}, \mathbf{s}) \setminus \mathcal{Z}$ has nonempty interior. This completes the proof of the theorem. ∎

4.6 Weierstrass preparation and division theorem

An important tool to study the local behavior of an analytic function defined near a point is the so called Weierstrass preparation theorem and Weierstrass division theorem. The first of these is helpful in relating a function of several complex variables to one whose Hartogs series is terminating, hence, can be thought of as a polynomial. The Weierstrass division theorem is the analogue of the Division algorithm for polynomials but in the context of functions of several complex variables.

Theorem 4.19. (Weierstrass preparation theorem.) *Suppose that F vanishes to degree k in z_d about a point $\boldsymbol{a}$. Then there is a polyradius $\boldsymbol{r}$, a monic polynomial $P(\boldsymbol{z}) \in \mathcal{H}(\Delta(\boldsymbol{a}', \boldsymbol{r}'))[z_d - a_d]$ of degree k and a function $C(\boldsymbol{z})$ analytic and zero-free in $\Delta(\boldsymbol{a}, \boldsymbol{r})$ such that*

$$F(\boldsymbol{z}) = P(\boldsymbol{z}) \cdot C(\boldsymbol{z}) \,, \text{ for all } z \in \Delta(\boldsymbol{a}, \boldsymbol{r}) \,. \tag{4.23}$$

Moreover, for each $\boldsymbol{z}' \in \Delta(\boldsymbol{a}', \boldsymbol{r}')$, the polynomial $P(\boldsymbol{z}', z_d)$ has exactly k zeroes, counting multiplicity, in the disk $\Delta(a_d, r_d)$.

The notation $H(\Delta(\mathbf{a}', \mathbf{r}'))[z_d]$ in Weierstrass' theorem refers to the *polynomial ring over $H(\Delta(\boldsymbol{a}', \boldsymbol{r}'))$ in the indeterminate z_d*. Thus, a generic element in this ring is a polynomial in the variable z_d whose coefficients are analytic functions of $\mathbf{z}'$ over the polydisk $\Delta(\mathbf{a}', \mathbf{r}')$.

The factorization in (4.23) is certainly unique. Because of this, $P(\mathbf{z})$ is called the *Weierstrass polynomial of F in z_d at $\boldsymbol{a}$.*

To prove theorem 4.19 we require first some definitions.

Given k complex-variables $w_1, \ldots, w_k$ a *power sum function* of them is any function of the form

$$s_n = s_n(w_1, ..., w_k) := \sum_{j=1}^{k} w_j^n \,,$$

with $n \geq 1$ an integer. Accordingly, we define $s_0 = s_0(w_1, ..., w_k) \equiv 1$. On the other hand, for $0 \leq n \leq k$, the *elementary symmetric functions* $\phi_n := \phi_n(w_1, ..., w_k)$ in the variables $w_1, \ldots, w_k$ are implicitly defined thorough the relation

$$\prod_{j=1}^{k} (\lambda - w_j) = \phi_0 \, \lambda^k + \sum_{j=1}^{k} (-1)^j \, \phi_j \, \lambda^{k-j} \,. \tag{4.24}$$

Thus, we see that $\phi_0 \equiv 1$ and

$$\begin{aligned} \phi_1 &= \sum_{j=1}^{k} w_j \,, \\ \phi_2 &= \sum_{1 \leq j < l \leq k} w_j \, w_l \,, \\ &\vdots \\ \phi_k &= \prod_{j=1}^{k} w_j \,. \end{aligned}$$

Lemma 4.20. *For each* $1 \leq n \leq k$, $\phi_n \in \mathbb{C}[s_1, ..., s_n]$. [3]

Proof. Given $w_1, \ldots, w_k$ consider the function $\varphi(\lambda) := \prod_{j=1}^{k} (1 - \lambda \cdot w_j)$ which is a polynomial in λ. Observe that the $(\log \varphi)$ is well-defined for all λ in a disk such that $|\lambda \cdot w_j| < 1$, for $j = 1, ..., d$. In particular, for all such λ, we have that

$$\begin{aligned} -\frac{\varphi'(\lambda)}{\varphi(\lambda)} &= -\frac{\partial}{\partial \lambda} \Big[\log \varphi(\lambda) \Big] \,, \\ &= \sum_{j=1}^{k} \frac{w_j}{1 - \lambda \cdot w_j} \,, \\ &= \sum_{j=0}^{\infty} s_{j+1} \, \lambda^j \,. \end{aligned} \tag{4.25}$$

[3] The set $\mathbb{C}[s_1, \ldots, s_n]$ denotes to the ring of polynomials in the variables $s_1, \ldots, s_n$ with constant complex coefficients.

On the other hand, using (4.24) it is almost immediate to see that

$$\varphi(\lambda) = \sum_{n=0}^{k} (-1)^n \, \phi_n \, \lambda^n \, .$$

As a result, if in (4.25) we multiply both sides by $\varphi(\lambda)$ and identify the coefficient of λ^{n-1} in each side, we obtain the recursive formula

$$\phi_n = -\frac{(-1)^n}{n} \cdot \sum_{j=0}^{n-1} (-1)^j \, \phi_j \cdot s_{n-j} \, , \tag{4.26}$$

for all $1 \leq n \leq k$.

Having established the above recursion, the proof of the lemma proceeds by induction. The result is trivial for $n = 1$. Next, suppose $1 < n \leq k$ and that $\phi_j \in \mathbb{C}[s_1, \ldots, s_j]$, for all $j = 1, \ldots, (n-1)$. Then, for all j in this range, $\phi_j \cdot s_{n-j} \in \mathbb{C}[s_1, \ldots, s_n]$ and (4.26) implies that $\phi_n \in \mathbb{C}[s_1, \ldots, s_n]$. This completes the proof of the lemma. □

We are ready to prove theorem 4.19. There is no loss of generality in assuming that $\mathbf{a} = \mathbf{0}$. In other words, we will suppose that F is analytic in a neighborhood of $\mathbf{0}$ where it vanishes to finite degree k in the variable z_d.

Due to lemma 4.16, for each $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$, we can enumerate the zeroes of $F(\mathbf{z}', z_d)$ according to their multiplicity as $x_1(\mathbf{z}'), \ldots, x_k(\mathbf{z}')$. Define

$$P(\mathbf{z}) := \prod_{j=1}^{k} \big(z_d - x_j(\mathbf{z}') \big) \, .$$

We cannot ensure (and indeed, it is not true in general) that the functions $x_j(\mathbf{z}')$ depend analytically nor even continuously on $\mathbf{z}'$. Despite this nuisance, we will show that P is analytic. For this, we first quote (4.24) to rewrite

$$P(\mathbf{z}) = z_d^k + \sum_{j=1}^{k} (-1)^j \, \phi_j(\mathbf{z}') \, z_d^{k-j} \, ,$$

where each $\phi_j(\mathbf{z}')$ is an elementary symmetric function of $x_1(\mathbf{z}'), \ldots, x_k(\mathbf{z}')$. To show that $P(\mathbf{z})$ is analytic for $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$ it is therefore enough to show that each $\phi_j(\mathbf{z}')$ is analytic over $\Delta(\mathbf{0}', \mathbf{r}')$. Lemma 4.20 reduces the problem to show that $s_n(\mathbf{z}') := \sum_{j=1}^{k} \{x_j(\mathbf{z}')\}^n$ is analytic for all $1 \leq n \leq k$. The key ingredient for this is the identity

$$s_n(\mathbf{z}') = \frac{1}{2\pi i} \int_{|\xi|=r_d} \xi^n \, \frac{\frac{\partial F}{\partial z_d}(\mathbf{z}', \xi)}{F(\mathbf{z}', \xi)} \, d\xi \,, \tag{4.27}$$

valid for all $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$. To prove (4.27) observe that the integral on the right-hand side above is convergent for all $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$ because, according to (4.21), all the zeroes of $F(\mathbf{z}', z_d)$ within $\Delta[0, r_d]$ belong to the interior of this disk. Next, fix $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$. The theory of one-complex variable together with lemma 4.16 implies that we can factor $F(\mathbf{z}', \cdot)$ in the form

$$F(\mathbf{z}', z_d) = P(\mathbf{z}', z_d) \cdot C(\mathbf{z}', z_d) \,, \tag{4.28}$$

where $C(\mathbf{z}', z_d)$ — regarded solely as a function of z_d — is certain analytic and zero-free function in an open neighborhood of the disk $\Delta[0, r_d]$. A simple computation then shows that

$$\frac{\frac{\partial F}{\partial z_d}(\mathbf{z}', \xi)}{F(\mathbf{z}', \xi)} = \frac{\frac{\partial C}{\partial z_d}(\mathbf{z}', \xi)}{C(\mathbf{z}', \xi)} + \sum_{j=1}^{k} \frac{1}{\xi - x_j(\mathbf{z}')} \,.$$

As a result, the residue theorem (see [Rud87]) allows us obtain that

$$\begin{aligned}
\frac{1}{2\pi i} \int_{|\xi|=r_d} \xi^n \, \frac{\frac{\partial F}{\partial z_d}(\mathbf{z}', \xi)}{F(\mathbf{z}', \xi)} \, d\xi &= \int_{|\xi|=r_d} \xi^n \, \frac{\frac{\partial C}{\partial z_d}(\mathbf{z}', \xi)}{C(\mathbf{z}', \xi)} \, d\xi + \sum_{j=1}^{k} \int_{|\xi|=r_d} \frac{\xi^n}{\xi - x_j(\mathbf{z}')} \, d\xi \,, \\
&= 0 + \sum_{j=1}^{k} \{x_j(\mathbf{z}')\}^n \,, \\
&= s_n(\mathbf{z}') \,.
\end{aligned}$$

This proves (4.27). Since the right-hand side in (4.27) is certainly an analytic function of $\mathbf{z}'$ in the polydisk $\Delta(\mathbf{0}', \mathbf{r}')$ we conclude that $s_n(\mathbf{z}')$ is analytic. Therefore, $P \in \mathcal{H}(\Delta(\mathbf{0}', \mathbf{r}'))[z_d]$.

Having established the analyticity of $P(\mathbf{z})$, to finalize the proof of theorem 4.19 it will be enough to show that the function $C(\mathbf{z})$, implicitly defined in (4.28), is analytic for $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$. For this consider the set $\mathcal{Z} := \{\mathbf{z} \in \Delta[\mathbf{0}, \mathbf{r}] : P(\mathbf{z}) = 0\}$. Lemma 4.16 implies that $\mathcal{Z} \subset \Delta(\mathbf{0}, \mathbf{r})$; in particular, since $C(\mathbf{z}) = \frac{F(\mathbf{z})}{P(\mathbf{z})}$, for $\mathbf{z} \notin \mathcal{Z}$, C must be analytic over $\Delta(\mathbf{0}, \mathbf{r}) \setminus \mathcal{Z}$ and continuous over $\Delta[\mathbf{0}', \mathbf{r}'] \times [z_d : |z_d| = r_d]$. In addition to this, the maximum modulus principle in one-complex variable (see [Rud87]) implies that the

$$\max_{\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})} |C(\mathbf{z})| = \max_{\mathbf{z} \in \Delta(\mathbf{0}', \mathbf{r}') \times [z_d : |z_d| = r_d]} |C(\mathbf{z})| \,.$$

Since the set $\Delta[\mathbf{0}', \mathbf{r}'] \times [z_d : |z_d| = r_d]$ is compact, we deduce that C is bounded over $\Delta(\mathbf{0}, \mathbf{r}) \setminus \mathcal{Z}$. Finally, since $\mathcal{Z}$ is certainly a thin subset of $\Delta(\mathbf{0}, \mathbf{r})$, the removable singularity theorem 4.18 implies that C can be analytically extended to $\Delta(\mathbf{0}, \mathbf{r})$. This completes the proof of theorem 4.19 because the factorization in (4.28) also applies to the analytic extension of C.

Some of the elements of the proof of the Weierstrass preparation theorem can now used to show the following useful result.

Theorem 4.21. (Weierstrass division theorem.) *Suppose that F is analytic near a point $\boldsymbol{a}$ where it vanishes to a finite degree k in z_d. Then, there is a polyradius $\boldsymbol{r}$, such that any function G analytic in an open neighborhood of $\Delta[\boldsymbol{a}, \boldsymbol{r}]$ admits a unique decomposition of the form*

$$G = E \cdot F + R \,, \tag{4.29}$$

where $E \in \mathcal{H}(\Delta(\boldsymbol{a}, \boldsymbol{r}))$ and $R \in \mathcal{H}(\Delta(\boldsymbol{a}', \boldsymbol{s}'))[z_d]$ is of degree less than k.

Proof. We first consider the case in which F is a Weierstrass polynomial itself. Using lemma 4.16, it follows that there is a polyradius $\mathbf{r}$ such that F is analytic in the polydisk $\Delta[\mathbf{a},\mathbf{r}]$ and for each $\mathbf{z}'\in\Delta(\mathbf{a}',\mathbf{r}')$, all the zeroes of $F(\mathbf{z}',\cdot)$ belong to the interior of $\Delta[a_d,r_d]$.

We claim that the decomposition in (4.29) applies for all $G(\mathbf{z})$ analytic in an open neighborhood of the closed polydisk $\Delta[\mathbf{a},\mathbf{r}]$. Indeed, under these circumstances we may define

$$\begin{aligned} E(\mathbf{z}) &:= \frac{1}{2\pi i}\int_{|\xi-a_d|=r_d}\frac{G(\mathbf{z}',\xi)}{F(\mathbf{z}',\xi)}\,\frac{d\xi}{\xi-z_d}\,,\\ R(\mathbf{z}) &:= G(\mathbf{z})-E(\mathbf{z})\cdot F(\mathbf{z})\,. \end{aligned}$$

The function E is certainly analytic over $\Delta(\mathbf{a},\mathbf{r})$; in particular, also is R on the same polydisk. To obtain (4.29), it only remains to show that $R(\mathbf{z})\in\mathcal{H}(\Delta(\mathbf{a}',\mathbf{r}'))[z_d]$ and is of degree less than k. For this, we may represent $G(\mathbf{z},\cdot)$ using Cauchy's integral formula (see [Rud87]) to obtain

$$R(\mathbf{z})=\frac{1}{2\pi i}\int_{|\xi-a_d|=r_d}\frac{G(\mathbf{z}',\xi)}{F(\mathbf{z}',\xi)}\,\frac{F(\mathbf{z}',\xi)-F(\mathbf{z}',z_d)}{\xi-z_d}\,d\xi\,.$$

For each fixed ξ, the ratio $\frac{F(\mathbf{z}',\xi)-F(\mathbf{z}',z_d)}{\xi-z_d}$ — thought of as a function of z_d — belongs to $\mathcal{H}(\Delta(\mathbf{a}',\mathbf{s}'))[z_d]$ and has degree less than k; in particular, the same can be concluded for $R(\mathbf{z})$. This shows (4.29) for the particular case in which F is a Weierstrass polynomial.

For the general case, let P be the Weierstrass polynomial of F. The Weierstrass preparation theorem 4.19 implies that there is a polyradius $\mathbf{r}>\mathbf{0}$ and a function C analytic and zero-free in $\Delta(\mathbf{a},\mathbf{r})$ such that $F(\mathbf{z})=C(\mathbf{z})\cdot P(\mathbf{z})$, for all $z\in\Delta(\mathbf{a},\mathbf{r})$. Lemma 4.16 lets us assume without loss of generality that for all $\mathbf{z}'\in\Delta(\mathbf{a}',\mathbf{r}')$, all the zeroes of $F(\mathbf{z}', z_d)$ are in the interior of $\Delta[a_d, r_d]$. As a result, the discussion on the previous paragraph implies that there is a polyradius $\mathbf{s} < \mathbf{r}$ such that any function G which is analytic in a neighborhood of $\Delta[\mathbf{a}, \mathbf{s}]$ has a decomposition of the form $G = E \cdot P + R$, with $E \in \mathcal{H}(\Delta(\mathbf{a}, \mathbf{s}))$ and $R \in \mathcal{H}(\Delta(\mathbf{a}', \mathbf{s}'))[z_d]$ of degree less than k. Since C is zero-free in the smaller polydisk, we can conclude that

$$G = \left(\frac{E}{C}\right) \cdot F + R\,.$$

This proves (4.29) for the general case.

To prove uniqueness suppose that $G = E \cdot F + R = \tilde{E} \cdot F + \tilde{R}$, where $\tilde{E}$ is analytic over $\Delta(\mathbf{a}, \mathbf{s})$ and $\tilde{R} \in \mathcal{H}(\Delta(\mathbf{a}', \mathbf{s}'))[z_d]$ is of degree less than k. Lemma 4.16 lets us assume without loss of generality that, for each $\mathbf{z}' \in \Delta(\mathbf{a}', \mathbf{s}')$, the zeroes of $R(\mathbf{z}', z_d)$ and $\tilde{R}(\mathbf{z}', z_d)$ are all contained in the interior of $\Delta[a_d, s_d]$.

For any fixed $\mathbf{z}' \in \Delta(\mathbf{a}', \mathbf{s}')$, it follows that

$$(R - \tilde{R})(\mathbf{z}', z_d) = (E - \tilde{E})(\mathbf{z}', z_d) \cdot F(\mathbf{z}', z_d)\,,$$

for all $z_d \in \Delta(a_d, s_d)$. But, observe that the right-hand side vanishes at k points counting multiplicity. Since the left-hand side is a polynomial in z_d of degree less than k it must be identically zero. Therefore, $R = \tilde{R}$ over $\Delta(\mathbf{a}, \mathbf{s})$. In particular, $(E - \tilde{E})(\mathbf{z}) \cdot F(\mathbf{z}) = 0$, for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{s})$. Since lemma 4.16 implies that $F(\mathbf{z})$, regarded as function of z_d, has a finite number of zeroes over $\Delta(a_d, r_d)$, the continuity of E and $\tilde{E}$ lets us conclude that $\tilde{E} = E$ over $\Delta(\mathbf{a}, \mathbf{s})$. This shows that the factorization in (4.29) is unique and completes the proof of the theorem. □

4.7 Implicit and inverse mapping theorem

The well-known implicit and inverse mapping theorem in the context of smooth functions in $\mathbb{R}^d$ have their analogue in the setting of analytic function of several variables.

To state the next results, the following notation will be used. Suppose that $\mathcal{D} \subset \mathbb{C}^d$ is an open set and $F : \mathcal{D} \to \mathbb{C}^m$ is an analytic function with coordinate functions $F = (F_1, \ldots, F_m)$. The *Jacobian matrix of* F at a point $\mathbf{a} \in \mathcal{D}$ is defined to be the matrix in $\mathbb{C}^{m \times d}$ with entries

$$\frac{\partial F}{\partial \mathbf{z}}(\mathbf{a}) = \frac{\partial(F_1 \ldots F_m)}{\partial(z_1 \ldots z_m)}(\mathbf{a}) := \begin{bmatrix} \frac{\partial F_1}{\partial z_1}(\mathbf{a}) & \ldots & \frac{\partial F_1}{\partial z_d}(\mathbf{a}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial z_1}(\mathbf{a}) & \ldots & \frac{\partial F_m}{\partial z_d}(\mathbf{a}) \end{bmatrix}.$$

Theorem 4.22. (Implicit mapping theorem.) *Let* $1 \le m \le d$ *and* $F : \mathcal{D} \subset \mathbb{C}^d \to \mathbb{C}^m$, *with* $\mathcal{D}$ *open, be analytic with coordinate functions* $F = (F_1, \ldots, F_m)$. *If* $\boldsymbol{a} \in \mathcal{D}$ *is such that* $F(\boldsymbol{a}) = (0, ..., 0)$ *and the matrix* $\frac{\partial(F_1,\ldots,F_m)}{\partial(z_{d-m+1},\ldots,z_d)}(\boldsymbol{a})$ *is non-singular then there is a polyradius* $\boldsymbol{r}$ *and an analytic map*

$$G : \Delta\big((a_1, \ldots, a_{d-m}), (r_1, \ldots, r_{d-m})\big) \to \Delta\big((a_{d-m+1}, \ldots, a_d), (r_{d-m+1}, \ldots, r_d)\big)$$

such that for all $\boldsymbol{z} \in \Delta(\boldsymbol{a}, \boldsymbol{r})$

$$F(\boldsymbol{z}) = (0, \ldots, 0) \text{ if and only if } (z_{d-m+1}, \ldots, z_d) = G(z_1, \ldots, z_{d-m})\,. \tag{4.30}$$

Proof. The proof proceeds by induction in m. For $m = 1$ the hypotheses of the theorem imply that $\mathbf{a}$ is a zero of order 1 of F in the variable z_d. As a result, the Weierstrass preparation theorem 4.19 implies that there is a polyradius $\mathbf{r}$ such that $F(\mathbf{z}) = \big\{p_0(\mathbf{z}') + p_1(\mathbf{z}') \cdot (z_d - a_d)\big\} \cdot C(\mathbf{z})$ and $p_1(\mathbf{z}') \cdot C(\mathbf{z}) \neq 0$, for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{r})$.

Thus, for $\mathbf{z}$ in this polydisk, we have that $F(\mathbf{z}) = 0$ if and only if $z_d = a_d - \frac{p_0(\mathbf{z}')}{p_1(\mathbf{z}')}$. This shows (4.30) for the case $m = 1$.

Suppose that $m \geq 2$ and that the theorem has already been proved for smaller values of m. Using a 1-to-1 linear transformation is $\mathbb{C}^m$ there is no loss of generality in assuming that $F(\mathbf{a}) = (0, \ldots, 0)$ and $\frac{\partial(F_1,\ldots,F_m)}{\partial(z_{d-m+1},\ldots,z_d)}(\mathbf{a})$ is an identity matrix. In particular, $\frac{\partial F_m}{\partial z_d}(\mathbf{a}) = 1$ and therefore using the inductive hypothesis it follows that there is a polyradius $\mathbf{r}$ and an a map $G : \Delta(\mathbf{a}', \mathbf{r}') \to \Delta(a_d, r_d)$ such that $F_m(\mathbf{z}) = 0$ if and only if $z_d = G(\mathbf{z}')$. Without loss of generality we may assume that $F \in \mathcal{H}(\Delta(\mathbf{a}, \mathbf{r}))$. Define $H : \Delta(\mathbf{a}', \mathbf{r}') \to \mathbb{C}^{m-1}$ by $H(\mathbf{z}') := \big(F_1(\mathbf{z}', G(\mathbf{z}')), \ldots, F_{m-1}(\mathbf{z}', G(\mathbf{z}'))\big)$. Then, for all $\mathbf{z} \in \Delta(\mathbf{a}, \mathbf{r})$ it follows that

$$(4.31) \qquad F(\mathbf{z}) = (0, \ldots, 0) \text{ if and only if } H(\mathbf{z}') = (0, \ldots, 0) \text{ and } z_d = G(\mathbf{z}')\,.$$

But, observe that $H(\mathbf{a}') = (0, \ldots, 0)$ and the last $(m-1)$ columns of the Jacobian matrix of H at $\mathbf{a}'$ form an identity matrix. Thus, from the inductive hypothesis, we can conclude that there is a polyradius $\mathbf{s}' < \mathbf{r}'$ and a holomorphic map $E : \Delta\big((a_1, \ldots, a_{d-m-1}), (s_1, \ldots, s_{d-m-1})\big) \to \Delta\big((a_{d-m}, \ldots, a_{d-1}), (s_{d-m}, \ldots, s_{d-1})\big)$ such that for all $\mathbf{z}' \in \Delta(\mathbf{a}', \mathbf{s}')$

$$(4.32) \qquad H(\mathbf{z}') = (0, \ldots, 0) \text{ if and only if } (z_{d-m}, \ldots, z_{d-1}) = E(z_1, \ldots, z_{d-m-1})\,.$$

From (4.31) and (4.32) it follows, for all $\mathbf{z} \in \Delta(\mathbf{a}', \mathbf{s}') \times \Delta(a_d, r_d)$, that

$$F(\mathbf{z}) = 0 \text{ iff } \begin{cases} (z_{d-m}, \ldots, z_{d-1}) &= E(z_1, \ldots, z_{d-m-1}), \text{ and} \\ z_d &= G\Big(z_1, \ldots, z_{d-m-1}, E(z_1, \ldots, z_{d-m-1})\Big)\,. \end{cases}$$

This completes the proof of the theorem. □

Theorem 4.23. (Inverse mapping theorem.) *Let $d \geq 1$ and $\mathcal{D} \subset \mathbb{C}^d$ be an open set. Suppose that $F : \mathcal{D} \subset \mathbb{C}^d \to \mathbb{C}^d$ is analytic and $\boldsymbol{a} \in \mathcal{D}$ is such that the Jacobian matrix $\frac{\partial F}{\partial \boldsymbol{z}}(\boldsymbol{a})$ is non-singular. Then, there is a polyradius $\boldsymbol{r}$ such that F is biholomorphic from $\Delta(\boldsymbol{a}, \boldsymbol{r})$ to an open neighborhood of $F(\boldsymbol{a})$.*

Proof. Let $\mathbf{s}$ be a polyradius such that $F \in \mathcal{H}(\Delta(\mathbf{a}, \mathbf{s}))$ and consider the mapping $G : \Delta(\mathbf{a}, \mathbf{s}) \times \mathbb{C}^d \to \mathbb{C}^d$ defined as $G(\mathbf{z}, \mathbf{w}) := F(\mathbf{w}) - \mathbf{z}$. Observe that $G(F(\mathbf{a}), \mathbf{a}) = 0$ and $\frac{\partial G}{\partial \mathbf{w}}(F(\mathbf{a}), \mathbf{a}) = \frac{\partial F}{\partial \mathbf{z}}(\mathbf{a})$ is non-singular. The implicit mapping theorem 4.22 lets us conclude that there are polyradii $\mathbf{r}_1 < \mathbf{s}$ and $\mathbf{r}_2$ and a holomorphic mapping $H : \Delta(F(\mathbf{a}), \mathbf{r}_2) \to \Delta(\mathbf{a}, \mathbf{r}_1)$ such that $F(\mathbf{w}) = \mathbf{z}$ if and only if $\mathbf{w} = H(\mathbf{z})$. In equivalent words, the restriction of H to the set $F(\Delta(\mathbf{a}, \mathbf{r}_1))$ — which according to the Open mapping theorem 4.11 is an open neighborhood of $F(\mathbf{a})$, because the hypotheses ensure that F cannot be constant on any neighborhood of $\mathbf{a}$ — is the inverse function of $F : \Delta(\mathbf{a}, \mathbf{r}_1) \to F(\Delta(\mathbf{a}, \mathbf{r}_1))$. This completes the proof of the theorem. □

4.8 Polynomial canonical representations.

Our discussion will focus on analytic functions of several variables which are analytic in a neighborhood of a particular point. Without loss of generality we will assume that this point is the origin in $\mathbb{C}^d$ for some $d \geq 2$ fixed.

A typical example of a canonical representation is the Weierstrass preparation theorem 4.19. Suppose that $F(\mathbf{z})$ is an analytic function near the origin in $\mathbb{C}^d$ such that for certain $k \geq 0$ the $\frac{\partial^k F}{\partial z_d^k}(\mathbf{0})$ is nonzero. If k is minimal with this property the Weierstrass preparation theorem 4.19 provides the existence of a unique decomposition of the form

$$F(\mathbf{z}) = G(\mathbf{z}) \cdot P(\mathbf{z})\,, \tag{4.33}$$

where both factors on the right-hand side are analytic near the origin but in addition $G(\mathbf{0}) \neq 0$ and P is the Weierstrass polynomial of F at $\mathbf{0}$ in the variable z_d (which is of degree k in the variable z_d). The decomposition in (4.33) implies that F near $\mathbf{z} = \mathbf{0}$ is practically a constant multiple of P. The question of whether F itself can be represented as a polynomial dates back to the work of Chester, Friedman and Ursell [CFU56] in 1956 who studied this problem for a special class of a two-complex variable functions. For an overview of their work refer to section 9.2 in [BleHan86].

Later, in 1961, Levinson [Lev61] provided a way to represent an analytic function (of several complex variables) near the origin canonically as a polynomial. In [Lev60b] he even showed that, for the special case of $d = 2$, it is possible to represent F itself as a polynomial in two-variables provided that the discriminant of the Weierstrass polynomial is not identically zero near the origin. [4]

The main result of Levinson in [Lev61] can be rephrased in our context as follows.

Theorem 4.24. (Levinson's canonical representation.) *Suppose that F is analytic in a neighborhood of the origin in $\mathbb{C}^d$ and that it vanishes to degree $k \geq 1$ in z_d about $\boldsymbol{0}$. Then, F admits a representation of the form*

$$F(\boldsymbol{z}) = \sum_{j=0}^{k} a_j(\boldsymbol{z}') \cdot x_d^j\,. \tag{4.34}$$

where $x_d := x_d(\boldsymbol{z})$ is certain analytic function near the origin such that $x_d(\boldsymbol{z}', 0) \equiv 0$,

[4] If $p(z)$ is a polynomial of degree k and $z_1, \ldots, z_k$ are its roots repeated according to their multiplicity then its discriminant (which is well defined up to a sign) is defined to be the quantity $D := \prod_{j<l} (z_l - z_j)$.

$\frac{\partial x_d}{\partial z_d}(\boldsymbol{z}', 0) \equiv 1$, *and the functions* $a_j(\boldsymbol{z}')$ *are analytic near the origin and such that* $a_j(\boldsymbol{0}') = 0$, *for all* $0 \leq j < k$, *however,* $a_k(\boldsymbol{0}') \neq 0$.

Hörmander (see [Hör90], theorem 7.5.13) provides an analogue to theorem 4.24 but in the context of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$. The proof he provides can be easily adapted to consider the analytic case. His canonical representation can be rephrased in our context as follows.

Theorem 4.25. (Hörmander's canonical representation.) *Suppose that* F *is analytic in a neighborhood of the origin in* $\mathbb{C}^d$ *and that it vanishes to degree* $k \geq 1$ *in* z_d *about* $\boldsymbol{0}$. *Then,* F *admits near the origin a representation of the form*

$$F(\boldsymbol{z}) = \frac{x_d^k}{k} + \sum_{j=0}^{k-2} a_j(\boldsymbol{z}') \cdot x_d^j \,. \tag{4.35}$$

where $x_d := x_d(\boldsymbol{z})$ *is analytic and such that* $x_d(\boldsymbol{0}) = 0$, $\frac{\partial x_d}{\partial z_d}(\boldsymbol{0}) \neq 0$, *and the coefficients* $a_j(\boldsymbol{z}')$ *are analytic and such that* $a_j(\boldsymbol{0}') = 0$, *for all* $0 \leq j < (k-1)$.

The above representation is certainly not unique. After all by multiplying x_d by a k^{th}-root of unity factor we may obtain k different representations. Furthermore, in the category of $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$ it is unclear that the above representation is unique. For instance, uniqueness does not apply for the Malagrange preparation theorem which is the analogue of the Weierstrass preparation theorem but in the class $\mathcal{C}^\infty$-functions (see theorem 7.5.5 in [Hör90]). Uniqueness does not apply either for the equivalent of the Weierstrass division theorem for $\mathcal{C}^\infty$-functions (see theorem 7.5.6 in [Hör90]).

In this section we will prove that Levinson's representation is unique and that Hörmander's representation is unique up to a k^{th}-root of unity factor. Both results of uniqueness are part of the research work of this dissertation. However, we remark that Dr. Jean-Pierre Rosay, from the Department of Mathematics of The University of Wisconsin in Madison, provided in a personal communication a more direct proof of the uniqueness of these representations using a version of the inverse mapping theorem over Banach Spaces.

The approach we will follow to show the uniqueness of these representations relies on two one-complex variable propositions which we state next.

Proposition 4.26. *Suppose that $P \in \mathbb{C}[z]$ and $Q \in \mathbb{C}[y]$ are polynomials of the same degree $k \geq 1$. If $R > 0$ is such that all the roots of $P(z)$ are contained in $\Delta(0, R)$ and there is a 1-to-1 analytic function $y : \Delta(0, R) \to \mathbb{C}$ such that for all $|z| < R$*

$$P(z) = Q(y(z))$$

then all the roots of $Q(y)$ are contained in the range of $y(z)$. Moreover, if $\xi_1, \ldots, \xi_k$ lists the roots of $P(z)$ repeated according to their multiplicity then the polynomial $\prod_{j=1}^{k} \{y - y(\xi_j)\}$ divides $Q(y)$.

Proposition 4.27. *Let $R > 0$. In the space of analytic function over $\Delta(0, R)$ consider the functional equation*

$$\begin{cases} \prod_{j=1}^{k} (z - z_j) = \prod_{j=1}^{k} \{y(z) - y(z_j)\}, \\ y(0) = 0, \end{cases} \tag{4.36}$$

with $z_1, \ldots, z_k \in \Delta(0, R)$. For all ρ and r such that $0 < 2\rho < r < R$ there exists a $\delta > 0$ such that for all $(z_1, \ldots, z_k) \in \mathbb{C}^k$ with $\max_{j=1 \ldots k} |z_j| \leq \rho$, $y(z) = z$ is the only solution of (4.36) satisfying $\sup_{|z| \leq r} |y(z) - z| \leq \delta$.

4.8.1 Proof of uniqueness for Levinson's representation.

We will to show that the canonical representation in (4.34) is unique. Thus, suppose that there are functions $x_d = x_d(\mathbf{z})$, $y_d = y_d(\mathbf{z})$ and $a_j = a_j(\mathbf{z}')$ and $b_j = b_j(\mathbf{z}')$ analytic near the origin such that

$$
\begin{aligned}
F(\mathbf{z}) &:= \sum_{j=0}^{k} a_j \cdot x_d^j\,, \\
&= \sum_{j=0}^{k} b_j \cdot y_d^j\,,
\end{aligned}
\tag{4.37}
$$

with $x_d(\mathbf{z}',0) \equiv y_d(\mathbf{z}',0) \equiv 0$, $\frac{\partial x_d}{\partial z_d}(\mathbf{z}',0) \equiv \frac{\partial y_d}{\partial z_d}(\mathbf{z}',0) \equiv 1$, and $a_j(\mathbf{0}') = b_j(\mathbf{0}') = 0$ for all $0 \leq j < k$, however, $a_k(\mathbf{0}') \cdot b_k(\mathbf{0}') \neq 0$. We want to show that $x_d = y_d$ and $a_j = b_j$, for all $0 \leq j \leq k$.

Consider the map $\Phi(\mathbf{z}) := (\mathbf{z}', x_d(\mathbf{z}))$. Since $\frac{\partial x_d}{\partial z_d}(\mathbf{0}) = 1$, the Jacobian matrix $\frac{\partial \Phi}{\partial \mathbf{z}}(\mathbf{0})$ is lower-triangular with nonzero entries along the diagonal. The inverse mapping theorem 4.23 ensures that there is a polydisk $\Delta(\mathbf{0}, \mathbf{r})$ on which Φ^{-1} is well defined. Letting $G(\mathbf{z}) := F(\Phi^{-1}(\mathbf{z}))$ and $w_d = w_d(\mathbf{z}) := y_d(\Phi^{-1}(\mathbf{z}))$ it follows from (4.37) that

$$
\begin{aligned}
G(\mathbf{z}) &= \sum_{j=0}^{k} a_j \cdot z_d^j\,, \\
&= \sum_{j=0}^{k} b_j \cdot w_d^j\,.
\end{aligned}
\tag{4.38}
$$

Observe that $w_d(\mathbf{z}',0) = 0$ and $\frac{\partial w_d}{\partial z_d}(\mathbf{z}',0) = 1$.

Since G vanishes to degree k about the origin in the variable z_d, lemma 4.16 lets us assume without loss of generality that, for each $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$, the zeroes of the transformation $z_d \to G(\mathbf{z}', z_d)$, with $z_d \in \Delta(0, r_d)$, can be listed (repeated according to their multiplicity) in the form $\xi_1(\mathbf{z}'), \ldots, \xi_k(\mathbf{z}')$. Furthermore, the inverse mapping theorem 4.23 lets us also assume that the restriction $w_d(\mathbf{z}', \cdot) : \Delta(0, r_d) \to \mathbb{C}$ is a

1-to-1 transformation. As a result, proposition 4.26 lets us conclude using (4.38) that

$$a_k(\mathbf{z}') \cdot \prod_{j=1}^{k} \{z_d - \xi_j(\mathbf{z}')\} = b_k(\mathbf{z}') \cdot \prod_{j=1}^{k} \Big\{ w_d(\mathbf{z}', z_d) - w_d(\mathbf{z}', \xi_j(\mathbf{z}')) \Big\}. \tag{4.39}$$

In addition, back in (4.38), the conditions $a_j(\mathbf{0}') = b_j(\mathbf{0}') = 0$ valid for all $0 \le j < k$ imply that $w_d(\mathbf{0}', z_d) = z_d \cdot \left\{ \frac{a_k(\mathbf{0}')}{b_k(\mathbf{0}')} \right\}^{1/k}$, provided that the k^{th}-root of unity is appropriately selected. Therefore, if we define

$$v_d(\mathbf{z}) := w_d(\mathbf{z}) \cdot \left\{ \frac{a_k(\mathbf{z}')}{b_k(\mathbf{z}')} \right\}^{-1/k}$$

then we may rewrite (4.39) in the form

$$\prod_{j=1}^{k} \{z_d - \xi_j(\mathbf{z}')\} = \prod_{j=1}^{k} \Big\{ v_d(\mathbf{z}', z_d) - v_d(\mathbf{z}', \xi_j(\mathbf{z}')) \Big\}. \tag{4.40}$$

Without loss of generality we may assume that $v_d(\mathbf{z})$ is analytic for all $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$.

