Double-dimers, the Ising model and the hexahedron recurrence

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June 19, 2015

Abstract

We define and study a recurrence relation in $\mathbb{Z}^3$, called the hexahedron recurrence, which is similar to the octahedron recurrence (Hirota bilinear difference equation) and cube recurrence (Miwa equation). Like these examples, solutions to the hexahedron recurrence are partition sums for edge configurations on a certain graph, and have a natural interpretation in terms of cluster algebras. We give an explicit correspondence between monomials in the Laurent expansions arising in the recurrence with certain double-dimer configurations of a graph. We compute limit shapes for the corresponding double-dimer configurations.

The Kashaev difference equation arising in the Ising model Y-Delta relation is a special case of the hexahedron recurrence. In particular this reveals the cluster nature underlying the Ising model. The above relation allows us to prove a Laurent phenomenon for the Kashaev difference equation.

1 Introduction

1.1 Overview

A function $a : \mathbb{Z}^3 \to \mathbb{C}$ is said to satisfy the *octahedron recurrence* or *Hirota bilinear difference equation* if at all points $v \in \mathbb{Z}^3$

$$a_{(1)}a_{(23)} = a_{(2)}a_{(13)} + a_{(3)}a_{(12)} \tag{1.1}$$

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†The research of RP was supported by the NSF Grant DMS-1209117

(here $a_{(S)} := a_{v+\sum_{i\in S} e_i}$ represents a evaluated at the translate of v by the basis vectors in S, e.g. $a_{(12)}$ represents $a_{v+e_1+e_2}$). The term "octahedron recurrence" was coined by Propp (see [17]), though the recurrence first appeared in Dodgson [3] as a means of recursively computing determinants: up to an affine change of indices (and some sign changes) it is the recurrence satisfied by determinants of contiguous submatrices. In this setting it is known as *Dodgson condensation*. The octahedron recurrence is fundamental in combinatorics, statistical mechanics, cluster algebras, and integrable systems, see e.g. [13].

There are several similar recurrences. The best known is the *cube recurrence* or *Miwa equation*. A function $g : \mathbb{Z}^3 \to \mathbb{C}$ is said to satisfy the Miwa equation or cube recurrence if

$$g_{(123)}g = g_{(1)}g_{(23)} + g_{(2)}g_{(13)} + g_{(3)}g_{(12)}. \tag{1.2}$$

This recurrence also has its roots in the 19^{th} century: Kennelly [10] discovered the so-called Y-Delta identity for resistor networks[1]. Under a certain change of variables (see [7, 8]) this transformation can be written as a cube recurrence. This is not quite the same Y-Delta transformation we study in Section 7, though it has the same underlying graph transformations.

In [7], see also [8], it was noticed that the cube recurrence is a specialization of a more fundamental recurrence, which we call the *cuboctahedron recurrence*. This is a recurrence on a function on the edges of the cubes of the $\mathbb{Z}^3$ lattice. If certain monomial equations are satisfied then the cuboctahedron recurrence reduces to the cube recurrence. Like the octahedron recurrence, the cuboctahedron recurrence arises from cluster algebras, and in fact is a composition of cluster algebra mutations on a certain planar graph. This leads to a cluster algebra interpretation of the cube recurrence, and in particular allows one to prove a Laurent phenomenon as shown in [7].

A less-well known recurrence is the *Kashaev recurrence* [9]. A function $f : \mathbb{Z}^3 \to \mathbb{C}$ is said to satisfy the Kashaev recurrence if the following expression evaluates to zero:

$$\begin{aligned} &f^2 f_{(123)}^2 + f_{(1)}^2 f_{(23)}^2 + f_{(2)}^2 f_{(13)}^2 + f_{(3)}^2 f_{(12)}^2 \\ &-2f_{(1)}f_{(2)}f_{(23)}f_{(13)} - 2f_{(1)}f_{(3)}f_{(23)}f_{(12)} - 2f_{(3)}f_{(2)}f_{(12)}f_{(13)} \\ &-2ff_{(123)}(f_{(1)}f_{(23)} + f_{(2)}f_{(13)} + f_{(3)}f_{(12)}) - 4ff_{(23)}f_{(13)}f_{(12)} - 4f_{(123)}f_{(1)}f_{(2)}f_{(3)}\,. \end{aligned} \tag{1.3}$$

This recurrence arises in the Y-Delta move (Yang-Baxter equation) for the Ising model.

Our main goal in this paper is to define another recurrence, generalizing the Kashaev recurrence, called the *hexahedron recurrence*[2]. This is a relation for a function defined on the faces and vertices of the $\mathbb{Z}^3$ cubic tiling. Given four functions

$$h, h^{(x)}, h^{(y)}, h^{(z)} \colon \mathbb{Z}^3 \to \mathbb{C}$$

[1] This transformation was used in [14] to show that resistance could be computed in linear time.

[2] The hexahedron is another name for the cube, but emphasizing the fact that it has 6 faces. We chose this nomenclature because our variables sit on both the vertices and faces of a cube.

