What You (Should) Get out of Freshman Calculus

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The majority of the students who take a mathematics course in college will not be concentrating their study in mathematics. These are the students whom I would like to address in this essay: the ones who will take one or two mathematics courses and whose careers will not rest on their deep and detailed knowledge of mathematics. The question of why one might study mathematics is somewhat philosophical, so instead I propose to answer the more concrete question, “What, normatively, should a student get out of his or her freshman calculus class?”

For some students and some instructors, the answer is self-evident. Look at the problems in the textbook (these are nearly identical across the most commonly used texts), look at the exam problems, and these will define the scope and purpose of the course. But this answer misses some of the most crucial aspects of freshman mathematics. To promote this view is to mislead students into concentrating their efforts on skills that will vanish outside a small arena and to leave them no more motivated or enlightened than they were on day one of the course. There is a hidden curriculum of cognitive and verbal skills, of abstraction and interpretation and of some very quantitative skills (estimation, bounding, determining approximate and qualitative solutions) that form a core of the syllabus equal in importance to the core that is represented by the table of contents of the textbook.

Students profit from being made explicitly aware of these course goals. Instructors too would benefit from keeping these goals in mind, especially when the pace is fast and the course is jammed with a technical to-do list that threatens to eclipse all other aspects. In what follows, I will begin by examining the freshman population and the broad reasons for inclusion of mathematics in their curriculum. After a brief discussion of pros and cons of curricula defined by assessments, I will give a detailed description of the hidden curriculum and conclude with implications for teaching practices.

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[3] Mathematics Subject classification (2010): 97B40, 97B70

Freshman populations and curricula

Mathematics offerings at the University of Pennsylvania are similar to those at most of the better US universities. The text we use[4] contains the material for roughly three semesters of calculus: introduction to differential and integral calculus, further study of calculus of one variable, and multivariate calculus. Many students arrive with either one or two semesters of calculus already under their belts. Because of this, the freshman curriculum may be considered to consist of either the first two of these semesters or the last two.

Students arrive at the university with a general sense of their intended area of study. Some are in STEM fields (Science, Technology, Engineering and Mathematics) and will have a long list of mathematics courses to take. For most of these students, much of what I will say below does not apply, because they already "get it." They have internalized the hidden curriculum. They can, at least to some degree, supply this for themselves, leaving the instructor to concentrate on communicating technical details and subtleties. However, even among the STEM fields, a substantial fraction of students are not and this level of sophistication and do indeed require instruction in more than just the technical side of mathematics. I recently served as our Director of Undergraduate Studies and in canvassing the faculty in other fields as to what they most needed the Math Department to provide, I was struck by the consensus from the biological sciences that their students had a huge disconnect between technique, at which they were pretty good, and application to science, about which many seemed not to have learned anything.

Many students who are not in STEM fields take at least one math course. In the social sciences, Economics majors require the complete calculus curriculum but many of the other disciplines have become increasingly mathematical and require or encourage study of mathematics. Pre-meds must take at least one calculus course regardless of whether their major is in a STEM field. Students in the business school are also required to take at least one math course. For students pursuing a classical liberal arts education, study of mathematics is not required, but even among these students the number who appear in calculus classes is not small.

Why students take freshman calculus varies with membership in one of the above groups. For STEM fields the answers are the most transparent. These students will be using mathematics to model systems that obey or nearly obey simple mathematical laws. They will be doing, verbatim, many of the computations they learn in calculus. Students in social sciences and business, to the extent that they use mathematics, will mostly be making "big-picture" models to gain some qualitative understanding of cause and effect.

Even for students in areas that are entirely non-quantitative, there are broad philosophical reasons for taking math courses. Indeed, twelve years of pre-college mathematics are widely held to

[4]Thomas Calculus Early Transcendentals, Custom Edition for the University of Pennsylvania Pearson, 2012. ISBN : 978-1-256-82329-2

train the mind for logic and abstraction, and to teach the language of science and of the modern world. The post-enlightenment world is described, at least in principle, by mathematics, with objects following mathematically described trajectories, assembling into macroscopic ensembles and playing cosmic games of roulette. Sometimes the evident mathematical objects melt away and are replaced by more abstract and convoluted forms such as waves in the ether or mega-dimensional strings. While practical life demands perhaps a sixth grade knowledge of mathematics, intellectual membership in the current century (or even the one before) can be argued to require an understanding of functions, limits and rates of change, which together form the core of freshman calculus.

Goals of freshman calculus

The above reasons for studying mathematics are important but they are so broad as to be generic. They apply almost equally to any conception of freshman mathematics and do not take into account the courses actually being taught. When advising students as to whether and which math course to take, the actual courses matter more than the in-principle status of mathematics in the intellectual pantheon. More importantly, when improving mathematics instruction, unless we plan to revolutionize the curriculum, we should be grappling with the question of how to deliver the best instruction for a course that shares, at least approximately, a syllabus with the one that was taught last year.