On the other hand, (4.38) implies that there is a polyradius $\mathbf{s}' < \mathbf{r}'$ such that $\xi_j(\mathbf{z}') \in \Delta\big(0, \frac{r_d}{4}\big)$, for all $1 \le j \le k$ and $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{s}')$. Thus, proposition 4.27 (with $\rho = r_d/4$, $r = 3r_d/4$ and $R = r_d$) let us conclude that there is $\delta > 0$ such that, for each $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{s}')$, $v_d(\mathbf{z}', z_d) = z_d$ is the only analytic function of $z_d \in \Delta(0, r_d)$ which satisfies (4.40) subjected to the condition

$$\sup_{|z_d| < \frac{3r_d}{4}} |v_d(\mathbf{z}', z_d) - z_d| \le \delta. \tag{4.41}$$

But, observe that $v_d(\mathbf{0}', z_d) = z_d$. Therefore, due to the uniform continuity of $v_d(\mathbf{z})$, we may ensure that (4.41) is satisfied for all $\mathbf{z}'$ sufficiently close to $\mathbf{0}'$. This let us conclude that $v_d(\mathbf{z}', z_d) = z_d$, for all $\mathbf{z}'$ sufficiently close to $\mathbf{0}'$. The identity theorem 4.9 implies that $v_d(\mathbf{z}) = z_d$, for all $\mathbf{z} \in \Delta(\mathbf{0}, \mathbf{r})$. In particular, $w_d(\mathbf{z}) = z_d \cdot \left\{ \frac{a_k(\mathbf{z}')}{b_k(\mathbf{z}')} \right\}^{1/k}$. Thus, since $\frac{\partial w_d}{\partial z_d}(\mathbf{z}', 0) = 1$, we conclude that $w_d(\mathbf{z}) = z_d$. This finding in (4.38) implies that $a_j = b_j$, for all $0 \le j \le k$. Furthermore, since $w_d(\mathbf{z}) := y_d(\Phi^{-1}(\mathbf{z}))$, with $\Phi(\mathbf{z}) := (\mathbf{z}', x_d(\mathbf{z}))$, we obtain that $x_d = y_d$. This shows that the polynomial canonical representation of Levinson is unique. □

4.8.2 There are k different Hörmander's representations.

We prove next that the canonical representation in (4.35) is unique once $\frac{\partial x_d}{\partial z_d}(\mathbf{0})$ has been specified. Thus, suppose that there are functions $x_d = x_d(\mathbf{z})$, $y_d = y_d(\mathbf{z})$ and $a_j = a_j(\mathbf{z}')$ and $b_j = b_j(\mathbf{z}')$ analytic near the origin such that

$$\begin{aligned} F(\mathbf{z}) &:= \frac{x_d^k}{k} + \sum_{j=0}^{k-2} a_j(\mathbf{z}') \cdot x_d^j\,, \\ &= \frac{y_d^k}{k} + \sum_{j=0}^{k-2} b_j(\mathbf{z}') \cdot y_d^j\,, \end{aligned} \tag{4.42}$$

with $x_d(\mathbf{0}) = y_d(\mathbf{0}) = 0$, $\frac{\partial x_d}{\partial z_d}(\mathbf{0}) = \frac{\partial y_d}{\partial z_d}(\mathbf{0}) \neq 0$ and $a_j(\mathbf{0}') = b_j(\mathbf{0}') = 0$, for all $0 \le j \le (k-2)$. We want to show that $x_d = y_d$ and $a_j = b_j$, for all $0 \le j \le (k-2)$.

Consider the map $\Phi(\mathbf{z}) := (\mathbf{z}', x_d(\mathbf{z}))$. Since $\frac{\partial x_d}{\partial z_d}(\mathbf{0}) \neq 0$, the same argument used in proving the uniqueness of Levinson's representation implies that there is a polydisk $\Delta(\mathbf{0}, \mathbf{r})$ on which Φ^{-1} is well defined. Letting $G(\mathbf{z}) := F(\Phi^{-1}(\mathbf{z}))$ and $w_d = w_d(\mathbf{z}) := y_d(\Phi^{-1}(\mathbf{z}))$ we obtain from (4.42) that

$$\begin{aligned} G(\mathbf{z}) &= \frac{z_d^k}{k} + \sum_{j=0}^{k-2} a_j(\mathbf{z}') \cdot z_d^j\,, \\ &= \frac{w_d^k}{k} + \sum_{j=0}^{k-2} b_j(\mathbf{z}') \cdot w_d^j\,. \end{aligned} \tag{4.43}$$

But, observe that $w_d(\mathbf{0}) = 0$ and $\frac{\partial w_d}{\partial z_d}(\mathbf{0}) = 1$. Since $a_j(\mathbf{0}') = b_j(\mathbf{0}') = 0$, for all $0 \le j \le (k-2)$, (4.43) implies that $w_d(\mathbf{0}', z_d) = z_d$. As a result, for all $0 < r < r_d$, the

$$\lim_{\mathbf{z}' \to \mathbf{0}'} \sup_{|z_d| \le r} |w_d(\mathbf{z}', z_d) - z_d| = 0\,.$$

Similarly as we argued to prove the uniqueness of Levinson's representation, propositions 4.26 and 4.27 together with the above condition imply that $w_d(\mathbf{z}) = z_d$. Hence, $x_d(\mathbf{z}) = y_d(\mathbf{z})$, for all $\mathbf{z}$ sufficiently close to the origin. Moreover, from (4.43) we can also conclude that, in some open neighborhood of the origin, $a_j = b_j$, for all $0 \leq j \leq (k-2)$. The uniqueness of Hörmander's canonical representation then follows from the identity theorem 4.9. $\square$

4.8.3 Proof of proposition 4.26.

All over this section it will be assumed that $P \in \mathbb{C}[z]$ and $Q \in \mathbb{C}[y]$ are polynomials of the same degree $k \geq 1$. We will let $U := \Delta(0, R)$, for some $R > 0$. Proposition 4.26 is the direct consequence of the following two lemmas.

Lemma 4.28. *Suppose that $y : U \to \mathbb{C}$ is a 1-to-1 analytic function such that $P(z) = Q(y(z))$, for all $z \in U$. If all roots of P are contained within U then all roots of Q are contained in the range of $y(z)$.*

Proof. Let $\xi_1, \ldots, \xi_k$ be a list of the roots of P repeated according to their multiplicity. Define $y_j := y(\xi_j)$. Certainly, $y = y_j$ is a root of $Q(y)$. Let n_j be the multiplicity of $z = \xi_j$ as a root of $P(z)$; in particular, the $\lim\limits_{z \to \xi_j} \frac{P(z)}{(z-\xi_j)^{n_j}}$ is finite and nonzero. It follows that

$$
\text{(4.44)} \qquad \begin{aligned} \lim_{y \to y_j} \frac{Q(y)}{(y - y_j)^{n_j}} &= \lim_{z \to \xi_j} \frac{Q(y(z))}{\{y(z) - y_j\}^{n_j}} \\ &= \left\{ \frac{1}{y'(\xi_j)} \right\}^{n_j} \cdot \lim_{z \to \xi_j} \frac{P(z)}{(z - \xi_j)^{n_j}} . \end{aligned}
$$

For the first identity above, we have used that $y_j = y(\xi_j)$ is in the interior of the range of $y(z)$ as warranted by the Open mapping theorem (see [Rud87]). Since $y'(\xi_j) \neq 0$, (4.44) implies that $y = y_j$ is a root of multiplicity n_j of $Q(y)$. But, the injectivity of $y(z)$ implies that the multiplicity of y_j in the list $y_1, \ldots, y_k$ is precisely n_j. As a result, $\prod_{j=1}^{k}(y - y_j)$ divides $Q(y)$, and this proves the lemma. $\square$

In lemma 4.28, the single condition that all roots of $P(z)$ are contained in U is not sufficient to ensure that all roots of $Q(y)$ are in the range of $y(z)$. Injectivity plays a fundamental role which is more than just technical. As a counterexample consider $R > 1$ and let ρ be a nonnegative real number such that $\rho^2 > 4(R^2 + 1)$. Define $P(z) := (z - 1) \cdot (z + 1)$ and $Q(y) := y \cdot (y - \rho)$. Setting $y(z) := \frac{\rho}{2} + \sqrt{\left(\frac{\rho}{2}\right)^2 + z^2 - 1}$ a simple computation reveals that $P(z) = Q(y(z))$, for all $|z| < R$. However, observe that $\{y - y(1)\} \cdot \{y - y(-1)\}$ is either equal to $(y - \rho)^2$ or y^2. Therefore, no matter which branch for the squared-root is chosen either $y = 0$ or $y = \rho$ will not belong to the range of $y(z)$. This is not in contradiction with lemma 4.28 because $y(z)$ is an even function of z.

Lemma 4.29. *Suppose that $y : U \to \mathbb{C}$ is an analytic function such that $P(z) = Q(y(z))$, for all $z \in U$, and all roots of Q are contained in the range of $y(z)$. If $\xi_1, \ldots, \xi_k$ is a list of the roots of P repeated according to their multiplicity then $y(\xi_j) = y(\xi_l)$ if and only if $\xi_j = \xi_l$, and the polynomial $\prod_{j=1}^{k}\{y - y(\xi_j)\}$ divides $Q(y)$.*

Proof. List the roots of $Q(y)$ in the form $y(\xi_1), \ldots, y(\xi_k)$ repeated according to their multiplicity in such a way that $y(\xi_j) = y(\xi_l)$ if and only if $\xi_j = \xi_l$. Define $y_j := y(\xi_j)$ and observe that ξ_j is a root of $P(z)$. To prove the lemma it will be enough to show that $\xi_1, \ldots, \xi_k$ lists the roots of P repeated according to its multiplicity.

Suppose that $y = y_j$ is a zero of order n_j of $Q(y)$; in particular, the $\lim_{y \to y_j} \frac{Q(y)}{(y - y_j)^{n_j}}$ is finite. Then, a similar argument as the one used in (4.44) lets us obtain this time that

$$\lim_{z\to\xi_j}\frac{P(z)}{(z-\xi_j)^{n_j}}=\{y'(\xi_j)\}^{n_j}\cdot\lim_{y\to y_j}\frac{Q(y)}{(y-y_j)^{n_j}}\,. \tag{4.45}$$

Since the right-hand side above is finite we can conclude that ξ_j is a root of multiplicity at least n_j of $P(z)$. As a result, for all j, the polynomial $(z-\xi_j)^{n_j}$ divides $P(z)$. In particular, since n_j is precisely the multiplicity of ξ_j in the list $\xi_1,\ldots,\xi_k$ we can deduce that $\prod_{j=1}^{k}(z-\xi_j)$ divides $P(z)$. Thus, being $P(z)$ of degree k, it follows that $\xi_1,\ldots,\xi_k$ lists the roots of P repeated according to their multiplicity. This completes the proof of the lemma. □

4.8.4 Proof of proposition 4.27.

To prove the proposition we first require three lemmas.

Lemma 4.30. *Suppose that $\alpha_0,\alpha_1,\alpha_2,\ldots$ is a sequence of complex numbers such that $\alpha_0=1$ and there is $0<\rho\le 1$ such that for all $j\ge 1$ the $|\alpha_j|\le\rho^j$. Then there is a sequence $\beta_0,\beta_1,\beta_2,\ldots$ such that $|\beta_j|\le(2\rho)^j$, for all $j\ge 0$, and*

$$\sum_{j=l}^{n}\beta_{n-j}\cdot\alpha_{j-l}=\begin{cases}0 & , \quad n>l\,,\\ 1 & , \quad n=l\,.\end{cases} \tag{4.46}$$

Proof. Define $A(z):=\sum_{j=0}^{\infty}\alpha_j\cdot z^j$. The condition imposed on the coefficients α_j's implies that $A(z)$ is analytic for all $|z|<\frac{1}{\rho}$. Furthermore, $\frac{1}{A(z)}$ is analytic near the origin because $A(0)=1$. As a result, if we let $\beta_j:=[z^j]\,\frac{1}{A(z)}$ then

$$\sum_{j=0}^{n}\beta_{n-j}\cdot\alpha_j=\begin{cases}0 & , \quad n>0\,,\\ 1 & , \quad n=0\,.\end{cases} \tag{4.47}$$

(4.46) follows almost immediately from (4.47). To complete the proof of the lemma we will show by induction that $|\beta_j| \leq (2\rho)^j$, for all $j \geq 0$. The case $j = 0$ is trivial because $\beta_0 = 1$. For the general case consider $j \geq 1$ and assume that $|\beta_l| \leq (2\rho)^l$, for all $0 \leq l < j$. (4.47) then implies that

$$|\beta_j| = \left| \sum_{l=1}^{j} \beta_{j-l} \cdot \alpha_l \right| \leq (2^j - 1) \cdot \rho^j \leq (2\rho)^j .$$

This completes the proof of the lemma. □

Before stating our next result we will introduce some notation. We will let U to denote the open disk $\Delta(0, R)$ with $R > 0$ and for all $z \in U$ we will define $id(z) := z$. The space of analytic functions over U will be denoted as $\mathcal{H}$. For each compact set $K \subset U$, we will consider the seminorm $\|\cdot\|_K$ on $\mathcal{H}$ defined as $\|f\|_K := \sup_{z \in K} |f(z)|$. For the particular case in which $K = \Delta[0, r]$ for some $0 < r < R$ the notation $\|f\|_r$ will be used instead of $\|f\|_{\Delta[0,r]}$.

We will embed $\mathcal{H}$ with the topology induced by the semi-norms $\|\cdot\|_K$ with $K \subset U$ compact. This is the so called topology of uniform convergence over compact subsets of U. This topology is induced by a metric under which $\mathcal{H}$ is a Banach Space. Moreover, a sequence $(f_j)_{j \geq 0} \subset \mathcal{H}$ converges to $f \in \mathcal{H}$ provided that $\lim_{j \to \infty} \|f - f_j\|_K = 0$, for all compact set $K \subset U$. (For a general reference on these facts see [Tay02], section 2.4.) With these remarks it should be clear that the vector subspace $\mathcal{H}_0 := \{f \in \mathcal{H} : f(0) = 0\}$ is closed and therefore embedded with the topology of uniform convergence it is also a Banach Space.

We will let $\mathcal{P}$ to denote the vector space of polynomials in the variable z. $\mathcal{P}_0$ will be the vector subspace of polynomials that vanish at the origin. Observe that $\mathcal{P}_0$ is a dense subset of $\mathcal{H}_0$ whereas $\mathcal{P}$ is a dense subset of $\mathcal{H}$.

Lemma 4.31. *Let* $\mathcal{L} : \mathcal{H}_0 \times U^k \to \mathcal{H}$ *be the operator defined as*

$$\mathcal{L}(f; z_1, \ldots, z_k)(z) := \frac{1}{k} \sum_{j=1}^{k} \frac{f(z) - f(z_j)}{z - z_j} \,. \tag{4.48}$$

If the $\max_{j=1\ldots k} |z_j| \le \rho < \frac{R}{2}$ *then the restriction* $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k) : \mathcal{H}_0 \to \mathcal{H}$ *is a linear isomorphism.*

Proof. The removable singularity theorem 4.18 implies that the map $\mathcal{L} : \mathcal{H}_0 \times U^k \to \mathcal{H}$ is well-defined, moreover, for each given $(z_1, \ldots, z_k) \in U^k$, the restriction $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k) : \mathcal{H}_0 \to \mathcal{H}$ is certainly linear. To show that this transformation is also continuous observe that

$$\mathcal{L}(f; z_1, \ldots, z_k)(z) = \frac{1}{k} \sum_{j=1}^{k} \int_0^1 f'(z_j + t(z - z_j))\, dt \,.$$

If the $\max_{j=1\ldots k} |z_j| < r_1 < R$ then $|\mathcal{L}(f; z_1, \ldots, z_k)(z)| \le \|f'\|_{r_1}$, for all $|z| = r_1$. On the other hand, if $r_1 < r_2 < R$ then the Cauchy's estimates together with the maximum modulus principle (see [Rud87]) imply that $\|f'\|_{r_1} \le \frac{\|f\|_{r_2}}{r_2 - r_1}$. As a result, if $f \in \mathcal{H}_0$ then

$$\|\mathcal{L}(f; z_1, \ldots, z_k)\|_{r_1} \le \frac{\|f\|_{r_2}}{r_2 - r_1} \,,$$

provided that the $\max_{j=1\ldots k} |z_j| < r_1 < r_2 < R$. The continuity of $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k)$ is now almost a direct consequence of the above inequality.

Define

$$p(z; z_1, \ldots, z_k) := \prod_{j=1}^{k} (z - z_j) \,. \tag{4.49}$$

To show that $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k)$ is 1-to-1, suppose that $\mathcal{L}(f; z_1, \ldots, z_k) \equiv 0$, with $f \in \mathcal{H}_0$. We will show this implies that $f \equiv 0$. Indeed, if $\mathcal{L}(f; z_1, \ldots, z_k) \equiv 0$ then a simple calculation reveals that $f(z) \cdot u(z) = v(z)$, where $u(z) := \frac{dp}{dz}(z; z_1, \ldots, z_k)$ and

$v(z)$ is certain polynomial of degree less or equal to $(k-1)$. The division algorithm for polynomials implies that there is $\alpha \in \mathbb{C}$ and a polynomial $r(z)$ of degree less than the degree of $u(z)$ such that

$$f(z) = \alpha + \frac{r(z)}{u(z)}\,, \tag{4.50}$$

provided that $z \in U$ and $u(z) \neq 0$.

Consider $\max\limits_{j=1\ldots k} |z_j| < r < R$. It is a well-known fact that the roots of $u(z)$ are all a convex linear combination of $z_1, \ldots, z_k$. [5] Hence, all the roots of $u(z)$ are contained in the disk $\Delta(0,r)$ and therefore, since the degree of $r(z)$ is less than the degree of $u(z)$, $\frac{r(z)}{u(z)}$ cannot be bounded in this disk unless $r(z) \equiv 0$. Since the left-hand side in (4.50) is bounded we conclude that $r(z) \equiv 0$. Thus, $f(z)$ must be constant. The condition $f(0) = 0$ finally implies that $f \equiv 0$. Consequently, $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k)$ is 1-to-1 as claimed.

To conclude that $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k) : \mathcal{H}_0 \to \mathcal{H}$ is an isomorphism we will use the condition $\max\limits_{j=1\ldots k} |z_j| \leq \rho < \frac{R}{2}$ to show that there is a continuous linear operator $\mathcal{T}_0 : \mathcal{H} \to \mathcal{H}_0$ such that $\mathcal{L}_0(\mathcal{T}_0 f; z_1, \ldots, z_k) = f$, for all $f \in \mathcal{H}$.

Define $\alpha_0 := 1$ and for all $n \geq 1$ let $\alpha_n := \frac{1}{k}\sum\limits_{j=1}^{k} z_j^n$. A simple calculation reveals that

$$\mathcal{L}_0(z^{n+1}; z_1, \ldots, z_k) = \sum_{j=0}^{n} \alpha_{n-j}\, z^j\,.$$

The above computation implies that the restriction $\mathcal{L}(\,\cdot\,; z_1, \ldots, z_k) : \mathcal{P}_0 \to \mathcal{P}$ is a well-defined linear operator. To invert this map, observe that $|\alpha_n| \leq \rho^n$, for all $n \geq 0$. Thus, with $(\beta_j)_{j\geq 0}$ as prescribed in Lemma 4.30, consider the linear mapping

[5] Indeed if $u(z) = 0$ and $z \neq z_j$, for all j, then $\left\{\sum\limits_{j=1}^{k} \alpha_j\right\} z = \sum\limits_{j=1}^{k} \alpha_j\, z_j$, with $\alpha_j := \frac{1}{|z-z_j|^2}$

$\mathcal{T}_0 : \mathcal{P} \to \mathcal{P}_0$ defined as

$$\mathcal{T}_0 f(z) := z \cdot \sum_{j=0}^{\infty} \left\{ \sum_{l=j}^{\infty} f_l \, \beta_{l-j} \right\} z^j \,, \text{ with } f_l := [z^l] f \,. \tag{4.51}$$

A simple calculation reveals that

$$\mathcal{L}_0 \left(\sum_{j=0}^{n} \beta_{n-j} \cdot z^{j+1}; z_1, \ldots, z_k \right) = z^n \,.$$

As a result, it is almost direct to verify that $\mathcal{L}_0(\mathcal{T}_0 \, z^n; z_1, \ldots, z_k) = z^n$, for all $n \geq 0$. Therefore,

$$\mathcal{L}_0(\mathcal{T}_0 \, f; z_1, \ldots, z_k) = f \,, \tag{4.52}$$

for all $f \in \mathcal{P}$.

To conclude that $\mathcal{L}_0 : \mathcal{H}_0 \to \mathcal{H}$ is an isomorphism it will be enough to show that $\mathcal{T}_0 : \mathcal{P} \to \mathcal{P}_0$ can be extended continuously to a linear map from $\mathcal{H}$ to $\mathcal{H}_0$. Thus consider $f \in \mathcal{H}$ and $2\rho < r_0 < r_1 < R$. Cauchy's estimates (see [Rud87]) imply that $|f_n| \leq \frac{\|f\|_{r_1}}{r_1^n}$, for all $n \geq 0$. Since $|\beta_j| \leq (2\rho)^j$, it follows that

$$\left| \sum_{l=j}^{\infty} f_l \cdot \beta_{l-j} \right| \leq \frac{r_1 \, \|f\|_{r_1}}{r_1 - 2\rho} \cdot r_1^{-j} \,. \tag{4.53}$$

The above inequality implies that the power series in (4.51) defines an analytic function for $|z| < r_1$. Since this is the case for all $2\rho < r_1 < R$ and $\mathcal{T}_0 f(0) = 0$ we can deduce that $\mathcal{T}_0 : \mathcal{H} \to \mathcal{H}_0$ is a well-defined linear operator. Furthermore, (4.53) implies that

$$\|\mathcal{T}_0 \, f\|_{r_0} \leq \frac{r_0 \, r_1}{(r_1 - 2\rho) \cdot (1 - r_0/r_1)} \cdot \|f\|_{r_1} \,, \tag{4.54}$$

for all $f \in \mathcal{H}$, provided that $2\rho < r_0 < r_1 < R$. As a result, $\mathcal{T}_0 : \mathcal{H} \to \mathcal{H}_0$ is a continuous linear operator and this completes the proof of the lemma. □

Lemma 4.32. *Let $\mathcal{L} : \mathcal{H}_0 \times U^k \to \mathcal{H}$ and $p(z; z_1, \ldots, z_k)$ be as defined in (4.48) and (4.49) respectively. If $0 \le 2\rho < r < R$ then there is a constant $c = c(\rho, r, R)$ such that for all $f \in \mathcal{H}_0$ and $(z_1, \ldots, z_k)$ such that the $\max_{j=1\ldots k} |z_j| \le \rho$ the*

$$\|p(\,\cdot\,; z_1, \ldots, z_k) \cdot \mathcal{L}(f; z_1, \ldots, z_k)\|_r \ge c \cdot \|f\|_r \,. \tag{4.55}$$

Proof. Consider the compact set

$$K := \Big\{ (z; z_1, \ldots, z_k) : |z| = r \text{ and } |z_j| \le \rho, \text{ for all } 1 \le j \le k \Big\}.$$

Since $|p(z; z_1, \ldots, z_k)| > 0$, for all $(z; z_1, \ldots, z_k) \in K$, the continuity of p implies the existence of a constant constant $c_0 > 0$ such that $|p(z; z_1, \ldots, z_k)| \ge c_0$, for all $(z; z_1, \ldots, z_k) \in K$. The maximum modulus principle ([Rud87]) then implies that

$$\|p(\,\cdot\,; z_1, \ldots, z_k) \cdot \mathcal{L}(f; z_1, \ldots, z_k)\|_r \ge c_0 \cdot \|\mathcal{L}(f; z_1, \ldots, z_k)\|_r \,. \tag{4.56}$$

To obtain a lower bound for $\|\mathcal{L}(f; z_1, \ldots, z_k)\|_r$ in terms of $\|f\|_r$ we will refine the estimate in (4.54). For this, reconsider $\mathcal{T}_0 : \mathcal{H} \to \mathcal{H}_0$ as given in (4.51). The condition $\mathcal{L}(\mathcal{T}_0 f; z_1, \ldots, z_k) = f$, for all $f \in \mathcal{H}$, implies that

$$\frac{dp}{dz}(z; z_1, \ldots, z_k) \cdot \mathcal{T}_0 f(z) = p(z; z_1, \ldots, z_k) \cdot \left\{ k \cdot f(z) + \sum_{j=1}^{k} \frac{\mathcal{T}_0 f(z_j)}{z - z_j} \right\}, \tag{4.57}$$

for all $|z| < R$.

On the other hand, all the zeroes of $\frac{dp}{dz}(z; z_1, \ldots, z_k)$, regarded as a function of z, are within the disk $\Delta[0, \rho]$. Since $2\rho < r$, it follows that there is a finite constant $c_1 = c_1(r, \rho, R) > 0$ such that $\left| \frac{p(z; z_1, \ldots, z_k)}{\frac{dp}{dz}(z; z_1, \ldots, z_k)} \right| \le c_1$, for all $(z; z_1, \ldots, z_k) \in K$. (4.57) followed by the inequality in (4.54) with $r_1 = r$ implies for all $2\rho < r_0 < r$ that

$$\begin{aligned} \|\mathcal{T}_0 f\|_r &\le k \cdot c_1 \cdot \left\{ \|f\|_r + \frac{\|T_0 f\|_\rho}{r - \rho} \right\}, \\ &\le k \cdot c_1 \cdot \left\{ 1 + \frac{r_0 \cdot r}{(r - \rho) \cdot (r - 2\rho) \cdot (1 - r_0/r)} \right\} \cdot \|f\|_r \,. \end{aligned}$$

We conclude that for all $2\rho < r < R$ there is a constant $c_2 = c_2(r, \rho, R) > 0$ such that

$$\|\mathcal{T}_0 f\|_r \leq c_2 \cdot \|f\|_r ,$$

for all $f \in \mathcal{H}$ and $(z_1, \ldots, z_k)$ such that $\max\limits_{j=1\ldots k} |z_j| \leq \rho$. In particular, we can deduce that $\|f\|_r = \|\mathcal{T}_0(\mathcal{L}_0(f; z_1, \ldots, z_k))\|_r \leq c_2 \cdot \|\mathcal{L}_0(f; z_1, \ldots, z_k)\|_r$. This in (4.56) completes the proof of the lemma. $\square$

We now prove proposition 4.27. Motivated by (4.36) consider the map $\mathcal{F} : \mathcal{H}_0 \times U^k \to \mathcal{H}$ defined as $\mathcal{F}(f; z_1, \ldots, z_k)(z) := \prod\limits_{j=1}^{k} \{f(z) - f(z_j)\}$.

A simple inductive argument then shows that

$$\mathcal{F}(f + id; z_1, \ldots, z_k)(z) = \sum_J \left\{ \prod_{j \in J} (z - z_j) \right\} \cdot \prod_{j \notin J} \{f(z) - f(z_j)\} , \tag{4.58}$$

where the index J in the summation includes all possible subsets of $\{1, \ldots, k\}$.[6] As a result, by defining

$$\begin{aligned} \mathcal{L}_1(f; z_1, \ldots, z_k) &:= \sum_{J:|J|=(k-1)} \left\{ \prod_{j \in J} (z - z_j) \right\} \cdot \prod_{j \notin J} \{f(z) - f(z_j)\} , \\ \mathcal{E}_1(f; z_1, \ldots, z_k) &:= \sum_{J:|J|\leq(k-2)} \left\{ \prod_{j \in J} (z - z_j) \right\} \cdot \prod_{j \notin J} \{f(z) - f(z_j)\} , \end{aligned}$$

it follows that

$$\mathcal{F}(f + id; z_1, \ldots, z_k) - \mathcal{F}(id; z_1, \ldots, z_k) = \mathcal{L}_1(f; z_1, \ldots, z_k) + \mathcal{E}_1(f; z_1, \ldots, z_k) . \tag{4.59}$$

Observe that $\mathcal{L}_1(f; z_1, \ldots, z_k)(z) = k \cdot p(z; z_1, \ldots, z_k) \cdot \mathcal{L}(f; z_1, \ldots, z_k)(z)$, with $p(z; z_1, \ldots, z_k)$ and $\mathcal{L}(f; z_1, \ldots, z_k)$ as given in lemma 4.32 respectively. In addition, a straightforward computation reveals that for all $2\rho < r < R$ there is a finite constant

[6] Given complex number $\xi_1, \ldots, \xi_k$ we will use the convention that $\prod\limits_{j \in \emptyset} \xi_j := 1$ and $\sum\limits_{j \in \emptyset} \xi_j := 0$.

$c_1 = c_1(r, \rho, R) > 0$ such that for all $f \in \mathcal{H}_0$ and $(z_1, \ldots, z_k)$ such that $\max_{j=1\ldots k} |z_j| \le \rho$ the

$$\|\mathcal{E}_1(f; z_1, \ldots, z_k)\|_r \le c_1 \cdot \sum_{j=2}^{k} \|f\|_r^j . \tag{4.60}$$

To prove proposition 4.27 suppose that $y \in \mathcal{H}_0$ and consider $f := (y - id) \in \mathcal{H}_0$. (4.59), (4.55), and (4.60) then let us obtain that

$$\begin{aligned}
\|\mathcal{F}(y; z_1, \ldots, z_k) - \mathcal{F}(id; z_1, \ldots, z_k)\|_r &= \|\mathcal{L}_1(f; z_1, \ldots, z_k) + \mathcal{E}_1(f; z_1, \ldots, z_k)\|_r , \\
&\ge k \cdot \|p(\cdot; z_1, \ldots, z_k) \cdot \mathcal{L}(f; z_1, \ldots, z_k)\|_r \\
&\qquad - \|\mathcal{E}_1(f; z_1, \ldots, z_k)\|_r , \\
&\ge \|f\|_r \cdot \left\{ k\, c - c_1 \cdot \sum_{j=1}^{k-1} \|f\|_r^j \right\} .
\end{aligned}$$

As a result, if we let $\delta := \min\left\{1, \frac{k\,c}{c_1(k-1)}\right\}$ then, for all $(z_1, \ldots, z_k)$ such that $\max_{j=1\ldots k} |z_j| \le \rho$, and all $y \in \mathcal{H}_0$ such that $0 < \|y - id\|_r < \delta$, we obtain that $\|\mathcal{F}(y; z_1, \ldots, z_k) - \mathcal{F}(id; z_1, \ldots, z_k)\| > 0$. This shows (4.36) and completes the proof of proposition 4.27. □

CHAPTER 5

TWO GENERALIZED SADDLE POINT METHODS

5.1 Introduction

This chapter is concerned with the asymptotic analysis of Fourier-Laplace integrals which depend upon a parameter. To start our introduction we will consider first the parameter-free case which is the most studied and well-understood.

A *parameter-free Fourier-Laplace integral* is of the form $\int_\gamma \exp\{-s \cdot f(z)\}\, a(z)\, dz$ where $s \geq 0$ and $f(z)$ and $a(z)$ are analytic functions in an open neighborhood of the contour γ. The function $f(z)$ in the exponential is usually referred to as *phase term* whereas $a(z)$ is the so called *amplitude term.* The term *Laplace integral* is preferred to describe the case where the phase term is real-valued along the contour of integration. However, if it takes only purely imaginary values the term *Fourier integral* is used instead.

To study the asymptotic behavior for big values of $s \geq 0$ of an integral of the form $\int_\gamma \exp\{-s \cdot f(z)\}\, a(z)\, dz$ one may use Cauchy's deformation theorem to reshape the contour of integration into a variety of other contours. The value of the integral will remain unchanged as long as the starting and ending points of the new contour coincide with the original contour γ. The basic idea of the *method of steepest descents* or also called *saddle point method* (see [BleHan86], section 7) is to deform the contour γ into a new contour which produces a Laplace integral.

The contour deformation as intended in the previous paragraph is not always possible. Bleistein and Handelsman [BleHan86] said it well: "[this contour deformation] is not only the pivotal step in the analysis [of a Fourier-Laplace integral, but] it is also often the most difficult to apply." Indeed, in the best scenario, one can only hope to replace the contour γ by an asymptotically equivalent contour which, piecewise, decomposes into smaller contours where the phase term remains purely real and/or purely imaginary. For clarification, two contours γ_1 and γ_2 are said to be *asymptotically equivalent* if the quantity

$$\left| \int_{\gamma_1} \exp\{-s \cdot f(z)\}\, a(z)\, dz - \int_{\gamma_2} \exp\{-s \cdot f(z)\}\, a(z)\, dz \right|$$

is a rapidly decreasing function of s.

The advantage of having to deal with Laplace integrals is that it is relatively simple to study their asymptotic behavior. The *Laplace method* (see chapter 5 in [BleHan86]) has been devised to handle precisely integrals of this form. It states that the asymptotic behavior of $\int_\gamma \exp\{-s \cdot f(z)\}\, a(z)\, dz$ is determined by the local behavior of $f(z)$ in a small neighborhood of the points which minimize the phase term along the contour of integration.

On the other hand, the *integration by parts method* (see chapter 3 in [BleHan86]) and the *stationary phase method* (see chapter 6 in [BleHan86]) are very suitable to deal with Fourier integrals. For Fourier integrals of the form $\int_\gamma \exp\{-s \cdot f(z)\}\, a(z)\, dz$ there are two types of points to consider to determine their asymptotic behavior. These are the boundary points of γ and the stationary points of the phase term lying over γ. A point z is said to be a *stationary point* or sometimes *saddle point* of $f(z)$ if $f'(z) = 0$. The contribution to an integral produced by stationary points can be obtained using the stationary phase method, however, the contribution of the boundary points of the contour of integration may be obtained using the method of integration by parts.

It is important to remark that the general theory of Fourier integrals is much more intriguing than the theory for Laplace integrals. This is mainly because the boundary points of the domain of integration of a Laplace integral do not contribute significantly to the integral unless, the phase term is minimized at the boundary. In contrast, for a general Fourier integral boundary points and stationary points may contribute equally to the integral. However, there are settings where the boundary points of Fourier integrals are of no relevance. This is the case when the amplitude term is compactly supported (see chapter VIII in [Ste93]). More specifically these are integrals of the form $\int_{\mathbb{R}} \exp\{-i \cdot s \cdot \phi(z)\}\, \psi(z)\, dz$ where $\phi(z)$ and $\psi(z)$ are real-valued C^∞-functions and $\psi(z)$ is compactly supported on $\mathbb{R}$. In this case, if $\phi'(z)$ is zero-free over the support of $\psi(z)$ then the $\int_{\mathbb{R}} \exp\{-i \cdot s \cdot \phi(z)\}\, \psi(z)\, dz$ is a rapidly decreasing function of s. However, if a stationary point of $\phi(z)$ belongs to the interior of the support of $\psi(z)$ then a full asymptotic expansion for the Fourier integral can be obtained by only considering the local behavior of $\phi(z)$ and $\psi(z)$ near this point.

The previous discussion pretty much accounts for all the known methods to deal with parameter-free Fourier-Laplace integrals. However, in many situations of interest one is forced to consider instead integrals of the form

$$I(t;s) := \int_\gamma \exp\{-s \cdot F(t,z)\}\, A(t,z)\, dz\,,$$

where $F(t, z)$ and $A(t, z)$ are analytic functions of (t, z) for all z in an open neighborhood of γ (which can be regarded as independent of t) for all t sufficiently close, say to, $t = 0$. For convenience we will assume without loss of generality that $z = 0 \in \gamma$.

The first attempt to study the asymptotic behavior of integrals of this form seems to have been done by Olver (see [Olv54a] and [Olv54b]). A systematization of his ideas then was provided by Chester et al. whose method became to be known as the *coalescing saddle point method* (see [CFU56] and/or chapter 9 in [BleHan86]). It is designed to find an asymptotic expansion for an integral such as $I(t; s)$ for a phase term in special class and a parameter-free amplitude. More specifically, they considered a phase term with a Hartogs series near the origin of the form

$$F(t, z) = u(t) \cdot z^2 + v(t) \cdot z^3 + \dots$$

where $u(0) = 0$, $v(0) \neq 0$, however, $u(t)$ is not identically zero near $t = 0$. This implies that for all t sufficiently small but nonzero, $F(t, z)$, regarded as a function of z, has two stationary points near the origin, namely $z = 0$ and another point $z = z(t)$ such that $\lim_{t \to 0} z(t) = 0$. (This is the motivation to refer to this method as the coalescing saddle point method.) Assume that (i) the boundary points of the integral do not contribute significantly, and (ii) at any other stationary point of $F(t, z)$ the real part of $F(t, z)$ is greater than the real part of $F(t, z)$ at $z = 0$ and $z = z(t)$. Then, Chester et al. [CFU56] conclude that $I(t; s)$ has an asymptotic expansion in terms of the Airy function which is uniform for all values of t sufficiently small, as $s \to \infty$. The hypotheses required on the phase term are very common in most discussions where the steepest descent method is used. Condition (i) is satisfied per se if the contour of integration has an empty boundary. Condition (ii) is usually troublesome but likely to be tested in cases where $F(t,z)$ is known explicitly and has a simple algebraic description. However, these conditions, especially condition (ii), are very limiting in situations where little quantitative information is available for the phase term.

A vast level of generalization of the results of Chester et al. can be found in the literature. Particularly interesting is the work of Ludwing [Lud67] who, in the context of a compactly supported amplitude term, studied the problem of determining the bandwidth for the parameter t so as to ensure that the stationary phase formula for $I(t;s)$ remains valid as if $F(t,z)$ did not expose a change of degree as $t \to 0$.

The greatest generalization of the work of Chester et al. was provided after the publication of the work of Levinson on canonical representations of functions of several complex variables [Lev61] (see section 4.8). Under conditions similar to (i) and (ii) just mentioned, Bleistein [Ble67] and latter Ursell [Urs72] generalized the coalescing saddle point method to consider the case of many coalescing saddle points and even a parameter varying amplitude term.