(where we think of h as being the function on the vertices of the cubes, $h_v^{(x)}$ as being the value of the function on the "yz"-face with vertices $\{v, v+e_2, v+e_2+e_3, v+e_3\}$, and similarly for $h_v^{(y)}$ and $h_v^{(z)}$ on the "xz" and "xy" faces respectively) we say they satisfy the hexahedron recurrence if the following equations are satisfied for all $v \in \mathbb{Z}^3$.

$$\begin{aligned}
h_{(1)}^{(x)}h^{(x)}h &= h^{(x)}h^{(y)}h^{(z)} + h_{(1)}h_{(2)}h_{(3)} + hh_{(1)}h_{(23)} && \text{(1.4a)}\\
h_{(2)}^{(y)}h^{(y)}h &= h^{(x)}h^{(y)}h^{(z)} + h_{(1)}h_{(2)}h_{(3)} + hh_{(2)}h_{(13)} && \text{(1.4b)}\\
h_{(3)}^{(z)}h^{(z)}h &= h^{(x)}h^{(y)}h^{(z)} + h_{(1)}h_{(2)}h_{(3)} + hh_{(3)}h_{(12)} && \text{(1.4c)}\\
h_{(123)}h^2h^{(x)}h^{(y)}h^{(z)} &= (h^{(x)}h^{(y)}h^{(z)})^2 \\
&\quad + h^{(x)}h^{(y)}h^{(z)}\left[2h_{(1)}h_{(2)}h_{(3)} + hh_{(1)}h_{(23)} + hh_{(2)}h_{(13)} + hh_{(3)}h_{(12)}\right] \\
&\quad + (h_{(1)}h_{(2)} + hh_{(12)})(h_{(1)}h_{(3)} + hh_{(13)})(h_{(2)}h_{(3)} + hh_{(23)}). && \text{(1.4d)}
\end{aligned}$$

Here again $h_{(1)} = h_{v+e_1}$ and so on. In Section 7 below we will show that the Kashaev recurrence is a special case of the hexahedron recurrence. Understanding the cluster algebra structure of the Y-Delta transformation for the Ising model was our initial motivation for defining the hexahedron recurrence.

The octahedron, cuboctahedron and hexahedron recurrences have an underlying cluster algebra structure in the sense of [6]. This is based on the local transformation (mutation) called *urban renewal*, see Figure 3. For example, the octahedron recurrence can be thought of as a "periodic" urban renewal step on the square grid graph; the cuboctahedron recurrence can be decomposed into a product of four periodic urban renewal steps, see [7]. The hexahedron recurrence can be decomposed into a product of 6 periodic urban renewals. In particular this allows us to write $h_{i,j,k}, h_{i,j,k}^{(x)}, h_{i,j,k}^{(y)}, h_{i,j,k}^{(z)}$, for $i+j+k$ large, as Laurent polynomials in the initial values $\{h_{i,j,k}\}_{0\le i+j+k\le 2}$ and $\{h_{i,j,k}^{(x)}, h_{i,j,k}^{(y)}, h_{i,j,k}^{(z)}\}_{i+j+k=0}$.

1.2 Organization

The paper falls into four parts: background and constructions (Sections 1–3), Combinatorial interpretation (Section 4), limit theorems (Sections 5–6), and Kashaev's recursion (Section 7). It may not be obvious how these parts fit together; for this reason we include the following guide.

In the remainder of this section we review the combinatorial and statistical mechanical models obeying the cube and octahedron recurrences. Speyer [17] and Carroll and Speyer [2] respectively gave combinatorial interpretations of the terms in the Laurent expansions arising in the octahedron and cube recurrences. Our first main result does the same for the hexahedron recurrence. We give an explicit bijection between the monomials of the Laurent polynomials which arise and certain "taut" double-dimer covers of a sequence of planar graphs Γ_n.

Section 2 gives rigorous definitions for dimer models and the double-dimer model and discusses connections to cluster algebras. There is space only for an extremely brief excerpt from the relevant theory of cluster algebras; for a proper introduction, the reader is referred to the seminal paper [6] or to the expository work [**?**]. To get from these locally defined recursions on general graphs to periodic recursions on infinite lattices requires *stepped surfaces*. These graphs, arising from order ideals on the cubic lattice and reminiscent of the Conway-Thurston height functions, are discussed in Section 3. This section finishes with the definition of taut double-dimer configurations.

Section 4 gives the precise statement and proof of our main result, the combinatorial interpretation for the monomials in the recursion.

