Non-universality for longest increasing subsequence of a random walk

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Abstract. The longest increasing subsequence of a random walk with mean zero and finite variance is known to be $n^{1/2+o(1)}$. We show that this is not universal for symmetric random walks. In particular, the symmetric Ultra-fat tailed random walk has a longest increasing subsequence that is asymptotically at least $n^{0.690}$ and at most $n^{0.815}$. An exponent strictly greater than $1/2$ is also shown for the symmetric stable-α distribution when α is sufficiently small.

1. Introduction

It is well known that the longest increasing subsequence (LIS) of a sequence of n IID non-atomic random variables has length $(2 + o(1))\sqrt{n}$ with high probability. (see Vershik and Kerov 1977; Logan and Shepp 1977). A different model was considered by Angel et al. (2014). Let $S_n := \sum_{k=1}^{n} X_k$ be the partial sums of random walk with mean zero and finite variance. Angel et al. show that the LIS of the partial sum sequence $(S_1, \ldots, S_n)$ has length $n^{1/2+o(1)}$. They do not shed any light on what happens when the finite variance hypothesis is removed. When the second moment, and possibly the first, are undefined, it makes sense to consider other ways to keep the walk from having a drift. Here we consider random walk trajectories whose increments are symmetric about zero. We show that random walks whose increments have fat tails will have a longer LIS than do those with finite variance.

The cleanest model in which this occurs is the so-called Ultra-fat tailed distribution, which is a distribution not on $\mathbb{R}$ but on a non-archimedean totally ordered space $\mathcal{S}$ described in Section 2 below. There, we are able to show that the LIS has length at least $n^{0.69}$; see Theorem 2.1 below. However, the result also holds

for real random walks with fat tails, such as the symmetric stable-α when α is sufficiently small. We also show that the LIS has length at most $n^{0.82}$. Neither of these exponents is believed to be sharp, however empirical studies suggest that the LIS exponent for the Ultra-fat tailed distribution is roughly 0.72, so nearer to our lower bound. Numerical evidence also suggests that for stable laws, the exponent varies, interpolating between this and $1/2$.

The organization of the rest of the paper is as follows. The next section contains definitions, notation and preliminary facts. Section 4 proves the $n^{\beta_0+o(1)}$ lower bound with an explicit constant β_0 slightly larger than 0.69. Section 5 proves the $n^{\beta_1+o(1)}$ upper bound, with an explicit constant β_1 slightly less than 0.9. Section 6 extends the lower bound from the Ultra-fat tail case to actual fat-tailed distributions. We conclude with some further remarks and questions.

2. Definitions and results

2.1. *Ultra-fat tailed distribution.* We begin by defining the state space $\mathcal{S}$, which is a free $\mathbb{Z}$-module with one generator x for each $x \in (0, 1)$. In other words, elements of $\mathcal{S}$ are finite formal linear combinations of the symbols $\{x : 0 < x < 1\}$ with coefficients in $\mathbb{Z}$. There should be no confusion between the formal symbol x and the real number x as coefficients take only integer values and are always written on the left.

Endow $\mathcal{S}$ with the lexicographic order. Formally, if $\alpha = \sum_{x \in F} a_x x$ and $\beta = \sum_{x \in G} b_x x$, we may define this order relation by induction on the minimum length $m \wedge n$ of α and β as follows. For $\alpha = \sum_{x \in F} a_x x \in \mathcal{S}$, define its degree by $|\alpha| := \sup\{t : a_t \neq 0\}$. By convention $|0| = 0$. We define comparisons to 0 by $\alpha > 0$ if and only if $|F| > 0$ and $a_{|\alpha|} > 0$ and $\alpha < 0$ if and only if $-\alpha > 0$. For elements $\alpha = \sum_{x \in F} a_x x$ and $\beta = \sum_{x \in G} b_x x$, assuming $|F|, |G| > 0$, inductively define $\alpha > \beta$ if and only if one of the following conditions holds.

(i) $|\alpha| = t > |\beta|$ and $a_t > 0$;
(ii) $|\beta| = t > |\alpha|$ and $b_t < 0$;
(iii) $|\alpha| = t = |\beta|$ and $a_t > b_t$;
(iv) $|\alpha| = t = |\beta|$ and $a_t = b_t$ and $\alpha - a_t t > \beta - b_t t$.