The main two results we present in this chapter are concerned with a parameter varying Fourier-Laplace integral such as $I(t;s)$ where the contour of integration has a nonempty boundary. Two cases of interest are considered in here. A generalized version of the stationary phase method is proposed to consider the asymptotic behavior of a parameter varying amplitude term and a phase term which vanishes to constant degree about its critical points for all sufficiently small t. Unlike the work of Bleistein and Ursell, our approach to deal with a parameter varying amplitude does not use the technique of integration by parts but rather Levinson's canonical forms. This has the advantage to provide a uniform asymptotic expansion for $I(t;s)$ depending on coefficients which relate more explicitly to the amplitude term. Our second result, a generalized version of the coalescing saddle point method, allows again the possibility of a parameter varying amplitude, however, this time, a phase term like the one in the discussion of Chester et al. is considered.

The demonstrations of our two main results in this chapter draw on the techniques of Chester et. al, the result of Levinson on canonical forms, and the work of Pemantle and Wilson in [PemWil01]. The later adapted the stationary phase method for a parameter-free Fourier-Laplace integral without requiring a compactly supported amplitude term.

In relation to applications, we remark that the main hypotheses required to make use of both main theorems are simple to verify. Furthermore, since they are of a "local sort", they are likely of verification in situations where little qualitative information is known about the phase and amplitude term. Moreover, they fit without alteration with the applications we have in mind to enumerative combinatorics.

The last section of this chapter before engaging in the proof of our main results, section 5.5, is devoted to an application. Our discussion in there is two-folded. We first motivate the appearance of parameter varying Fourier-Laplace integrals as natural objects to be encountered in the context of asymptotic enumeration. Our findings are then applied to a concrete example related to the problem of counting the number of three-connected components of a non-separable rooted map. Our generalized version of the coalescing saddle point method is then applied as an alternative approach to obtain a local limit of the map Airy-type as fist presented by Banderier et al. in [BFSS00]. Several other applications of the main two results of this chapter will be discussed in chapter 6 to consider the problem of determining the asymptotic behavior of the coefficients of meromorphic functions along directions specified by smooth points in the zero set of their denominator.

5.2 A generalized stationary phase method

To state our main result in this and the following section the next definition will be of use.

Definition 5.1. Given nonnegative integers $n < m$ and a function $H(t,z)$ analytic in an open neighborhood of the origin we will say that $H(t,z)$ *has an n-to-m change of degree at $z = 0$ as $t \to 0$* provided that $H(t,z)$ has a Hartogs series about the origin of the form $H(t,z) = h_n(t) \cdot z^n + \ldots + h_m(t) \cdot z^m + \ldots$ with $h_j(0) = 0$, for all $n \leq j \leq (m-1)$, $h_m(0) \neq 0$, however, $h_n(t)$ is not identically zero in any neighborhood of $t = 0$.

On the contrary, if $H(t,z) = h_n(t) \cdot z^n + \ldots$ with $h_n(0) \neq 0$ we will say that $H(t,z)$ has an *n-to-n change of degree about $z = 0$ as $t \to 0$*. Alternatively, we will sometimes say that $H(t,z)$ *vanishes to constant degree n at $z = 0$ for all t nearby $t = 0$*.

Our main results in this section concern with Fourier-Laplace integrals of the form

$$I(t;s) := \int_0^1 e^{-s \cdot F(t,z)} \, A(t,z) \, dz \,,$$
$$J(t;s) := \int_{-1}^1 e^{-s \cdot F(t,z)} \, A(t,z) \, dz \,,$$

where s is a nonnegative real number which we later let to tend to infinity. $F(t,z)$ and $A(t,z)$ are assumed to be analytic in a neighborhood of $(0,0)$ that contains points of the form (t,z) with $|z| \leq 1$ for all sufficiently small t.

The following result generalizes theorem 5.2 in [PemWil01] to consider a Fourier-Laplace integral with an amplitude term which does not necessarily vanish to constant degree about its critical points.

Theorem 5.2. (Generalized stationary phase method.) *Let $I(t;s)$, etc. be as defined before. In addition, suppose that*

(a) $\Re\{F(0,z)\} > \Re\{F(0,0)\}$, *for all* $z \in [0,1]$, *but* $z = 0$,

(b) there is an integer $n \geq 1$ *such that:* $F(t,z) - F(t,0) = u(t) \cdot z^n + \ldots$ *with* $u(0) \neq 0$, *and*

(c) there are nonnegative integers $p \leq q$ *such that* $A(t,z)$ *has a p-to-q change of degree about* $z = 0$ *as* $t \to 0$.

Then there are functions $A_k(t)$ *such that* $A_k(0) = 0$, *for all* $p \leq k \leq (q-1)$, *however,* $A_q(0) \neq 0$, *and functions* $I_k(t;s)$ *analytic in* t *near* $t = 0$ *and a constant* $c > 0$ *such that*

$$I(t;s) = e^{-s \cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot I_k(t;s) + O(e^{-s \cdot c}) \right\}, \tag{5.1}$$

uniformly for all t *sufficiently close to* 0 *and all* $s \geq 0$. *Moreover, for each* $p \leq k \leq q$, *there is an asymptotic expansion of the form*

$$I_k(t;s) \approx \sum_{j=k}^{\infty} \frac{c_k(t;j)}{n} \cdot \Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}, \tag{5.2}$$

which is uniform for all t *sufficiently close to* 0, *as* $s \to \infty$. *The coefficients* $c_k(t;j)$ *are analytic in* t *near* $t = 0$ *and*

$$c_k(t;k) = \{u(t)\}^{-(k+1)/n}. \tag{5.3}$$

Remark 5.3. We remark that the multiplicative constant for the big-O in (5.1) is independent of t and s, provided that t is sufficiently close to $t = 0$ and $s \geq 0$. (See section 3.1 for further reference on our terminology.) (5.1) might be of little use if s is not sufficiently big. However, our interest is in providing an asymptotic description for $I(t; s)$ valid for rather big-values of s. Under this perspective, (5.1) is a very useful identity.

Remark 5.4. It is cumbersome and in fact obfuscating to provide general formulae for the coefficients $A_k(t)$ and $c_k(t; s)$ appearing in (5.1) and (5.2) respectively. However, the following characterizations of these coefficients will be more than enough in the applications of theorem 5.2 we have in mind.

The coefficients $A_k(t)$ together with an auxiliary function $y = y(t, z)$ are uniquely characterized through the relations

$$\begin{cases} \int_0^z A(t,\xi)\, d\xi = \sum\limits_{k=p}^{q} \frac{A_k(t)}{k+1} \cdot y^{k+1}\,, \\ A_k(0) = 0, \text{ for all } p \leq k < q,\ A_q(0) \neq 0\,, \\ y = y(t,z) = z + \ldots \end{cases} \tag{5.4}$$

We will define the transformations

$$\begin{cases} (t, y) & = & \Psi(t, z) & := & (t, y(t, z))\,, \\ x & = & \phi(t, y) & := & y \cdot \{u(t)\}^{1/n} \cdot \left\{1 + \frac{F(\Psi^{-1}(t,y)) - F(t,0) - u(t)\cdot y^n}{u(t)\cdot y^n}\right\}^{1/n} \end{cases}, \tag{5.5}$$

where the selection of the n^{th}-root is in the principal sense. Observe that $y = z + \ldots$ and $x = y \cdot \{u(t)\}^{1/n} + \ldots$ The coefficients $c_k(t; j)$ are then characterized through the relation

$$\sum_{j=k}^{\infty} c_k(t; j) \cdot x^j = y^k \cdot \frac{\partial y}{\partial x}\,. \tag{5.6}$$

Remark 5.5. The asymptotic notation used in (5.2) is in the standard sense where the sequence $(s^{-(j+1)/n})_{j\geq k}$ is the so called "auxiliary asymptotic sequence." (See [BleHan86] section 1.5.) By this we mean that the difference between $I_k(t;s)$ and the partial sum on the right-hand side up to the term $j = m$ is $O(s^{-(m+2)/n})$ uniformly for all t sufficiently small as $s \to \infty$.

Remark 5.6. It is remarkable that condition (a) in theorem 5.2 is as weaker as requesting that for the particular value of $t = 0$, $z = 0$ is the dominant critical point of the Fourier-Laplace integral $I(0;s) = \int_0^1 e^{-s\cdot F(0,z)}\, A(0,z)\, dz$. Condition (a) does not necessarily imply that for all t sufficiently small, $z = 0$ minimizes the $\Re\{F(t,z)\}$ over the interval $[0,1]$. Indeed, if $\Re\{u(0)\} = 0$ then the Open Mapping theorem for harmonic functions (see [Rud87]) lets us state that on any neighborhood of $t = 0$ there are infinitely many points where the $\Re\{u(t)\} < 0$. For such values of t there will be $z_0 \in (0,1)$ such that that $\Re\{F(t,z_0) - F(t,0)\} = \Re\{u(t)\} \cdot z_0^n + \ldots < 0$ and therefore the

$$\min_{z\in[0,1]} \Re\{F(t,z)\} < \Re\{F(t,0)\}\,.$$

This is not in contradiction with (5.1) because points where this minimum is attained will approach $z = 0$ as $t \to 0$ and hence they are not stationary points of the phase term (unless they coincide with $z = 0$). Thus, the impression that $e^{-s\cdot \min_{z\in[0,1]} \Re\{F(t,z)\}}$ is the right exponential order of $I(t;s)$ is mistaken.

Remark 5.7. The condition that $A(t,z)$ has a p-to-q change of degree about $z = 0$ as $t \to 0$ with $q < \infty$ can be weakened. Indeed, unless $A(t,z)$ is identically zero near the origin, there is a unique factorization of the form $A(t,z) = t^N \cdot B(t,z)$, where N is a nonnegative integer and $B(t,z)$ has a p-to-q change of degree about $z = 0$ as

$t \to 0$, with $0 \le p \le q < \infty$. In particular,

$$I(t;s) = t^N \cdot \int_0^1 e^{-s\cdot F(t,z)}\, B(t,z)\, dz\,,$$

and the integral on the right-hand side above can be studied using theorem 5.2. To amplify, if $\sum_{k=0}^{\infty} a_k(t)\, z^k$ is the Hartogs series of $A(t,z)$ near the origin in powers of z then N is the minimum among the degrees of vanishing (about $t=0$) of the coefficient functions $a_k(t)$ which are not identically zero (in any neighborhood of $t=0$). One then can show that

$$B(t,z) = \frac{1}{(N-1)!}\int_0^1 (1-\tau)^{N-1}\cdot \frac{\partial^N A}{\partial t^N}(\tau\cdot t, z)\, d\tau\,.$$

Remark 5.8. Before we engage in some applications of theorem 5.2 we pause to comment on an interesting phenomena introduced by the presence of a parameter in the amplitude term of a Fourier-Laplace integral. At a first glance, since these integrals can be thought as linear operators in their amplitude term, it may seem that the presence of a parameter, for example, that produces a change of degree in the amplitude term, should be of little relevance. The aim of this remark is to show that this view is not quite right. We will refer to this anomalous aspect as *the issue with linearity.*

To fix ideas consider a Fourier-Laplace integral $I(t;s) := \int_0^1 e^{-s\cdot z^2}\, A(t,z)\, dz$ where $A(t,z)$ is certain entire function of t and z whose Hartogs series about the origin is of the form $A(t,z) = t + z^q + \ldots$ with $q > 0$. This implies that $A(t,z)$ has a 0-to-q change of degree about $z=0$ as $t \to 0$. Since the phase term of $I(t;s)$ is parameter-free the classical version of the stationary phase method implies that

$$I(t;s) \approx \frac{t}{2}\cdot \Gamma\left(\frac{1}{2}\right)\cdot s^{-1/2} + \frac{1}{2}\cdot \Gamma\left(\frac{q+1}{2}\right)\cdot s^{-(q+1)/2} + \ldots \tag{5.7}$$

uniformly for all t sufficiently small, as $s \to \infty$. Furthermore, if cancellation of the first two terms is ruled out, the leading order of $I(t; s)$ should be found among them, and we could conclude that

$$I(t; s) \sim \frac{t}{2} \cdot \Gamma\left(\frac{1}{2}\right) \cdot s^{-1/2} + \frac{1}{2} \cdot \Gamma\left(\frac{q+1}{2}\right) \cdot s^{-(q+1)/2},$$

uniformly for all t sufficiently small, as $s \to \infty$. Assume that $t = t(s) \to 0$ as $s \to \infty$ at a sufficiently fast rate so that the second term above is the dominant asymptotic order. What we have decided to describe as *the issue with linearity* is concerned with the problem of determining the second leading order. Indeed, does not take much of an effort to realize that the term of order $t \cdot s^{-1/2}$ above may have nothing to do with the second dominant asymptotic order of $I(t; s)$. For example, if $A(t, z) = t + z^q + z^{q+1} + \ldots$ then

$$I(t; s) \approx \frac{t}{2} \cdot \Gamma\left(\frac{1}{2}\right) \cdot s^{-1/2} + \frac{1}{2} \cdot \Gamma\left(\frac{q+1}{2}\right) \cdot s^{-(q+1)/2} + \frac{1}{2} \cdot \Gamma\left(\frac{q+2}{2}\right) \cdot s^{-(q+2)/2} + \ldots$$

uniformly for all t sufficiently small, as $s \to \infty$. In particular, if $t = t(s) = o(s^{-(q+1)/2})$ then

$$I(t(s); s) = \frac{1}{2} \cdot \Gamma\left(\frac{q+1}{2}\right) \cdot s^{-(q+1)/2} + \frac{1}{2} \cdot \Gamma\left(\frac{q+2}{2}\right) \cdot s^{-(q+2)/2} + o(s^{-(q+2)/2}),$$

as $s \to \infty$. As a result, the information that $A(t, z) = t + z^q + \ldots$ may not be sufficient to detect the second dominant asymptotic order of $I(t; s)$ when t is allowed to depend on s: some orders produced by the tail of the Hartogs series of $A(t, z)$ may "sneak-in" between the terms of order $t \cdot s^{-1/2}$ and $s^{-(q+1)/2}$ in (5.7).

The example we have just discussed should serve as a model of much more complicated situations. It shows that there is combinatorial problem behind the determination of the different bandwidths for the parameter $t = t(s)$ which specify radically different behaviors for $I(t(s); s)$. In this respect theorem 5.2 contributes with a qualitative understanding of $I(t; s)$ stating that at most $(q - 2)$ non-trivial asymptotic terms could "sneak-in" between the two orders exposed in (5.7); this, no matter how well or badly possed are the terms hidden in the Hartogs series of $A(t, z)$.

The following example is an application of theorem 5.2 which, due to the small change degree of the amplitude term, can be worked out almost explicitly. It is by no means an interesting application of theorem 5.2, however, it illustrates well how to apply this theorem to a concrete situation.

Example 5.9. (1-to-2 amplitude, 2-to-2 phase.)

Consider the integral

$$I(t; s) := \int_0^1 e^{-s \cdot z^2} \tan(t \cdot z + z^2)\, dz\,.$$

In the notation of theorem 5.2 we have $F(t, z) := z^2$ and $A(t, z) := \tan(t \cdot z + z^2) = t \cdot z + z^2 + \ldots$ and therefore $A(t, z)$ has a 1-to-2 change of degree about $z = 0$ as $t \to 0$. Conditions (a), (b) and (c) of theorem 5.2 are easily verified and therefore there are functions $A_1(t)$, $A_2(t)$, $I_1(t; s)$ and $I_2(t; s)$ analytic near $t = 0$ and a nonnegative constant $c > 0$ such that

$$\begin{aligned} I(t; s) &= A_1(t) \cdot I_1(t; s) + A_2(t) \cdot I_2(t; s) + O(e^{-s \cdot c})\,, \\ I_1(t; s) &\approx \frac{1}{2s} + \ldots \\ I_2(t; s) &\approx \frac{\sqrt{\pi}}{4s^{3/2}} + \ldots \end{aligned}$$

The coefficient functions $A_1(t)$ and $A_2(t)$ are uniquely determined via the relation

$$\int_0^z \tan(t \cdot \xi + \xi^2)\, d\xi = \frac{A_1(t)}{2} \cdot y^2 + \frac{A_2(t)}{3} \cdot y^3\,, \tag{5.8}$$

with $A_1(0) = 0$, $A_2(0) \neq 0$ and $y(t,z) = z + \ldots$ This last condition implies that $\lim_{z\to 0} \frac{y}{z} = 1$, for all t sufficiently small. As a result, if we divide both sides above by z^2 and then let $z \to 0$ it follows that

$$A_1(t) = \lim_{z\to 0} \frac{2}{z^2} \int_0^z \tan(t \cdot \xi + \xi^2) d\xi = t\,. \tag{5.9}$$

In addition, if in (5.8) we evaluate both sides at $t = 0$, then divide by z^3, and let $z \to 0$ we obtain that

$$A_2(0) = \lim_{z\to 0} \frac{3}{z^3} \int_0^z \tan(t \cdot \xi + \xi^2) d\xi = 1\,.$$

The above computation will prove useful to determine $A_2(t)$. Indeed, as we will show, $A_2(t)$ is the solution of a second-order polynomial equation. To obtain this relation we first differentiate both sides in (5.8) with respect to z. We obtain

$$\tan(t \cdot z + z^2) = y \cdot \Big(t + A_2(t) \cdot y\Big) \cdot \frac{\partial y}{\partial z}(t,z)\,.$$

The left-hand side above vanishes if $t \cdot z + z^2 = 0$. Indeed, if for all sufficiently small t we think of the equation: $\tan(t \cdot z + z^2) = 0$ as an equation in z near $z = 0$ then the origin and $z = -t$ are its only solutions. On the other hand, the right-hand side above vanishes at $y = 0$ and $y = \frac{-t}{A_2(t)}$. Thus, since for all sufficiently small t the transformation $z \longrightarrow y(t,z)$ is 1-to-1 near the origin and $y(t,0) = 0$, we must have $y(t,-t) = \frac{-t}{A_2(t)}$. Plugging in these values of z and y in (5.8) let us conclude that

$$A_2(t) = \frac{1}{\sqrt{\frac{6}{t^3}\int_0^{-t} \tan(t \cdot \xi + \xi^2) d\xi}}\,, \tag{5.10}$$

$$= 1 - \frac{1}{140}\, t^4 - \frac{263}{3880800}\, t^8 + \ldots \tag{5.11}$$

where the principal branch of the square-root has to be chosen in order to be consistent with the condition $A_2(0) = 1$. Thus, we finally obtain that

$$(5.12)\quad I(t;s) = t\cdot\left(\frac{1}{2s} + \dots\right) + \frac{1}{\sqrt{\frac{6}{t^3}\int_0^{-t}\tan(t\cdot\xi+\xi^2)d\xi}}\cdot\left(\frac{\sqrt{\pi}}{4s^{3/2}} + \dots\right) + O(e^{-s\cdot c})\,.$$

The small change of degree of the amplitude term made possible the exact determination of $A_1(t)$ and $A_2(t)$. However, it is clear that theorem 5.2 is highly complex from the numerical point of view. In this regard, it might not be worth using when the amplitude term undergoes a small change of degree.

An alternative asymptotic development for $I(t;s)$ could have been obtained using the Hartogs series of $\tan(t\cdot z + z^2)$ about $(t,z) = (0,0)$ and in powers of z. Although a general formula for the terms in this series is far to be trivial we may list some few of them to obtain

$$(5.13)\qquad I(t;s) \approx \frac{t}{2s} + \frac{\sqrt{\pi}}{4s^{3/2}} + \frac{t^3}{6s^2} + \frac{3\sqrt{\pi}}{8s^{5/2}} + \frac{t}{s^3}\left(1 + \frac{2t^4}{15}\right) + \dots$$

uniformly for all t sufficiently small, as $s \to \infty$.

Observe that our findings in (5.12) and (5.13) are consistent. It is difficult to judge which representation is better than the other in this case. If t is allowed to tend to zero with s at slow rate namely so that $s^{-1/2} = o(t)$ then both series are equally good, for example, to conclude that $I(t;s) \sim \frac{t}{2s}$. Similarly, if the dependence of t with respect to s is such that $t = o(s^{-1/2})$ then using either expansion one obtains that $I(t;s) \sim \frac{\sqrt{\pi}}{4s^{3/2}}$. On the other hand, if t is restricted to be away from the negative real-axis, for example, requesting that for some very small value of $\epsilon > 0$ the $|\arg t| \leq (\pi - \epsilon)$, then the leading orders of the each of the two terms on the right-hand side of (5.12) cannot cancel each other and we may assert that

$$I(t;s) = \left\{\frac{t}{2s} + \frac{1}{\sqrt{\frac{6}{t^3}\int_0^{-t}\tan(t\cdot\xi+\xi^2)d\xi}}\cdot\frac{\sqrt{\pi}}{4s^{3/2}}\right\}\cdot(1 + o(1))\,,$$

uniformly for all t sufficiently small in the sector $|\arg t| < (\pi - \epsilon)$, as $s \to \infty$. $\square$

Theorem 5.2 can be used to provide the asymptotic expansion of a two-sided integral. In some sense, the following result generalizes the statement and proof of corollary 5.3 in [PemWil01] to consider a FL-integral with a parameter varying amplitude term.

Corollary 5.10. *Let $J(t;s)$, etc. be as defined before and suppose that*

(a) $\Re\{F(0,z)\} > \Re\{F(0,0)\}$, *for all* $z \in [-1,1]$, *but* $z = 0$,

(b) there is an integer $n \geq 2$ *such that:* $F(t,z) - F(t,0) = u(t) \cdot z^n + \ldots$ *with* $u(0) \neq 0$, *and*

(c) there are nonnegative integers $p \leq q$ *such that* $A(t,z)$ *has a* p*-to-*q *change of degree about* $z = 0$ *as* $t \to 0$.

If $A_k(t)$ *with* $k = p, \ldots, q$ *are as defined in (5.4) then there are coefficients* $J_k(t;s)$ *and a constant* $c > 0$ *such that*

$$J(t;s) = e^{-s \cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot J_k(t;s) + O(e^{-s \cdot c}) \right\}, \tag{5.14}$$

uniformly for all t *sufficiently close to* 0 *and all* $s \geq 0$. *Moreover, each term* $J_k(t;s)$ *in the above summation admits an asymptotic expansion involving the coefficients* $c_k(t;j)$ *as defined in (5.6). More precisely,*

$$J_k(t;s) \approx \sum_{j=k}^{\infty} c_k(t;j) \cdot \left\{ 1 + (-1)^j \cdot D(j,n) \right\} \cdot \frac{1}{n} \Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}, \tag{5.15}$$

uniformly for all t *sufficiently small, as* $s \to \infty$, *where we have defined*

$$D(j,n) := \begin{cases} 1 & , \quad n \text{ even} \quad , \\ \exp\left(-\frac{i\pi(j+1)}{n} \cdot sgn\{i \cdot u(0)\}\right) & , \quad n \text{ odd} \quad . \end{cases}$$

Proof. The two-sided integral $J(t;s)$ is the sum of two one-sided integrals on the interval $[0,1]$ and $[-1,0]$ respectively. The interval of integration of this last integral can be transformed to be $[0,1]$ by means of a substitution which takes into account the transformation $z \longrightarrow (-z)$. This let us rewrite

$$J(t;s) = \int_0^1 e^{-s\cdot F(t,z)}\, A(t,z)\, dz + \int_0^1 e^{-s\cdot F(t,-z)}\, A(t,-z)\, dz\,.$$

Each integral on the right-hand side above admits an asymptotic expansion as prescribed in theorem 5.2. More precisely, there is a constant $c > 0$ and functions $A_k(t)$, $c_k(t;j)$, $\tilde{A}_k(t)$ and $\tilde{c}_k(t;j)$ analytic near $t = 0$ such that

$$\int_0^1 e^{-s\cdot F(t,z)}\, A(t,z)\, dz \;=\; e^{-s\cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot I_k(t;s) + O(e^{-s\cdot c}) \right\}, \tag{5.16}$$

$$I_k(t;s) \;\approx\; \sum_{j=k}^{\infty} \frac{c_k(t;j)}{n} \cdot \Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}, \tag{5.17}$$

$$\int_0^1 e^{-s\cdot F(t,-z)}\, A(t,-z)\, dz \;=\; e^{-s\cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} \tilde{A}_k(t) \cdot \tilde{I}_k(t;s) + O(e^{-s\cdot c}) \right\}, \tag{5.18}$$

$$\tilde{I}_k(t;s) \;\approx\; \sum_{j=k}^{\infty} \frac{\tilde{c}_k(t;j)}{n} \cdot \Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}. \tag{5.19}$$

The terms $A_k(t)$ and $\tilde{A}_k(t)$ together with certain auxiliary functions $y = y(t,z)$ and $\tilde{y} = \tilde{y}(t,z)$ are uniquely characterized through the relations

$$\begin{cases} \int_0^z A(t,\xi)\, d\xi = \sum_{k=p}^{q} \frac{A_k(t)}{k+1} \cdot y^{k+1} \quad , \quad y = y(t,z) = z + \ldots \\ \int_0^z A(t,-\xi)\, d\xi = \sum_{k=p}^{q} \frac{\tilde{A}_k(t)}{k+1} \cdot \tilde{y}^{k+1} \quad , \quad \tilde{y} = \tilde{y}(t,z) = z + \ldots \end{cases}$$

with the additional restriction that $A_k(0) = \tilde{A}_k(0) = 0$, for all $p \leq k < q$, and $A_q(0) \cdot \tilde{A}_q(0) \neq 0$. But, the identity on the second row above can be written in the equivalent form

$$\int_0^z A(t,\eta)\, d\eta = \sum_{k=p}^{q} \frac{(-1)^k \cdot \tilde{A}_k(t)}{k+1} \cdot \{-\tilde{y}(t,-z)\}^{k+1}\,.$$

The uniqueness then implies that $\tilde{A}_k(t) = (-1)^k \cdot A_k(t)$ and $\tilde{y}(t,z) = -y(t,-z)$. In particular, if we define

$$J_k(t;s) := I_k(t;s) + (-1)^k \cdot \tilde{I}_k(t;s)\,,$$

then using (5.16) and (5.18) we obtain that

$$J(t;s) = e^{-s\cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot J_k(t;s) + O(e^{-s\cdot c}) \right\}. \tag{5.20}$$

Moreover, using (5.17) and (5.19) it follows that

$$J_k(t;s) \approx \sum_{j=k}^{\infty} \left\{ c_k(t;j) + (-1)^k \cdot \tilde{c}_k(t;j) \right\} \cdot \frac{1}{n}\,\Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}. \tag{5.21}$$

To complete the proof of the corollary all we are required to do is to specify a (linear) relation between $c_k(t;j)$ and $\tilde{c}_k(t;j)$ which is consistent with (5.15). This will be done using the remark in (5.6) according to which $c_k(t;j) = [x^j]\, y^k \cdot \frac{\partial y}{\partial x}$ and $\tilde{c}_k(t;j) = [\tilde{x}^j]\, \tilde{y}^k \cdot \frac{\partial \tilde{y}}{\partial \tilde{x}}$. Here x and $\tilde{x}$ relate respectively to y and $\tilde{y}$ through the transformations

$$\begin{aligned} x &:= \psi(t,z) &&:= y \cdot \{u(t)\}^{1/n} \cdot \left\{ 1 + \tfrac{F(t,z) - F(t,0) - u(t)\cdot y^n}{u(t)\cdot y^n} \right\}^{1/n}, \\ \tilde{x} &:= \tilde{\psi}(t,z) &&:= \tilde{y} \cdot \{(-1)^n \cdot u(t)\}^{1/n} \cdot \left\{ 1 + \tfrac{F(t,-z) - F(t,0) - (-1)^n\cdot u(t)\cdot \tilde{y}^n}{(-1)^n\cdot u(t)\cdot \tilde{y}^n} \right\}^{1/n}. \end{aligned}$$

We first consider the case in which n is even. In this case, using the identity $\tilde{y}(t,z) = -y(t,-z)$, it is almost immediate to see that $\tilde{\psi}(t,z) = -\psi(t,-z)$. Furthermore, since $\tilde{c}_k(t;j)$ is the coefficient of $\tilde{x}^j$ in the series $\tilde{y}^k \cdot \frac{\partial \tilde{y}}{\partial \tilde{x}}$ we can rewrite

$$\begin{aligned} (-1)^k \cdot \tilde{c}_k(t;j) &= \frac{(-1)^k}{2\pi i} \int \frac{\tilde{y}^k}{\tilde{x}^{j+1}} \frac{\partial \tilde{y}}{\partial \tilde{x}}\, d\tilde{x}\,, \\ &= \frac{(-1)^j}{2\pi i} \int \frac{\{y(t,-z)\}^k}{\{\psi(t,-z)\}^{j+1}} \frac{\partial y}{\partial z}(t,-z)\, d(-z)\,, \\ &= \frac{(-1)^j}{2\pi i} \int \frac{y^k}{x^{j+1}} \frac{\partial y}{\partial x}\, dx\,, \\ &= (-1)^j \cdot c_k(t;j)\,. \end{aligned}$$

The above identity in (5.21) shows (5.15) for the case in which n is even.

Consider now the case n odd. Then, condition (a) in the corollary implies that $u(0)$ is purely imaginary and, as a result, a simple calculation reveals that $\{-u(t)\}^{1/n} = e^{\frac{i\pi\cdot sgn\{i\cdot u(0)\}}{n}} \cdot \{u(t)\}^{1/n}$. This implies that $\tilde{\psi}(t,z) = -e^{\frac{i\pi\cdot sgn\{iu(0)\}}{n}} \cdot \psi(t,-z)$. As a result, pretty much by repeating the argument used for the case in which n is even, this time we obtain

$$(-1)^k \cdot \tilde{c}_k(t;j) = (-1)^j \cdot \exp\left\{-i\pi \cdot \text{sgn}\{i \cdot u(0)\} \cdot \frac{j+1}{n}\right\} \cdot c_k(t;j)\,.$$

(5.15) then follows using the above identity in (5.21). This completes the proof of the corollary. □

Example 5.11. (Two-sided integral, 1-to-2 amplitude, 3-to-3 phase.)

We will use corollary 5.10 to determine the leading asymptotic order of

$$J(t;s) := \int_{-1}^{1} e^{-s\cdot(i\cdot z^3 + z^4)} \tan(t\cdot z + z^2)\, dz\,,$$

for values of t sufficiently close to 0 and of $s \geq 0$ sufficiently big. Part of the work done in example 5.9 can be reused in here. Indeed, using corollary 5.10, it follows that there is a constant $c > 0$ and coefficients $J_1(t;s)$ and $J_2(t;s)$ such that

$$\begin{aligned} J(t;s) &= t\cdot J_1(t;s) + A_2(t)\cdot J_2(t;s) + O(e^{-s\cdot c})\,, \\ A_2(t) &:= \frac{1}{\sqrt{\frac{6}{t^3}\int_0^{-t}\tan(t\cdot\xi+\xi^2)\,d\xi}}\,, \\ &= 1 - \frac{1}{140}\,t^4 + \ldots \end{aligned}$$

We will show that

$$\begin{aligned} J_1(t;s) &\sim -\frac{i\sqrt{3}}{3}\Gamma\left(\frac{2}{3}\right)\cdot s^{-2/3}\,, \\ J_2(t;s) &\sim \frac{4\sqrt{3}}{9}\cdot\Gamma\left(\frac{4}{3}\right)\cdot s^{-4/3}\,. \end{aligned}$$

The leading orders of $J_1(t;s)$ and $J_2(t;s)$ can be obtained from (5.15). They depend on certain coefficients $c_1(t;1)$ and $c_2(t;3)$, however, using (5.3) it follows immediately that $c_1(t;1) = e^{-\pi i/3}$. Therefore, one finds that

$$\begin{aligned} J_1(t;s) &= -\frac{i\sqrt{3}}{3}\Gamma\left(\frac{2}{3}\right)\cdot s^{-2/3} + O(s^{-4/3})\,, \\ J_2(t;s) &= c_2(t;3)\cdot\frac{3+i\sqrt{3}}{6}\cdot\Gamma\left(\frac{4}{3}\right)\cdot s^{-4/3} + O(s^{-5/3})\,. \end{aligned}$$

The above shows the order claimed for $J_1(t;s)$.

In what remains of this example we will determine $c_2(t;3)$ as explicitly as possible. We start observing that the remark in (5.6) states that $c_2(t;3) = [x^3]\,y^2\frac{\partial y}{\partial x}$. But, from the remarks in (5.4) and (5.5) it follows that

$$\begin{aligned} y &= y(t,z) = z + \dots \\ x &= x(t,y) = i^{1/3}\cdot y + \dots \end{aligned}$$

This implies that $c_2(t;3) = 4\,i^{-2/3}\,[x^2]\,y$. To compute $[x^2]\,y$ we need to look more closely the relation between the variables x, y and z as established in (5.4) and (5.5). In our context these relations are described through the transformations

$$\frac{t}{2}\cdot y^2 + \frac{A_2(t)}{3}\cdot y^3 = \int_0^z \tan(t\cdot\xi + \xi^2)\,d\xi \tag{5.22}$$

$$x = i^{1/3}\cdot y\cdot\left(1 + \frac{i\cdot z^3 + z^4 - i\cdot y^3}{i\cdot y^3}\right)^{1/3} \tag{5.23}$$

To find $[x^2]\,y$ we use z as an intermediate variable between x and y. For this, define $\alpha(t) := [y^2]\,z$, and write

$$z = y + \alpha(t)\cdot y^2 + \dots$$

Back in (5.22) if we make explicit the first few terms of the series on the right-hand side and then write z as a series in y we obtain that

$$\begin{aligned} \frac{t}{2} \cdot y^2 + \frac{A_2(t)}{3} \cdot y^3 &= \frac{t}{2} \cdot z + \frac{z^3}{3} + \dots \\ &= \frac{t}{2} \cdot y^2 + \left\{ t \cdot \alpha(t) + \frac{1}{3} \right\} \cdot y^3 + \dots \end{aligned}$$

Therefore, we can conclude that

$$\begin{aligned} \alpha(t) &= \frac{A_2(t) - 1}{3t}, \\ &= \frac{-1}{420} \cdot t^3 + \frac{-263}{11642400} \cdot t^7 + \frac{-817}{2542700160} \cdot t^{11} + \dots \end{aligned}$$

With $[y^2]\, z$ available we seek for the term $[x^2]\, y$. This is done using the identity $(1+w)^{1/3} = 1 + \frac{w}{3} + \dots$ in (5.23). We obtain

$$x = i^{1/3} \cdot y + i^{1/3} \cdot \frac{3\alpha(t) - i}{3} \cdot y^2 + \dots$$

This last series is easy to reverse. Its first few terms are recognized to be

$$y = \frac{x}{i^{1/3}} + \frac{i - 3\alpha(t)}{3i^{2/3}} \cdot x^2 + \dots$$

implying that $[x^2]\, y = \frac{i - 3\alpha(t)}{3i^{2/3}}$. Finally, using the relation determined between $\alpha(t)$ and $A_2(t)$ we obtain that

$$\begin{aligned} c_2(t;3) &= 4 \cdot e^{-2\pi i/3} \cdot \frac{1 + i \cdot t - A_2(t)}{3t}, \\ &= \frac{4 \cdot e^{-\pi i/6}}{3} + \frac{e^{-2\pi i/3}}{105} \cdot t^3 + \frac{263 \cdot e^{-2\pi i/3}}{2910600} \cdot t^7 + \dots \end{aligned}$$

and from this the claim made on the leading order of $J_2(t;s)$ follows almost immediately. $\square$

5.3 A brief review of the Airy-function

In this section we briefly review of some basic properties of the Airy function that will be of use in the coming sections. An *Airy-function* is any nontrivial solution of the second-order linear differential equation

$$a''(x) - x \cdot a(x) = 0\,,\;\; x \in \mathbb{C}\,. \tag{5.24}$$

If $a(x)$ is a nontrivial solution of (5.24) then $a(x)$ and $a'(x)$ cannot have common zeroes in the complex plane. Indeed, by repeated differentiation of (5.24), it follows that $a^{(n)}(x) = (n-2)\cdot a^{(n-3)}(x) + x \cdot a^{(n-2)}(x)$, for all $n \geq 3$. In particular, if there is $x_0 \in \mathbb{C}$ such that $a(x_0) = a'(x_0) = 0$ then $a^{(n)}(x_0) = 0$, for all $n \geq 0$. But, being $a(x)$ an entire analytic function, this implies that $a(x) \equiv 0$ and this proves the claim.

Bleistein and Handelsman (see [BleHan86], section 2.5) give a deductive approach to find nontrivial solutions of (5.24). Instead, we claim that

$$a(x) := \frac{1}{2\pi} \int_\gamma \exp\left\{i \cdot \left(x\,\zeta + \zeta^3/3\right)\right\} d\zeta\,, \tag{5.25}$$

is a particular solution for the Airy-equation provided that γ is a contour going through infinity (in the Riemann sphere) and eventually contained in a set of the form $\left\{\xi \in \mathbb{C} :\; Re\{i \cdot \zeta^3\} \leq -c \cdot |\zeta|^3\right\}$, for some arbitrary constant $c > 0$. To verify the assertion, observe that

$$\begin{aligned} x \cdot a(x) - a''(x) &= \frac{1}{2\pi} \int_\gamma e^{i\cdot(x\,\zeta + \zeta^3/3)}\,(x + \zeta^2)\,d\zeta\,, \\ &= \frac{1}{2\pi i} \int_\gamma \frac{d}{d\zeta} \left\{e^{i\cdot(x\,\zeta+\zeta^3/3)}\right\} d\zeta\,, \\ &= 0\,. \end{aligned}$$

For example, the contour γ in (5.25) could be taken to be any of the contours γ_1, γ_2 or γ_3 depicted in figure 5.1.