Some students and instructors alike believe that the course content is defined by the skills one must learn to get a good grade in the course. This is increasingly common and is part of an educational movement that defines knowledge in terms of objective assessments. There are some very good reasons to do this, among which are the fact that bad instruction can more easily hide behind a veil of subjectivity. For this reason, mainstream thinking in K-12 education is to verbally articulate the educational goals, then translate them into a carefully constructed battery of assessments. After this, one can foget about the originally formulated goals and operate as if high scores on the assessments is the primary goal.

In this view it is easy to state the main goals of Freshman calculus. In so-called AB Calculus, named after the College Board exam for high school students to show they have mastered a semester of college level mathematics, the main goals are to identify and apply as set of rules for determining limits, learn half a dozen rules for differentiation culminating in the Chain Rule, solve word problems to do with rates of change and optimization, apply anti-differentiaion to compute areas and sums, and apply a few basic rules relating the shape of a graph of a function to the behavior of its derivatives. In BC calculus (also named for the College Board exam) the goals are to compute various lengths, areas and volumes of geometrically defined regions, to learn a battery of less obvious techniques for computing anti-derivatives, to compile and apply criteria for the convergence of infinite series, and to compute series and series approximations to functions composed algebraically from polynomial and well known transcedental functions. Multivariate calculus has a similar set of goals which, for the sake of space, I will not list here.

Is it obvious or is it surprising that these skills are not definitionally equivalent to success in first year calculus? In fact, for example, when the College Board exams were first instituted, the AP exams were good surrogates for successful learning in calculus. High school calculus teachers taught to a small, elite segment of the student population. They could teach what they wanted, as they were playing with house money. A student who understood what was being taught would naturally be able to do the computations involved and perform well on the AP exam. Conversely, while good performance on the computational exam need not logically imply understanding on other fronts, it may be presumed to do so because the alternative would be ridiculous: teaching students exactly those computational skills that appear on the exams while carefully avoiding the underlying concepts, applications, interpretations and so forth. Yet this ridiculous scenario has largely come to pass. Optimizing long term learning and optimizing performance on the next test are overlapping but distinct goals, and if the outcome of the next test is viewed as important *per se*, then teaching strategy will shift from long term to short term goals. This is not due to a deficiency in these particular standardized tests, which are in fact pretty good as tests go. Any exam that recurs and defines a curriculum can be taught to, and if the stakes are high enough, it will be.

The hidden curriculum

Many college professors I have talked to describe their meta-curriculum as a kind of pushback against the procedurally focused learning that has been paramount in their students' prior education. Their view of the most important lessons of freshman calculus encompasses many of the following.

Exponential behavior. The exponential function is introduced in high school, along with some rules for manipulating expressions involving exponentials and logarithms. Only with calculus does the relataion become apparent between an exponential function and its infinitesimal behavior. From a physical viewpoint, the units are a big tipoff: when the rate of change is in units of the quantity itself, exponential behavior ensues. When the feedback is negative, it is exponential decay or approach. When the feedback is positive, one sees exponential growth. Perhaps this one topic, learned properly, underlies half the applications of calculus to biological and ecological sciences at the undergraduate level.

Orders of growth, estimates and bounds. Which functions grow faster than which others and which approach zero faster? Our present calculus text has one section of one chapter on "Order of Growth", but in fact this concept is almost ubiquitious in first year college mathematics. It allows us to identify limits easily and to test integrability. More importantly, it gives us an immediate idea of the approximate size of an expression without a numerical computation. Many students bog down in complicated computations because they fail to think it through with the function replaced by a simple one of the same approximate size. Exponential growth should immediately be identified as rapid (e.g., outpacing any polynomial). The big-O and little-o notation is useful, easy, and vastly underutilized. Expressions involving these can be manipulated and understood. Furthermore they represent how scientists really think when faced with complicated expressions. Related to these issues is the concept of finding not just an estimate for a quantity but a bound: an estimate that you know to be greater (or less) than the quantity being estimated. Finding a good bound can be tricky, but finding some nontrivial bound is usually easy. Nevertheless, because it does not fit the usual mold of procedural mathematics, students asked to find a nontrivial bound on, say $\cos(\pi/5)$, will often give up before they have even tried to think.

Logic and abstraction. This is really a continuation of a theme in pre-college mathematics, only at a more sophisticated level. Students must formulate concepts precisely, in a way they have not before.[5] The definitions of limit, continuity and derivative all involve an alternation of quantifiers. Often in calculus, the function f becomes the basic object of study rather than the value $f(x)$, and this in iteslf is a level of abstraction that challenges the abstraction capabilities of many students.

Problem solving. Our students consider it unfair to expect them to do something when they have not explicitly been shown how. In fact this is a trend in textbooks and curricula at all levels. In tandem with assessment-based curricula it leads to the shrinking of the curriculum to an exhaustive list of procedures. Problem solving is the opposite of this. Students learn basic strategems[6] for attacking a problem that does not fit into an obvious category. The extent to which freshman calculus courses teach problem solving varies greatly. When asked, most instructors would agree that it is important to get students to think in this way; certainly instructors in courses that make use of freshman calculus believe this skill to be important and widely to be lacking. I list it as as a goal of freshman calculus in the normative sense, because its inclusion is usually consistent with stated syllabi whether or not it is present in practice.