This defines a total order on $\mathcal{S}$ consistent with addition: $\alpha > \beta$ and $\gamma \geq \delta$ implies $\alpha + \gamma \geq \beta + \delta$.

2.2. *Ultra-fat tailed random walk.* Define $F : (-1, 1) \to \mathcal{S}$ by $F(0) = 0$ and $F(x) = \operatorname{sgn}(x)|x|$ for $x \neq 0$. Let $\{U_n : n \geq 1\}$ be an IID collection of real random variables uniform on $(-1, 1)$. The law of $F(U_1)$ is called the **Ultra-fat tailed distribution**. Let $X_n := F(U_n)$ and $S_n := \sum_{k=1}^{n} X_k$, with $S_0 := 0$. The sequence $\{S_n : n \geq 0\}$ is called the so-called **Ultra-fat tailed random walk**. A sequence $n_1 < \cdots < n_k$ is an increasing subsequence if $S_{n_i} < S_{n_j}$ for all $1 \leq i < j \leq k$. The Ultra-fat tailed distribution has been used elsewhere, without a formal definition; see, e.g., Limic and Pemantle (2004). The following theorem is the main result of this note.

2.3. *Main result.* Let $\{S_n\}$ be a random walk on $\mathcal{S}$ with increments from the Ultra-fat tailed distribution. Let $L(t)$ denote the length of the LIS of $(S_1, \ldots, S_t)$. We remark that, by convention, we have not allowed S_0 to be an element of the LIS, hence the increment X_1 will play no role.

Theorem 2.1. *There are numbers $1/2 < \gamma < \delta < 1$ such that as $t \to \infty$,*

$$\mathbb{P}(t^\gamma \leq L(t) \leq t^\delta) \to 1\,.$$

In particular, one can take $\gamma = 0.690$ and $\delta = 0.815$.

Remark. It can be shown (see Section 7) that neither exponent is sharp.

3. Preliminary results

We look at the growth rate of various deterministic random functions going between the time variable for the random walk and the length variable for the corresponding LIS. Because of the proliferation of notation, we will organize by using t and nearby letters when possible for quantities in the time domain and ℓ and nearby letters for quantities in the length domain. As usual, we use upper case letters such as L and T for random quantities.

The random variables $\{L(t) : t \geq 1\}$ have already been introduced and follow this notational scheme. For $\ell \geq 1$ let $T(\ell) := \inf\{t : L(t) \geq \ell\}$ denote the random time that the LIS first reaches length ℓ. Thus $L(T(\ell)) = \ell$ and $T(L(t)) \leq t$.

The magnitudes of the steps are the values $|U_1|, |U_2|, \ldots$. Trivially, the order type of the first t of these is uniform on all $t!$ possible orders and independent of the sign vector, which is also uniform on $\{\pm 1\}^t$. This allows for the usual conditioning identities. For example, if the variable of the greatest magnitude is U_σ then the order types of $(|U_1|, \ldots, |U_{\sigma-1}|)$ and $(|U_{\sigma+1}|, \ldots, |U_t|)$ are independent and uniform. Also immediate is the following Markov property. Construct the random variables $\{U_n\}$ as the coordinate functions on the canonical space $\Omega := [-1, 1]^\infty$ with normalized Lebesgue measure. Let $\theta : \Omega \to \Omega$ be the shift $(U_1, U_2, U_3, \ldots) \mapsto (U_2, U_3, \ldots)$. Let $\mathcal{F}_t := \sigma(U_1, \ldots, U_t)$ and let τ be a stopping time with respect to the filtration $\{\mathcal{F}_t\}$. Then conditionally on $\mathcal{F}_\tau$, the sequence $\{X_{\tau+n}\}$ is distributed as the unconditional sequence $\{X_n\}$.

This is all pretty trivial but it allows us to state two important relationships, one sub-additive and super-additive:

$$L(s + t) \quad \leq \quad L(s) + L(t) \circ \theta^s\,; \tag{3.1}$$

$$T(\ell + m) \quad \geq \quad T(\ell) + T(m) \circ \theta^{T(\ell)}\,. \tag{3.2}$$

Intuitively, the first of these holds because any increasing subsequence of $(S_1, \ldots, S_{s+t})$ has at most $L(s)$ entries in $[s]$ and $L(t) \circ \theta^s$ elements in $\{s+1, \ldots, s+t\}$. The second holds because to get an increasing subsequence of length $\ell + m$ one first needs one of length ℓ, and must then find one of length m among the remainder of the sequence. These properties do not rely on the Ultra-fat tailed distribution and hold for the LIS of any random walk.