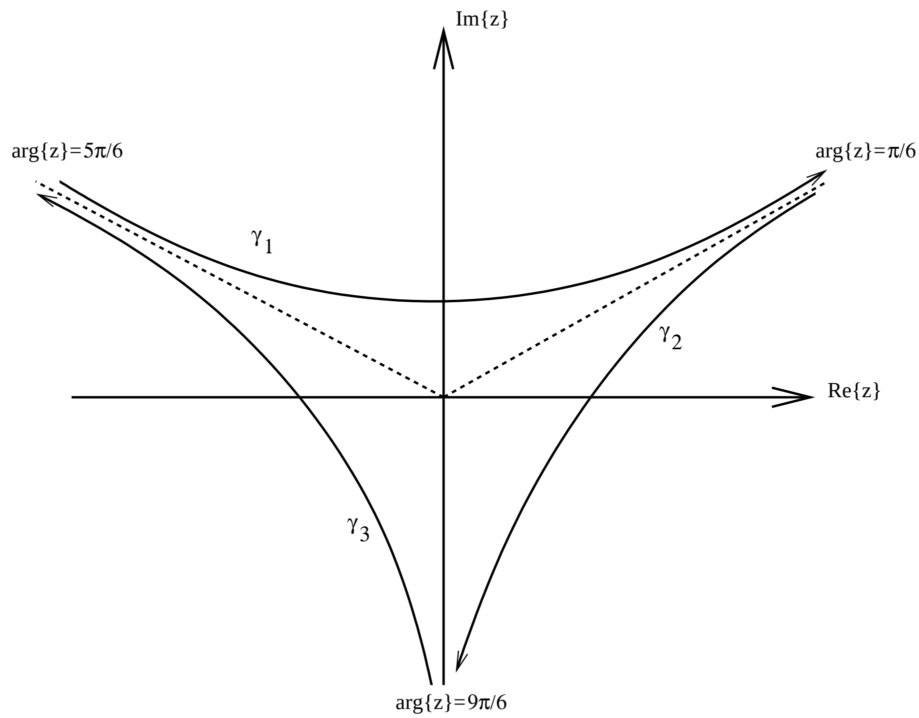

Figure 5.1: Any function of the form $a(x) := \frac{1}{2\pi}\int_\gamma \exp\left\{i\cdot\left(x\,\zeta + \zeta^3/3\right)\right\}d\zeta$, with $\gamma = \gamma_1$, γ_2 or γ_3 is a non-trivial solution of the Airy-equation. The Airy function corresponds to the selection of $\gamma = \gamma_1$.

A useful remark involving solutions of the Airy equation such as in (5.25) is that

$$a(x) + \omega\cdot a(\omega\,x) + \omega^2\cdot a(\omega^2\,x) = 0\,, x\in\mathbb{C}\,, \tag{5.26}$$

provided that ω is a nontrivial cube-root of unity. If $a(x)$ is any solution of (5.24) then it follows almost immediately that $a(\omega{\cdot}x)$ is also a solution provided that $\omega^3 = 1$. As a result, the above identity reveals explicitly the linear relation between $a(x)$, $a(\omega\,x)$ and $a(\omega^2\,x)$. To show (5.26), suppose that $a(x)$ is of the particular form in (5.25).

Substituting: $\xi = \omega^n \cdot \zeta$, one obtains that

$$\omega^n \cdot a(\omega^n x) = \frac{1}{2\pi} \int_{\omega^n \cdot \gamma} e^{i(x\,\xi + \xi^3/3)}\, d\xi\,.$$

The claim in (5.26) follows using Cauchy's theorem as shown in the following computation

$$\begin{aligned} a(x) + \omega \cdot a(\omega\, x) + \omega^2 \cdot a(\omega^2\, x) &= \frac{1}{2\pi} \left\{ \int_\gamma + \int_{\omega \cdot \gamma} + \int_{\omega^2 \cdot \gamma} \right\} e^{i \cdot (x\,\zeta + \zeta^3/3)}\, d\zeta\,, \\ &= \frac{1}{2\pi} \int_{\gamma + \omega \cdot \gamma + \omega^2 \cdot \gamma} e^{i \cdot (x\,\zeta + \zeta^3/3)}\, d\zeta\,, \\ &= 0\,. \end{aligned}$$

The *Airy function* is defined as

$$\mathbf{Ai}(x) := \frac{1}{2\pi} \int_{\gamma_1} \exp\Big\{ i \cdot \big(x\,\zeta + \zeta^3/3 \big) \Big\}\, d\zeta\,, \tag{5.27}$$

where the contour γ_1 is any contour as the one represented in figure 5.1. In particular, the derivative of the Airy function can be represented also in an integral form, namely

$$\mathbf{Ai}'(x) := \frac{i}{2\pi} \int_{\gamma_1} \exp\Big\{ i \cdot \big(x\,\zeta + \zeta^3/3 \big) \Big\}\, \zeta\, d\zeta\,. \tag{5.28}$$

Selecting $\gamma_1 = (e^{i\pi/6} \cdot \mathbb{R}_+ - e^{5\pi i/6} \cdot \mathbb{R}_+)$ one easily obtains the identities

$$\begin{aligned} \mathbf{Ai}(0) &= \frac{3^{-1/6}}{2\pi} \cdot \Gamma\left(\frac{1}{3}\right)\,, \\ \mathbf{Ai}'(0) &= -\frac{3^{1/6}}{2\pi} \cdot \Gamma\left(\frac{2}{3}\right)\,, \end{aligned}$$

and these reaffirm that the Airy function is effectively a nontrivial solution of (5.24).

In general, for an arbitrary $x \in \mathbb{C}$, it is not possible to deform the contour γ_1 in (5.27) or (5.28) to the real ζ-axis. However, the contour γ_1 is equivalent to any contour of the form $(\mathbb{R} + i\eta)$ provided that $\eta > 0$.

Thus, we have the alternative representations

$$\text{(5.29)} \qquad \mathbf{Ai}(x) = \frac{1}{2\pi}\int_{\mathbb{R}+i\eta} \exp\Big\{ i\cdot\big(x\,\zeta+\zeta^3/3\big)\Big\}\,d\zeta\,,$$

$$\text{(5.30)} \qquad \mathbf{Ai}'(x) = \frac{i}{2\pi}\int_{\mathbb{R}+i\eta} \exp\Big\{ i\cdot\big(x\,\zeta+\zeta^3/3\big)\Big\}\,\zeta\,d\zeta\,,$$

valid for all $x\in\mathbb{C}$ and $\eta>0$.

These last representations will be of great use to relate the Airy function to integrals of the form $\int_{-\infty}^{\infty} e^{-\lambda\cdot\xi^2+i\cdot\xi^3/3}\,d\xi$ and $\int_{-\infty}^{\infty} e^{-\lambda\cdot\xi^2+i\cdot\xi^3/3}\,\xi\,d\xi$. The ideas that will follow are inspired by the discussion of Hörmander in section 7.6 in [Hör90]. Given $\lambda>0$ consider $x=\lambda^2$ in (5.29) and (5.30). Observe that $\frac{d}{d\zeta}\big[x\,\zeta+\zeta^3/3\big]=0$, if $\zeta=i\cdot\lambda$. Thus, to eliminate the linear term in ζ in the exponential terms in (5.29) and (5.30) it will be enough to integrate along the contour $(\mathbb{R}+i\lambda)$. If we parametrize this contour in the form: $\zeta=\xi+i\lambda$, with $\xi\in\mathbb{R}$, a simple computation reveals that

$$\begin{aligned} 2\pi\cdot\mathbf{Ai}(\lambda^2) &= e^{-2\lambda^3/3}\cdot\int_{-\infty}^{\infty}\exp\Big\{-\lambda\,\xi^2+i\,\xi^3/3\Big\}\,d\xi\,,\\ -2\pi i\cdot\mathbf{Ai}'(\lambda^2) &= e^{-2\lambda^3/3}\cdot\int_{-\infty}^{\infty}\exp\Big\{-\lambda\,\xi^2+i\,\xi^3/3\Big\}\,\big(\xi+i\lambda\big)\,d\xi\,. \end{aligned}$$

Each integral on the right-hand side above can be deformed back to the contour γ_1 originally used to define the Airy function. Over this new contour both integrals define now an entire function of λ. Since the functions on the left-hand side are also entire and equality holds for all $\lambda>0$ then equality must hold for all $\lambda\in\mathbb{C}$. A simple algebra computation now leads to the identities

$$\text{(5.31)} \qquad \int_{\gamma_1} e^{-\lambda\,\xi^2+i\,\xi^3/3}\,d\xi = 2\pi\cdot\exp\big(2\lambda^3/3\big)\cdot\mathbf{Ai}\big(\lambda^2\big)\,,$$

$$\text{(5.32)} \qquad \int_{\gamma_1} e^{-\lambda\,\xi^2+i\,\xi^3/3}\,\xi\,d\xi = -2\pi i\cdot\exp\big(2\lambda^3/3\big)\cdot\big(\lambda\cdot\mathbf{Ai}\big(\lambda^2\big)+\mathbf{Ai}'\big(\lambda^2\big)\big)\,,$$

for all $\lambda\in\mathbb{C}$.

To end this section we will use these identities to provide some few basic and well-known properties of the Airy function. Letting $\lambda = \sqrt{x}$ (the principal squared-root of x) we obtain from (5.31) the identity

$$\mathbf{Ai}(x) = \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi} \int_{-\infty}^{\infty} \exp\Big\{ -\sqrt{x}\,\xi^2 + i\,\xi^3/3 \Big\}\, d\xi\,, \tag{5.33}$$

provided of course that the $|\arg(x)| < \pi$. The above identity is very helpful to obtain asymptotics for the Airy function as $x \to \infty$ along directions away from the negative real axis. This is can be done using the stationary phase method (see [BleHan86], chapter 6) which implies that

$$\mathbf{Ai}(x) \approx \frac{e^{-\frac{2}{3}x^{3/2}}}{2\pi} \cdot x^{-1/4} \cdot \sum_{k=0}^{\infty} (-9)^{-k} \cdot \Gamma\left(3k + \frac{1}{2}\right) \cdot \frac{x^{-3k/2}}{(2k)!}\,, \tag{5.34}$$

as $x \to \infty$ over any sector of the form $|\arg(x)| \leq (\pi - \epsilon)$, with $\epsilon > 0$ as small as wanted. Moreover, as remarked by Hörmander, we see that $\mathbf{Ai}(x)$ is exponentially decreasing when $|\arg(x)| < \pi/3$, oscillatory when $\arg(x) = \pm\pi/3$, and exponentially increasing when $\pi/3 < |\arg(x)| < \pi$. We can not use (5.33) to obtain asymptotics for $\pi \leq |\arg(x)| \leq (\pi - \epsilon)$. However, (5.26) lets us rewrite

$$\mathbf{Ai}(x) = e^{i\pi/3} \cdot \mathbf{Ai}\big(-e^{i\pi/3}x\big) + e^{-i\pi/3} \cdot \mathbf{Ai}\big(-e^{-i\pi/3}x\big)\,,$$

and in this form a full asymptotic expansion for the two terms on the right-hand side above can be obtained using (5.34).

Since $\mathbf{Ai}(x)$ is a nontrivial solution of (5.24), the remarks at the beginning of this section imply that $\mathbf{Ai}(x)$ and $\mathbf{Ai}'(x)$ cannot have common zeroes in the complex plane. Moreover, it can be shown that they are zero-free over the complex plane slit along the negative real axis and indeed each has a countable number of zeros along the negative real axis.

Another nontrivial solutions of equation (5.24) are

$$a_2(x) \;:=\; e^{-2\pi i/3}\cdot \mathbf{Ai}\big(e^{-2\pi i/3}x\big)\,,$$
$$a_3(x) \;:=\; e^{2\pi i/3}\cdot \mathbf{Ai}\big(e^{2\pi i/3}x\big)\,.$$

Indeed, they are of the form given in (5.25) with the selection of $\gamma = \gamma_2$ and $\gamma = \gamma_3$ respectively, with γ_2 and γ_3 as given in figure 5.1. The function $\mathbf{Bi}(x) := i \cdot \{a_2(x) - a_3(x)\}$ is called the *Airy function of the second kind.* Using (5.26) a simple computation reveals that $\mathbf{Ai}(x)$ and $\mathbf{Bi}(x)$ are linearly independent. Because of this, in most applications, a general solution to the Airy equation is written as a linear combination of $\mathbf{Ai}(x)$ and $\mathbf{Bi}(x)$.

5.4 A generalized coalescing saddle point method

In this section we continue our discussion on parameter varying Fourier-Laplace integrals. We are interested in the asymptotic behavior of an integral of the form

$$J(t;s) := \int_{-1}^{1} e^{-s\cdot F(t,z)}\,A(t,z)\,dz\,,$$

for sufficiently small values of t and big values of $s \geq 0$. $F(t,z)$ and $A(t,z)$ are assumed to be analytic in a neighborhood of $(0,0)$ which contains points of the form (t,z) with $z \in [-1,1]$ for all t sufficiently small. The case covered by the following result is the simplest one for which corollary 5.10 cannot be applied.

Theorem 5.12. (Generalized coalescing saddle point method.) *Let $J(t;s)$, etc. be as defined before. Suppose that*

(a) $\Re\{F(0,z)\} > \Re\{F(0,0)\}$, *for all* $z \in [-1,1]$, *but* $z = 0$,

(b) *$F(t,z) - F(t,0) = t^N \cdot u(t) \cdot z^2 - i \cdot v(t) \cdot z^3 + \ldots$ with $N \geq 1$ an integer and $u(t)$ and $v(t)$ analytic near $t = 0$ and such that $u(0) \neq 0$ and $v(0) > 0$, and*

(c) *there are nonnegative integers $p \leq q$ such that $A(t,z)$ has a p-to-q change of degree about $z = 0$ as $t \to 0$.*

Then there are functions $A_k(t)$ such that $A_k(0) = 0$, for all $p \leq k \leq (q-1)$, however, $A_q(0) \neq 0$ and functions $J_k(t;s)$ analytic in t near $t = 0$ and a constant $c > 0$ such that

$$J(t;s) = e^{-s \cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot J_k(t;s) + O(e^{-s \cdot c}) \right\}, \tag{5.35}$$

uniformly for all t sufficiently close to 0 and all $s \geq 0$. Moreover, if for all $\epsilon > 0$ sufficiently small it is defined

$$\mathcal{T}_\epsilon := \left\{ t : t = 0 \text{ or }, t \neq 0 \text{ and } |\arg\{t^N \cdot u(t)\}| < \left(\frac{\pi}{2} - \epsilon \right) \right\},$$

then, for each $p \leq k \leq q$, there is an asymptotic expansion of the form

$$J_k(t;s) \approx \left\{ \sum_{l=0}^{\infty} \frac{R_k(t;2l)}{s^{l+1/3}} \right\} \cdot \mathcal{L}_0(t;s) + \left\{ \sum_{l=0}^{\infty} \frac{R_k(t;2l+1)}{s^{l+2/3}} \right\} \cdot \mathcal{L}_1(t;s), \tag{5.36}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small, as $s \to \infty$, where

$$\mathcal{L}_0(t;s) := \frac{2\pi}{\{3\,V(t)\}^{1/3}} \cdot e^{2\lambda(t;s)^3/3} \cdot \mathbf{Ai}(\lambda(t;s)^2), \tag{5.37}$$

$$\mathcal{L}_1(t;s) := \frac{-2\pi\, i}{\{3\,V(t)\}^{2/3}} \cdot e^{2\lambda(t;s)^3/3} \cdot \left\{ \lambda(t;s) \cdot \mathbf{Ai}(\lambda(t;s)^2) + \mathbf{Ai}'(\lambda(t;s)^2) \right\}, \tag{5.38}$$

$$\lambda(t;s) := \frac{u(t) \cdot t^N}{\{3\,V(t)\}^{2/3}} \cdot s^{1/3}, \tag{5.39}$$

and $V(t)$ is certain analytic function of t near $t = 0$ such that $V(0) = v(0)$.

Remark 5.13. (5.35) is possibly of little use if s is not sufficiently big; after all, the big-O term may hide a very complicated function of t and s. However, recall that, our interest is in providing an asymptotic description for $J(t;s)$ valid for big-values of s. (See section 3.1 for further clarification on our terminology relating asymptotics.)

Remark 5.14. The coefficients $A_k(t)$ together with certain auxiliary function $y = y(t,z)$ are uniquely characterized through the relations

$$(5.40)\qquad \begin{cases} \int_0^z A(t,\xi)\,d\xi = \sum_{k=p}^{q} \frac{A_k(t)}{k+1}\cdot y^{k+1}\,, \\ A_k(0)=0,\ \text{for all } p\le k<q,\ A_q(0)\neq 0\,, \\ y(t,z) = z+\dots \end{cases}$$

We will define the 1-to-1 transformation

$$(t,y) = \Psi(t,z) := (t, y(t,z))\,.$$

The coefficient $V(t)$ together with an auxiliary function $x = x(t,y)$ are uniquely characterized by the relations

$$(5.41)\qquad \begin{cases} F(\Psi^{-1}(t,y)) = t^N\cdot u(t)\cdot x^2 - i\cdot V(t)\cdot x^3\,, \\ x = x(t,y) = y+\dots \end{cases}$$

Remark 5.15. The coefficients $R_k(t;l)$ in (5.36) together with certain auxiliary functions $B_k(t,x;l)$, with $l\ge 0$, analytic in t and x near $t=0$ and $x=0$ can be defined recursively by means of the Weierstrass division theorem 4.21 as follows

$$(5.42)\qquad \begin{cases} B_k(t,x;0) &:= & y^k\cdot \frac{\partial y}{\partial x}(t,x)\,, \\ B_k(t,x;l) &=: & R_k(t;2l) + R_k(t;2l+1)\cdot x \\ & & \quad + B_k(t,x;l+1)\cdot \frac{\partial}{\partial x}\left\{t^N\cdot u(t)\cdot x^2 - i\cdot V(t)\cdot x^3\right\}\,. \end{cases}$$

In particular, if we let $x(t) := -2\,i\,u(t)\,t^N/\{3\,V(t)\}$ then

$$\begin{aligned} R_k(t;2l) &= B_k(t,0;l)\,, \\ R_k(t;2l+1) &= \frac{B_k(t,x(t);l) - B_k(t,0;l)}{x(t)}\,. \end{aligned}$$

Remark 5.16. In (5.39), observe that $\lambda(t;s) = \lambda(t) \cdot s^{1/3}$, where $\lambda(t) := \frac{t^N \cdot u(t)}{\{3\,V(t)\}^{2/3}}$. Since $F(\Psi^{-1}(t,y))$ has a 2-to-3 change of degree at $y = 0$ as $t \to 0$ then, near $y = 0$, $F(\Psi^{-1}(t,y))$ has only one nontrivial stationary point which will be denoted $y(t)$. It turns out that $F(\Psi^{-1}(t, y(t)))$ vanishes to degree $3N$ about $t = 0$, and therefore $\lambda(t)$ is uniquely characterized by the relations

$$(5.43)\qquad \begin{cases} 4\,[\lambda(t)]^3 + 3\,F(\Psi^{-1}(t, y(t))) = 0\,, \\ \lambda(t) \sim t^N \cdot u(0) \cdot \{3v(0)\}^{-2/3}\,. \end{cases}$$

Remark 5.17. The asymptotic expansion in (5.36) is with respect to the asymptotic sequence $\left(\frac{|\mathcal{L}_0(t;s)|}{s^{n+1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{n+2/3}}\right)_{n\geq 0}$. By this we mean that given any $n \geq 0$ there is $c_1 > 0$ such that the quantity

$$\left| J_k(t;s) - \left\{ \sum_{l=0}^{n} \frac{R_k(t;2l)}{s^{l+1/3}} \right\} \cdot \mathcal{L}_0(t;s) - \left\{ \sum_{l=0}^{n} \frac{R_k(t;2l+1)}{s^{l+2/3}} \right\} \cdot \mathcal{L}_1(t;s) \right| \leq \frac{c_1}{s} \cdot \left\{ \frac{|\mathcal{L}_0(t;s)|}{s^{n+1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{n+2/3}} \right\}\,,$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$ sufficiently big.

Theorem 5.12 provides a uniform asymptotic expansion for $J(t;s)$ in terms of the two special functions $\mathcal{L}_0(t;s)$ and $\mathcal{L}_1(t;s)$. These are, up to a multiplicative factor, the evaluation of $e^{-2\lambda^3/3} \cdot \mathbf{Ai}(\lambda^2)$ and $e^{-2\lambda^3/3} \cdot \left\{\lambda \cdot \mathbf{Ai}(\lambda^2) + \mathbf{Ai}(\lambda^2)\right\}$ at $\lambda = \lambda(t;s)$. The following result provides a better understanding of the asymptotic behavior of $\mathcal{L}_0(t;s)$ and $\mathcal{L}_1(t;s)$ for the cases in which $|\lambda(t;s)|$ is either of a big or small size.

Corollary 5.18. (Bandwidth characterization.) *Let $\mathcal{L}_0(t;s)$, $\mathcal{L}_1(t;s)$, etc. be as defined in theorem 5.12. Then*

$$(5.44)\qquad \mathcal{L}_0(t;s) \;\approx\; \sum_{n=0}^{\infty} \alpha_0(t;n) \cdot (st^{3N})^{-(6n+1)/6}\,,$$

$$(5.45)\qquad \mathcal{L}_1(t;s) \;\approx\; \sum_{n=0}^{\infty} \alpha_1(t;n) \cdot (st^{3N})^{-(6n+5)/6}\,,$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to \infty$. Above it has been defined

$$\begin{aligned}\alpha_0(t;n) &:= \frac{(-1)^n \cdot \{3\,V(t)\}^{2n}}{(2n)! \cdot 9^n \cdot \{u(t)\}^{(6n+1)/2}} \cdot \Gamma\left(\frac{6n+1}{2}\right), \\ \alpha_1(t;n) &:= \frac{(-1)^n \cdot i \cdot \{3\,V(t)\}^{(18n+11)/6}}{3 \cdot (2n+1)! \cdot 9^n \cdot \{u(t)\}^{(6n+5)/2}} \cdot \Gamma\left(\frac{6n+1}{2}\right).\end{aligned}$$

On the contrary,

$$\mathcal{L}_0(t;s) \approx \sum_{n=0}^{\infty} \beta_0(t;n) \cdot (st^{3N})^{n/3}, \tag{5.46}$$

$$\mathcal{L}_1(t;s) \approx \sum_{n=0}^{\infty} \beta_1(t;n) \cdot (st^{3N})^{n/3}, \tag{5.47}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to 0$. Above it has been defined

$$\begin{aligned}\beta_0(t;n) &:= \frac{(-1)^n \cdot 3^{(2n+1)/3} \cdot \{u(t)\}^n}{n! \cdot \{3\,V(t)\}^{(2n+1)/3}} \cdot \frac{e^{(2n+1)\pi i/6} - e^{5(2n+1)\pi i/6}}{3} \cdot \Gamma\left(\frac{2n+1}{3}\right), \\ \beta_1(t;n) &:= \frac{(-1)^n \cdot 3^{2(n+1)/3} \cdot \{u(t)\}^n}{n! \cdot \{3\,V(t)\}^{2(n+1)/3}} \cdot \frac{e^{(n+1)\pi i/3} - e^{5(n+1)\pi i/3}}{3} \cdot \Gamma\left(\frac{2n+2}{3}\right).\end{aligned}$$

Remark 5.19. According to (5.39), $|\lambda(t;s)| \sim |t|^N \cdot s^{1/3}$. Thus, the condition $|\lambda(t;s)| \to \infty$ allows the possibility of t to depend on s so that $t(s) \to 0$, as $s \to \infty$, at a rate not faster than $s^{-1/3}$.

The leading orders of $\mathcal{L}_l(t;s)$ are easy recognized from (5.44) – (5.47). One finds that

$$\mathcal{L}_0(t;s) \sim \frac{\sqrt{\pi}}{\{u(t)\}^{1/2}} \cdot (st^{3N})^{-1/6} \cdot \left(1 + O(\lambda(t;s)^{-3})\right), \tag{5.48}$$

$$\mathcal{L}_1(t;s) \sim \frac{i\sqrt{\pi} \cdot \{3\,V(t)\}^{11/6}}{3\{u(t)\}^{5/2}} \cdot (st^{3N})^{-5/6} \cdot \left(1 + O(\lambda(t;s)^{-3})\right), \tag{5.49}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to \infty$.

On the other hand,

$$\mathcal{L}_0(t;s) \sim \frac{1}{\{3\sqrt{3}\,V(t)\}^{1/3}}\Gamma\left(\frac{1}{3}\right)\cdot\big(1+O(\lambda(t;s))\big)\,, \tag{5.50}$$

$$\mathcal{L}_1(t;s) \sim \frac{i}{\sqrt{3}\{V(t)\}^{2/3}}\Gamma\left(\frac{2}{3}\right)\cdot\big(1+O(\lambda(t;s))\big)\,, \tag{5.51}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to 0$.

Proof of corollary 5.18: Using the identities in (5.31) and (5.32) in (5.37) and (5.38) respectively, we find that

$$\mathcal{L}_l(t;s) = \frac{1}{\{3\,V(t)\}^{(l+1)/3}}\cdot\int_{\gamma_1} e^{-\lambda(t;s)\cdot\xi^2+i\cdot\xi^3/3}\,\xi^l\,d\xi\,, \tag{5.52}$$

for $l = 0$ and $l = 1$.

Fix $\epsilon > 0$ sufficiently small. The remark in (5.41) implies that $V(0) = v(0) > 0$. As a result, for all t sufficiently small, $|\arg\{V(t)\}| < \epsilon$. Therefore, the definition of $\lambda(t;s)$ in (5.39) implies, for all $t \in \mathcal{T}_\epsilon$ sufficiently small, that $|\arg\{\lambda(t;s)\}| \leq |\arg\{t^N\cdot u(t)\}| + 2\epsilon/3$; in particular, for all such t, $|\arg\{\lambda(t;s)\}| \leq \theta := (\pi/2 - \epsilon/3)$. Since $\cos(\theta) > 0$, we conclude that, for all $t \in \mathcal{T}_\epsilon$ sufficiently small, the $\Re\{\lambda(t;s)\} \geq |\lambda(t;s)|\cdot\cos(\theta)$. This lets us replace the contour γ_1 in the last representation for $\mathcal{L}_l(t;s)$ by the real ξ-axis. Thus,

$$\begin{aligned}\mathcal{L}_l(t;s) &= \frac{1}{\{3\,V(t)\}^{(l+1)/3}}\cdot\int_{-\infty}^{\infty} e^{-\lambda(t;s)\cdot\xi^2}\cdot e^{i\cdot\xi^3/3}\,\xi^l\,d\xi\,,\\ &= \frac{1}{\{3\,V(t)\}^{(l+1)/3}}\cdot\int_{-\sqrt{2}}^{\sqrt{2}} e^{-\lambda(t;s)\cdot\xi^2}\cdot e^{i\cdot\xi^3/3}\,\xi^l\,d\xi + O(e^{-\lambda(t;s)})\,,\end{aligned}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to \infty$.

On the other hand, the stationary phase method (see [BleHan86], chapter 6) implies that

$$\int_{-\sqrt{2}}^{\sqrt{2}} e^{-\lambda\cdot\xi^2}\cdot e^{i\cdot\xi^3/3}\,\xi^l\,d\xi \approx \sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{i}{3}\right)^n\cdot\lambda^{-(l+3n+1)/2}\cdot\frac{1+(-1)^{l+3n}}{2}\cdot\Gamma\left(\frac{l+3n+1}{2}\right),$$

uniformly for all λ such that $\arg\{\lambda\}$ remains in a compact subset of $(-\pi/2, \pi/2)$, as $|\lambda| \to \infty$. By this we mean that the difference between the integral and the summation on the right-hand side truncated in the k^{th}-term is $O\big(\Re\{\lambda^{1/2}\}^{-(l+3k+4)}\big)$, as $|\lambda| \to \infty$.

Since for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$ the $|\arg\{\lambda(t;s)\}| \leq (\pi/2 - \epsilon/3)$, the remarks on the previous paragraph imply that

$$\begin{aligned}
\mathcal{L}_0(t;s) &\approx \frac{1}{\{3\,V(t)\}^{1/3}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)! \cdot 9^n} \cdot \lambda(t;s)^{-(6n+1)/2} \cdot \Gamma\left(\frac{6n+1}{2}\right), \\
\mathcal{L}_1(t;s) &\approx \frac{1}{\{3\,V(t)\}^{2/3}} \cdot \sum_{n=0}^{\infty} \frac{(-1)^n \cdot i}{3 \cdot (2n+1)! \cdot 9^n} \cdot \lambda(t;s)^{-(6n+5)/2} \cdot \Gamma\left(\frac{6n+5}{2}\right),
\end{aligned}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, as $|\lambda(t;s)| \to \infty$. Using the definition of $\lambda(t;s)$ given in (5.39), (5.44) and (5.45) follow almost immediately from the above identities.

To study the case $|\lambda(t;s)| \to 0$ we first quote the identity

$$e^{-a} - \sum_{n=0}^{k} \frac{(-a)^n}{n!} = (-1)^{k+1} \cdot e^{-a} \int_0^a \frac{e^\tau \cdot \tau^k}{k!}\, d\tau\,,$$

which is a simple exercise of integration by parts. It implies that $\left| e^{-a} - \sum_{n=0}^{k} \frac{(-a)^n}{n!} \right| \leq \frac{|a|^{k+1} \cdot e^{|a|}}{k!}$, for all $a \in \mathbb{C}$ and integer $k \geq 0$. As a result, back in (5.52) and if we select $\gamma_1 = (e^{\pi i/6} \cdot \mathbb{R}_+) - (e^{5\pi i/6} \cdot \mathbb{R}_+)$ we can conclude that

$$\begin{aligned}
&\left| \mathcal{L}_l(t;s) - \frac{1}{\{3\,V(t)\}^{(l+1)/3}} \cdot \sum_{n=0}^{k} \frac{(-1)^n}{n!} \cdot \{\lambda(t;s)\}^n \int_{\gamma_1} e^{i\xi^3/3}\, \xi^{l+2n}\, d\xi \right| \\
&\qquad\qquad \leq \frac{|\lambda(t;s)|^{k+1}}{k! \cdot |3\,V(t)|^{(l+1)/3}} \cdot \int_{\gamma_1} e^{|\lambda(t;s)||\xi|^2 + \Re\{i\xi|^3/3\}} |\xi|^{2(k+1)+l}\, d|\xi|\,.
\end{aligned}$$

But, observe that for $\xi \in \gamma_1$, the $\Re\{i\xi^3\} = -|\xi|^3$. This implies that the integral term on the right-hand side above is convergent. Moreover, it is bounded as long as the

$|\lambda(t;s)|$ remains bounded. Accordingly, if we define

$$\gamma_l(t;n) := \frac{(-1)^n \cdot 3^{(2n+l+1)/3}}{n! \cdot \{3\,V(t)\}^{(l+1)/3}} \cdot \frac{e^{(2n+l+1)\pi i/6} - e^{5(2n+l+1)\pi i/6}}{3} \cdot \Gamma\left(\frac{2n+l+1}{3}\right)$$

then the above inequality implies that

$$\mathcal{L}_l(t;s) \approx \sum_{n=0}^{\infty} \gamma_l(t;n) \cdot \lambda(t;s)^n \,, \tag{5.53}$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, provided that $|\lambda(t;s)|$ remains bounded. (5.46) and (5.47) follow almost directly from the above identity after some few calculations. This completes the proof of the corollary. □

5.5 Applications to big powers of generating functions

In this section we outline a method that could be of use to study the asymptotic behavior of the coefficient of z^n of a power series of the form

$$f(z)^n \cdot g(z)^m \cdot h(z) \,,$$

where $f(z)$, $g(z)$ and $h(z)$ are analytic in some disk containing the origin.

Our goal is to provide a uniform asymptotic expansion valid for all n and m sufficiently big but with $\frac{m}{n}$ restricted to a compact subset of $[0,\infty)$. The discussion that will follow could be adapted to also consider the case in which $\frac{n}{m}$ is restricted to a compact set of $[0,\infty)$, however, this case will not be covered.

The motivation to study this problem is the invitation to research of Banderier et al. at the end of section 2 in [BFSS01].

The following definition is inspired by the work of Pemantle and Wilson in [PemWil01].

Definition 5.20. Suppose that $f(z)$ and $g(z)$ are nonzero analytic functions in a disk of the form $[z : |z| < R]$ with $R > 0$ and let $d \geq 0$. We will say that a point z_0 in this disk is a *strictly minimal critical point associated to the direction* d provided that

(i) $|f(z)| \cdot |g(z)|^d$ is solely maximized along the circle $[z : |z| = |z_0|]$ at $z = z_0$, and

(ii) $z = z_0$ is a solution to the equation: $\frac{z \cdot f'(z)}{f(z)} + d \cdot \frac{z \cdot g'(z)}{g(z)} = 1$.

Observe that, if z_0 is a strictly minimal critical point then $z_0 \neq 0$. Moreover, the hypothesis that neither $f(z)$ nor $g(z)$ are identically zero implies that $f(z_0) \cdot g(z_0) \neq 0$.

We remark that the conditions in the above definition are somehow related. Indeed condition (i) implies that: $\frac{z \cdot f'(z)}{f(z)} + d \cdot \frac{z \cdot g'(z)}{g(z)} \in \mathbb{R}$. This follows by noticing that for all $\theta \in \mathbb{R}$ sufficiently small,

$$\frac{|f(z_0\, e^{i\theta})| \cdot |g(z_0\, e^{i\theta})|^d}{|f(z_0)| \cdot |g(z_0)|^d} = \exp\left(-\theta \cdot \Im \left\{ \frac{z_0 \cdot f'(z_0)}{f(z_0)} + d \cdot \frac{z_0 \cdot g'(z_0)}{g(z_0)} \right\} + O(\theta^2) \right) .$$

The occurrence of strictly minimal critical points is far to be uncommon. For example, suppose that the series coefficients of $f(z)$ and $g(z)$ are nonnegative real numbers. [1] Define $d(z) := \left\{ 1 - \frac{z \cdot f'(z)}{f(z)} \right\} \cdot \left\{ \frac{z \cdot g'(z)}{g(z)} \right\}^{-1}$. The restriction on the sign of the coefficients of $f(z)$ and $g(z)$ implies that each $z \in (0, R)$ such that $d(z) \geq 0$ is a strictly minimal critical point associated to the direction $d(z)$.

Our next result demonstrates the natural occurrence of parameter varying Fourier-Laplace integrals in relation to the problem of determining the asymptotic behavior of $[z^n]\, f(z)^n \cdot g(z)^m \cdot h(z)$.

[1] This is a fairly typical situation if $f(z)$ and $g(z)$ are the power series associated to a combinatorial problem, however, we emphasize that this is not the only context on which strictly minimal points may be encountered.

Proposition 5.21. *Suppose that $\mathcal{D} \subset [0, \infty)$ is a compact set and, for each $d \in \mathcal{D}$, z_d is a strictly minimal critical point associated to the direction d which depends continuously on d. If $\epsilon > 0$ is sufficiently small then, for all $d \in \mathcal{D}$, the functions*

$$\begin{aligned} F(d,\theta) &:= i \cdot \theta - \ln\left\{\frac{f(z_d \cdot e^{i\theta})}{f(z_d)}\right\} - d \cdot \ln\left\{\frac{g(z_d \cdot e^{i\theta})}{g(z_d)}\right\}, \\ A(d,\theta) &:= h(z_d \cdot e^{i\theta}), \end{aligned}$$

are analytic on the disk $[\theta : |\theta| \leq \epsilon]$. Furthermore, $F(d,0) = \frac{\partial F}{\partial \theta}(d,0) = 0$ and the $\Re\{F(d,\theta)\} > 0 = \Re\{F(d,0)\}$, for all $-\epsilon \leq \theta \leq \epsilon$, but $\theta = 0$. In addition, there is a constant $c > 0$ such that

$$(5.54) \quad [z^n]\, f(z)^n \cdot g(z)^m \cdot h(z) = \left\{\frac{f(z_d)}{z_d}\right\}^n \cdot \frac{g(z_d)^m}{2\pi} \cdot \left\{\int_{-\epsilon}^{\epsilon} e^{-n \cdot F(d,\theta)}\, A(d,\theta)\, d\theta + O(e^{-n \cdot c})\right\},$$

uniformly for all integers $n > 0$ and $m \geq 0$ such that $\frac{m}{n} \in \mathcal{D}$.

Remark 5.22. Under additional hypothesis it is possible to rewrite the problem of estimating the coefficient of $[z^n]\, f(z)^n \cdot g(z)^m \cdot h(z)$ as a problem of estimating the diagonal coefficients of a parameter-dependent bivariate power series. For example, if $g(0) \neq 0$ then

$$\begin{aligned} [z^n]\, f(z)^n \cdot g(z)^m \cdot h(z) &= [u^n\, v^n]\, F\left(u, v, \frac{m}{n}\right), \\ F(u,v,w) &:= \frac{h(u)}{1 - v \cdot f(u) \cdot g(u)^w}. \end{aligned}$$

If z_0 is a strictly minimal critical point associated to the direction d then $(u_0, v_0) := \left(z_0, \frac{1}{f(z_0) \cdot g(z_0)^d}\right)$ is the only solution to the equation: $1 - v \cdot f(u) \cdot g(u)^d = 0$ on the polydisk $[u : |u| \leq |u_0|] \times [v : |v| \leq |v_0|]$. Furthermore, (u_0, v_0) turns out to be a *strictly minimal simple pole of* $F\left(u, v, \frac{m}{n}\right)$ (as defined in [PemWil01]) and

$\text{dir}(u_0, v_0) = \Big\{(r, s) : r = s\Big\}$. As a result, if $\frac{m}{n}$ remains constant then the scheme developed in [PemWil01] could be used to determine asymptotics for the coefficients $[u^n\, v^n]\, F\left(u, v, \frac{m}{n}\right)$, as $n \to \infty$.