Modeling and verbal skills. Word problems are a difficulty throughout the mathematics curriculum. Those who easily learn mathematical procedures are often frustrated that translating a scenario into equations on which the procedures may be performed is a far less algorithmic task. Mathematical modeling with calculus is similar but at a higher level. Interpretations of variables must be clearly described. Equations depend on assumptions that must be articulated as well. Through high school, students often have not been made to provide explanations, let alone ones that employ correct grammar and can be understood by someone other than the teacher. By emphasizing problems with a verbal component and enforcing some level of verbal standard, not only do students become better communicators but also they are able to use the verbal realm to clarify their own thoughts.

Notation. Mathematical notation is a hybrid of abstraction and linguistic skills. As with foreign languages, notation is best taught by immersion and mimicry. It is not an end in itself, but it is hard

[5]Except perhaps in euclidean geometry, which to many students feels more like a formal game than an arena in which formal justification is related to convincing an actual skeptic.

[6]These are explicitly discussed in texts such as G. Pólya's *How to Solve It*, but their application is more of an art than a science.

for students to be fluent in mathematics if they stumble over notation. This includes most notably summation notation, but also limits, composition of functions and inverse functions, all of which condense a lot of information into a few symbols. A common notational theme with subscripts, summations and integrals is the role of free variables versus bound variables. Students should know, when looking at a summation, which of the variables will appear in the final answer; yet this point is seldom emphasized or recognized as a point of learning in calculus.

Teaching the hidden curriculum

Mastery of a set of procedures can be taught in a number of ways. One of them is to provide clear and careful instruction, perhaps accompanied by diagrams or videos, along with plenty of opportunity for practice. Depending on the difficulty, it may also be helpful to provide a coach to facilitate some of the practice. This is in fact how most calculus courses are organized, with the lecture providing the explanation, plenty of homework for the students to do on their own, and a recitation section that provides not one on one coaching but some kind of interactive coaching through the harder aspects of the weekly homework.

This format is not as good in teaching the hidden curriculum. Concepts such as exponential behavior and orders of growth are more amenable than the others to inclusion in lectures. Skills of a linguistic nature such as notation and modeling can be exhibited by the lecturer but must be practiced repeately by the student. Problem solving too must be continually practiced. Logic and abstraction are not skills one addressed on their own, but are reinforced through the types and levels of problems that students are asked to tackle. For this reason, one might look to a different model of instruction.

The tutorial system, common in the UK, comes close to this. Lectures are held, but the bulk of the work for both students and instructors is in the tutorials, which are like recitations but smaller and even more interactive. The flipped classroom is another possibility. There, students spend class time primarily on homework, which may include deeper problems due to the availability of coaching and peer instruction. The explanation of procedures no longer occurs in class but students are given ways to obtain it outside of class, from reading the book, to watching videotaped lectures, to the online instructional platforms constructed by textbook publishers. A third possibility, which we employ in some of the more advanced undergraduate courses at Penn, is to replace the recitation in the lecture-recitation format by a lab which interpolates between a recitation and a tutorial.

For those instructors who don't have the luxury of altering the lecture format there are still ways to address the hidden curriculum. Some of these involve tradeoffs because the time in lecture is finite and freshman calculus courses are typically very tightly packed. Is it worth placing a greater emphasis on word problems and problem solving? is it worth pressing students harder on abstraction? These are individual choices to be made in the context of what is feasible at a given school. However there is some room for improvement in hidden areas at almost no cost in time and topic coverage. This is in the area of linguistic skills: notation, argumentation, verbalization, and to some degree, logic and abstraction. The immersion and mimcry approach works almost as well in a lecture format, with the caveat that one must actually elicit the mimicry. Notation for summations, inverse functions, composition, subscripts, so forth, may be introduced, emphasized and dwelt on for a few seconds, re-introduced a few lectures later, and inserted into the homework. If it is important for the students to learn that $\int_0^\infty$ means $\lim_{M\to\infty} \int_0^M$ then insert that extra step every time, then begin requiring the students to do so. These are not the biggest elements of the hidden curriculum, but their correct use keeps the students' minds clearer and can be achieved with minimal cost.

Conclusion

In the end, the question of why one should study mathematics, and freshman calculus in particular, is inextricably bound up with the question of what one gets out of these courses. The syllabi for these courses are largely dictated by the science and engineering students, and therefore the technical content of these courses is not likely to change. Within these parameters, however, there is ample room for changing the delivery and the emphasis and for addressing hidden curriculum. If we do a good job with these, then our answer to the question will be this. Freshman calculus will forever change how you look at the world, how you think and make sense of quantitative models of humans and nature; it will give you the tools to handle abstraction and the modeling skills to apply these to diverse areas such as finance, psychlogy, logistic operations, epidemiology and statistical inference, not to mention the usual bailywicks of mathematics such as physical science and engineering; it will give you the analytical capability to abstract and generalize, to dissect an argument and to evaluate evidence, which is crucial in areas such as law and public policy; you will understand all these things for decades to come: your experience in calculus will be, literally, unforgettable.