Definition 3.1 (NBU). *Say that a random variable X is new better than used (NBU) if for every pair of positive integers a and b,*

$$\mathbb{P}(X \geq a + b) \leq \mathbb{P}(X \geq a)\mathbb{P}(X \geq b)\,.$$

The terminology comes from reliability theory Barlow et al. (1963) and Barlow and Proschan (1965), where the inequality rewritten as $\mathbb{P}(X \geq a+b | X \geq a) \leq \mathbb{P}(X \geq b)$ says that a new light bulb has a better chance of surviving b units of time, than does a light bulb that has been used for a units of time.

We recall a basic property of NBU variables.

Lemma 3.2. *If the random variable X is NBU and $\mathbb{P}(X < q) = \epsilon$, then $\mathbb{E}[X] \leq q/\epsilon$.*

PROOF: The NBU assumption implies that $\mathbb{P}(X \geq kq) \leq (1 - \epsilon)^k$, so X/q is stochastically dominated by a Geometric Variable of mean $1/\epsilon - 1$.

Proposition 3.3. *For LIS of any random walk, each random variable $L(t)$ is NBU.*

PROOF: This follows from (3.1) and (3.2). The event $\{L(t) \geq a + b\}$ is the intersection of the events $\{T(a) \leq t\}$ and $\{T(b) \circ \theta^{T(a)} \leq t - T(a)\}$. Because $T(b) \circ \theta^{T(a)}$ is independent of $\mathcal{F}_{T(a)}$,

$$\mathbb{P}\big(L(t) \geq a + b\big) = \mathbb{P}(T(a) \leq t) \cdot \mathbb{P}\Big(T(b) \circ \theta^{T(a)} \leq t - T(a)\Big)$$

$$\leq \mathbb{P}(L(t) \geq a) \cdot \mathbb{P}(L(t) \geq b).$$

$\square$

Proposition 3.4. *Let X be NBU with mean μ and let Y be geometric started from zero with mean μ (that is, one less than a geometric of mean $\mu + 1$). Denote*

$$a_n := \mathbb{P}(X \geq n)$$

$$A_n := \sum_{k=n}^{\infty} a_k$$

$$g_n := \mathbb{P}(Y \geq n) = \left(\frac{\mu}{1 + \mu}\right)^n$$

$$G_n := \sum_{k=n}^{\infty} g_k = G_0 g_n$$

where $G_0 = 1 + \mu$. Then for all n we have $A_n \leq G_n$.

PROOF: Let t be the least integer such that $a_t < g_t$. Then t is at least 1 because $a_0 = g_0 = 1$. Also t is finite unless X and Y have the same distribution because

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} g_n = 1 + \mu.$$

Suppose first that $n \leq t$. Then

$$A_n = 1 + \mu - \sum_{k=0}^{n-1} a_k \leq 1 + \mu - \sum_{k=0}^{n-1} g_k = G_n.$$

Now suppose that $n > t$ and assume for induction that $A_m \leq G_m$ for all $m < n$. Then using the NBU property and induction,

$$A_n = \sum_{k \geq n} a_k \leq a_t \sum_{k \geq n-t} a_k = a_t A_{n-t} \leq g_t G_{n-t} = G_n,$$

completing the induction. $\square$

Corollary 3.5. *Let X be NBU and Y be geometric started from zero with the same mean. Then for any convex function ϕ,*

$$\mathbb{E}\phi(X) \leq \mathbb{E}\phi(Y).$$

Remark: An equivalent conclusion is that X has the distribution of some conditional expectation of Y.