Proof. For each $d \in \mathcal{D}$ consider the contours

$$\begin{aligned}
\gamma_1(d) &:= \{z : |z| = |z_d| \text{ and } |\arg\{z/z_d\}| \le \epsilon\}\,, \\
\gamma_2(d) &:= \{z : |z| = |z_d| \text{ and } |\arg\{z/z_d\}| \ge \epsilon\}\,.
\end{aligned}$$

For each $n > 0$ and $m \ge 0$ such that $d = d(n, m) := \frac{m}{n} \in \mathcal{D}$ we may use Cauchy's integral formula to obtain that

$$[z^n]\, f(z)^n \cdot g(z)^m \cdot h(z) = J_1(d; n) + J_2(d; n)\,, \tag{5.55}$$

where

$$\begin{aligned}
J_1(d; n) &:= \frac{1}{2\pi} \int_{z \in \gamma_1(d)} \left\{\frac{f(z)}{z}\right\}^n \cdot g(z)^{n \cdot d} \cdot h(z)\, \frac{dz}{i\, z}\,, \\
J_2(d; n) &:= \frac{1}{2\pi} \int_{z \in \gamma_2(d)} \left\{\frac{f(z)}{z}\right\}^n \cdot g(z)^{n \cdot d} \cdot h(z)\, \frac{dz}{i\, z}\,.
\end{aligned}$$

Define $\Lambda_2 := \Big\{z : \text{ there exists } d \in \mathcal{D} \text{ such that } z \in \gamma_2(d)\Big\}$. Observe that

$$|J_2(d; n)| \le \sup_{z : z \in \Lambda_2} |h(z)| \cdot \sup_{z : z \in \Lambda_2} \left|\frac{f(z)}{z}\right|^n \cdot |g(z)|^{n \cdot d}\,,$$

uniformly for all $n > 0$ and $d \in \mathcal{D}$. The compactness of $\mathcal{D}$ and the continuous dependence of z_d on d implies that Λ_2 is a compact set. As a result, since the transformation

$$(d, z) \in \mathcal{D} \times \Lambda_2 \longrightarrow \left\{\left|\frac{f(z)}{z}\right| \cdot |g(z)|^d\right\} \cdot \left\{\left|\frac{f(z_d)}{z_d}\right| \cdot |g(z_d)|^d\right\}^{-1}\,,$$

continuous it follows that there is $c_2 > 0$ and $\delta_2 \in (0,1)$ such that

$$|J_2(d;n)| \le c_2 \cdot (1-\delta_2)^n \cdot \left|\frac{f(z_d)}{z_d}\right|^n \cdot |g(z_d)|^{n\cdot d}\,, \tag{5.56}$$

uniformly for all $n > 0$ such that $d \in \mathcal{D}$.

To deal with $J_1(d;n)$ it is convenient to parametrize the integral using polar coordinates. If we then normalize the integrand by $\left\{\frac{f(z_d)}{z_d}\right\}^n \cdot g(z_d)^{n\cdot d}$ then we obtain

$$J_1(d;n) := \left\{\frac{f(z_d)}{z_d}\right\}^n \cdot \frac{g(z_d)^m}{2\pi} \int_{-\epsilon}^{\epsilon} e^{-n\cdot F(d,\theta)}\, A(d,\theta)\, d\theta\,, \tag{5.57}$$

with $F(d,\theta)$ and $A(d,\theta)$ as defined in the enunciate of the proposition. The compactness of $\mathcal{D}$ and the continuity of z_d as a function of d let us chose $\epsilon > 0$ sufficiently small so that for each $d \in \mathcal{D}$, $F(d,\theta)$ and $A(d,\theta)$ are analytic functions of θ for $|\theta| \le \epsilon$. (5.54) follows now immediately from (5.56) and (5.57) in (5.55).

Finally, observe that $F(d,0) = 0$. Furthermore, the strict minimality of z_d for $d \in \mathcal{D}$ implies that $\frac{\partial F}{\partial \theta}(d,0) = 0$ and the $\Re\{F(d,\theta)\} > 0$, for all $|\theta| \le \epsilon$ but $\theta = 0$. This completes the proof of the proposition. □

Example 5.23. (Rediscovering the Airy Phenomena.)

Let M_r be the number of non-separable rooted maps with $(r+1)$-edges and C_s be the number of three-connected non-separable rooted maps with $(s+1)$-edges. Tutte [Tut89] showed that the generating function associated to $(M_r)_{r\ge 0}$ is Lagrangian and obtained that

$$\begin{aligned} M_r &= \frac{4\,(3\,r)!}{r!\,(2\,r+2)!}\,, \\ &= \frac{\sqrt{3}}{2\,\sqrt{\pi}} \left(\frac{27}{4}\right)^r r^{-5/2} \cdot \left\{1 + O(r^{-1})\right\}. \end{aligned} \tag{5.58}$$

On the other hand, Banderier et al. [BFSS00] used singularity analysis to show that

$$C_s = \frac{8}{243\sqrt{\pi}}\, 4^s\, s^{-5/2} \cdot \left\{1 + O(s^{-1})\right\}. \tag{5.59}$$

A quantity of interest is $p_{r,s}$ defined to be the probability that a NSR-map with $(r+1)$-edges has a 3CNSR-submap with $(s+1)$-edges. Using Tutte's work one finds that

$$p_{r,s} = \frac{s \cdot C_s}{r \cdot M_r} \cdot [z^r]\, \phi(z)^r \cdot \psi(z)^s \cdot \frac{z\,\psi'(z)}{\psi(z)}, \tag{5.60}$$

where $\phi(z) := (1+z)^3$ and $\psi(z) := z \cdot (1-z)$. The coefficients $p_{r,s}$ were studied by Banderier et al. in [BFSS00] for the difficult case in which $\frac{s}{r}$ is close to $\frac{1}{3}$. Their starting point was to represent $p_{r,s}$ by an integral using Cauchy's formula in (5.60), namely

$$p_{r,s} = \frac{s\cdot C_s}{r\cdot M_r} \cdot \frac{1}{2\pi} \int_\gamma z^{s-r}\,(1-z)^s\,(1+z)^{3r}\,\frac{1-2z}{1-z}\,\frac{dz}{iz}. \tag{5.61}$$

Above, γ can be chosen to be any closed contour encircling the origin and contained within the punctured disk $[z : 0 < |z| < 1]$. To determine the asymptotic order of $p_{r,s}$, Banderier et al. analyzed the asymptotic behavior of the above integral using the coalescing saddle point method by choosing an appropriate contour γ which goes through the double saddle $z = \frac{1}{2}$.

They obtain (see [BFSS00], theorem 3) that the coefficients $p_{r,s}$ satisfy a *local limit law of the map Airy-type*. More precisely, they determine for all finite real numbers $a \le b$ that

$$\lim_{(r,s)\to\infty} \sup \left\{ \left| r^{2/3} \cdot p_{r,s} - \frac{16}{81} \cdot \frac{3^{4/3}}{4} \cdot \mathbf{A}\left(\frac{3^{4/3}}{4} \cdot \frac{s - r/3}{r^{2/3}} \right) \right| : \; a \le \frac{s - r/3}{r^{2/3}} \le b \right\} = 0.$$

The function $\mathbf{A}(\lambda) := 2\,e^{-2\lambda^3/3}\big(x\cdot\mathbf{Ai}(\lambda^2)-\mathbf{Ai}'(\lambda^2)\big)$ is the density function of a standard Airy-map distribution. This is a zero-free function over the real line (see lemma 5.26 ahead) and therefore $p_{r,s}$ is of order $r^{-2/3}$ provided that: $\frac{s}{r}=\frac{1}{3}+O(r^{1/3})$.

An alternative approach to obtain this local limit law is to make use of the link between $p_{r,s}$ and $[z^r]\,\phi(z)^r\cdot\psi(z)^s\cdot\frac{z\cdot\psi'(z)}{\psi(z)}$ in (5.60). These last coefficients can be related to a parameter varying Fourier-Laplace integral by means of the following result.

Proposition 5.24. *For all $d\geq 0$ sufficiently close to $\frac{1}{3}$, $z_d := \min\left\{\frac{1}{2},\frac{1-d}{1+d}\right\}$ is a strictly minimal critical point associated to d.*

Using (5.60), proposition 5.21 implies the existence of $\epsilon>0$ and a constant $c>0$ such that

$$(5.62)\qquad p_{r,s} \;=\; \frac{s\cdot C_s}{r\cdot M_r}\cdot\left\{\frac{\phi(z_d)}{z_d}\right\}^r\cdot\frac{\psi(z_d)^s}{2\pi}\cdot\left\{\int_{-\epsilon}^{\epsilon}e^{-r\cdot F(d,\theta)}\,A(d,\theta)\,d\theta+O(e^{-r\cdot c})\right\},$$

where

$$(5.63)\qquad F(d,\theta) \;:=\; i\cdot\theta-3\cdot\ln\left\{\frac{1+z_d\,e^{i\theta}}{1+z_d}\right\}-d\cdot\ln\left\{e^{i\theta}\cdot\frac{1-z_d\,e^{i\theta}}{1-z_d}\right\},$$

$$(5.64)\qquad A(d,\theta) \;:=\; \frac{1-2\,z_d\,e^{i\theta}}{1-z_d\,e^{i\theta}},$$

uniformly for all $r,s>0$ provided that $d:=\frac{s}{r}$ is sufficiently close to $\frac{1}{3}$.

The integral term on the right-hand side in (5.62) can be studied using theorem 5.12. Indeed, as we shall see promptly, $F(d,\theta)$ has a 2-to-3 change of degree about $\theta=0$ as $d\to\frac{1}{3}$, however, $A(d,\theta)$ has a 0-to-1 change of degree at $\theta=0$ as $d\to\frac{1}{3}$. Although both of these functions do not depend analytically in d they do depend analytically on (z_d,θ). The generalized coalescing-saddle points method let us obtain the following result.

Corollary 5.25. *For each (r,s) such that $\frac{s}{r}$ is sufficiently close to $\frac{1}{3}$ define $\Theta = \Theta(r,s) := \left|\frac{s}{r}-\frac{1}{3}\right|$. The asymptotic behavior of $p_{r,s}$ for $\frac{s}{r}$ nearby $\frac{1}{3}$ is determined by the quantity $\Delta = \Delta(r,s) := r\cdot\left|\frac{s}{r}-\frac{1}{3}\right|^3$ as follows.*

There is $\alpha > 0$ such that

$$p_{r,s} = \frac{4^{2+r+s}}{\sqrt{2\pi}\cdot 27^{r+1}}\cdot\left\{\frac{\phi(z_d)}{z_d}\right\}^r\cdot\psi(z_d)^s\cdot r^{-1/2}\cdot(3d-1)^{1/2} \cdot\left\{1+O\big(\max\left\{\Delta^{-1/6},\Theta,r^{-1}\right\}\big)\right\}, \tag{5.65}$$

uniformly for all (r,s) such that Θ is sufficiently small and $\Delta(r,s)\geq\alpha$, as $(r,s)\to\infty$.

On the contrary, for all $\beta\geq 0$,

$$p_{r,s} = r^{-2/3}\cdot\frac{16}{81}\cdot\frac{3^{3/4}}{4}\cdot\mathbf{A}\left(\frac{3^{3/4}}{4}\cdot\frac{s-r/3}{r^{2/3}}\right)\cdot\left\{1+O(r^{-1/3})\right\}, \tag{5.66}$$

uniformly for all (r,s) such that Θ is sufficiently small and $\Delta(r,s)\leq\beta$, as $(r,s)\to\infty$.

Proof of proposition 5.24: A simple calculation reveals that $z=\frac{1}{2}$ and $z=\frac{1-d}{1+d}$ are solutions to the equation: $\frac{z\cdot\phi'(z)}{\phi(z)}+d\cdot\frac{z\cdot\psi'(z)}{\psi(z)}=1$. Thus, it only remains to show that, for all d sufficiently close to $\frac{1}{3}$, the function $M(d,\theta):=|f(z_d\cdot e^{i\theta})|\cdot|g(z_d\cdot e^{i\theta})|^d$ is solely maximized at $\theta=0$ over the interval $-\pi\leq\theta\leq\pi$. This assertion is trivial if $d=\frac{1}{3}$ because

$$\frac{\partial\log(M)}{\partial\theta}\left(\frac{1}{3},\theta\right) = -\frac{80\sin(\theta)\cdot\{1-\cos(\theta)\}}{3\cdot\{25-16\cos^2(\theta)\}}.$$

To show that $M(d,\theta)$ is maximized at $\theta=0$, it will be enough to show there is $\delta>0$ (independent of d) such that $\theta=0$ maximizes $M(d,\theta)$ over the interval $-\delta\leq\theta\leq\delta$. But, observe that

$$\frac{M(d,\theta)}{M(d,0)} = \exp\left(\Re\left\{P(d,\theta)\right\}\right), \tag{5.67}$$

where $P(d,\theta) := \ln\left\{\frac{\phi(z_d\cdot e^{i\theta})}{\phi(z_d)}\right\} + d\cdot\ln\left\{\frac{\psi(z_d\cdot e^{i\theta})}{\psi(z_d)}\right\}$ is continuous in d, for all d nearby $d = 1/3$, and analytic in θ, for all $|\theta| \leq \epsilon$ provided that $\epsilon > 0$ is selected small enough. It is now matter of routine to verify that there are continuous functions $u(d)$ and $v(d)$ such that $u(\frac{1}{3}) < 0$, $v(\frac{1}{3}) < 0$, and

$$\Re\{P(d,\theta)\} = \begin{cases} \left(d-\frac{1}{3}\right)\cdot\theta^2 + u(d)\cdot\theta^4 + O(\theta^6) & , \quad d \leq \frac{1}{3} \\ \frac{3(d^2-1)}{8d}\left(d-\frac{1}{3}\right)\cdot\theta^2 + v(d)\cdot\theta^4 + O(\theta^6) & , \quad d \geq \frac{1}{3} \end{cases}$$

uniformly for all d nearby $d = \frac{1}{3}$ and θ in an interval of the form $\theta \in (-\delta, \delta)$, with $0 < \delta < \epsilon$. If necessary by reducing the size of $\delta > 0$, the above identities imply that the $\Re\{P(d,\theta)\} < 0$, for all d sufficiently close to $d = \frac{1}{3}$ and nonzero $\theta \in (-\delta, \delta)$. This in (5.67) implies the desired conclusion and completes the proof of the proposition.□

Proof of corollary 5.25: Only the case $d \geq \frac{1}{3}$ will be considered for a similar (yet simpler) argument will show the remaining case. If $d \geq \frac{1}{3}$ then $z_d = \frac{1-d}{1+d}$ and, without loss of generality, we may assume the identity holds for all $d \in \mathbb{C}$ sufficiently close to $d = \frac{1}{3}$. (5.62) together with the definitions in (5.63) and (5.64) imply that

$$p_{r,s} = \frac{s\cdot C_s}{r\cdot M_r}\cdot\left\{\frac{\phi(z_d)}{z_d}\right\}^r\cdot\frac{\psi(z_d)^s}{2\pi}\cdot\left\{J(d;r) + O(e^{-r\cdot c})\right\}, \tag{5.68}$$

uniformly for all (r,s) such that $d = d(r,s) := \frac{s}{r}$ is sufficiently close and less or equal to $d = \frac{1}{3}$ where, accordingly, one determines that

$$\begin{aligned} J(d;r) &:= \int_{-\epsilon}^{\epsilon} e^{-s\cdot F(d,\theta)}\, A(d,\theta)\, d\theta\,, \\ F(d,\theta) &= (3d-1)\cdot u(d)\cdot\theta^2 - i\cdot v(d)\cdot\theta^3 + \ldots \\ A(d,\theta) &= \frac{3d-1}{2d} + \frac{i(d^2-1)}{4d^2}\cdot\theta + \ldots \end{aligned}$$

with $u(d) := \frac{1-d^2}{8d}$ and $v(d) := \frac{(3d^3-1)\cdot(d^2-1)}{24d^2}$; in particular, $u(\frac{1}{3}) = \frac{1}{8}$ and $v(\frac{1}{3}) = \frac{8}{27} > 0$. $F(d,\theta)$ and $A(d,\theta)$ therefore have a 2-to-3 and 0-to-1 change of degree about $\theta = 0$

as $d \to \frac{1}{3}$. Furthermore, observe that $\arg\{(3d-1)\cdot u(d)\} = 0$, for all $d \geq \frac{1}{3}$. As a result, theorem 5.12 can be used to analyze the asymptotic behavior of $J(d;r)$, for all $d \geq \frac{1}{3}$ sufficiently close to $d = \frac{1}{3}$. Using the statement of the theorem and the remarks that follow one obtains that

$$
\begin{aligned}
J(d;r) &= A_0(d)\cdot J_0(d;r) + A_1(d)\cdot J_1(d;r) + O(e^{-r\cdot c})\,,\\
A_0(d) &= \frac{(3d-1)}{2d}\,,\\
A_1(d) &\sim -2i\,,\ \text{as } d \to \tfrac{1}{3}\,,\\
J_0(d;r) &= \frac{\mathcal{L}_0(d;r)}{r^{1/3}} + \frac{R_0(d;1)}{r^{2/3}}\cdot \mathcal{L}_1(d;r) + O\left(\frac{|\mathcal{L}_0(d;r)|}{r^{1+1/3}} + \frac{|\mathcal{L}_1(d;r)|}{r^{1+2/3}}\right)\,,\\
J_1(d;r) &= \frac{R_1(d;1)}{r^{2/3}}\cdot \mathcal{L}_1(d;r) + O\left(\frac{|\mathcal{L}_0(d;r)|}{r^{1+1/3}} + \frac{|\mathcal{L}_1(d;r)|}{r^{1+2/3}}\right)\,,\\
R_0(d;1) &\sim \frac{i}{6}\,,\ \text{as } d \to \tfrac{1}{3}\,,\\
R_1(d;1) &\sim 1\,,\ \text{as } d \to \tfrac{1}{3}\,,\\
\mathcal{L}_0(d;r) &= \frac{2\pi}{\{3V(d)\}^{1/3}}\cdot e^{2\lambda(d;r)^3/3}\cdot \mathbf{Ai}(\lambda(d;r)^2)\,,\\
\mathcal{L}_1(d;r) &= \frac{-2\pi i}{\{3V(d)\}^{2/3}}\cdot e^{2\lambda(d;r)^3/3}\cdot \left\{\lambda(d;r)\cdot \mathbf{Ai}(\lambda(d;r)^2) + \mathbf{Ai}'(\lambda(d;r)^2)\right\}\,,\\
\lambda(d;r) &= \frac{1-d^2}{8d\{3V(d)\}^{2/3}}\cdot (3d-1)\, r^{1/3}\,,\\
V(d) &\sim \frac{8}{27}\,,\ \text{as } d \to \tfrac{1}{3}\,.
\end{aligned}
$$

Let $\epsilon > 0$ be such that the all big-O terms above are uniform for all d such that $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$, as $r \to \infty$. If necessary, by reducing the size of $\epsilon > 0$, we can assume that all terms above are analytic functions of d in the disk $[d : |d - \frac{1}{3}| \leq \epsilon]$. The above identities then imply that

$$
\begin{aligned}
(5.69)\quad J(d;r) &= \frac{A_0(d)}{r^{1/3}}\cdot \mathcal{L}_0(d;r) + \frac{A_1(d)\cdot R_1(d;1)}{r^{2/3}}\cdot \mathcal{L}_1(d;r)\\
&\quad + O\left(\frac{3d-1}{r^{2/3}}\cdot \mathcal{L}_1(d;r) + \frac{|\mathcal{L}_0(d;r)|}{r^{1+1/3}} + \frac{|\mathcal{L}_1(d;r)|}{r^{1+2/3}}\right) + O(e^{-r\cdot c})\,,
\end{aligned}
$$

uniformly for all $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$, as $r \to \infty$.

Next we show that the asymptotic behavior of $J(d;r)$ for $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and for big values of r is determine by the quantity $\Delta(d;r) := r \cdot \left(d - \frac{1}{3}\right)^3$. For this, observe that $\Delta(d;r)$ is of the same order as $|\lambda(d;r)|^3$, this provided that $\epsilon > 0$ is sufficiently small.

We first consider the case in which $\Delta(d;r)$ is of a big size. This forces the $|\lambda(d;r)|$ to also be large and the remarks in (5.48) and (5.49) imply that $\mathcal{L}_0(d;r)$ is of order $|\lambda(d;r)|^{-1/2}$ whereas $\mathcal{L}_1(d;r)$ is of order $|\lambda(d;r)|^{-5/2}$. In particular,

$$\frac{\mathcal{L}_1(d;r)}{r^{2/3}} = \frac{A_0(d)}{r^{1/3}} \cdot \mathcal{L}_0(d;r) \cdot O\big(|\lambda(d;r)|^{-3}\big)\,,$$

uniformly for all $|\lambda(d;r)|$ sufficiently big. These findings in (5.69) together with the remark in (5.48) imply that there is $\alpha > 0$ such that

$$\begin{aligned} J(d;r) &= \frac{A_0(d)}{r^{1/3}} \cdot \mathcal{L}_0(d;r) \cdot \left\{1 + O(|\lambda(d;r)|^{-1/2})\right\}, \\ &= \sqrt{\frac{\pi}{u(d)}} \cdot A_0(d) \cdot r^{-1/2} \cdot (3d-1)^{-1/2} \cdot \left\{1 + O(|\lambda(d;r)|^{-1/2})\right\}, \\ &= 3\sqrt{2\pi} \cdot r^{-1/2} \cdot (3d-1)^{1/2} \cdot \left\{1 + O(|\lambda(d;r)|^{-1/2})\right\} \cdot \left\{1 + O|3d-1|\right\}, \end{aligned}$$

uniformly for all $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and $r > 0$ sufficiently big such that $\Delta(d;r) \geq \alpha$. We see that $J(d;r)$ is of order $r^{-1/2} \cdot (3d-1)^{-1/2}$. This in (5.68) implies that $e^{-r \cdot c} = J(d;r) \cdot O(|\lambda(d;r)|^{-1/2})$ uniformly for all $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and $r > 0$ such that $\Delta(d;r) \geq \alpha$, as $r \to \infty$. As a result, using (5.68) and the asymptotic formulas in (5.58) and (5.59), we obtain that

$$\begin{aligned} p_{r,s} &= 3 \cdot \frac{s \cdot C_s}{r \cdot M_r} \cdot \left\{\frac{\phi(z_d)}{z_d}\right\}^r \cdot \frac{\psi(z_d)^s}{\sqrt{2\pi}} \cdot r^{-1/2} \cdot (3d-1)^{1/2} \\ &\qquad \cdot \left\{1 + O(|\lambda(d;r)|^{-1/2})\right\} \cdot \left\{1 + O|3d-1|\right\}, \end{aligned}$$

$$= \frac{4^{2+r+s}}{\sqrt{2\pi}\cdot 27^{r+1}} \cdot \left\{\frac{\phi(z_d)}{z_d}\right\}^r \cdot \psi(z_d)^s \cdot r^{-1/2} \cdot (3d-1)^{1/2}$$

$$\cdot\left\{1+O(\Delta(r,s)|^{-1/6})\right\} \cdot \left\{1+O|3d-1|\right\} \cdot \left\{1+O(r^{-1})\right\},$$

uniformly for all $0 \le \left(d-\frac{1}{3}\right) \le \epsilon$ and $r > 0$ such that $\Delta(d;r) \ge \alpha$, as $r \to \infty$. This implies (5.65).

To complete the proof of the corollary it remains to determine the asymptotic order of $J(d;r)$ for $0 \le \left(d-\frac{1}{3}\right) \le \epsilon$ and $r > 0$ sufficiently large such that $\Delta(d;r) \le \beta$, where $\beta > 0$ is a fixed constant. This implies that $(3d-1) = O(r^{-1/3})$, hence $\lambda(d;r)$ is bounded and therefore $\mathcal{L}_0(d;r)$ and $\mathcal{L}_1(d;r)$ are uniformly bounded. These findings in (5.69) imply, after some few calculations, that

$$\begin{aligned} J(d;r) &= \frac{A_0(d)}{r^{1/3}} \cdot \mathcal{L}_0(d;r) + \frac{A_1(d)\cdot R_1(d;1)}{r^{2/3}} \cdot \mathcal{L}_1(d;r) + O(r^{-1}), \\ &= \frac{3(3d-1)}{2r^{1/3}} \cdot \mathcal{L}_0(d;r) - \frac{2i}{r^{2/3}} \cdot \mathcal{L}_1(d;r) + O(r^{-1}), \\ &= \frac{3^{4/3}\cdot\pi}{2\cdot r^{2/3}} \cdot e^{4\lambda(d;r)^3/3} \cdot \mathbf{A}\big(\lambda(d;r)\big) + O(r^{-1}), \end{aligned}$$

uniformly for all $0 \le \left(d-\frac{1}{3}\right) \le \epsilon$ and all $r > 0$ sufficiently big such that $\Delta(d;r) \le \beta$. To conclude that $J(d;r)$ is of order $r^{-2/3}$ we require the following result.

Lemma 5.26. $\mathbf{A}(\lambda)$ *is zero-free for* $\lambda \in \mathbb{R}$.

Proof. The proof follows by contradiction. Thus, suppose that there is $\lambda \in \mathbb{R}$ such that $\mathbf{A}(\lambda) = 0$. Since $\mathbf{A}(\lambda)$ is a probability distribution over the real-line (see section 1.1 in [BFSS00]) then it must be the case that $\mathbf{A}'(\lambda) = 0$. But, the conditions $\mathbf{A}(\lambda) = 0$ and $\mathbf{A}'(\lambda) = 0$ are equivalent to say that

$$\begin{cases} \mathbf{Ai}'(\lambda^2) &= \lambda \cdot \mathbf{Ai}(\lambda^2), \\ \mathbf{Ai}(\lambda^2) &= 2\lambda \cdot \big(\mathbf{Ai}''(\lambda^2) - \lambda \cdot \mathbf{Ai}'(\lambda^2)\big). \end{cases}$$

(5.24) now implies that $\mathbf{Ai}''(\lambda^2) = \lambda^2 \cdot \mathbf{Ai}(\lambda^2)$. The above identities let us conclude $\mathbf{Ai}(\lambda^2) = 0$ and this is not possible because $\mathbf{Ai}(x)$ is zero-free for all $x \geq 0$ (see end of section 5.3). This shows the lemma. $\square$

Since the argument of $\lambda(d;r)$ can be made as small as wanted by reducing the size of $\epsilon > 0$, the lemma implies that, for a sufficiently small choice of $\epsilon > 0$, $A(\lambda(d;r))$ is zero-free for $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and r such that $\Delta(d;r) \leq \beta$. This shows that $J(d;r)$ is effectively of order $r^{-2/3}$ in the stated range for $(d;r)$. (5.68) lets us then to conclude that

$$p_{r,s} = \frac{3^{4/3}}{2} \cdot \frac{s \cdot C_s}{r \cdot M_r} \cdot \left\{\frac{\phi(z_d)}{z_d}\right\}^r \cdot \psi(z_d)^s \cdot r^{-2/3} \cdot e^{4\lambda(d;r)^3/3} \mathbf{A}(\lambda(d;s)) \cdot \left\{1 + O(r^{-1/3})\right\},$$

uniformly for all $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and all sufficiently large r such that $\Delta(d;r) \leq \beta$. To deduce (5.66) all what remains is to make more explicit the order of the factor $\left\{\frac{\phi(z_d)}{z_d}\right\}^r \cdot \psi(z_d)^s$ above. Indeed, a simple calculation reveals that

$$\begin{aligned}\left\{\frac{\phi(z_d)}{z_d}\right\}^r \cdot \psi(z_d)^s &= 4^{-s} \cdot \left(\frac{27}{4}\right)^r \cdot e^{-r(3d-1)^3/16 + O(r(3d-1)^4)}, \\ &= 4^{-s} \cdot \left(\frac{27}{4}\right)^r \cdot e^{-4 \cdot \lambda(d;r)^3/3} \cdot \left\{1 + O(r^{-1/3})\right\}.\end{aligned}$$

The asymptotic formulas in (5.58) and (5.59) then imply that

$$\begin{aligned}p_{r,s} &= 4 \cdot d^{-3/2} \cdot 3^{-25/6} \cdot r^{-2/3} \cdot \mathbf{A}(\lambda(d;r)) \cdot \left\{1 + O(r^{-1/3})\right\}, \\ &= \frac{16}{81} \cdot \frac{3^{3/4}}{4} \cdot r^{-2/3} \cdot \mathbf{A}(\lambda(d;r)) \cdot \left\{1 + O(r^{-1/3})\right\},\end{aligned}$$

uniformly for all $0 \leq \left(d - \frac{1}{3}\right) \leq \epsilon$ and $\Delta(d;r) \leq \beta$, as $r \to \infty$. (5.66) follows from the fact that $\lambda(d;r) \sim \frac{3^{4/3}}{4} \cdot \frac{s - r/3}{r^{2/3}}$ uniformly for all (r,s) such that $\Delta(r,s) \leq \beta$, as $(r,s) \to \infty$. This completes the proof of corollary 5.25. $\square$

5.6 Preliminary results

The following two results concern with analytic functions of several complex variables. They will be used for the particular case of two-complex variables to prove theorems 5.2 and 5.12. The notation used in chapter 4 will be continued in here.

The following result gives more precise information about Levinson's canonical representation (see theorem 4.24) for analytic functions that expose a p-to-q change of degree.

Lemma 5.27. *Let $d \geq 1$ be an integer and suppose that $F(\boldsymbol{z})$ is analytic in a neighborhood of the origin in $\mathbb{C}^d$. Moreover, suppose that $F(\boldsymbol{z})$ has a p-to-q change of degree about $z_d = 0$ as $\boldsymbol{z}' \to \boldsymbol{0}'$. Then, F admits near the origin a unique representation of the form*

$$F(\boldsymbol{z}) = \sum_{k=p}^{q} F_k(\boldsymbol{z}') \cdot x_d^k\,, \tag{5.70}$$

where $F_k(\boldsymbol{0}') = 0$, for $p \leq k \leq (q-1)$, $F_q(\boldsymbol{0}') \neq 0$ and $x_d = x_d(\boldsymbol{z})$ is such that $x_d(\boldsymbol{z}', 0) \equiv 0$ and $\frac{\partial x_d}{\partial z_d}(\boldsymbol{z}', 0) \equiv 1$. Indeed, this implies that

$$F_p(\boldsymbol{z}') = \frac{1}{p!}\frac{\partial^p F}{\partial z_d^p}(\boldsymbol{z}', 0)\,. \tag{5.71}$$

Proof. Using Levinson's canonical representation theorem 4.24 and our result in section 4.8.1, it follows that there is a unique representation of the form $F(\mathbf{z}) = \sum_{k=0}^{q} F_k(\mathbf{z}') \cdot x_d^k$, where $F_k(\mathbf{0}') = 0$, for $0 \leq k \leq (q-1)$, $F_q(\mathbf{0}') \neq 0$ and $x_d = x_d(\mathbf{z})$ is such that $x_d(\mathbf{z}', 0) \equiv 0$ and $\frac{\partial x_d}{\partial z_d}(\mathbf{z}', 0) \equiv 1$. Suppose that this representation applies for all $\mathbf{z}$ in an open neighborhood of a polydisk $\Delta[\mathbf{0}, \mathbf{r}]$.

Since $F(\mathbf{z})$ has a p-to-q change of degree about $z_d = 0$ as $\mathbf{z}' \to \mathbf{0}'$, its Hartogs series in powers of z_d near the $\mathbf{0}$ must be of the form $\sum_{k=p}^{\infty} f_k(\mathbf{z}')\, z_d^k$, with $f_p(\mathbf{z}')$ not identically zero in any neighborhood of $\mathbf{0}'$.

Consider the map $\Phi(\mathbf{z}) = (\mathbf{z}', x_d(\mathbf{z}))$. The conditions imposed over $x_d(\mathbf{z})$ imply that the Jacobian matric $\frac{\partial \Phi}{\partial \mathbf{z}}(\mathbf{0})$ is lower-triangular with all entries equal to 1 along the diagonal. Since $\Phi(\mathbf{0}) = \mathbf{0}$, the Inverse mapping theorem 4.23 lets us assume, without loss of generality, that Φ is holomorphic and 1-to-1 over $\Delta[\mathbf{0}, \mathbf{r}]$.

Fix $\mathbf{z}' \in \Delta[\mathbf{0}', \mathbf{r}']$. Since $x_d(\mathbf{z}', 0) = 0$, there is $\rho_1 > 0$ such that $[x_d : |x_d| \leq \rho_1] \subset x_d(\mathbf{z}', [z_d : |z_d| < r_d])$. Moreover, since $x_d(\mathbf{z}', \cdot)$ is 1-to-1 we may find $\rho_2 > 0$ such that $[z_d : |z_d| \leq \rho_2]$ is contained the pre-image of $[x_d : |x_d| < \rho_1]$. Using Cauchy's theorem, then substituting: $x_d = x_d(\mathbf{z}', z_d)$, and finally using that the Hartogs series of F is uniformly convergent toward F over compact sets (see theorem 4.13) it follows, for all $0 \leq j \leq q$, that

$$\begin{aligned} F_j(\mathbf{z}') &= \frac{1}{2\pi i} \int_{|x_d|=\rho_1} \frac{1}{x_d^{j+1}} \sum_{k=0}^{q} F_k(\mathbf{z}') \cdot x_d^k \, dx_d \,, \\ &= \frac{1}{2\pi i} \int_{|z_d|=\rho_2} \frac{F(\mathbf{z}', z_d)}{\{x_d(\mathbf{z}', z_d)\}^{j+1}} \frac{\partial x_d}{\partial z_d}(\mathbf{z}', z_d)\, dz_d \,, \\ &= \frac{1}{2\pi i} \sum_{k=p}^{\infty} f_k(\mathbf{z}') \cdot \int_{|z_d|=\rho_2} \frac{z_d^k}{\{x_d(\mathbf{z}', z_d)\}^{j+1}} \frac{\partial x_d}{\partial z_d}(\mathbf{z}', z_d)\, dz_d \,. \end{aligned}$$

Observe that $x_d(\mathbf{z}', z_d) = z_d \cdot h(\mathbf{z}', z_d)$ where, for each fixed $\mathbf{z}' \in \Delta(\mathbf{0}', \mathbf{r}')$, $h(\mathbf{z}', 0) \equiv 1$ and $h(\mathbf{z}', \cdot)$ is analytic and zero-free in an open neighborhood of $[z_d : |z_d| \leq \rho_2]$. This implies that, for all $j < p \leq k$, the function $\frac{z_d^k}{\{x_d(\mathbf{z}', z_d)\}^{j+1}} \frac{\partial x_d}{\partial z_d}(\mathbf{z}', z_d)$ is analytic in z_d in an open neighborhood of $[z_d : |z_d| \leq \rho_2]$. Consequently, in the above summation, each integral term vanishes and therefore $F_j(\mathbf{z}') \equiv 0$, for all $j < p$. This proves (5.70).

On the other hand, for the same reasons, if $j = p$ then the integral terms in the above summation all vanish for $k > p$. This implies that

$$F_p(\mathbf{z}') = \frac{f_p(\mathbf{z}')}{2\pi i} \cdot \int_{|z_d|=\rho_2} \frac{z_d^p}{\{x_d(\mathbf{z}', z_d)\}^{p+1}} \frac{\partial x_d}{\partial z_d}(\mathbf{z}', z_d)\, dz_d\,.$$

Observe that the integrand above has a simple pole at $z_d = 0$ and the residue there is equal to the

$$\lim_{z_d \to 0} \frac{z_d^{p+1}}{\{x_d(\mathbf{z}', z_d)\}^{p+1}} \frac{\partial x_d}{\partial z_d}(\mathbf{z}', z_d) = \{h(\mathbf{z}', 0)\}^{-(p+1)} = 1\,.$$

As a result, $F_p(\mathbf{z}') = f_p(\mathbf{z}')$ and this shows (5.71). This completes the proof of the lemma. □

The next result could be proved using Differential equations (see proof of theorem 2 in [CFU56], theorem I in [Ble67], or theorem 9.2.2 in [BleHan86].) They all proceed to show a result somehow reminiscent of ours arguing that the derivatives up to order $(p-1)$ of a non-trivial solution of a linear homogeneous ordinary differential equation of order p cannot all vanish simultaneously. (This argument was used for the particular case of the Airy function in section 5.3.) However, here we present with an alternative and, to the best of our knowledge, also original proof which uses rather more elementary and well-known techniques.

Lemma 5.28. (Functions with no common zeroes.) *Let $d \geq 1$ be an integer and $\gamma \subset \mathbb{C}$ be an infinite contour such that for an appropriate constant $c > 0$ the $\Re\{x^{d+1}\} \geq c \cdot |x|^{d+1}$, for all $x \in \gamma$ sufficiently big. For each $k \geq 0$ let $F_k : \mathbb{C}^d \to \mathbb{C}$ be the entire analytic function defined as*

$$F_k(\boldsymbol{z}) := \int_\gamma \exp\left\{-x^{d+1} - \sum_{j=1}^{d} z_j \cdot x^j\right\} x^k\, dx\,. \tag{5.72}$$

If the

$$\int_\gamma \exp\left\{-x^{d+1}\right\}\, dx \neq 0 \tag{5.73}$$

then the $\sum_{k=0}^{d-1} |F_k(\boldsymbol{z})| > 0$ *for all* $\boldsymbol{z} \in \mathbb{C}^d$*; in other words, the functions* F_k *with* $k = 0, \ldots, (d-1)$ *do not share a common zero in* $\mathbb{C}^d$*.*

Remark 5.29. Observe that the conditions in the lemma apply in particular for any contour γ of the form $\big(u \cdot \mathbb{R}_+ - v \cdot \mathbb{R}_+\big)$ with $u \neq v$ and $u^{d+1} = v^{d+1} = 1$. More generally, for all nonzero u and v situated in different connected components of the set $\big\{x \in \mathbb{C} : \Re\{x^{d+1}\} > 0\big\}$.