PROOF: As before, let a_n and A_n be tail probabilities for X and their tail sums. Letting $\Delta h(n)$ denote $h(n+1) - h(n)$, we sum by parts twice to obtain

$$\mathbb{E}\phi(X) = \phi(0) + \sum_{n=1}^{\infty} \Delta\phi(n-1)a_n = \phi(0) + \Delta\phi(0)A_1 + \sum_{n=2}^{\infty} \Delta\Delta\phi(n-2)A_n \, .$$

Similarly,

$$\mathbb{E}\phi(Y) = \phi(0) + \Delta\phi(0)G_1 + \sum_{n=2}^{\infty} \Delta\Delta\phi(n-2)G_n \, .$$

Because ϕ is convex, $\Delta\Delta\phi(k) \geq 0$ for $k \geq 0$. Together with $A_k \leq G_k$ for all k and $A_1 = G_1 = \mu$, this proves the corollary. $\qquad\square$

4. Proof of the lower bound

In this section we prove the lower bound in Theorem 2.1. In terms of universality, this direction is the more interesting, as it shows the Ultra-fat tailed walk to be in a different LIS-universality class from mean zero finite variance walks.

For $1 \leq m < n$, let $L(m, n) := L(n-m) \circ \theta^m$, in other words, it is the length of the LIS of $(S_m, \ldots, S_n)$ (recall that, by convention, the LIS cannot include the initial element, S_m). Of course $L(m, n)$ has the same distribution as $L(n-m)$. Define $\sigma(n)$ to be the almost surely unique $k \in [n] \setminus \{1\}$ such that $|U_k| = \max_{2 \leq j \leq n} |U_j|$. In other words, $\sigma(n)$ is the time at which the random walk completed its largest magnitude step among those occuring after time 1 and before time n. Let $\mathsf{up}_n := \{U_{\sigma(n)} > 0\}$ denote the event that this greatest magnitude increment was positive. The complementary event is denoted down_n. On up_n, one has the inequality $S_j > S_i$ whenever $j \geq \sigma(n) > i$. Therefore, the increasing subsequences of $[n]$ are precisely the unions $A \cup B$ where A is an increasing subsequence of $[\sigma(n) - 1]$ and B is an increasing subsequence of $[n] \setminus [\sigma(n) - 1]$. It follows that $L(n) = L(\sigma(n) - 1) + L(\sigma(n) - 1, n)$ On down_n, one has $S_j < S_i$ whenever $j \geq \sigma(n) > i$, hence the increasing subsequences of $[n]$ are precisely the sets that are either an increasing subsequence of $[\sigma(n) - 1]$ or of $[n] \setminus [\sigma(n) - 1]$. We have therefore proved:

Proposition 4.1. *The sequence of random variables $\{L(n)\}$ satisfies the recursion*

$$\begin{aligned} L(n) \quad = \quad & \mathbf{1}_{\mathsf{up}_n} \left[L(\sigma(n) - 1) + L(\sigma(n) - 1, n) \right] \\ & + \mathbf{1}_{\mathsf{down}_n} \max \left\{ L(\sigma(n) - 1), L(\sigma(n) - 1, n) \right\} \, . \end{aligned}$$

$\qquad\square$

We will prove the lower bound in Theorem 2.1 by using the recursion to obtain the following lower bound on $\mathbb{E}L(n)$.

Lemma 4.2. *Let β_0 be the positive solution to $x + 2^{-1-x} = 1$, whose decimal expansion begins 0.690069. Then $\mathbb{E}L(n) \geq n^{\beta_0 - o(1)}$.*

PROOF: Let $a_n = \mathbb{E}L(n)$. Note that, conditional on $\sigma(n)$, the three random variables $L(\sigma(n) - 1)$, $L(\sigma(n) - 1, n)$ and $\mathbf{1}_{\mathsf{up}_n}$ are all independent and distributed respectively as $L(k-1), L(n-k+1)$ and Bernoulli$(1/2)$, where $k = \sigma(n)$. Next, use

$\mathbb{E} \max\{L(k-1), L(k-1,n)\} \geq \max\{\mathbb{E}L(k-1), \mathbb{E}L(k-1,n)\} = \max\{a_{k-1}, a_{n-k+1}\} = a_{\max\{k-1,n-k+1\}}$ because $\{a_n\}$ is nondecreasing. This gives

$$
\begin{aligned}
a_n &= \frac{1}{2(n-1)} \sum_{k=2}^{n} (a_{k-1} + a_{n-k+1}) + \frac{1}{2(n-1)} \sum_{k=2}^{n} \mathbb{E} \max\{L(k-1), L(n-k+1)\} \\
&\geq \frac{1}{n-1} \sum_{k=1}^{n-1} a_k + \frac{1}{n-1} \sum_{k=\lceil n/2 \rceil}^{n-1} a_k \left(1 - \frac{1}{2}\delta_{k,n/2}\right).
\end{aligned}
\tag{4.1}
$$

The key observation is that for $\beta < \beta_0$ and sufficiently large n,

$$
n^\beta \leq \frac{1}{n-1} \sum_{k=1}^{n-1} k^\beta + \frac{1}{n-1} \sum_{k=\lceil n/2 \rceil}^{n-1} k^\beta \left(1 - \frac{1}{2}\delta_{k,n/2}\right).
\tag{4.2}
$$

Indeed, dividing (4.2) through by n^β, the right-hand side is a Riemann sum approximation for

$$
c_\beta := \int_0^1 x^\beta \, dx + \int_{1/2}^1 x^\beta \, dx
$$

which evaluates to

$$
\frac{1}{\beta+1} \left(2 - 2^{-\beta-1}\right).
$$

As a function of β, the quantity c_β decreases as β varies over $[0,1]$, passing through the value 1 at $\beta = \beta_0$. Therefore, for $\beta < \beta_0$, we have

$$
\frac{1}{n-1} \sum_{k=1}^{n-1} k^\beta + \frac{2}{n-1} \sum_{k=\lceil n/2 \rceil}^{n-1} k^\beta \left(1 - \frac{1}{2}\delta_{k,n/2}\right) = n^\beta \left[c_\beta - o(1)\right] > n^\beta
$$

provided that $n > N(\beta)$, where $N(\beta)$ is sufficiently large so that the $o(1)$ term is less than $c_\beta - 1$.

The rest is easy. Fixing $\beta < \beta_0$, we may pick $C = C(\beta)$ such that $a_n \geq Cn^\beta$ for all $n \leq N(\beta)$. We claim, by induction, that this is true for all $n > N(\beta)$ as well. Indeed, assuming it to be true for $n-1$, we see that the right-hand side of (4.1), which is a lower bound for a_n, is at least $C(\beta)$ times the right-hand side of (4.2). Because $n > N(\beta)$, we see from (4.2) that this is at least Cn^β, proving the claim.

We have shown that for all $\beta < \beta_0$ there exists a C such that $\mathbb{E}L(n) \geq Cn^\beta$ for all n. This completes the proof of Lemma 4.2. □

PROOF OF LOWER BOUND IN THEOREM 2.1: Fix $\gamma < \beta < \beta_0$. The preceding lemma gives $\mathbb{E}L(n) \geq Cn^\beta$. By Lemma 3.2, we have $\mathbb{P}[L(n) < n^\gamma] \leq n^{\gamma-\beta} \to 0$. □

5. Proof of upper bound

The proof of the upper bound in Theorem 2.1 is analogous to the proof Lemma 4.2 but in the reverse direction. It reduces to the following result.

Lemma 5.1. *Let β_1 be the positive solution to*

$$
\frac{2}{1+\beta} - \int_0^{1/2} \frac{x^\beta(1-x)^\beta}{x^\beta + (1-x)^\beta} = 1,
$$

whose decimal expansion begins 0.814834. Then $\mathbb{E}L_n \leq n^{\beta_1 + o(1)}$.

Before proving this, we record the following lemma.

Lemma 5.2. *Let X_1 and X_2 be independent random variables both of which are NBU. Let $a := \mathbb{E}X_1 \leq b := \mathbb{E}X_2$. Then*

$$\mathbb{E}(X_1 \wedge X_2) \geq \frac{ab}{a + b + 1}\,.$$

PROOF: Let Y_1 and Y_2 be independent geometric random variables (taking values $0, 1, 2, \ldots$) with respective means a and b. Corollary 3.5 applied to $-(X_1 \wedge s)$ shows that $\mathbb{E}(X_1 \wedge s) \geq \mathbb{E}(Y_1 \wedge s)$ for each fixed s. It follows that

$$\mathbb{E}(X_1 \wedge X_2 \,|\, X_2) \geq \mathbb{E}(Y_1 \wedge X_2 \,|\, X_2)$$

and hence that $\mathbb{E}(X_1 \wedge X_2) \geq \mathbb{E}(Y_1 \wedge X_2)$. Similar reasoning shows that $\mathbb{E}(Y_1 \wedge X_2) \geq \mathbb{E}(Y_1 \wedge Y_2)$. This last quantity may be computed exactly. This is one less than the minimum of two geometrics (started from 1) with respective success probabilities $1/(a + 1)$ and $1/(b + 1)$, which means a combined success probability of $(a + b + 1)/(ab + a + b + 1)$. One less than the mean is $ab/(a + b + 1)$, proving the lemma. □