Proof. The proof proceeds by contradiction. Therefore, suppose that there is $\mathbf{a} \in \mathbb{C}^d$ such that $F_k(\mathbf{a}) = 0$, for all $0 \leq k \leq (d-1)$. This is equivalent to say that, for all $0 \leq k \leq (d-1)$, the

$$\int_\gamma e^{-p(x)}\, x^k\, dx = 0\,, \tag{5.74}$$

where for convenience we have defined $p(x) := x^{d+1} + \sum_{j=1}^{d} a_j \cdot x^j$.

We will use the above condition to show that for all $q \in \mathbb{C}[x]$ the

$$\int_\gamma e^{-p(x)}\, q(x)\, dx = 0\,. \tag{5.75}$$

To show this, define $q_0 := q$. Then, using the Division algorithm for polynomials, we may recursively define $r_k, e_k, q_k \in \mathbb{C}[x]$, for all $k \geq 1$, to satisfy: $q_{k-1} = r_k + p' \cdot e_k$ with the $\deg[r_k] < \deg[p'] = d$ and $q_k = e_k'$; in particular, the $\int_\gamma e^{-p(x)}\, r_k(x)\, dx = 0$, for all $k \geq 1$. With q_k defined in this way, a simple inductive argument, using integration by parts, shows that

$$\int_\gamma e^{-p(x)}\, q_k(x)\, dx = \int_\gamma e^{-p(x)}\, q_{k+1}(x)\, dx\,.$$

Observe that unless $e_{k+1} \equiv 0$ the $\deg[q_k] > \deg[q_{k+1}]$. Otherwise, if $e_{k+1} \equiv 0$ then the $\deg[q_k] < d$. This shows that for a sufficiently large k the $\deg[q_k] < d$. Thus, for any such k and since $q_0 = q$, we can deduce that the $\int_\gamma e^{-p(x)}\, q(x)\, dx = \int_\gamma e^{-p(x)}\, q_k(x)\, dx = 0$. This proves (5.75).

Observe that each function $F_k : \mathbb{C}^d \to \mathbb{C}$, as defined in (5.72), is an entire analytic function. Moreover, for all multi-index $\mathbf{n}$, a simple inductive argument shows that

$$\frac{\partial^{\mathbf{n}} F_k}{\partial \mathbf{z}^{\mathbf{n}}}(\mathbf{a}) = (-1)^{\langle \mathbf{n} \rangle} \cdot \int_\gamma e^{-p(x)}\, x^{k+\sum_{j=1}^{d} j\cdot n_j}\, dx\,.$$

(5.75) then implies that $F_k \equiv 0$, for all k. In particular, $F_0(\mathbf{0}) = \int_\gamma \exp\left\{-x^{d+1}\right\} dx = 0$, but this is in contradiction with condition (5.73). As a result, there cannot be $\mathbf{a} \in \mathbb{C}^d$ such that $F_k(\mathbf{a}) = 0$, for all $0 \leq k \leq (d-1)$. This completes the proof of the lemma. □

5.7 Proof of the generalized stationary phase method

In this section we prove theorem 5.2. Our interest is to provide a uniform asymptotic expansion for an integral of the form

$$I(t;s) = \int_0^1 e^{-s\cdot F(t,z)}\, A(t,z)\, dz\,,$$

uniformly valid for all t sufficiently small and all $s \geq 0$ sufficiently large. The phase and amplitude term of $I(t;s)$ are assumed to be analytic in an open polydisk centered at the origin containing points of the form (t,z), with $|z| \leq 1$, for all t sufficiently small. It is also assumed that there are integers $n \geq 1$ and $q \geq p \geq 0$ such that $\{F(t,z) - F(t,0)\}$ and $A(t,z)$ have respectively an n-to-n and p-to-q change of degree about $z = 0$ as $t \to 0$. Furthermore, we will work with the assumption that

$\Re\{F(0,z)-F(0,0)\}>0$, for all $z\in[0,1]$, but $z=0$, and write

$$F(t,z)-F(t,0)=u(t)\cdot z^n+\dots$$

with $u(0)\neq 0$.

Observe that the hypotheses on $F(t,z)$ imply that the $\Re\{u(0)\}\geq 0$.

To prove theorem 5.2 it will be crucial to localize $I(t;s)$ in an interval of the form $[0,r]$, for all $r\in(0,1)$ sufficiently small. This is required in order to exploit the local behavior of the phase and amplitude term of $I(t;s)$ for (t,z) near the origin. For this, fix $r\in(0,1)$. Observe that $\Re\{F(t,z)-F(t,0)\}\to\Re\{F(0,z)-F(0,0)\}$ uniformly for all $z\in[r,1]$, as $t\to 0$. As a result, since the $\Re\{F(0,z)-F(0,0)\}>0$, for all $z\in[r,1]$, then there is a constant $c_0>0$ such that the $\Re\{F(t,z)-F(t,0)\}\geq c_0$, for all t sufficiently small and all $z\in[r,1]$.

With r and c_0 as in the previous paragraph, we obtain that

$$I(t;s)=e^{-s\cdot F(t,0)}\cdot\left\{\int_0^r e^{-s\cdot\{F(t,z)-F(t,0)\}}\,A(t,z)\,dz+O\big(e^{-s\cdot c_0}\big)\right\}, \tag{5.76}$$

uniformly for all t sufficiently small and all $s\geq 0$. Motivated by the above identity we will define

$$I(t,r;s):=\int_0^r e^{-s\cdot\{F(t,z)-F(t,0)\}}\,A(t,z)\,dz\,. \tag{5.77}$$

Our next goal is to transform $I(t,r;s)$ to a Fourier-Laplace integral with a monomial phase term and a polynomial like amplitude term. This will be done through a change of coordinates which will take t into account. It will be of the form $(t,x)=\Phi(t,z)$ where Φ is a biholomorphic map defined on a neighborhood of $(t,z)=(0,0)$ and taking values in a neighborhood of $(t,x)=(0,0)$. The map Φ will be the result of the composition of two maps of the form $(t,y)=\Psi_1(t,z)$ and

$(t,x) = \Psi_2(t,y)$. The first of these maps will be used to put the amplitude term in the desired form whereas Ψ_2 will produce a monomial phase term.

Lemma 5.27 states that there is a unique representation of the form

$$\int_0^z A(t,\xi)\,d\xi = \sum_{k=p}^{q} \frac{A_k(t)}{k+1} \cdot y^{k+1}\,,$$

where $A_k(0) = 0$, for all $p \leq k \leq (q-1)$, $A_q(0) \neq 0$, and $y = y(t,z) = z + \ldots$ In particular, by differentiating both sides with respect to z we obtain that

$$A(t,z) = \left\{ \sum_{k=p}^{q} A_k(t) \cdot y^k \right\} \cdot \frac{\partial y}{\partial z}(t,z)\,. \tag{5.78}$$

The motivation we had for the above representation is the following. If for each fixed t sufficiently small, we let $y = y(t,z)$, then: $A(t,z)\,dz = \left\{ \sum_{k=p}^{q} A_k(t) \cdot y^k \right\} dy$. As a result, in the new variable y, $I(t,r;s)$ has a polynomial amplitude term. This insight motivates to consider the mapping

$$\begin{aligned} (t,y) &= \Psi_1(t,z) \\ &:= (t, y(t,z)) = (t, z + \ldots)\,. \end{aligned}$$

Ψ_1 can be thought of as a local change of coordinates. Indeed, observe that the conditions given on $y(t,z)$ imply that the Jacobian matrix $\frac{\partial \Psi_1}{\partial(t,z)}(0,0)$ is lower-triangular with nonzero entries along the diagonal. Thus, using the Inverse mapping theorem 4.23, we may conclude that Ψ_1 is a biholomorphic map between an open neighborhood of $(t,z) = (0,0)$ and an open neighborhood of $(t,y) = (0,0)$.

Based on remarks in the previous paragraph it follows that $F(\Psi_1^{-1}(t,y))$ is analytic in some open neighborhood of $(t,y) = (0,0)$. Moreover, its Hartogs series about $(t,y) = (0,0)$ in powers of y can be easily shown to be of the form

$$F(\Psi_1^{-1}(t,y)) = F(t,0) + u(t) \cdot y^n + \ldots$$

Consider the map

$$\phi(t,y) := y \cdot \{u(t)\}^{1/n} \cdot \left\{1 + \frac{F(\Psi_1^{-1}(t,y)) - F(t,0) - u(t) \cdot y^n}{u(t) \cdot y^n}\right\}^{1/n} .$$

Above, the principal branch of the n^{th}-root has been selected. This is possible because the hypotheses of theorem 5.2 imply that $u(0) \neq 0$ and the $\Re\{u(0)\} \geq 0$. Furthermore, it follows that $\phi(t,y)$ is analytic near $(t,y) = (0,0)$ and, from the way it was defined, we see that: $F(\Psi_1^{-1}(t,y)) - F(t,0) = \{\phi(t,y)\}^n$. This motivates to define the map

$$\begin{aligned} (t,x) &= \Psi_2(t,y) \\ &:= (t, \phi(t,y)) = (t, y \cdot \{u(t)\}^{1/n} + \ldots) . \end{aligned}$$

Ψ_2 like Ψ_1 can be shown to be a biholomorphic map between an open neighborhood of $(t,y) = (0,0)$ and an open neighborhood of $(t,x) = (0,0)$. Moreover, in the coordinate system (t,x), we may rewrite

$$F(\Psi_1^{-1}(t,y)) - F(t,0) = x^n . \tag{5.79}$$

As we anticipated we will define

$$\begin{aligned} (t,x) &= \Phi(t,z) \\ &:= (\Psi_2 \circ \Psi_1)(t,z) = (t, z \cdot \{u(t)\}^{1/n} + \ldots) . \end{aligned}$$

We will write $\Phi = (\Phi_1, \Phi_2)$ to refer to the coordinates functions of Φ, and thus, we have that $\Phi_1(t,z) = t$ and $\Phi_2(t,z) = z \cdot \{u(t)\}^{1/n} + \ldots$ Since Φ is biholomorphic we may find $r \in (0,1)$ such that points of the form (t,z), with t sufficiently small and $|z| \leq 2r$, are contained in the domain of definition of $\Phi(t,z)$, $F(t,z)$ and $A(t,z)$.

For each such t we may then perform the substitution: $x = \Phi_2(t,z)$, in $I(t,r;s)$. In particular, z and x relate to each other trough an auxiliary variable y specified by the relations $(t,x) = \Psi_2(t,y)$ and $(t,y) = \Psi_1(t,z)$. Back in (5.77) and using (5.79) then (5.78), the substitution: $x = \Phi_2(t,z)$, in $I(t,r;s)$, produces

$$\begin{aligned} I(t,r;s) &= \int_{\Phi_2(t,[0,r])} e^{-s\cdot x^n} \cdot A\Big(\Phi^{-1}(t,x)\Big) \cdot \frac{\partial z}{\partial x}(t,x)\, dx\,, \\ &= \sum_{k=p}^{q} A_k(t) \cdot \int_{\Phi_2(t,[0,r])} e^{-s\cdot x^n} \cdot \left\{ y^k \cdot \frac{\partial y}{\partial x}(t,x) \right\} dx\,. \end{aligned}$$

As a result, if we define

$$I_k(t,r;s) := \int_{\Phi_2(t,[0,r])} e^{-s\cdot x^n} \cdot \left\{ y^k \cdot \frac{\partial y}{\partial x}(t,x) \right\} dx\,, \tag{5.80}$$

then back in (5.76) we obtain that

$$I(t;s) = e^{-s\cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot I_k(t,r;s) + O(e^{-s\cdot c_0}) \right\}\,, \tag{5.81}$$

uniformly for all t sufficiently close to 0 and all $s \geq 0$. This shows (5.1) in theorem 5.2.

A uniform asymptotic expansion for $I(t;s)$ will follow provided that we can determine an asymptotic expansion for each term $I_k(t,r;s)$ in (5.81). For this we will show that the contour of integration of $I_k(t,r;s)$ can be replaced by an interval of the form $[0,\delta]$, for some $\delta > 0$, by incurring in a negligible error.

Since $F(t,z) - F(t,0) = x^n$, the hypotheses of theorem 5.2 imply that $\Re\{x^n\} > 0$ if $x = \Phi_2(0,r)$. But, recall that $\Phi_2(0,r) = r\cdot\{u(0)\}^{1/n} + O(r^2)$, with the $\Re\{u(0)\} \geq 0$. Since the selection of the n^{th}-root was in the principal sense, this implies that the

$$|\arg \Phi_2(0,r)| < \frac{\pi}{2n}\,. \tag{5.82}$$

Observe that $\Phi_2(0, r)$ is the end-point of $\Phi_2(0, [0, r])$. On the other hand, as $t \to 0$, the end-point of $\Phi_2(t, [0, r])$ converges to the end-point of $\Phi_2(0, [0, r])$. Thus, due to (5.82), we can replace the contour $\Phi_2(t, [0, r])$, used first to define $I_k(t, r; s)$ in (5.80), by the interval $[0, \delta]$, with $\delta := \Re\{\Phi_2(0, r)\} > 0$. The error incurred by this contour replacement is a $O(e^{-s \cdot c_1})$, for an appropriate constant $c_1 > 0$, uniformly for all t sufficiently small. Hence, back in (5.81), we may assume without any loss of generality that

$$I_k(t, r; s) = \int_0^\delta e^{-s \cdot x^n} \cdot \left\{ y^k \cdot \frac{\partial y}{\partial x}(t, x) \right\} dx \,, \tag{5.83}$$

provided that the constant c_0 in the big-O term is replaced with the $\min\{c_0, c_1\} > 0$. In the above form an asymptotic expansion for $I_k(t, r; s)$ can be easily obtained using the standard stationary phase method (see [BleHan86], chapter 6). Indeed, since the amplitude term of $I_k(t, r; s)$ has at $(t, x) = (0, 0)$ a Hartogs series in powers of x of the form

$$\begin{aligned} y^k \cdot \frac{\partial y}{\partial x}(t, x) &=: \sum_{j=k}^{\infty} c_k(t; j) \cdot x^j \\ &= \{u(t)\}^{-(k+1)/n} \cdot x^k + \ldots \end{aligned}$$

then it follows that $c_k(t; k) = \{u(t)\}^{-(k+1)/n}$ and

$$I_k(t, r; s) \approx \sum_{j=k}^{\infty} \frac{c_k(t; j)}{n} \cdot \Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n} \,,$$

uniformly for all t sufficiently close to 0, as $s \to \infty$. This shows (5.2) and (5.3) and completes the proof of theorem 5.2. □

5.8 Proof of the generalized coalescing saddle point method

In this section we prove theorem 5.12. Our interest is in providing a uniform asymptotic expansion for integrals of the form

$$J(t;s) = \int_{-1}^{1} e^{-s\cdot F(t,z)}\, A(t,z)\, dz\,,$$

valid for all $t \in \mathcal{T}_\epsilon$ sufficiently small and $s \geq 0$ sufficiently large. The hypothesis on the amplitude term are the same as in the previous section. However, this time, $\{F(t,z) - F(t,0)\}$ is assumed to have a 2-to-3 change of degree about $z = 0$ as $t \to 0$ and the $\Re\{F(0,z) - F(0,0)\} > 0$, for all $z \in [-1,1]$, but $z = 0$. Furthermore, we assume that

$$F(t,z) - F(t,0) = t^N \cdot u(t) \cdot x^2 - i \cdot v(t) \cdot x^3 + \ldots$$

where $N \geq 1$ is an integer, $u(0) \neq 0$ and $v(0) > 0$.

The localization of $J(t;s)$ to an interval of the form $[-r,r]$, for all $r \in (0,1)$, proceeds by an argument similar to the one used to localize $I(t;s)$ to an interval of the form $[0,r]$. (See the argument that led to (5.76).) Accordingly, if we define

$$J(t,r;s) := \int_{-r}^{r} e^{-s\cdot\{F(t,z)-F(t,0)\}}\, A(t,z)\, dz\,, \tag{5.84}$$

it follows, for all $r \in (0,1)$, that there is a constant $c_0 > 0$ such that

$$J(t;s) = e^{-s\cdot F(t,0)} \cdot \left\{ J(t,r;s) + O(e^{-s\cdot c_0}) \right\}, \tag{5.85}$$

uniformly for all t sufficiently small and all $s \geq 0$.

As in (5.78) we may represent $A(t,z)$ near the origin in the form

$$A(t,z) = \left\{ \sum_{k=p}^{q} A_k(t) \cdot y^k \right\} \cdot \frac{\partial y}{\partial z}(t,z)\,,$$

where $A_k(0) = 0$, for all $p \leq k < q$, however, $A_q(0) \neq 0$, and $y = y(t,z) = z + \ldots$ is analytic near the origin. This motivates to consider again the biholomorphic map $(t,y) = \Psi_1(t,z) = (t, z + \ldots)$. We observe that $F(\Psi_1^{-1}(t,y))$ also exposes a 2-to-3 change of degree about $y = 0$ as $t \to 0$. Thus, using lemma 5.27, we conclude that there is a unique representation of the form

$$F(\Psi_1^{-1}(t,y)) = \sum_{j=2}^{3} F_j(t) \cdot x^j \,,$$

where $F_2(0) = 0$, $F_3(0) \neq 0$, and $x = x(t,y) = y + \ldots$ In particular, we see that the transformation $(t,x) = \Psi_2(t,y) := (t, x(t,y)) = (t, y + \ldots)$ is biholomorphic between an open neighborhood of $(t,y) = (0,0)$ and an open neighborhood of $(t,x) = (0,0)$. We will define $\Phi := \Psi_2 \circ \Psi_1$. Observe that Φ is of the form

$$\Phi(t,z) = (t, \phi(t,z)) = (t, z + \ldots)\,.$$

Since Φ is biholomorphic, we may find $r \in (0,1)$ such that points of the form (t,z) with t sufficiently small and $|z| \leq 2r$ are contained in the domain of definition of Φ, F and A. If for each such t we substitute: $x = \phi(t,z)$, in $J(t,r;s)$, we obtain that

$$\begin{aligned} J(t,r;s) &= \int_{\phi(t,[-r,r])} e^{-s\cdot \sum_{j=2}^{3} F_j(t)\cdot x^j} \cdot A\Big(\Phi^{-1}(t,x)\Big) \cdot \frac{\partial z}{\partial x}(t,x)\, dx\,, \\ &= \sum_{k=p}^{q} A_k(t) \cdot \int_{\phi(t,[-r,r])} e^{-s\cdot \sum_{j=2}^{3} F_j(t)\cdot x^j} \cdot \left\{ y^k \cdot \frac{\partial y}{\partial x}(t,x) \right\} dx\,. \end{aligned}$$

Accordingly, if we define

$$J_k(t,r;s) := \int_{\phi(t,[-r,r])} e^{-s\cdot \sum_{j=2}^{3} F_j(t)\cdot x^j} \cdot \left\{ y^k \cdot \frac{\partial y}{\partial x}(t,x) \right\} dx\,, \tag{5.86}$$

we may rewrite (5.85) in the form

$$J(t;s) = e^{-s\cdot F(t,0)} \cdot \left\{ \sum_{k=p}^{q} A_k(t) \cdot J_k(t,r;s) + O(e^{-s\cdot c_0}) \right\}\,, \tag{5.87}$$

uniformly for all t sufficiently small and all $s \geq 0$. The above identity shows (5.35) in theorem 5.12.

To obtain an expansion for each term $J_k(t, r; s)$ appearing in (5.87) it will show important to relate the coefficient functions $F_2(t)$ and $F_3(t)$ with the former coefficient terms $u(t)$ and $v(t)$. For this, observe that the relation $(t, z) = \Phi^{-1}(t, x) = (t, x + ...)$ implies that

$$\begin{aligned} F(t,z) - F(t,0) &= t^N \cdot u(t) \cdot z^2 - i \cdot v(t) \cdot z^3 + \dots \\ &= t^N \cdot u(t) \cdot x^2 - i \left\{ v(t) + i \cdot t^N \cdot u(t) \cdot \frac{\partial^2 z}{\partial x^2}(t,0) \right\} \cdot x^3 + \dots \\ &= F_2(t) \cdot x^2 + F_3(t) \cdot x^3 \,. \end{aligned}$$

The last two identities above imply that $F_2(t) = t^N \cdot U(t)$ and $F_3(t) = -i \cdot V(t)$ where $U(t) \equiv u(t)$ and $V(0) = v(0) > 0$.

To complete the proof of theorem 5.12 we require the following lemma which we plan to use to obtain an asymptotic expansion for each term $J_k(t, r; s)$ in (5.87).

Lemma 5.30. *Suppose that $F(t, z)$ is analytic in an open neighborhood of $(0, 0)$ which contains points of the form (t, z), with $|z| \leq r$, for all t is sufficiently small. Furthermore, suppose that*

(a) $\Re\{F(0, z) - F(0, 0)\} > 0$, for all $z \in [-r, r]$, but $z = 0$, and

(b) there is a biholomorphic map Φ, defined on the domain of $F(t, z)$ and of the form $\Phi(t, z) = (t, \phi(t, z)) = (t, z + \dots)$, such that in the new coordinate system $(t, x) = \Phi(t, z)$ it applies that

$$F(t, z) - F(t, 0) = t^N \cdot U(t) \cdot x^2 - i \cdot V(t) \cdot x^3 \,,$$

where $U(0) \neq 0$ and $V(0) > 0$.

If for an arbitrary function $B(t,x)$ analytic in the range of Φ it is defined

$$(5.88) \qquad J(t,r;s) := \int_{\phi(t,[-r,r])} \exp\big\{ -s\cdot(t^N\cdot U(t)\cdot x^2 - i\cdot V(t)\cdot x^3)\big\}\, B(t,x)\,dx\,,$$

then there is a constant $c_0 > 0$ such that

$$(5.89) \qquad J(t,r;s) = \frac{R_0(t)}{s^{1/3}}\cdot\mathcal{L}_0(t;s) + \frac{R_1(t)}{s^{2/3}}\cdot\mathcal{L}_1(t;s) + \frac{1}{s}\cdot\tilde{J}(t,r;s) + O\left(e^{-s\cdot c_0}\right)\,,$$

uniformly for all t sufficiently close to 0 and all $s\geq 0$. Above, the terms $R_0(t)$ and $R_1(t)$ together with an auxiliary function $\tilde{B}(t,x)$ analytic for t and x near $t=0$ and $x=0$ are uniquely characterized by the relation

$$B(t,x) = R_0(t) + R_1(t)\cdot x + \tilde{B}(t,x)\cdot\frac{\partial}{\partial x}\big\{t^N\cdot U(t)\cdot x^2 - i\cdot V(t)\cdot x^3\big\}\,.$$

Furthermore,

$$\begin{aligned}
(5.90) \qquad \mathcal{L}_0(t;s) &:= \frac{2\pi}{\{3\,V(t)\}^{1/3}}\cdot e^{2\lambda(t;s)^3/3}\cdot\mathbf{Ai}(\lambda(t;s)^2)\,,\\
(5.91) \qquad \mathcal{L}_1(t;s) &:= \frac{-2\pi\, i}{\{3\,V(t)\}^{2/3}}\cdot e^{2\lambda(t;s)^3/3}\cdot\Big\{\lambda(t;s)\cdot\mathbf{Ai}(\lambda(t;s)^2) + \mathbf{Ai}'(\lambda(t;s)^2)\Big\}\,,\\
(5.92) \qquad \lambda(t;s) &:= \frac{U(t)}{\{3\,V(t)\}^{2/3}}\cdot(st^{3P})^{1/3}\,,\\
(5.93) \qquad \tilde{J}(t,r;s) &:= \int_{\phi(t,[-r,r])} \exp\big\{-s\cdot(t^N\cdot U(t)\cdot x^2 - i\cdot V(t)\cdot x^3)\big\}\,\frac{\partial\tilde{B}}{\partial x}(t,x)\,dx\,.
\end{aligned}$$

In addition, for all nonnegative integer n, all $c_1\geq 0$ and $c_2\geq 0$, and all $\epsilon > 0$ sufficiently small there is a constant $c_3 > 0$ such that

$$(5.94) \qquad |\tilde{J}(t,r;s)| + s^{n+1}\cdot c_1\, e^{-s\cdot c_2} \leq c_3\cdot\left\{\frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}}\right\}\,,$$

uniformly for all $t\in\mathcal{S}_\epsilon$ sufficiently small and all $s\geq 0$ sufficiently big, where it has been defined

$$\mathcal{S}_\epsilon := \big\{t : t = 0 \text{ or, } t\neq 0 \text{ and } |\arg\{t^N\cdot U(t)\}| < (\pi/2-\epsilon)\big\}\,.$$

We will use lemma 5.30 to obtain a full asymptotic expansion for the coefficients $J_k(t,r;s)$ as defined in (5.86). Recall that we seek for expansions which are uniformly valid for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$, where

$$\mathcal{T}_\epsilon := \Big\{ t : |\arg\{u(t) \cdot t^N\}| < \Big(\frac{\pi}{2} - \epsilon\Big) \Big\} \,.$$

We will define recursively, for each nonnegative integer l, functions $B_k(t,x;l)$ and $R_k(t;l)$ analytic in t and x near $t=0$ and $x=0$. This is done using the Weierstrass division theorem 4.21 which, without loss of generality, lets us assert that there are unique functions $B_k(t,x;l)$ and $R_k(t;l)$, analytic for all t sufficiently small and all $x \in \phi(t,[|z| \leq r])$, such that

$$\begin{aligned} B_k(t,x;0) &:= y^k \cdot \frac{\partial y}{\partial x}(t,x)\,, \\ B_k(t,x;l) &=: R_k(t;2l) + R_k(t;2l+1) \cdot x \\ &\qquad + B_k(t,x;l+1) \cdot \frac{\partial}{\partial x}\big\{t^P \cdot U(t) \cdot x^2 - i \cdot V(t) \cdot x^3\big\}\,. \end{aligned}$$

For each $l \geq 0$ consider the integral

$$\tilde{J}_k(t,r;l;s) := \int_{\phi(t,[-r,r])} \exp\Big\{ - s \cdot (t^P \cdot U(t) \cdot x^2 - i \cdot V(t) \cdot x^3)\Big\} B_k(t,x;l)\, dx\,.$$

Each of the preceding integrals is in the context of lemma 5.30. Therefore, by its repeated use, we obtain that there is a constant $c > 0$ such that the following equalities hold

$$\begin{aligned} \tilde{J}_k(t,r;0;s) &= \frac{R_k(t;0)}{s^{1/3}} \cdot \mathcal{L}_0(t;s) + \frac{R_k(t;1)}{s^{2/3}} \cdot \mathcal{L}_1(t;s) + \frac{\tilde{J}_k(t,r;1;s)}{s} + O(e^{-s\cdot c})\,, \\ \tilde{J}_k(t,r;1;s) &= \frac{R_k(t;2)}{s^{1/3}} \cdot \mathcal{L}_0(t;s) + \frac{R_k(t;3)}{s^{2/3}} \cdot \mathcal{L}_1(t;s) + \frac{\tilde{J}_k(t,r;2;s)}{s} + O(e^{-s\cdot c})\,, \\ &\;\;\vdots \end{aligned}$$

$$\tilde{J}_k(t,r;n;s) = \frac{R_k(t;2n)}{s^{1/3}} \cdot \mathcal{L}_0(t;s) + \frac{R_k(t;2n+1)}{s^{2/3}} \cdot \mathcal{L}_1(t;s) + \frac{\tilde{J}_k(t,r;n+1;s)}{s} + O(e^{-s\cdot c}),$$

uniformly for all t sufficiently small and all $s \geq 0$. Above, $\mathcal{L}_0(t;s)$ and $\mathcal{L}_1(t;s)$ are as defined in (5.90) and (5.91) respectively. Back tracking the above identities and noticing that $J_k(t,r;s) = \tilde{J}_k(t,r;0;s)$ we obtain that

$$J_k(t,r;s) = \left\{\sum_{l=0}^{n} \frac{R_k(t;2l)}{s^{l+1/3}}\right\} \cdot \mathcal{L}_0(t;s) + \left\{\sum_{l=0}^{n} \frac{R_k(t;2l+1)}{s^{l+2/3}}\right\} \cdot \mathcal{L}_1(t;s) + \frac{\tilde{J}_k(t,r;n+1;s)}{s^{n+1}} + O(e^{-s\cdot c}).$$

Finally, the inequality in (5.94) implies that there is a constant $c_1 > 0$ such that

$$\frac{|\tilde{J}_k(t,r;n+1;s)|}{s^{n+1}} + O(e^{-s\cdot c}) = O\left(\frac{|\mathcal{L}_0(t;s)|}{s^{(n+1)+1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{(n+1)+2/3}}\right),$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small and all $s \geq 0$ sufficiently big. This implies that

$$J_k(t,r;s) \approx \left\{\sum_{l=0}^{\infty} \frac{R_k(t;2l)}{s^{l+1/3}}\right\} \cdot \mathcal{L}_0(t;s) + \left\{\sum_{l=0}^{\infty} \frac{R_k(t;2l+1)}{s^{l+2/3}}\right\} \cdot \mathcal{L}_1(t;s),$$

uniformly for all $t \in \mathcal{T}_\epsilon$ sufficiently small, as $s \to \infty$. Furthermore, the above expansion is with respect to the auxiliary asymptotic sequence $\left(\frac{|\mathcal{L}_0(t;s)|}{s^{n+1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{n+2/3}}\right)_{n\geq 0}$ and this completes the proof of theorem 5.12. $\square$

Proof of lemma 5.30: For simplicity we will write

$$J(t,r;s) = \int_{\gamma(t,r)} \exp\{-s \cdot P(t,x)\}\, B(t,x)\, dx,$$

where $\gamma(t,r) := \phi(t,[-r,r])$ and $P(t,x) := t^N \cdot U(t) \cdot x^2 - i \cdot V(t) \cdot x^3$.

Our first observation is that to obtain a uniform asymptotic expansion for $J(t, r; s)$ valid for all t sufficiently close to 0 we may replace the contour $\gamma(t, r)$ with $\gamma(0, r)$ by only incurring in an exponentially small error. To amplify on this observe that the end-points of $\gamma(t, r)$ converge to the end-points of $\gamma(0, r)$, as $t \to 0$. On the other hand, Cauchy's theorem (see [Rud87]) states that $J(t, r; s)$ is determined by the end-points of the contour of integration. Therefore, since $P(t, x)$ converges uniformly (on compact sets) to $P(0, x)$, as $t \to 0$, to show the claim it will be enough to prove that on some open neighborhood of $\partial\gamma(0, r)$, $P(t, x)$ has positive real part. This is almost a direct consequence of hypothesis (a) in the lemma. Indeed, observe that, for all $x \in \gamma(0, r)$, but $x = 0$, we have

$$\Re\{P(0, x)\} = \Re\{F(\Phi^{-1}(0, x)) - F(0, 0)\} > 0 \,. \tag{5.95}$$

The above inequality applies in particular for $x \in \partial\gamma(0, r)$ and from this the claim follows.

The discussion in the preceding paragraph implies that there is a constant $c_0 > 0$ such that

$$J(t, r; s) = \int_{\gamma(0,r)} \exp\{-s \cdot P(t, x)\} \; B(t, x) \, dx + O(e^{-s \cdot c_0}) \,, \tag{5.96}$$

uniformly for all t sufficiently small and all $s \geq 0$.

On the other hand, since $\frac{\partial P}{\partial x}(t, x)$ vanishes to degree 2 in the variable x about the origin, the Weierstrass division theorem 4.21 implies that there are functions $R_0(t)$, $R_1(t)$ and $\tilde{B}(t, x)$, analytic in t and x near $t = 0$ and $x = 0$, such that

$$B(t, x) = R_0(t) + R_1(t) \cdot x + \tilde{B}(t, x) \cdot \frac{\partial P}{\partial x}(t, x) \,. \tag{5.97}$$

By selecting $r > 0$ sufficiently small in (5.96) we may assume that (5.97) applies for all t sufficiently small and all x in an open disk containing $\gamma(0,r)$. This is useful to integrate by parts in (5.96) and leads us to the identity

$$\begin{aligned}J(t,r;s) &= \sum_{l=0}^{1} R_l(t)\cdot\int_{\gamma(0,r)} \exp\{-s\cdot P(t,x)\}\, x^l\,dx \\ &\qquad + \frac{1}{s}\cdot\int_{\gamma(0,r)} \exp\{-s\cdot P(t,x)\}\,\frac{\partial\tilde{B}}{\partial x}(t,x;1)\,dx \\ &\qquad - \frac{1}{s}\cdot\exp\{-s\cdot P(t,x)\}\,\tilde{B}(t,x)\Big|_{x\in\partial\gamma(0,r)} \\ &\qquad + O(e^{-s\cdot c_0})\,.\end{aligned}$$

The boundary term produced by the integration by parts above is also $O(e^{-s\cdot c_0})$ uniformly for all $s\geq 0$. Therefore, by defining

$$\mathcal{L}_l(t,r;s) := s^{(l+1)/3}\cdot\int_{\gamma(0,r)} \exp\{-s\cdot P(t,x)\}\, x^l\,dx\,, \tag{5.98}$$

$$\tilde{J}(t,r;s) := \int_{\gamma(0,r)} \exp\{-s\cdot P(t,x)\}\,\frac{\partial\tilde{B}}{\partial x}(t,x)\,dx\,, \tag{5.99}$$

we may rewrite the last expression for $J(t,r;s)$ in the form

$$J(t,r;s) = \sum_{l=0}^{1}\frac{R_l(t)}{s^{l+1/3}}\cdot\mathcal{L}_l(t,r;s) + \frac{1}{s}\cdot\tilde{J}(t,r;s) + O\left(e^{-s\cdot c_0}\right)\,, \tag{5.100}$$

uniformly for all t sufficiently small and all $s\geq 0$.

The proof of (5.89) will be established from (5.100) once a relation between the terms $\mathcal{L}_l(t,r;s)$ and the Airy function is shown to be as stated in (5.90) and (5.91). This will be done by identifying the location of the end-points of $\gamma(0,r)$. For this, we reuse (5.95) which lets us conclude that the $\Re\{-i\cdot V(0)\cdot x^3\} > 0$, for $x\in\partial\gamma(0,r)$. Since $V(0)>0$, this implies that

$$\partial\gamma(0,r)\subset\left\{x\in\mathbb{C} : \Re\{-i\cdot x^3\} > 0\right\}.$$

Let P and Q be respectively the starting and ending point of $\gamma(0,r)$. Since this contour is the conformal image of the interval $[-r,r]$ under a map of the form $x=\phi(0,z)=z+\ldots$ it follows that

$$P\in\Big\{x:\Re\{-i\cdot x^3\}>0,\Re\{x\}<0 \text{ and } \Im\{x\}>0\Big\}, \tag{5.101}$$

$$Q\in\Big\{x:\Re\{-i\cdot x^3\}>0,\Re\{x\}>0 \text{ and } \Im\{x\}>0\Big\}. \tag{5.102}$$

This let us replace the contour $\gamma(0,r)$ by any contour γ going through infinity (in the Riemann sphere) and eventually contained in a the set $\Big\{x:\Re\{-i\cdot x^3\}\geq c|x|^3 \text{ and } \Im\{x\}>0\Big\}$, for certain constant $c>0$. The error produced by this contour replacement is exponentially decreasing in s. We will define

$$\mathcal{L}_l(t;s):=s^{(l+1)/3}\cdot\int_\gamma \exp\{-s\cdot P(t,x)\}\,x^l\,dx\,. \tag{5.103}$$

Back in (5.100), we may thus replace the terms $\mathcal{L}_l(t,r;s)$ with $\mathcal{L}_l(t;s)$ and the identity will still hold possibly with a new constant in c_0 in the big-O; however, without loss of generality we will assume this constant remains unchanged. Moreover, with the selection made over γ it is now easy to relate the term $\mathcal{L}_l(t;s)$ to the Airy function. Indeed, if one substitutes: $x=\{3\,V(t)\cdot s\}^{-1/3}\cdot\xi$, in (5.103), then one obtains that

$$\mathcal{L}_l(t;s)=\frac{1}{\{3\,V(t)\}^{(l+1)/3}}\int_\gamma e^{-\lambda(t;s)\cdot\xi^2+i\xi^3/3}\,\xi^l\,d\xi\,, \tag{5.104}$$

where accordingly it has been defined

$$\lambda(t;s):=\frac{U(t)}{\{3\,V(t)\}^{2/3}}\cdot(st^{3P})^{1/3}\,. \tag{5.105}$$

The identities in (5.31) and (5.32) let us conclude that

$$\begin{aligned}\mathcal{L}_0(t;s) &= \frac{2\pi}{\{3\,V(t)\}^{1/3}} \cdot e^{2\lambda(t;s)^3/3} \cdot \mathbf{Ai}(\lambda(t;s)^2)\,,\\ \mathcal{L}_1(t;s) &= \frac{-2\pi\, i}{\{3\,V(t)\}^{2/3}} \cdot e^{2\lambda(t;s)^3/3} \cdot \Big\{\lambda(t;s) \cdot \mathbf{Ai}(\lambda(t;s)^2) + \mathbf{Ai}'(\lambda(t;s)^2)\Big\}\,.\end{aligned}$$

This shows the relations in (5.90), (5.91) and (5.92) completing the proof of (5.89).