PROOF OF LEMMA 5.1: Again let a_n denote $\mathbb{E}L(n)$. Fix n and again let $\sigma = \sigma(n)$ denote the time of the largest magnitude step up to time n. The identity $\max\{a, b\} = a + b - \min\{a, b\}$ gives

$$L(n) = L(\sigma - 1) + L(\sigma - 1, n) - \mathbf{1}_{\mathsf{down}_n} \min\{L(\sigma - 1), L(\sigma - 1, n)\}\,. \tag{5.1}$$

The random variables $\mathbf{1}_{\mathsf{down}_n}$, $L(\sigma - 1)$ and $L(\sigma - 1, n)$ are conditionally independent given σ. Now use Lemma 5.2 with $X = L(k)$ and $Y = L(n - k)$ where k is the minimum of $\sigma - 1$ and $n - \sigma + 1$. This gives

$$\mathbb{E}\left[\mathbf{1}_{\mathsf{down}_n} \min\{L(\sigma) - 1, L(\sigma - 1, n)\}\right] \geq \frac{1 - o(1)}{2} \frac{\mathbb{E}L(k) \cdot \mathbb{E}L(n - k)}{\mathbb{E}[L(k) + L(n - k)]}$$

where the $o(1)$ term is uniform in k as $n \to \infty$, coming from the ratio of $a + b + 1$ and $a + b$ when $a = \mathbb{E}L(k)$ and $b = \mathbb{E}L(n - k)$. Plugging this in to (5.1) after taking expectations gives

$$a_n \leq \frac{2}{n - 1} \sum_{k=2}^{n} a_k - \frac{2}{n - 1} \sum_{k=2}^{\lfloor n/2 \rfloor} \frac{1 - o(1)}{2} \frac{\mathbb{E}L(k) \cdot \mathbb{E}L(n - k)}{\mathbb{E}[L(k) + L(n - k)]}\,. \tag{5.2}$$

Again we play the trick of replacing a_k by k^β and approximating the sum by an integral. Pulling out a factor of n^β, the right hand side becomes

$$n^\beta \left[2 \int_0^1 x^\beta \, dx - \int_0^{1/2} \frac{x^\beta (1 - x)^\beta}{x^\beta + (1 - x)^\beta}\right] dx + o(1)\,. \tag{5.3}$$

The expression (5.3) is decreasing on $[0, 1]$ and passes through the value 1 at β_1. Now fix $\beta > \beta_1$, let $N(\beta)$ be large enough so that (5.3) is less than n^β for all $n \geq N(\beta)$. Choosing C so that $a_n \leq Cn^\beta$ for $n \leq N(\beta)$, the integral approximation then shows by induction that $a_n \leq Cn^\beta$ for all n, finishing the proof of Lemma 5.1. Invoking Markov's inequality then ayields the upper bound in Theorem 2.1. □

6. Real random walks

For whose who don't accept the Ultra-fat tailed distribution as a true random walk, we include the following result.

Theorem 6.1. *For any $\gamma < \beta_0$ there are real $\alpha, C > 0$ such that the length $L(n)$ of the LIS up to time n of the symmetric stable walk with index α has expectation at least Cn^γ.*

We begin with a lemma.

Lemma 6.2. *Fix any index $\alpha \in (0, 2)$. Let $\hat{\ell}(n) = \hat{\ell}_\alpha(n)$ denote the median of $L(n)$ for the symmetric stable walk of index α. Then the family $\{L(n)/\hat{\ell}(n) : n \geq 1\}$ is uniformly integrable. In particular,*

$$\mathbb{E}\left[\frac{L(n)}{\hat{\ell}(n)}\, \mathbf{1}_A\right] \leq h(\mathbb{P}(A))$$

for some function h with $\lim_{\varepsilon \downarrow 0} h(\varepsilon) = 0$.

PROOF: It suffices to show that $L(n)/(\hat{\ell}(n) + 1)$ has uniform exponential tails. Subadditivity (3.1) holds for any random walk, implying for any integer k the inequality $\mathbb{P}(L(n)/(\hat{\ell}(n) + 1) \geq k) \leq \mathbb{P}(L(n) \geq \hat{\ell}(n) + 1)^k \leq (1/2)^k$. $\qquad\square$

PROOF OF THEOREM 6.1: Let $W_n := \max\{|X_k| : 1 \leq k \leq n\}$, let σ_n be the almost surely unique argmax , i.e., $W_n = |X_{\sigma_n}|$, and let $Z_n := \sum_{k=1}^n |X_k|$. Let up'_n be the event that up_n occurs and $W_n > Z_n - W_n$. On up'_n, the recursion in Proposition 4.1 is satisfied at n.

Lemma 6.3.
$$\frac{W_n}{Z_n} \to 1 \ \text{as } \alpha \downarrow 0, \ \text{uniformly in } n. \tag{6.1}$$
Consequently, $\mathbb{P}(\mathsf{up}_n \setminus \mathsf{up}'_n) \to 0$ as $\alpha \to 0$, uniformly in n.

PROOF OF THEOREM 6.1 from Lemma 6.3: On down_n, the inequality is favorable: we can still choose to use only the longer segment, hence $\mathbb{E}L(n) \geq \mathbb{E}L[\max\{\sigma(n) - 1, n - \sigma(n) + 1\}]$. On up_n the inequality goes the wrong way, but the difference is bounded above by $(L(\sigma(n)-1)+L(\sigma(n)-1,n))(\mathbf{1}_{\mathsf{up}_n}-\mathbf{1}_{\mathsf{up}'_n})$. Taking expectations,

$$\mathbb{E}L_n \geq \frac{1}{n-1}\sum_{k=1}^{n-1}\mathbb{E}L_k + \frac{1}{n-1}\sum_{k=\lceil n/2\rceil}^{n-1}\mathbb{E}L_k - \mathbb{E}\Big(L(\sigma(n)-1)+L(\sigma(n)-1,n)\Big)(\mathbf{1}_{\mathsf{up}_n}-\mathbf{1}_{\mathsf{up}'_n}).$$

The subtracted term $\mathbb{E}\Big(L(\sigma(n) - 1) + L(\sigma(n) - 1, n)\Big)(\mathbf{1}_{\mathsf{up}_n} - \mathbf{1}_{\mathsf{up}'_n})$ is $o(\hat{\ell}(n))$ by (6.1) and Lemma 6.2. The approximation by Riemann sum and the resulting inequality then finish the proof as before. $\qquad\square$

Remark. This lemma is the only place we use specific properties of the step distribution, other than symmetry. The conclusion of the theorem will therefore hold for any symmetric distribution satisfying (6.1). There are many more extreme distributions, such as $Z = Re^X$ where X is Cauchy and R is Rademacher, for which the ratio of the greatest of n picks to the sum of the magnitudes of the other $n-1$ goes to infinity in probability. For such distributions, the conclusion holds for all $\gamma < \beta_0$.

PROOF OF LEMMA 6.3: It will be convenient to construct random walk steps $\{X_n^{(\alpha)}\}$ simultaneously for all $n \geq 1$ and $\alpha \in (0, 1)$, such that for fixed α, the collection $\{X_n^{(\alpha)}\}$ is IID symmetric stable-α. To do so, let $\mathcal{B}$ denote the Borel sets of $\mathbb{R}$ and choose a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting an IID sequence of Poisson point

processes $N_n : \Omega \times \mathcal{B} \to \mathbb{Z}^+$ with intensity $|x|^{-2}$. Mapping each point x to its signed $1/\alpha$ power $x^{(1/\alpha)} := \mathrm{sgn}\,(x) \cdot |x|^{1/\alpha}$ produces a Poisson process of intensity $c|x|^{-1-\alpha}$. Recall (see, e.g., Durrett 2010, Chapter 3, Section 7) that the sum $\int x\,dN(\omega, \cdot)(x)$ of ordinates of a Poisson process with intensity $|x|^{-1-\alpha}$ is a.s. absolutely convergent for $\alpha \in (0,1)$ and has the law of a symmetric stable-α. Therefore, the construction is completed by defining