To end the proof of the lemma it remains to show the inequality in (5.94) which is sought for t sufficiently small in a sector of the form

$$\mathcal{S}_\epsilon := \Big\{t : t = 0 \text{ or, } t \neq 0 \text{ and } |\arg\{t^N \cdot U(t)\}| < (\pi/2 - \epsilon)\Big\}\,.$$

First, a contour replacement will be performed to deal better with $\tilde{J}(t,r;s)$. Observe that the starting point of $\gamma(0,r)$ may be replaced by any point $\tilde{P}$ in the same component as P in (5.101), provided that $\tilde{P}$ is of a small size. Similarly, Q may be replaced by any point $\tilde{Q}$ of a sufficiently small size in set in (5.102). The requirement of both $\tilde{P}$ and $\tilde{Q}$ to be small in size is to ensure that the amplitude term of $\tilde{J}(t,r;s)$ is well defined on the contour $[\tilde{P},0]+[0,\tilde{Q}]$. Moreover, we can select these points so that $0 < \arg\{\tilde{P}^2\} < \epsilon/2$ and $0 < \arg\{\tilde{Q}^2\} < \epsilon/2$. In this way, we can ensure that

$$\max\Big\{\Big|\arg\{t^N \cdot U(t) \cdot \tilde{P}^2\}\Big|, \Big|\arg\{t^N \cdot U(t) \cdot \tilde{Q}^2\}\Big|\Big\} \leq \frac{\pi - \epsilon}{2}\,, \tag{5.106}$$

provided that $t \neq 0$ and $|\arg\{t^N \cdot U(t)\}| < (\pi/2 - \epsilon)$.

On the other hand, observe that the $|\arg\{-i\cdot\tilde{P}^3\}| < \pi/2$ and $|\arg\{-i\cdot\tilde{Q}^3\}| < \pi/2$ for the $\Re\{-i \cdot \tilde{P}^3\} > 0$ and $\Re\{-i \cdot \tilde{Q}^3\} > 0$. As a result, since the $V(0) > 0$, we can also ensure that the

$$\max\Big\{\Big|\arg\{-i \cdot V(t) \cdot \tilde{P}^3\}\Big|, \Big|\arg\{-i \cdot V(t) \cdot \tilde{Q}^3\}\Big|\Big\} \leq \frac{\pi - \epsilon}{2}\,, \tag{5.107}$$

provided that $t \neq 0$ is sufficiently small. Inequalities (5.106) and (5.107) together let us conclude that there is a constant $c_1 > 0$ such that the

$$\Re\{P(t,x)\} \geq c_1 \cdot \{|t|^N \cdot |x|^2 + |x|^3\}\,,$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and $x \in (\mathbb{R}_+ \cdot \tilde{Q}) \cup (\mathbb{R}_+ \cdot \tilde{P})$. In particular, if c_2 is a bound for the amplitude term of $\tilde{J}(t,r;s)$, valid for all sufficiently small t and all $|x| \leq \max\{|\tilde{P}|, |\tilde{Q}|\}$, then

$$|\tilde{J}(t,r;s)| \leq 2\,c_2 \cdot \max\{|\tilde{P}|, |\tilde{Q}|\} \cdot \int_0^\infty e^{-s\cdot c_1(|t|^N\cdot x^2 + x^3)}\,dx\,,$$

uniformly for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$. To show (5.94) is therefore enough to show that, for all $c > 0$, there is a constant $c_3 > 0$ such that

$$\left|\int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\,dx\right| \;\leq\; c_3 \cdot \left\{\frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}}\right\}, \tag{5.108}$$

$$e^{-s\cdot c} \;\leq\; c_3 \cdot \left\{\frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}}\right\}, \tag{5.109}$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ sufficiently big. This is certainly possible if $t = 0$, therefore, without loss of generality, we may assume that $t \neq 0$. Under this last assumption, if on the integral on the left-hand side in (5.108) we substitute: $x = |t|^N \cdot \xi$, then we obtain that

$$\int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\,dx = |t|^N \cdot \int_0^\infty e^{-(s\cdot|t|^{3N})\cdot(\xi^2 + \xi^3)}\,d\xi\,.$$

The Laplace method (see [BleHan86], section 5.1) let us conclude that the integral on the right-hand side above is of order $(s \cdot |t|^{3N})^{-1/2}$, if $s \cdot |t|^{3N}$ is sufficiently large. In particular, there is $\delta_1 > 0$ and $c_3 > 0$ such that

$$\left|\int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\,dx\right| \leq c_3 \cdot |t|^N \cdot (s \cdot |t|^{3N})^{-1/2}\,,$$

for all s and t such that $s \cdot |t|^{3N} \geq \delta_1$. On the other hand, corollary 5.18 can be used to conclude that $\mathcal{L}_0(t; s)$ is of order $(s \cdot |t|^{3N})^{-1/6}$, if $s \cdot |t|^{3N}$ is sufficiently big. In particular, there are $\delta_2 > 0$ and $c_4 > 0$ such that $c_4 \cdot (s \cdot |t|^{3N})^{-1/6} \leq |\mathcal{L}_0(t; s)|$, for all nonzero $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ such that $s \cdot |t|^{3N} \geq \delta_2$. Using this lower bound for $|\mathcal{L}_0(t; s)|$, it follows that

$$\begin{aligned} \left| \int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\, dx \right| &\leq \frac{c_3}{c_4} \cdot \frac{|t|^N \cdot (s \cdot |t|^{3N})^{-1/2}}{(s \cdot |t|^{3N})^{-1/6}} \cdot |\mathcal{L}_0(t; s)|\,, \\ &\leq \frac{c_3}{c_4} \cdot \left\{ \frac{|\mathcal{L}_0(t; s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t; s)|}{s^{2/3}} \right\}, \end{aligned}$$

for all nonzero $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ such that $s \cdot |t|^{3N} \geq \delta := \max\{\delta_1, \delta_2\}$. This proves (5.108) for the case $s \cdot |t|^{3N} \geq \delta$. Furthermore, the lower bound obtained for $|\mathcal{L}_0(t; s)|$ also implies that

$$e^{-s\cdot c} \leq \frac{|t|^{N/2} \cdot s^{1/2} e^{-s\cdot c}}{c_4} \cdot \frac{|\mathcal{L}_0(t; s)|}{s^{1/3}}\,,$$

for all nonzero $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ such that $s \cdot |t|^{3N} \geq \delta$. This shows (5.109) for the case $s \cdot |t|^{3N} \geq \delta$

To finalize the proof of the lemma we will show that (5.108) and (5.109) apply for $s \cdot |t|^{3N} \leq \delta$. To deal with this case, we substitute: $x = s^{-1/3} \cdot \xi$ in the integral on the left-hand side in (5.108), to obtain

$$\int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\, dx = s^{-1/3} \cdot \int_0^\infty e^{-(s\cdot|t|^{3N})^{1/3}\xi^2 - \xi^3}\, d\xi\,.$$

As a result, there is a constant $c_5 > 0$ such that

$$\left| \int_0^\infty e^{-s\cdot(|t|^N\cdot x^2 + x^3)}\, dx \right| \leq c_5 \cdot s^{-1/3}\,, \tag{5.110}$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ such that $s \cdot |t|^{3N} \leq \delta$. On the other hand, using (5.104), we see that there is a constant $c_6 > 0$ such that

$$|\mathcal{L}_l(t;s)| \geq c_6 \cdot \left| \int_\gamma e^{-\lambda(t;s)\cdot\xi^2 + i\xi^3/3}\, \xi^l \, d\xi \right| , \tag{5.111}$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$. But, observe that lemma 5.28 implies, for all fixed s and t, that the $\min_{l=0,1} \left| \int_\gamma e^{-\lambda(t;s)\cdot\xi^2+i\xi^3/3}\, \xi^l \, d\xi \right| > 0$. On the other hand, the condition $s \cdot |t|^{3N} \leq \delta$ is equivalent to request that $\lambda(t;s)$, as defined in (5.105), remains in a compact set of the complex plane. As a result, we can conclude that there is a constant $c_7 > 0$ such that the

$$\min_{l=0,1} \left| \int_\gamma e^{-\lambda(t;s)\cdot\xi^2 + i\xi^3/3}\, \xi^l \, d\xi \right| > c_7 \, ,$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s \geq 0$ such that $s \cdot |t|^{3N} \leq \delta$. Thus, using (5.111), we obtain that

$$\frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}} \geq c_6 \cdot c_7 \cdot \left\{ \frac{1}{s^{1/3}} + \frac{1}{s^{2/3}} \right\} , \tag{5.112}$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s > 0$ such that $s \cdot |t|^{3N} \leq \delta$. Back in (5.110), the above inequality implies that

$$\left| \int_0^\infty e^{-s\cdot(|t|^N \cdot x^2 + x^3)} \, dx \right| \leq \frac{c_5}{c_6 \cdot c_7} \cdot \left\{ \frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}} \right\} ,$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s > 0$ such that $s \cdot |t|^{3N} \leq \delta$. This proves (5.108) for the case $s \cdot |t|^{3N} \leq \delta$. Furthermore, the inequality in (5.112) also implies that

$$e^{-s\cdot c} \leq \frac{s^{1/3} e^{-s\cdot c}}{c_6 \cdot c_7} \cdot \left\{ \frac{|\mathcal{L}_0(t;s)|}{s^{1/3}} + \frac{|\mathcal{L}_1(t;s)|}{s^{2/3}} \right\} ,$$

for all $t \in \mathcal{S}_\epsilon$ sufficiently small and all $s > 0$ such that $s \cdot |t|^{3N} \leq \delta$. This proves (5.109) for the case $s \cdot |t|^{3N} \leq \delta$ and completes the proof of lemma 5.30. □

CHAPTER 6

ASYMPTOTICS FOR THE COEFFICIENTS OF BIVARIATE MEROMORPHIC FUNCTIONS

6.1 Introduction

Suppose that $G(z,w)$ and $H(z,w)$ are analytic functions of z and w in a polydisk containing a particular point (z_0, w_0). Furthermore, assume that $H(0,0) \neq 0$. Then, the meromorphic function

$$F(z,w) := \frac{G(z,w)}{H(z,w)}$$

is analytic in a neighborhood of the origin in $\mathbb{C}^2$; in particular, it has a power series representation of the form $\sum\limits_{r,s\geq 0} f_{r,s}\, z^r\, w^s$.

We will say that (z_0, w_0) *is a simple zero of* H provided that $H(z_0, w_0) = 0$, however, the complex gradient $\nabla H(z_0, w_0) \neq 0$. On the other hand, we will say that (z_0, w_0) *is a strictly minimal zero of* H provided that $z_0 \cdot w_0 \neq 0$ and (z_0, w_0) is the only zero of $H(z,w)$ in the polydisk $[|z| \leq |z_0|] \times [|w| \leq |w_0|]$.

The work of Pemantle and Wilson [PemWil01] implies that, associated to each strictly minimal simple zero of H, it is possible to determine an asymptotic expansion for the coefficients of F along certain direction in the (r,s)-lattice: $f_{r,s}$ admits an asymptotic expansion for $(r,s)\in \mathrm{dir}(z_0,w_0)$, as $(r,s)\to\infty$, where it is defined

$$\mathrm{dir}(z_0,w_0) := \Big\{(r,s)\in\mathbb{R}^2 : r\cdot w_0\,H_w(z_0,w_0) = s\cdot z_0\,H_z(z_0,w_0)\Big\}\,. \tag{6.1}$$

This set is a line in the (r,s)-lattice. Furthermore if, for example, $H_w(z_0,w_0)\neq 0$ then $(r,s)\in\mathrm{dir}(z_0,w_0)$ if and only if $r = d(z_0,w_0)\cdot s$, with $d(z_0,w_0) := \frac{z_0\cdot H_z(z_0,w_0)}{w_0\cdot H_w(z_0,w_0)}$. The strict minimality of (z_0,w_0) implies that $d(z_0,w_0)\geq 0$. (See lemma 2.1 in [PemWil01].)

In the remaining of our discussion we will assume that (z_0,w_0) is a strictly minimal simple zero of H such that $H_w(z_0,w_0)\neq 0$. Under these conditions, Pemantle and Wilson show that there are integers $n = n(z_0,w_0)\geq 2$ and $p = p(z_0,w_0)\geq 0$ and coefficients $c_j = c_j(z_0,w_0)$, with $j\geq p$ and $c_p\neq 0$, such that

$$f_{r,s} \approx \frac{z_0^{-r}\cdot w_0^{-s}}{2\pi}\sum_{j=p}^{\infty} c_j\cdot s^{-(j+1)/n}\,, \tag{6.2}$$

for all $(r,s)\in\mathrm{dir}(z_0,w_0)$, as $(r,s)\to\infty$. The quantities n and p are found to be respectively the degrees of vanishing, about $\theta = 0$, of certain analytic functions $f(\theta;z_0,w_0)$ and $a(\theta;z_0,w_0)$. Loosely speaking we will refer to these functions as the *associated phase* and *amplitude term* respectively. We will provide an explicit formulation for them in the section ahead, however, for the moment, we shall just say they are determined, in almost an explicit manner, taking into account the local behavior of $H(z,w)$ and $G(z,w)$ near the point (z_0,w_0).

The technique used to obtain (6.2) proceeds by relating the asymptotic behavior of the coefficients $f_{r,s}$ along the direction $\mathrm{dir}(z_0,w_0)$ to a Fourier-Laplace integral whose phase and amplitude term correspond to $f(\theta;z_0,w_0)$ and $a(\theta;z_0,w_0)$ respectively. The main steps in this process can be summarized as follows. To start, the coefficients of F are represented as an integral over over a 2-dimensional torus in $\mathbb{C}^2$. The strict minimality of (z_0, w_0) allows to expand the torus across the point (z_0, w_0) collecting a residual term. This lets to approximate $f_{r,s}$ by a 1-dimensional integral. The error incurred in the approximation can be shown to be negligible due to the fact that (z_0, w_0) is a simple zero. The asymptotic behavior of the resulting Fourier-Laplace integral is then obtained via an adapted version of the stationary phase method, namely theorem 5.2 in [PemWil01]. The asymptotic series in (6.2), up to the exponential factor $\frac{z_0^{-r} \cdot w_0^{-s}}{2\pi}$, corresponds to the asymptotic expansion of the integral

$$\int \exp\{-s \cdot f(\theta; z_0, w_0)\}\, a(\theta; z_0, w_0)\, d\theta\,,$$

as $s \to \infty$. The expansion in (6.2) is in powers of $s^{-1/n}$ because the phase term of the above integral vanishes to degree n about $\theta = 0$, which happens to be the dominant critical point of the integral. For the same reasons, since the amplitude term vanishes to degree p about $\theta = 0$, the leading order of this integral is $s^{-(p+1)/n}$.

It is therefore expected that the asymptotic expansion in (6.2) is uniform as (z_0, w_0) varies over a compact set of strictly minimal simple zeroes of H and the quantities n and p remain constant. Indeed, it is shown in [PemWil01] that if $\mathcal{K}$ is a compact set of points of this form and the associated functions $f(\theta; z, w)$ and $a(\theta; z, w)$ vanish to constant degree n and p respectively about $\theta = 0$, independently of $(z, w) \in \mathcal{K}$, then

$$f_{r,s} \approx \frac{z^{-r} \cdot w^{-s}}{2\pi} \sum_{j=p}^{\infty} c_j(z, w) \cdot s^{-(j+1)/n}\,, \tag{6.3}$$

uniformly for all $(r, s) \in \text{dir}(z, w)$ and $(z, w) \in \mathcal{K}$, as $(r, s) \to \infty$. In particular, since $c_p(z, w)$ can be shown to depend continuously on (z, w), the compactness of $\mathcal{K}$

implies that

$$f_{r,s} \sim c_p(z,w) \cdot \frac{z^{-r} \cdot w^{-s}}{2\pi} \cdot s^{-(p+1)/n}\,, \tag{6.4}$$

uniformly for all $(r,s) \in \text{dir}(z,w)$ and $(z,w) \in \mathcal{K}$, as $(r,s) \to \infty$.

The asymptotic expansions in (6.3) and (6.4) fully describe the asymptotic behavior of the coefficients of F for (r,s) in the cone

$$\Lambda := \left\{(r,s) : \text{ there exists } (z,w) \in \mathcal{K} \text{ such that } \frac{r}{s} = d(z,w)\right\}.$$

However, the restriction to have the quantities n and p to remain constant, as (z,w) varies over $\mathcal{K}$, is primarily technical. Indeed, there are examples of interest where these restrictions are violated.

Two pathological cases are of interest. One is when, at a particular point in $\mathcal{K}$, say (z_0,w_0), the amplitude term vanishes to degree q yet, for all $(z,w) \in \mathcal{K}$ nearby, the associated amplitude vanishes to some degree $p < q$. With these premises, (6.2) implies that

$$f_{r,s} \text{ is of order } \begin{cases} s^{-(q+1)/n} & , \quad \text{if } \frac{r}{s} = d(z_0,w_0) \\ s^{-(p+1)/n} & , \quad \text{if } \frac{r}{s} \neq d(z_0,w_0) \end{cases}$$

as $(r,s) \in \Lambda \to \infty$, provided that $\frac{r}{s}$ remains constant. A problem of interest is to determine the order of $f_{r,s}$ for big values of r and s, as $\frac{r}{s} \to d(z_0,w_0)$. This is required to have a full asymptotic description of the coefficients of F along the cone Λ.

A worse scenario is when both the associated amplitude and phase term do not vanish to constant degree about $\theta = 0$ as (z,w) varies over $\mathcal{K}$. In this situation, the easiest case of study is when for all $(z,w) \in \mathcal{K}$ nearby (z_0,w_0), the associated phase term vanishes to degree 2, however, at (z_0,w_0), this degree is instead 3. Under these assumptions, the work of Pemantle and Wilson implies that

$$(6.5) \qquad f_{r,s} \text{ is of order } \begin{cases} s^{-(q+1)/3} & , \quad \text{if } \frac{r}{s} = d(z_0, w_0) \\ s^{-(p+1)/2} & , \quad \text{if } \frac{r}{s} \neq d(z_0, w_0) \end{cases}$$

as $(r, s) \in \Lambda \to \infty$, provided that $\frac{r}{s}$ remains constant. In this regard, to complete the asymptotic description of the coefficients of $F(z, w)$ in the cone Λ, we are required to answer questions of the sort:

> Is it possible to provide an asymptotic expansion for the coefficients $f_{r,s}$ uniformly valid for all $(r, s) \in \Lambda$, as $(r, s) \to \infty$?
>
> What is the fastest rate at which $\frac{r}{s}$ may approach $d(z_0, w_0)$, with $(r, s) \in \Lambda$, so that $f_{r,s}$ remains of order $s^{-(p+1)/2}$?

Questions like above could not be treated in [PemWil01] due to the limitation of the stationary phase method to handle the asymptotic behavior of parameter varying Fourier-Laplace integrals where either the phase or amplitude term does not vanish to constant degree at the critical points of the integral. However, the methods we developed in chapter 5 were designed precisely to handle questions like these in as much generality as possible.

We will provide a fairly general and affirmative answer to the first question. The second question cannot be answered in its generality. Indeed, the uniform expansions we will provide show that the answer to the second question is too sensitive of the local behavior of $G(z, w)$ and $H(z, w)$ near the point (z_0, w_0). Yet, in the examples we will see, it will be relatively simple to determine the bandwidths on the cone Λ along which $f_{r,s}$ reflects an asymptotic behavior as if $\frac{r}{s} = d(z_0, w_0)$ versus $\frac{r}{s}$ was bounded away from $d(z_0, w_0)$.

6.2 Statement of results, with examples

All over this section it will be assumed that $F(z,w) = G(z,w)/H(z,w)$, with $G(z,w)$ and $H(z,w)$ analytic in a polydisk $\mathcal{D}$ centered at $(0,0)$ and $H(0,0) \neq 0$.

The coefficient of $z^r\, w^s$ in the power series expansion of $F(z,w)$ about the origin will be denoted as $f_{r,s}$.

We will assume as given a compact set $\mathcal{K} \subset \mathcal{D}$ of strictly minimal simple zeroes of H containing a particular point (z_0, w_0) such that $H_w(z_0, w_0) \neq 0$.

The above condition, on the partial derivative of H at (z_0, w_0), lets us use the implicit mapping theorem 4.22 to parametrize the zero set of H near (z_0, w_0) in the form $w = g(z)$, where $g(z)$ is certain analytic function of z near $z = z_0$. In particular, for all θ sufficiently small and for all z sufficiently close to z_0, we can define the functions

$$\text{(6.6)} \qquad a(z,\theta) \;:=\; \frac{-G(z\cdot e^{i\theta}, g(z\cdot e^{i\theta}))}{g(z\cdot e^{i\theta})\cdot H_w(z\cdot e^{i\theta}, g(z\cdot e^{i\theta}))}\,,$$

$$\text{(6.7)} \qquad f(z,\theta) \;:=\; \ln\left\{\frac{g(z\cdot e^{i\theta})}{g(z)}\right\} - i\cdot\theta\cdot\frac{z\cdot g'(z)}{g(z)}\,.$$

Theorem 6.1. *Let $F(z,w) = G(z,w)/H(z,w)$, (z_0,w_0), $\mathcal{K}$, etc. be as defined before. Suppose that there are nonnegative integers $p \leq q$ such that $a(z,\theta)$ has a p-to-q change of degree about $\theta = 0$ as $z \to z_0$, however, $f(z,\theta)$ vanishes to constant degree n about $\theta = 0$ for all z nearby z_0. Then there is a constant $c > 0$ and functions $A_k(z)$ and $B_k(z;s)$, with $p \leq k \leq q$ and $s \geq 0$, analytic in z near $z = z_0$ such that $A_k(z_0) = 0$, for all $p \leq k \leq (q-1)$, $A_q(z_0) \neq 0$, and*

$$\text{(6.8)} \qquad f_{r,s} = \frac{z^{-r} w^{-s}}{2\pi}\cdot\left\{\sum_{k=p}^{q} A_k(z)\cdot B_k(z;s) + O(e^{-s\cdot c})\right\}\,,$$

uniformly for all $(r,s) \in dir(z,w)$ *and* $(z,w) \in \mathcal{K}$ *sufficiently close to* (z_0, w_0)*. Furthermore, each coefficient* $B_k(z;s)$ *above admits an asymptotic expansion in powers of* $s^{-1/n}$ *depending one certain coefficients of the form* $c_k(z;j)$*, with* $j \geq k$*, which are analytic in* z *near* $z = z_0$ *and such that*

$$c_k(z;k) = \left\{[\theta^n]\, f(z,\theta)\right\}^{-(k+1)/n}.$$

More precisely,

$$B_k(z;s) \approx \sum_{j=k}^{\infty} c_k(z;j) \cdot \left\{1 + (-1)^j \cdot D(j,n)\right\} \cdot \frac{1}{n}\,\Gamma\left(\frac{j+1}{n}\right) \cdot s^{-(j+1)/n}, \tag{6.9}$$

uniformly for all $(z,w) \in \mathcal{K}$ *sufficiently close to* (z_0,w_0)*, as* $s \to \infty$*, where we have defined*

$$D(j,n) := \begin{cases} 1 & , \quad n \text{ even} \quad , \\ \exp\left(-\frac{i\pi(j+1)}{n} \cdot sgn\left\{i \cdot [\theta^n] f(z_0,\theta)\right\}\right) & , \quad n \text{ odd} \quad . \end{cases}$$

Remark 6.2. The coefficients $A_k(z)$ in (6.8) together with an auxiliary function $y = y(z,\theta)$ are the unique analytic solutions (near $\theta = 0$ and $z = z_0$) to the following system

$$\begin{cases} \int_0^\theta a(z,\xi)\, d\xi = \sum_{k=p}^{q} \frac{A_k(z)}{k+1}\, y^{k+1}, \\ A_k(z_0) = 0, \text{ for } p \leq k \leq (q-1),\ A_q(z_0) \neq 0, \\ y = y(z,\theta) = \theta + \dots \end{cases} \tag{6.10}$$

The above relations imply that

$$A_p(z) = [\theta^p]\, a(z,\theta), \tag{6.11}$$

$$A_q(z_0) = [\theta^q]\, a(z_0,\theta). \tag{6.12}$$

Furthermore, the uniqueness of the above system implies that $A_k(z) = [\theta^k]\, a(z,\theta)$ and $y(z,\theta) = \theta$ whenever $a(z,\theta)$ is a polynomial in θ with analytic functions of z as coefficients.

Remark 6.3. Write $f(z,\theta) = u(z) \cdot \theta^n + \dots$ where $u(z)$ is analytic in z near $z = z_0$. The coefficients $c_k(z;j)$ are characterized by the identity $c_k(z;j) = [x^j]\, y^k \frac{\partial y}{\partial x}$ where, for all z sufficiently close to $z = z_0$, the variables y and x are related to each other through the original variable θ according to the relations

$$\left\{ \begin{array}{rcl} y &=& y(z,\theta)\,, \\ x &=& y \cdot \{u(z)\}^{1/n} \cdot \left\{1 + \frac{f(z,\theta) - u(z)\cdot y^n}{u(z)\cdot y^n}\right\}^{1/n} . \end{array} \right. \tag{6.13}$$

Remark 6.4. The asymptotic notation used in (6.9) is in the standard sense. It means that the difference between $B_k(z;s)$ and the partial sum up to the term $j = l$ is $O(s^{-(l+2)/n})$, uniformly for all $(z,w) \in \mathcal{K}$ sufficiently close to (z_0,w_0), as $s \to \infty$.

Example 6.5. (Lattice paths.)

Consider the generating function

$$F(z,w) := \frac{z - w}{1 - z - w - zw}\,.$$

It is related to the *Delannoy numbers* $(d_{r,s})_{r,s\geq 0}$ (see [Sta99], pp. 185) whose generating function is precisely $\sum_{r,s\geq 0} d_{r,s}\, z^r\, w^s = \frac{1}{1-z-w-zw}$. The coefficient $d_{r,s}$ counts the number of paths in the lattice $\mathbb{Z} \times \mathbb{Z}$ going from $(0,0)$ to the position (r,s) but moving only North, East and Northeast.

If $f_{r,s}$ denotes the coefficient of $z^r\, w^s$ of $F(z,w)$ then a simple calculation reveals that: $f_{r,s} = d_{r-1,s} - d_{r,s-1}$, for all $r, s \geq 1$. Thus, $f_{r,s}$ counts the difference between the number of paths from $(0,0)$ to (r,s) which arrived to this last point from the West and those that arrived from the South. The symmetry implies that $f_{r,s} = -f_{s,r}$; in particular,

$$f_{r,s} = 0 \text{ whenever } r = s\,. \tag{6.14}$$

We are interested in the asymptotic behavior of the coefficients $f_{r,s}$ near the diagonal direction $r = s$, as $(r,s) \to \infty$. For this purpose, we observe that points of the form (z,w), with $z \in (0,1)$ and $w = g(z) := \frac{1-z}{1+z}$, are strictly minimal simple poles of F. Using the definitions in (6.6) and (6.7) we obtain that

$$\begin{aligned} a(z,\theta) &= \frac{1 - 2ze^{i\theta} - z^2 e^{2i\theta}}{z^2 e^{2i\theta} - 1} \\ &= \frac{1-2z-z^2}{z^2-1} + \frac{2iz(1+z^2)}{(z^2-1)^2}\,\theta + \ldots \\ f(z,\theta) &= \ln\left\{ \frac{(1-ze^{i\theta})\cdot(1+z)}{(1+ze^{i\theta})\cdot(1-z)} \right\} - \frac{2iz}{z^2-1}\,\theta \\ &= \frac{z(1+z^2)}{(z^2-1)^2}\,\theta^2 - \frac{iz(1+6z^2+z^4)}{(z^2-1)^3}\,\theta^3 + \ldots \end{aligned}$$

Thus, $a(z,\theta)$ vanishes to degree 0 at $\theta = 0$ at all strictly minimal simple poles (z,w) but $(z,w) = (\sqrt{2}-1, \sqrt{2}-1)$ where this degree is instead 1. Expectedly, $\mathrm{dir}(\sqrt{2}-1, \sqrt{2}-1) = (1,1)$. By contrast, $f(z,\theta)$ vanishes to constant degree 2 about $\theta = 0$.

More generally, we have that

$$\begin{aligned} \mathrm{dir}(z,w) &= \left\{(r,s) \in \mathbb{R}^2 : r = s \cdot d(z)\right\}\,, \\ d(z) &:= \frac{2z}{1-z^2} \end{aligned}$$

Hence, as z varies over the interval $(0,1)$, $\mathrm{dir}(z,w)$, with $w = g(z)$, covers all possible directions in $\mathbb{RP}^1$. Indeed, as remarked in example 3.2 in [PemWil01], the minimal point that solves $(r,s) \in \operatorname{dir}(z,w)$ is given by $z = \frac{\sqrt{r^2+s^2}-s}{r}$ and $w = \frac{\sqrt{r^2+s^2}-r}{s}$. Using theorems 3.1 and 3.3 in [PemWil01] we obtain respectively that

$$(6.15) \qquad a_{r,s} \sim \left(\frac{\sqrt{r^2+s^2}-s}{r}\right)^{-r} \cdot \left(\frac{\sqrt{r^2+s^2}-r}{s}\right)^{-s} \cdot \frac{r-s}{\sqrt{2\pi\, r\, s \cdot \sqrt{r^2+s^2}}}$$

uniformly as $(r,s) \to \infty$ with $\frac{r}{s} \neq 1$ and $\frac{s}{r} \neq 1$ bounded.

Equivalently, if $d = d(r,s) := \frac{r}{s}$ then we may rewrite the above expression in the form

$$a_{r,s} \sim \left(\frac{\sqrt{d^2+1}-1}{d}\right)^{-r} \cdot \left(\sqrt{d^2+1}-d\right)^{-s} \cdot \left\{\frac{(d-1)}{\sqrt{2\pi\, d\sqrt{d^2+1}}} \cdot s^{-1/2} + O\big(s^{-3/2}\big)\right\},$$

uniformly as $(r,s) \to \infty$ with $0 < d \neq 1$ bounded. The $O\big(s^{-3/2}\big)$ term is not a $o\left(\frac{(d-1)}{\sqrt{2\pi\, d\sqrt{d^2+1}}} \cdot s^{-1/2}\right)$ if d is allowed to depend on s in such a way that $d \to 1$. As a result, the leading order of $f_{r,s}$ is not necessarily $\frac{(d-1)}{\sqrt{2\pi\, d\sqrt{d^2+1}}} \cdot s^{-1/2}$, as $\frac{r}{s} \to 1$ and $(r,s) \to \infty$.

For each (r,s), with $\frac{r}{s}$ sufficiently close to 1, we will let $\zeta = \zeta(r,s) := \frac{\sqrt{r^2+s^2}-s}{r}$. Observe that, ζ is the only solution of the equation: $d(\zeta) = \frac{r}{s}$, $\zeta \in (0,1)$. Theorem 6.1 implies that there is a constant $c > 0$ such that

$$f_{r,s} \; := \; \frac{\zeta^r \cdot [g(\zeta)]^s}{2\pi} \cdot \big\{A_0(\zeta) \cdot B_0(\zeta;s) + A_1(\zeta) \cdot B_1(\zeta;s) + O(e^{-s \cdot c})\big\},$$

uniformly for all (r,s) such that $\frac{r}{s}$ is sufficiently close to 1, where one can determine that

$$\begin{aligned}
A_0(z) &:= \frac{1-2z-z^2}{z^2-1}, \\
B_0(z;s) &\sim \sqrt{\pi}\left\{\frac{z(1+z^2)}{(z^2-1)^2}\right\}^{-1/2} \cdot s^{-1/2}, \\
A_1(z) &\sim i\sqrt{2}, \\
B_1(z;s) &= O(s^{-5/2}),
\end{aligned}$$

There are strong reasons to believe that $B_1(z;s)$ is rapidly decreasing in s (uniformly for z sufficiently close to $\sqrt{2}-1$), however, for now this remains as a conjecture. $\square$

Theorem 6.6. *Let $F(z,w) = G(z,w)/H(z,w)$, (z_0,w_0), $\mathcal{K}$, etc. be as defined before. Suppose that there are nonnegative integers $p \le q$ such that $a(z,\theta)$ has a p-to-q change of degree about $\theta = 0$ as $z \to z_0$, however, $f(z,\theta)$ has a 2-to-3 change of degree about $\theta = 0$ as $z \to z_0$. Write*

$$f(z,\theta) = (z-z_0)^N \cdot u(z) \cdot \theta^2 - i \cdot v(z) \cdot \theta^3 + \dots$$

where $N \ge 1$ is a nonnegative integer and $u(z)$ and $v(z)$ are analytic in z near $z = z_0$ and such that $u(z_0) \ne 0$ and $v(z_0) > 0$. If there is $\epsilon > 0$ such that

$$\mathcal{K} \subset \left\{ (z,w) : z = z_0 \text{ or , } z \ne z_0 \text{ and } \arg\{(z-z_0)^N \cdot u(z)\} < \tfrac{\pi-\epsilon}{2} \right\}$$

then there is a constant $c > 0$ and functions $A_k(z)$ and $B_k(z;s)$, with $p \le k \le q$, analytic in z near $z = z_0$ such that $A_k(z_0) = 0$, for all $p \le k \le (q-1)$, $A_q(z_0) \ne 0$, and

$$f_{r,s} = \frac{z^{-r} w^{-s}}{2\pi} \cdot \left\{ \sum_{k=p}^{q} A_k(z) \cdot B_k(z;s) + O(e^{-s\cdot c}) \right\},$$

uniformly for all $(r,s) \in dir(z,w)$ and $(z,w) \in \mathcal{K}$ sufficiently close to (z_0,w_0). Furthermore, each coefficient $B_k(z;s)$ admits an asymptotic expansion depending on certain coefficients $R_k(z;l)$, with $l \ge 0$, analytic in z near $z = z_0$, of the form

$$B_k(z;s) \approx \left\{ \sum_{l=0}^{\infty} \frac{R_k(z;2l)}{s^{l+1/3}} \right\} \cdot \mathcal{L}_0(z;s) + \left\{ \sum_{l=0}^{\infty} \frac{R_k(z;2l+1)}{s^{l+2/3}} \right\} \cdot \mathcal{L}_1(z;s), \tag{6.16}$$

valid for all $(z,w) \in \mathcal{K}$ sufficiently close to (z_0,w_0), as $s \to \infty$. Above it has been defined

$$\mathcal{L}_0(z;s) := \frac{2\pi}{\{3\,V(z)\}^{1/3}} \cdot e^{2\lambda(z;s)^3/3} \cdot \mathbf{Ai}(\lambda(z;s)^2), \tag{6.17}$$

$$(6.18)\quad \mathcal{L}_1(z;s) \;:=\; \frac{-2\pi i}{\{3\,V(z)\}^{2/3}}\cdot e^{2\lambda(z;s)^3/3}\cdot\left\{\lambda(z;s)\cdot \mathbf{Ai}(\lambda(z;s)^2)+\mathbf{Ai}'(\lambda(z;s)^2)\right\},$$

$$(6.19)\quad \lambda(z;s) \;:=\; \frac{(z-z_0)^N\cdot u(z)}{\{3\,V(z)\}^{2/3}}\cdot s^{1/3},$$

where $V(z)$ is certain analytic function of z near $z = z_0$ and such that $V(z_0) = v(z_0)$.

Remark 6.7. The coefficients $A_k(z)$ together with an auxiliary function $y = y(z,\theta)$ are uniquely characterized by the relations in (6.10). We will define the 1-to-1 transformation

$$(z,y) = \Psi(z,\theta) := (z, y(z,\theta))\,.$$

The function $V(z)$ in (6.17)–(6.19) together with an auxiliary function $x = x(z,y)$ are uniquely characterized by the relations

$$(6.20)\qquad \begin{cases} F(\Psi^{-1}(z,y)) = (z-z_0)^N\cdot u(z)\cdot x^2 - i\cdot V(z)\cdot x^3\,, \\ x = x(z,y) = y + \dots \end{cases}$$

Remark 6.8. The coefficients $R_k(z;l)$ in (6.16) together with certain auxiliary functions $C_k(z,x;l)$, with $l \geq 0$, analytic in z and x near $z = z_0$ and $x = 0$, can be defined recursively by means of the Weierstrass division theorem 4.21 as follows

$$(6.21)\qquad \begin{cases} C_k(z,x;0) = y^k\cdot \frac{\partial y}{\partial x}(z,x)\,, \\ C_k(z,x;l) = R_k(z;2l) + R_k(z;2l+1)\cdot x + C_k(z,x;l+1) \\ \qquad\qquad\qquad \cdot\frac{\partial}{\partial x}\left\{(z-z_0)^N\cdot u(z)\cdot x^2 - i\cdot V(z)\cdot x^3\right\}. \end{cases}$$

In particular, if we let $x(z) := -2\,i\,(z-z_0)^N\,u(z)/\{3\,V(z)\}$ then

$$\begin{aligned} R_k(z;2l) &= C_k(z,0;l)\,, \\ R_k(z;2l+1) &= \frac{C_k(z,x(z);l) - C_k(t,0;l)}{x(z)}\,. \end{aligned}$$

Remark 6.9. The asymptotic notation used in (6.16) means that for all $n \geq 0$ there is a constant $c_1 > 0$ such that

$$\left| B_k(z;s) - \left\{ \sum_{l=0}^{n} \frac{R_k(z;2l)}{s^{l+1/3}} \right\} \cdot \mathcal{L}_0(z;s) - \left\{ \sum_{l=0}^{n} \frac{R_k(z;2l+1)}{s^{l+2/3}} \right\} \cdot \mathcal{L}_1(z;s) \right| \leq \frac{c_1}{s} \left\{ \frac{|\mathcal{L}_0(z;s)|}{s^{n+1/3}} + \frac{|\mathcal{L}_1(z;s)|}{s^{n+2/3}} \right\},$$

uniformly for all $(z, w) \in \mathcal{K}$ sufficiently close to (z_0, w_0) and all $s \geq 0$ sufficiently large.

Remark 6.10. The special functions $\mathcal{L}_0(z;s)$ and $\mathcal{L}_1(z;s)$ have two distinguishable asymptotic regimes according the size of $\lambda = \lambda(z;s)$, which is of the same order as $|z - z_0|^N \cdot s^{1/3}$. Their leading orders in each regime are as follows.

There is $\alpha > 0$ such that

$$\text{(6.22)} \quad \mathcal{L}_0(z;s) = \sqrt{\frac{\pi}{u(z)}} \cdot \{(z-z_0)^{3N} \cdot s\}^{-1/6} \cdot \left(1 + O(\lambda^{-3})\right),$$

$$\text{(6.23)} \quad \mathcal{L}_1(z;s) = \frac{i\sqrt{\pi} \cdot \{3\,V(z)\}^{11/6}}{3\{u(z)\}^{5/2}} \cdot \{(z-z_0)^{3N} \cdot s\}^{-5/6} \cdot \left(1 + O(\lambda^{-3})\right),$$

uniformly for all $(z, w) \in \mathcal{K}$ sufficiently close to (z_0, w_0) and $s \geq 0$ such that $|\lambda| \geq \alpha$.

On the contrary, for all $\beta > 0$,

$$\text{(6.24)} \quad \mathcal{L}_0(z;s) = \frac{1}{\{3\sqrt{3}\,V(z)\}^{1/3}} \cdot \Gamma\left(\frac{1}{3}\right) \cdot \left(1 + O(\lambda)\right),$$

$$\text{(6.25)} \quad \mathcal{L}_1(z;s) = \frac{i}{\sqrt{3}\{V(z)\}^{2/3}} \cdot \Gamma\left(\frac{2}{3}\right) \cdot \left(1 + O(\lambda)\right),$$

uniformly for all $(z, w) \in \mathcal{K}$ sufficiently close to (z_0, w_0) and $s \geq 0$ such that $|\lambda| \leq \beta$.

Example 6.11. (Cube root asymptotics.)

Consider the rational generating function

$$F(z,w) := \frac{1}{z^2 - 3z - w + 3},$$

and let $f_{r,s}$ be the coefficient of $z^r\,w^s$ in its Taylor expansion about $(z,w)=(0,0)$. These coefficients were partially studied by Pemantle and Wilson (see example 3.4 in [PemWil01]). For $t\in(0,1]$, they state that points of the form $(t,g(t))$, with $g(t):=t^2-3t+3$, are strictly minimal simple poles of F. Moreover, they compute that

$$\begin{aligned} \mathrm{dir}(t,g(t)) &= \{(r,s): r=s\cdot d(t)\}\,, \\ d(t) &:= \frac{t(3-2t)}{t^2-3t+3}\,. \end{aligned}$$

Observe that $d(t)$ is a strictly increasing function of $t\in[0,1]$, with $d(0)=0$ and $d(1)=1$. This implies that $\mathrm{dir}(t,g(t))$ covers all possible directions contained within the cone $\big\{(r,s)\,:\,0<r\le s\big\}$ in the (r,s)-lattice as t varies from $t=0$ to $t=1$. Further, for each (r,s) in this cone, the equation: $d(t)=\frac{r}{s}$, with unknown $t\in(0,1]$, has only one solution which will be denoted $\tau=\tau(r,s)$.

Using theorem 3.3 in [PemWil01] it follows that

$$f_{r,s}\sim\frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{\sqrt{6\pi\tau(3-\tau)}}\cdot[s\cdot(1-\tau)]^{-1/2}\,, \tag{6.26}$$

as $(r,s)\to\infty$ provided that $\frac{r}{s}$ remains in a compact subset of $(0,1)$. However,

$$f_{s,s}\sim\frac{1}{\pi\sqrt{12}}\cdot\Gamma\left(\frac{1}{3}\right)\cdot s^{-1/3}\,. \tag{6.27}$$

Pemantle and Wilson leave as an open problem to complete the asymptotic description of $f_{r,s}$ as $(r,s)\to\infty$ in such a way that $\frac{r}{s}\uparrow 1$. Intuitively, if $\frac{r}{s}\uparrow 1$ at a slow rate then (6.26) should remain valid. However, if $\frac{r}{s}\uparrow 1$ at a sufficiently fast rate then (6.27) should be the correct asymptotic description.

The following result determines the bandwidth to discriminate between the behavior of $f_{r,s}$ as prescribed in (6.26) from the one in (6.27).

Corollary 6.12. *For each (r,s), with $0<r\le s$, define $\Theta=\Theta(r,s):=\left(1-\frac{r}{s}\right)^{1/2}$ and let $\tau=\tau(r,s)$ be the only solution of the equation: $\frac{r}{s}=d(\tau)$, $\tau\in[0,1]$. The asymptotic behavior of $f_{r,s}$ nearby the line $r=s$ in cone $\{(r,s):0\le r\le s\}$ is determined by the quantity $\Delta=\Delta(r,s):=s\cdot\left(1-\frac{r}{s}\right)^{3/2}$ as follows.*

There is $\alpha>0$ such that

$$f_{r,s}=\frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{\sqrt{12\pi}}\cdot[s\cdot(1-\tau)]^{-1/2}\cdot\Big(1+O\big(\max\{\Delta^{-1},\Theta,s^{-1/3}\}\big)\Big)\;, \tag{6.28}$$

uniformly for all (r,s) such that Θ is sufficiently small and $\Delta\ge\alpha$, as $(r,s)\to\infty$.

On the contrary, for all $\beta\ge 0$,

$$f_{r,s}=3^{-1/3}\cdot\mathbf{Ai}\left(3^{-1/3}\cdot\frac{s-r}{s^{1/3}}\right)\cdot s^{-1/3}\cdot\Big(1+O\big(\max\{\Theta,s^{-1/3}\}\big)\Big)\;, \tag{6.29}$$

uniformly for all (r,s) such that Θ is sufficiently small and $\Delta\le\beta$, as $(r,s)\to\infty$.

Remark 6.13. The corollary implies that for a given sequence (r,s) such that $\frac{r}{s}\uparrow 1$, as $(r,s)\to\infty$, the asymptotic behavior of $f_{r,s}$ is described by the limit $l:=\lim\limits_{(r,s)\to\infty}\frac{s-r}{s^{1/3}}$. If $\frac{r}{s}\uparrow 1$ at a rate such that $l=\infty$ then the asymptotic description of $f_{r,s}$ in (6.28) is equivalent to the one provided by Pemantle and Wilson in (6.26). However, if $\frac{r}{s}\uparrow 1$ at a relatively fast rate so that $l=0$ then (6.29) implies that $f_{r,s}\sim\frac{1}{\pi\sqrt{12}}\cdot\Gamma\left(\frac{1}{3}\right)\cdot s^{-1/3}$, which corresponds to the asymptotic description in (6.27).

Observe that the case $0<l<\infty$ is equivalent to the existence of a constant $0<C<\infty$ such that: $\frac{r}{s}=1-C\cdot s^{-2/3}+o(s^{-2/3})$. Therefore, a bandwidth of size $s^{-2/3}$ in the (r,s)-lattice is what separates the behavior of $f_{r,s}$ as prescribed in (6.26) from the one in (6.27).

Remark 6.14. The expansion in (6.29) can be interpreted as a local limit theorem where the standard normal distribution has been replaced by the probability measure

$3\,\mathbf{Ai}(x)\,dx$ on $x \geq 0$, and the more traditional scale of $s^{1/2}$ is replaced by the new scale $s^{1/3}$.

Proof of corollary 6.12: The terms in (6.6) and (6.7) are easily found to be

$$\begin{aligned} a(t,\theta) &= \frac{1}{t^2-3t+3} + \ldots \\ f(t,\theta) &= (1-t)\cdot u(t)\cdot\theta^2 - i\cdot v(t)\cdot\theta^3 + \ldots \end{aligned}$$

with $u(t) := \frac{3t(3-t)}{2(t^2-3t+3)^2}$ and $v(t) = \frac{t(t^2-5t+3)(t^2-3)}{2(t^2-3t+3)^3}$; in particular, $u(1) = 3$ and $v(1) = 1 > 0$. Thus $a(t,\theta)$ and $f(t,\theta)$ have a respectively a 0-to-0 and a 2-to-3 change of degree about $\theta = 0$ as $t \to 1$. Furthermore, the $\arg\{(1-t)\cdot u(t)\} = 0$, for all $t \in (0,1)$. As a result, theorem 6.6 implies that, for all $\epsilon \in (0,1)$ sufficiently small, there is $c > 0$ such that

$$f_{r,s} = \frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{2\pi}\cdot\left\{A_0(\tau)\cdot B_0(\tau;s) + O(e^{-s\cdot c})\right\}, \tag{6.30}$$

uniformly for all (r,s) such that $(1-\epsilon) \leq \tau \leq 1$. The terms above are found to satisfy the conditions

$$\begin{aligned}
&(6.31) & A_0(1) &= 1\,, \\
&(6.32) & B_0(t;s) &= \frac{\mathcal{L}_0(t;s)}{s^{1/3}} + O\left(\frac{|\mathcal{L}_1(t;s)|}{s^{2/3}}\right) + O\left(\frac{|\mathcal{L}_0(t;s)|}{s^{1+1/3}}\right), \\
&(6.33) & \mathcal{L}_0(t;s) &= \frac{2\pi}{\{3V(t)\}^{1/3}}\cdot e^{2\lambda(t;s)^3/3}\cdot\mathbf{Ai}(\lambda(t;s)^2) \\
&(6.34) & \mathcal{L}_1(t;s) &= \frac{-2\pi i}{\{3V(t)\}^{2/3}}\cdot e^{2\lambda(t;s)^3/3}\cdot\left\{\lambda(t;s)\cdot\mathbf{Ai}(\lambda(t;s)^2) + \mathbf{Ai}'(\lambda(t;s)^2)\right\}, \\
&(6.35) & V(1) &= 1\,, \\
&(6.36) & \lambda(t;s) &= \frac{3t(3-t)}{2(t^2-3t+3)^2\cdot\{3V(t)\}^{2/3}}\cdot(1-t)\cdot s^{1/3}\,,
\end{aligned}$$

uniformly for all t such that $(1-\epsilon) \leq t \leq 1$, as $s \to \infty$. Without loss of generality we may chose $\epsilon > 0$ sufficiently small so that all terms above are analytic functions of t

for $|1-t| \le \epsilon$. Furthermore, by possibly reducing the size of $\epsilon > 0$, we may assume also that $|\lambda(t;s)|$ is of the same order as $|1-t| \cdot s^{1/3}$ and that the $\Re\{\lambda(t;s)\} \ge 0$, for all $(1-\epsilon) \le t \le 1$ and all $s \ge 0$.

We claim that, possibly by reducing further the size of $\epsilon > 0$, there is a constant $c' > 0$ such that

$$
\begin{aligned}
(6.37) \qquad B_0(t;s) &= \frac{\mathcal{L}_0(t;s)}{s^{1/3}} \cdot \Big(1 + O(s^{-1/3})\Big)\,, \\
(6.38) \qquad e^{-s\cdot c} &= B_0(t;s) \cdot O(e^{-s\cdot c'})\,,
\end{aligned}
$$

uniformly for all $(1-\epsilon) \le t \le 1$, as $s \to \infty$. Indeed, for $|\lambda(t;s)|$ sufficiently big the remarks in (6.22) and (6.23) imply that $\mathcal{L}_0(t;s)$ is of order $|\lambda(t,s)|^{-1/2}$ whereas $\mathcal{L}_1(t;s)$ is of order $|\lambda(t,s)|^{-5/2}$. These findings can be used in (6.32) to conclude that (6.37) applies for all $|\lambda(t;s)|$ sufficiently large. In particular, for large values of $|\lambda(t;s)|$, $B_0(t;s)$ is of order $s^{-1/3} \cdot |\lambda(t;s)|^{-1/2}$ and (6.38) follows using that $|\lambda(t;s)|$ is of a size comparable to $|1-t| \cdot s^{-1/3}$. On the other hand, if $|\lambda(t;s)|$ remains bounded then $\mathcal{L}_0(t;s)$ and $\mathcal{L}_1(t;s)$ remain bounded. But, recall that, from the choice on $\epsilon > 0$, the $\Re\{\lambda(t;s)\} \ge 0$. As a result, since the Airy function is zero-free in the first and forth quadrant of the complex plane, $\mathcal{L}_0(t;s)$ is indeed bounded away from zero. These findings in (6.32) imply (6.37) for the case in which $|\lambda(t;s)|$ is bounded. In particular, when $|\lambda(t;s)|$ remains bounded, $B_0(t;s)$ is of order $s^{-1/3}$ and from this (6.38) follows immediately. This completes the proof of (6.37) and (6.38).

Identities (6.37) and (6.38) in (6.30) imply that

$$
(6.39) \qquad f_{r,s} = \frac{\tau^{-r} \cdot (\tau^2 - 3\tau + 3)^{-s}}{2\pi} \cdot \frac{A_0(\tau) \cdot \mathcal{L}_0(\tau;s)}{s^{1/3}} \cdot \left\{1 + O(s^{-1/3})\right\},
$$

uniformly for all (r,s) such that $(1-\epsilon) \le \tau \le 1$, as $(r,s) \to \infty$.

The above expansion will be exploited to describe the asymptotic behavior of $f_{r,s}$ in terms of the size $\Delta(r,s)$. For this, we first observe that

$$(6.40)\qquad \begin{aligned} 1-\tfrac{r}{s} &= 1-d(\tau)\,,\\ &= 3(1-\tau)^2\cdot\big\{1+O|1-\tau|\big\}\,, \end{aligned}$$

uniformly for all (r,s) such that $(1-\epsilon)\le\tau\le 1$, provided that $\epsilon>0$ is sufficiently small. As a result, $|1-\tau|$ is of the same order as $\Theta(r,s)$. Moreover, it also follows that $\Delta(r,s)$ is of the same order as $\lambda(\tau;s)^3$, for all (r,s) such that $(1-\epsilon)\le\tau\le 1$.

The remark in (6.22) together with the insights on the previous paragraph let us conclude, using (6.39), that there is $\alpha>0$ such that

$$\begin{aligned} f_{r,s} &= \frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{2\sqrt{\pi}}\cdot\frac{A_0(\tau)}{\sqrt{u(\tau)}}\cdot\frac{[s\cdot(1-\tau)^3]^{-1/6}}{s^{1/3}}\\ &\qquad\cdot\big\{1+O(\lambda(\tau;s)^{-3})\big\}\cdot\big\{1+O(s^{-1/3})\big\},\\ &= \frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{\sqrt{12\pi}}\cdot[s\cdot(1-\tau)]^{-1/2}\\ &\qquad\cdot\big\{1+O(\Delta(r,s)^{-1})\big\}\cdot\big\{1+O(s^{-1/3})\big\}\cdot\big\{1+O(\Theta(r,s))\big\}, \end{aligned}$$

uniformly for all (r,s) such that $(1-\epsilon)\le\tau\le 1$ and $\Delta(r,s)\ge\alpha$. This shows (6.28).

It remains to consider the case in which $|\Delta(r,s)|\le\beta$, for some fixed $\beta>0$. This is equivalent to request that $\lambda(\tau;s)$ remains bounded. (6.39) together with the definition of $\mathcal{L}_0(t;s)$ in (6.33) implies that

$$\begin{aligned} f_{r,s} &= \frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{s^{1/3}}\cdot\frac{A_0(\tau)}{\{3V(\tau)\}^{1/3}}\cdot e^{2\lambda(\tau;s)^3/3}\cdot\mathbf{Ai}(\lambda(\tau;s)^2)\cdot\big\{1+O(s^{-1/3})\big\},\\ &= \frac{\tau^{-r}\cdot(\tau^2-3\tau+3)^{-s}}{3^{1/3}\cdot s^{1/3}}\cdot e^{2\lambda(\tau;s)^3/3}\cdot\mathbf{Ai}(\lambda(\tau;s)^2)\cdot\big\{1+O(s^{-1/3})\big\}\\ &\qquad\cdot\big\{1+O|1-\tau|\big\},\\ &= \frac{e^{-2s(1-\tau)^3+2\lambda(\tau;s)^3/3+O(s(1-\tau)^4)}}{3^{1/3}\cdot s^{1/3}}\cdot\mathbf{Ai}(\lambda(\tau;s)^2)\cdot\big\{1+O(s^{-1/3})\big\}\cdot\big\{1+O|1-\tau|\big\},\\ &= \frac{\mathbf{Ai}(\lambda(\tau;s)^2)}{3^{1/3}\cdot s^{1/3}}\cdot\big\{1+O(s^{-1/3})\big\}\cdot\big\{1+O|1-\tau|\big\}, \end{aligned}$$

uniformly for all (r, s) such that $(1-\epsilon) \leq \tau \leq 1$ and $\Delta(r, s) \leq \beta$. (6.29) follows using (6.40), which let us conclude that

$$\lambda(\tau; s)^2 = 3^{-1/3} \cdot \frac{s-r}{s^{1/3}} \cdot \left\{1 + O|1-\tau|\right\},$$

uniformly for all (r, s) such that $(1-\epsilon) \leq \tau \leq 1$, provided that $\epsilon > 0$ is sufficiently small. This completes the proof of the corollary. □

6.3 Proofs of main results

In this section we prove the main two results of this chapter, namely theorems 6.1 and 6.6. We will assume that $G(z, w)$ and $H(z, w)$ are analytic functions on an open polydisk $\mathcal{D}$ centered at $(0, 0)$. In addition, we assume as given a compact set $\mathcal{K} \subset \mathcal{D}$ of strictly minimal simple zeroes of H containing a particular point (z_0, w_0). To prove our main results we require the following lemmas.

Lemma 6.15. *For all $\epsilon_1 > 0$ sufficiently small there exists $\delta_1 > 0$ such that, for all $(z, w) \in \mathcal{K}$, $H(\xi, \zeta)$ is zero-free on the set*

$$\left\{\xi : |\xi| = |z|, |\arg(\xi/z)| \geq \epsilon_1\right\} \times \left\{\zeta : |\zeta| \leq (1+\delta_1) \cdot |w|\right\}.$$

Proof. Consider, for all $\epsilon > 0$ and $\delta > 0$ sufficiently small, the sets

$$\begin{aligned}
\Lambda_1 \;&:=\; \big\{(\xi, \zeta) : \text{ there exits } (z, w) \in \mathcal{K} \text{ such that} \\
&\qquad\qquad |\arg(\xi/z)| \geq \epsilon,\ |\xi| = |z|\,,\ \text{and } |w| \leq |\zeta| \leq (1+\delta) \cdot |w|\big\}\,, \\
\Lambda_2 \;&:=\; \big\{(\xi, \zeta) : \text{ there exits } (z, w) \in \mathcal{K} \text{ such that} \\
&\qquad\qquad |\arg(\xi/z)| \geq \epsilon,\ |\xi| = |z| \text{ and } |\zeta| = |w|\big\}\,.
\end{aligned}$$

Observe that Λ_1 and Λ_2 are contained within $\mathcal{D}$ provided that $\delta > 0$ is sufficiently small.

Due to the minimality of each $(z, w) \in \mathcal{K}$, to prove the lemma, it will be enough to show that $H(\xi, \zeta)$ is zero-free over Λ_1 provided that ϵ and δ are selected appropriately. For this, consider $(\xi, \zeta) \in \Lambda_1$ and let $(z, w) \in \mathcal{K}$ be such that $|\xi| = |z|$ and $|w| \le |\zeta| \le (1 + \delta) \cdot |w|$. Then, it follows that

$$\begin{aligned} |H(\xi,\zeta)| &= \left| H\left(\xi, \frac{|w|\,\zeta}{|\zeta|}\right) + \frac{|w|\,\zeta}{|\zeta|} \cdot \int_1^{|\zeta|/|w|} H_w\left(\xi, \frac{|w|\,\zeta}{|\zeta|} \cdot \rho\right) d\rho \right| , \\ &\ge \left| H\left(\xi, \frac{|w|\zeta}{|\zeta|}\right)\right| - \int_1^{1+\delta} \left|\rho\, w \cdot H_w\left(\xi, \frac{|w|\zeta}{|\zeta|} \cdot \rho\right)\right| d\rho\,, \\ &\ge \inf_{\Lambda_2} |H| - \delta \cdot \sup_{(u,v)\in\Lambda_1} |v \cdot H_w(u,v)|\,. \end{aligned}$$

Observe that Λ_1 is a compact set which increases with δ whereas Λ_2 is independent of this last quantity. As a result, to conclude the lemma, it will be enough to show that the $\inf_{\Lambda_2} |H| > 0$. But, this is obvious because Λ_2 is a compact set and H is zero-free over Λ_2. This completes the proof of the lemma. □

To state our next result and in consistence with the hypotheses of theorems 6.1 and 6.6, it will be assumed that $H_w(z_0, w_0) \neq 0$. The implicit mapping theorem 4.22 implies that there is an open neighborhood $\mathcal{Z}_0 \times \mathcal{W}_0 \subset \mathcal{D}$ of (z_0, w_0) and a biholomorphic map $g : \mathcal{Z}_0 \to \mathcal{W}_0$ such that, for all $(z, w) \in \mathcal{Z}_0 \times \mathcal{W}_0$, it applies

$$H(z, w) = 0 \text{ if and only if } w = g(z)\,. \tag{6.41}$$

At some point in our discussion we will have to deal with the multiplicative inverse of $g(z)$. Because of this we will assume without any loss of generality that $0 \notin \mathcal{W}_0$. This does not reduce the generality of our exposition for the strict minimality of (z_0, w_0) requires that $z_0 \cdot w_0 \neq 0$.

Lemma 6.16. *For all $\epsilon_2 > 0$ sufficiently small there exists $\delta_2 > 0$ such that the equation: $H(\xi,\zeta) = 0$ has, for each fixed ξ such that $|\xi - z_0| < \epsilon_2$, only one solution satisfying $|\zeta| \leq (1+\delta_2)\cdot|g(\xi)|$.*

Proof. Observe that $H(z_0,\cdot)$ cannot be a identically zero as a function of its second argument (see [Rud87], theorem 10.18). In particular, its zero set cannot have accumulation points. The minimality of (z_0, w_0) then implies that there is $\delta_1 > 0$ such that $\zeta = w_0$ is the only zero of $H(z_0,\zeta)$ in the disk $\mathcal{B} := \big\{|\zeta| \leq (1+\delta_1)\cdot|w_0|\big\}$. Let $0 < \delta_2 < \delta_1$ be such that the disk $\mathcal{B}_w := \big\{\zeta : |\zeta - w_0| \leq \delta_2 \cdot |w_0|\big\} \subset \mathcal{W}_0$. Observe that $H(z_0,\zeta)$ is nonzero for $\zeta \in \overline{(\mathcal{B}-\mathcal{B}_w)}$. The uniform continuity of H can now be used to conclude that, for a sufficiently small $\epsilon > 0$ and for all $\xi \in \mathcal{B}_z := \big\{z : |z - z_0| \leq \epsilon\big\}$, $H(\xi,\cdot)$ is zero-free on the set $\overline{(\mathcal{B}-\mathcal{B}_w)}$. Without loss of generality we may assume that $\mathcal{B}_z \subset \mathcal{Z}_0$. Since we had $\mathcal{B}_w \subset \mathcal{W}_0$, it follows that $\zeta = g(\xi)$ is the only zero of $H(\xi,\zeta)$ within $\mathcal{B}_w$.

The discussion in the previous paragraph shows that, for all $|\xi - z_0| < \epsilon$, the equation: $H(\xi,\zeta) = 0$, with $|\zeta| \leq (1+\delta_1)\cdot|w_0|$, has $\zeta = g(\xi)$ as its only solution; in particular, for all sufficiently small $\epsilon > 0$, the continuity of $g(z)$ implies that this zero is indeed contained in the disk $\big\{\zeta : |\zeta| \leq (1+\delta_1/2)\cdot|w_0|\big\}$. The lemma will follow if, for a sufficiently small choice of $\epsilon > 0$, we can find $\delta > 0$ such that

$$\left(1+\frac{\delta_1}{2}\right)\cdot \sup_{\xi:|\xi-z_0|\leq\epsilon}\left|\frac{w_0}{g(\xi)}\right| \leq (1+\delta) \leq (1+\delta_1)\cdot \inf_{\xi:|\xi-z_0|\leq\epsilon}\left|\frac{w_0}{g(\xi)}\right| .$$

Since this is certainly possible for a sufficiently small choice of $\epsilon > 0$, the lemma follows. $\square$

Our next result and its proof in many ways resembles the one of lemma 4.1 in [PemWil01]. Theorem 6.1 is a direct consequence of the generalized stationary phase method (see corollary 5.10 in chapter 5) to study the asymptotic behavior of the parameter varying Fourier-Laplace integral, $\Xi(z;s)$, in the following lemma. Similarly, theorem 6.6 results from the application of the generalized coalescing saddle point method (see theorem 5.12 in chapter 5).

Lemma 6.17. *For a sufficiently small choice of $\epsilon > 0$ and for all $|\theta| \le \epsilon$ and z sufficiently close to z_0 define*

$$f(z,\theta) := \ln\left\{\frac{g(z\cdot e^{i\theta})}{g(z)}\right\} - i\cdot\theta\cdot\frac{z\cdot g'(z)}{g(z)}, \tag{6.42}$$

$$a(z,\theta) := \frac{-G(z\cdot e^{i\theta}, g(z\cdot e^{i\theta}))}{g(z\cdot e^{i\theta})\cdot H_w(z\cdot e^{i\theta}, g(z\cdot e^{i\theta}))}. \tag{6.43}$$

In particular, $f(z,0) = \frac{\partial f}{\partial\theta}(z,0) = 0$, for all z sufficiently close to z_0. Furthermore, the $\Re\{f(z,\theta)\} > \Re\{f(z,0)\}$, for all $(z,w) \in \mathcal{K}$ and all nonzero $\theta \in [-\epsilon,\epsilon]$. In addition, if we define

$$\Xi(z;s) := \int_{-\epsilon}^{\epsilon} e^{-s\cdot f(z,\theta)}\, a(z,\theta)\, d\theta\,, \tag{6.44}$$

then there is a constant $c > 0$ such that

$$f_{r,s} = \frac{z^{-r}\, w^{-s}}{2\pi}\cdot\left\{\Xi\,(z;s) + O(e^{-s\cdot c})\right\}, \tag{6.45}$$

uniformly for all $(r,s) \in dir(z,w)$ and $(z,w) \in \mathcal{K}$ sufficiently close to (z_0,w_0).

Proof. Let $\epsilon_2 > 0$ and $\delta_2 > 0$ be as in lemma 6.16. Consider $\epsilon > 0$ sufficiently small so that the functions in (6.42) and (6.43) are well-defined for all $|\theta| \le \epsilon$ and z sufficiently close to z_0.

Define

$$\gamma_1(z) := \left\{\xi : |\xi| = |z| \text{ and } |\arg\{\xi/z\}| \le \epsilon\right\},$$

$$\gamma_2(z) := \left\{\xi : |\xi| = |z| \text{ and } |\arg\{\xi/z\}| \ge \epsilon\right\}.$$

Without loss of generality we may assume that $\epsilon > 0$ is small enough so that $\gamma_1(z) \subset \{\xi : |\xi - z_0| < \epsilon_2\}$, for all $(z,w) \in \mathcal{K}$ sufficiently close to (z_0, w_0). Furthermore, we may also assume that the conclusion in lemma 6.15 applies for $\epsilon_1 = \epsilon$ and $\delta_1 > 0$. Motivated by this, we will select $0 < \delta < \min\{\delta_1, \delta_2, 1\}$. Observe that $\delta \in (0,1)$ and, for all $(z,w) \in \mathcal{K}$ sufficiently close to (z_0, w_0), it follows that

$$H(\xi,\zeta) \text{ is zero-free on the set } \gamma_2(z) \times \{\zeta : |\zeta| \leq (1+\delta)\cdot|w|\}. \tag{6.46}$$

Furthermore, for all $\xi \in \gamma_1(z)$, it applies that

$$\zeta = g(\xi) \text{ is the only zero of } H(\xi,\zeta) \text{ within } \{\zeta : |\zeta| \leq (1+\delta)\cdot|g(\xi)|\}. \tag{6.47}$$

With $\epsilon > 0$ and $\delta \in (0,1)$ as in the previous paragraph observe that the strict minimality of $(z,w) \in \mathcal{K}$ implies that H is zero-free on the polydisk $\{\xi : |\xi| \leq |z|\} \times \{\zeta : |\zeta| \leq (1-\delta)\cdot|w|\}$; in particular, F is analytic within this polydisk and continuous up to the boundary. Cauchy's formula (see [Rud87]) then can be used to represent the coefficients of F in the integral form

$$\begin{aligned} f_{r,s} &= \frac{1}{2\pi}\int_{|\xi|=|z|} \frac{1}{\xi^r}\left\{\frac{1}{2\pi i}\int_{|\zeta|=(1-\delta)\cdot|w|} \frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right\}\frac{d\xi}{i\,\xi}, \\ &= \frac{1}{2\pi}\int_{\xi\in\gamma_1(z)}\int_{|\zeta|=(1-\delta)\cdot|w|} \frac{1}{\xi^r}\left\{\frac{1}{2\pi i}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right\}\frac{d\xi}{i\,\xi} \\ &\qquad + \frac{1}{2\pi}\int_{\xi\in\gamma_2(z)}\int_{|\zeta|=(1-\delta)\cdot|w|} \frac{1}{\xi^r}\left\{\frac{1}{2\pi i}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right\}\frac{d\xi}{i\,\xi}. \end{aligned}$$

We will show that the integral over $\gamma_2(z) \times \{\zeta : |\zeta| = (1-\delta)\cdot|w|\}$ is negligible compared to $|z|^{-r}\cdot|w|^{-s}$, which is the expected exponential order of $f_{r,s}$. Indeed, (6.46) implies, for all $\xi \in \gamma_2(z)$, that

$$\int_{|\zeta|=(1-\delta)\cdot|w|} \frac{1}{\xi^r}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta = \int_{|\zeta|=(1+\delta)\cdot|w|} \frac{1}{\xi^r}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta.$$

As a result, we obtain that

$$\left|\int_{\xi\in\gamma_2(z)}\int_{\zeta:|\zeta|=(1-\delta)\cdot|w|}\frac{1}{\xi^r}\left\{\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right\}\frac{d\xi}{i\,\xi}\right|\leq|z|^{-r}\cdot\{(1+\delta)\cdot|w|\}^{-s}\cdot\sup_{\Lambda_1}|F|\,,$$

where it has been defined

$$\Lambda_1:=\Big\{(\xi,\zeta):\text{ there exits }(z,w)\in\mathcal{K}\text{ such that }\xi\in\gamma_2(z)\text{ and }|\zeta|\leq(1+\delta)\cdot|w|\Big\}\,.$$

Since Λ_1 is compact and H is zero-free over Λ_1, it follows that F continuous over this set and therefore the $\sup_{\Lambda_1}|F|$ must me finite. As a result, back in the last identity determined for $f_{r,s}$, we obtain that

$$\begin{aligned}(6.48)\quad f_{r,s} &= \frac{1}{2\pi}\int_{\xi\in\gamma_1(z)}\frac{1}{\xi^r}\left\{\frac{1}{2\pi i}\int_{|\zeta|=(1-\delta)\cdot|w|}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right\}\frac{d\xi}{i\,\xi}\\ &\qquad + O\Big(|z|^{-r}\cdot|w|^{-s}\cdot\{1+\delta\}^{-s}\Big)\,,\end{aligned}$$

uniformly for all $(z,w)\in\mathcal{K}$ and $r,s\geq 0$.

To deal with the integral term in (6.48) we will use (6.47). It implies that, for all $\xi\in\gamma_1(z)$ with $(z,w)\in\mathcal{K}$ sufficiently close to (z_0,w_0), $\zeta=g(\xi)$ is the only singularity of $F(\xi,\zeta)$ within the annulus $\big\{\zeta:(1-\delta)\cdot|w|\leq|\xi|\leq(1+\delta)\cdot|g(\xi)|\big\}$. As a result, the Residue theorem (see [Rud87]) let us conclude that

$$\begin{aligned}(6.49)\quad \frac{1}{2\pi i}\int_{|\zeta|=(1-\delta)\cdot|w|}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta &= -\mathrm{Res}\left(\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}};\zeta=g(\xi)\right)\\ &\quad + \frac{1}{2\pi i}\int_{|\zeta|=(1+\delta)\cdot|g(\xi)|}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\,.\end{aligned}$$

The residue term above is computed to be

$$\begin{aligned}\mathrm{Res}\left(\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}};\zeta=g(\xi)\right) &= \lim_{\zeta\to g(\xi)}\big(\zeta-g(\xi)\big)\cdot\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,,\\ &= \frac{G(\xi,g(\xi))}{\{g(\xi)\}^{s+1}\cdot H_w\big(\xi,g(\xi)\big)}\,.\end{aligned}$$

On the other hand, the integral term remaining on the right-hand side in (6.49) is bounded from above by the quantity $|g(\xi)|^{-s}\cdot\{1+\delta\}^{-s}\cdot\sup_{\Lambda_2}|F|$, where it has been defined

$$\Lambda_2 := \big\{(\xi,\zeta):\ \text{there exits } (z,w)\in\mathcal{K} \text{ such that } \xi\in\gamma_1(z) \text{ and } |\zeta| = (1+\delta)\cdot|g(\xi)|\big\}\,.$$

But, observe that (6.47) implies that F is continuous over Λ_2; in particular, being this last a compact set, it follows that the $\sup_{\Lambda_2}|F|$ is finite. Moreover, the minimality of $(z,w)\in\mathcal{K}$ implies that $|g(\xi)|>|w|$, for all $\xi\in\gamma_1(z)$. Therefore, for each $\xi\in\gamma_1(z)$, it applies that

$$\left|\frac{1}{2\pi i}\int_{|\zeta|=(1+\delta)\cdot|g(\xi)|}\frac{G(\xi,\zeta)}{H(\xi,\zeta)\cdot\zeta^{s+1}}\,d\zeta\right| \le |w|^{-s}\cdot\{1+\delta\}^{-s}\cdot\sup_{\Lambda_2}|F|\,.$$

Thus, if we define

$$\Xi(r,s;z) := \frac{1}{2\pi}\int_{\xi\in\gamma_1(z)}\frac{1}{\xi^r}\cdot\frac{-G(\xi,g(\xi))}{\{g(\xi)\}^{s+1}\cdot H_w\big(\xi,g(\xi)\big)}\,\frac{d\xi}{i\,\xi}\,,$$

we conclude from (6.48) that

$$f_{r,s} = \Xi(r,s;z) + O\big(|z|^{-r}\cdot|w|^{-s}\cdot\{1+\delta\}^{-s}\big)\,, \tag{6.50}$$

uniformly for all $(z,w)\in\mathcal{K}$ sufficiently close to (z_0,w_0) and all $r,s\ge 0$.

Since $\gamma_1(z)$ is a circular arc, we may easily parametrize it using polar coordinates. Indeed, substituting: $\xi = z\cdot e^{i\theta}$, with $\theta\in[-\epsilon,\epsilon]$, in $\Xi(r,s;z)$, a simple calculation reveals that

$$\Xi(r,s;z) = \frac{z^{-r}\,w^{-s}}{2\pi}\int_{-\epsilon}^{\epsilon} e^{-s\cdot f(\theta;z,r/s)}\,a(z,\theta)\,d\theta\,, \tag{6.51}$$

where $a(z,\theta)$ is as defined in (6.43), and

$$f(\theta;z,\lambda) := \ln\left\{\frac{g(z\cdot e^{i\theta})}{g(z)}\right\} + i\cdot\lambda\cdot\theta\,.$$

The lemma will follow after the following remarks. First, observe that

$$\frac{\partial f}{\partial \theta}\left(0; z, \frac{r}{s}\right) = i\left\{\frac{z \cdot g'(z)}{g(z)} + \frac{r}{s}\right\} .$$

As a result, since

$$\frac{z \cdot g'(z)}{g(z)} = -\frac{z \cdot H_z(z, w)}{w \cdot H_w(z, w)} ,$$

it follows that $\theta = 0$ is a stationary point of $f(\theta; z, r/s)$ provided that $(r, s) \in \operatorname{dir}(z, w)$. (6.45) follows from (6.50) and (6.51) after noticing that, for $(r, s) \in \operatorname{dir}(z, w)$, $f(\theta; z, r/s) = f(z, \theta)$, with $f(z, \theta)$ as defined in (6.42).

Finally, consider $(z, w) \in \mathcal{K}$ as fixed. Since $w = g(z)$, the strict minimality of (z, w) implies that $|g(z \cdot e^{i\theta})| > |g(z)|$, for all nonzero $\theta \in [-\epsilon, \epsilon]$. As a result, if $(r, s) \in \operatorname{dir}(z, w)$ then

$$\begin{aligned} \Re\{f(z, \theta)\} = \Re\{f(\theta; z, r/s)\} &> 0 \\ &= \Re\{f(0; z, r/s)\} = \Re\{f(z; 0)\} , \end{aligned}$$

for all nonzero $\theta \in [-\epsilon, \epsilon]$. This completes the proof of lemma 6.17. □

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