$$X_n^{(\alpha)} := \int x^{(1/\alpha)}\,dN_n(\omega, \cdot)(x)\,. \tag{6.2}$$

Let M_n denote the maximum magnitude of a point in the support of the random measure $\Xi_n := \sum_{k=1}^n N_k$. The most that either $W_n^{(\alpha)}$ or $Z_n^{(\alpha)}$ can differ from $M_n^{1/\alpha}$ is the sum of the $1/\alpha$ powers of all other magnitudes of points in Ξ_n. Thus,

$$\left| \frac{W_n^{(\alpha)}}{M_n^{1/\alpha}} - 1 \right| \;\leq\; \int \left(\frac{|x|}{M_n} \right)^{1/\alpha}\,d\Xi_n^*(x) \tag{6.3}$$

$$\left| \frac{Z_n^{(\alpha)}}{M_n^{1/\alpha}} - 1 \right| \;\leq\; \int \left(\frac{|x|}{M_n} \right)^{1/\alpha}\,d\Xi_n^*(x) \tag{6.4}$$

where Ξ_n^* is Ξ_n without the atom at $\pm M_n$. The point process Ξ_n is a scaled copy of the process Ξ_1 and the same is true of Ξ_n^* and Ξ_1^*. It follows that the law of the self-scaled point process $M_n^{-1}\Xi_n$ is the same for all n. Consequently, the law of the random integral on the right-hand side of (6.3) and (6.4) does not depend on n.

For fixed sample point ω and fixed $\alpha \in (0,1)$, the atoms x of $\Xi_n^*(\omega, \cdot)$ are absolutely summable and each has maximum modulus less than M_n. Hence, each $(|x|/M_n)^{1/\alpha}$ goes to zero as $\alpha \downarrow 0$. By dominated convergence, the common right-hand side of (6.3) and (6.4) goes to zero. Pointwise convergence implies convergence in probability; because the law of the right-hand side of (6.3) and (6.4) is independent of n, the convergence is uniform in n. Thus, uniformly in n, both $Z_n^{(\alpha)}/M_n^{1/\alpha}$ and $W_n^{(\alpha)}/M^{1/\alpha}$ converge in probability to 1, implying (6.1). $\square$

7. Further remarks and questions

One natural question is to prove that the exponent $\lim \log L(n)/\log n$ exists.

Another is whether we can obtain better bounds on the exponent by finding a functional form for the distribution which yield an inequality when passed through the recursion.

Neither exponent β_0 nor β_1 is sharp. The proof of

$$\liminf \frac{\log \mathbb{E}L(n)}{\log n} \geq \beta_0$$

in fact computes the correct exponent, namely β_0, for the length of the greedy increasing subsequence. The GIS is defined by splitting the sequence at the location σ of the maximum step, and, if the step is a downward step, throwing away the smaller interval rather than the interval with the shorter LIS. The length Z_n of the greedy increasing subsequence obeys the recursion of Proposition 4.1 but with the max taken on the inside. This seems likely to give an exponent not too far from the correct exponent, but it gives up a non-negligible amount in the recursion and cannot be sharp.

The proof of

$$\limsup \frac{\log \mathbb{E}L(n)}{\log n} \leq \beta_1$$

does not, as far as we know compute anything natural. This bound could be improved by finding the correct function $\phi(n,k)$ that computes a better lower bound on $\mathbb{E}\min\{L(k), L(n-k)\}$. Lemma 5.2 is best possible assuming only the NBU property, as the geometric random variable is the extreme case. However, we know more about $L(k)$. For example, when j and k/j are integers then

$$\mathbb{P}(L(k) < \varepsilon\mathbb{E}L(k)) \leq \mathbb{P}(L(k/j) < \varepsilon\mathbb{E}L(k))^j .$$

If $\mathbb{E}L(j) = j^{\beta+o(1)}$ then taking choosing j so that $\varepsilon = cj^{-\beta}$ makes $\mathbb{P}(L(k/j) < \varepsilon\mathbb{E}L(k)) < 1/2$ and results in

$$\mathbb{P}(L(k) < \varepsilon\mathbb{E}L(k)) \leq 2^{-\varepsilon^{-(1+o(1))/\beta}} .$$

The lower tails on $L(k)$ are thus expected to be very small; this ought to lead to a better lower bound on $\mathbb{E}L(k) \wedge L(n-k)$, hence a better exponent in Lemma 5.1 and in Theorem 2.1.

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