Sections 5 and 6 concern limit shape theorems for the double-dimer ensemble. These results are in some sense an application of known technology: periodic recursions give rise to rational generating functions. These in turn yield limit theorems. The limit shapes are algebraic and may be read off relatively easily from the recursion. The methods of proof, however, require big machines and some technical hypotheses, as well as some methodology which is not adequately documented. Section 5 addresses these gaps in the literature. Sections 5.1 and 5.2 outline a general method to get from an algebraic recurrence on a lattice to a linear recurrence for macroscopic quantities of the related ensemble. Section 5.3 shows how these linear recurrences yield shape theorems. This section condenses quite a bit of machinery from multivariate analytic combinatorics. As with the cluster algebra theory, only the briefest excerpt is possible; the reader is referred to [15] for a more complete introduction. Section 6 then applies the results of Section 5 to the specific case of the hexahedron recurrence. This section is mostly computational and is self contained, assuming the results from Section 5.

The final section, Section 7, shows how Kashaev's recursion for the Ising model, a particularly nicely parametrized version of the Y-Delta moves for the Ising model, may be embedded as a special case of the hexahedron recursion when certain other periodic relations are imposed. As a result, Kaashaev's quadratic recursion is replaced by the linear recursion ensuing from the hexahedron model. This is summed up in Propositions 7.5 and 7.6, after which further consequences are stated. Our main result from this section is a limit shape theorem for random taut double-dimer covers of Γ_n. The arctic boundary is an algebraic curve defined from the characteristic polynomial of the recurrence relation.

As the discussion of cluster algebras, combinatorial gadgets, statistical mechanical limits and application to the Ising model is already substantial, we will not discuss the integrable nature of these systems. The hexahedron recurrence is a special case of a dimer integrable system. Integrable properties of such systems are discussed in [8].

1.3 Combinatorial and statistical mechanical results

We may picture the values of $h^{(x)}$, $h^{(y)}$ and $h^{(z)}$ as each sitting in the middle of a square face of the integer lattice. Letting

$$\mathbb{Z}^3_{1/2} := \{(x, y, z) \in (1/2)\mathbb{Z}^3 : x + y + z \in \mathbb{Z}\}$$

we then interpret $h^{(x)}, h^{(y)}, h^{(z)}$ as extending h to $\mathbb{Z}^3_{1/2}$ via $h^{(x)}(i, j, k) = h(i, j+1/2, k+1/2)$, $h^{(y)}(i, j, k) = h(i + 1/2, j, k + 1/2)$, and $h^{(z)}(i, j, k) = h(i + 1/2, j + 1/2, k)$.

Note that the equations (1.4a)–(1.4d) are not only homogeneous, but are 1-homogeneous: the sum of all indices of each monomial is 3, e.g., the first monomial is the product of three variables with respective indices $(1, 1/2, 1/2)$, $(0, 1/2, 1/2)$ and $(0, 0, 0)$. Also, given the values of h on seven of the corners and their three included faces of a cube, the values on the eighth corner and the three remaining faces are determined as rational functions of these; the locations of the new values are precisely those obtained if one increases a pile by a single cube. The cluster nature of the hexahedron recurrence immediately implies:

Theorem 1.1. *The values $\{h(v) : v \in \mathbb{Z}^3_{1/2}, v_1 + v_2 + v_3 \geq 0\}$ are Laurent polynomials in the values of $h(v)$ with v varying over integer vectors with $0 \leq v_1 + v_2 + v_3 \leq 2$ and half-integer vectors with $v_1 + v_2 + v_3 = 1$.*

Our main result, Theorem 4.1 below, gives a combinatorial interpretation of the terms of these Laurent polynomials: the monomials of the Laurent polynomial h_{iii} are in bijection with taut double-dimer configurations of the graph Γ_{3i}.

This result parallels results for the octahedron and cube recurrences. The octahedron recurrence may be used to express a_v as a Laurent polynomial in the values a_w as w ranges over variables in an initial graph. When the initial graph is the grid $\mathbb{Z}^2$ this Laurent polynomial is a generating function the ensemble of perfect matchings of the **Aztec diamond** graph, associated with the region in the initial graph lying in the shadow of v; see, e.g., [5, 17]. Setting the initial indeterminates all equal to one allows us to count perfect matchings; in general the indeterminates represent multiplicative weights, which we may change in certain natural ways to study further properties of the ensemble of perfect matchings. The cube recurrence (1.2) also has a combinatorial interpretation. Its monomials are in bijection with **cube groves**, first defined and studied in [2, 16] where they were called simply "groves". In a cube grove, each edge of a large triangular region in the planar triangular lattice is either present or absent. The allowed configurations are those in which the edge subsets contain no cycles and no islands (thus they are **essential spanning forests**), and the connectivity of boundary points has a prescribed form.

Both Aztec diamond matchings and cube groves have limiting shapes. Specifically, as the size of the box goes to infinity, there is a boundary, which is an algebraic curve, outside of which there is no entropy (the system is periodic almost surely) and inside of which there is positive entropy per site (each type of local configuration occurs with positive probability).

In the case of the Aztec diamond and cube grove, the algebraic curve is the inscribed circle, but for related ensembles, much more general algebraic curves are obtained; for example in the **fortress** tiling model shown on the right of Figure 1, the bounding curve is a degree-8 algebraic curve[3], see [11].

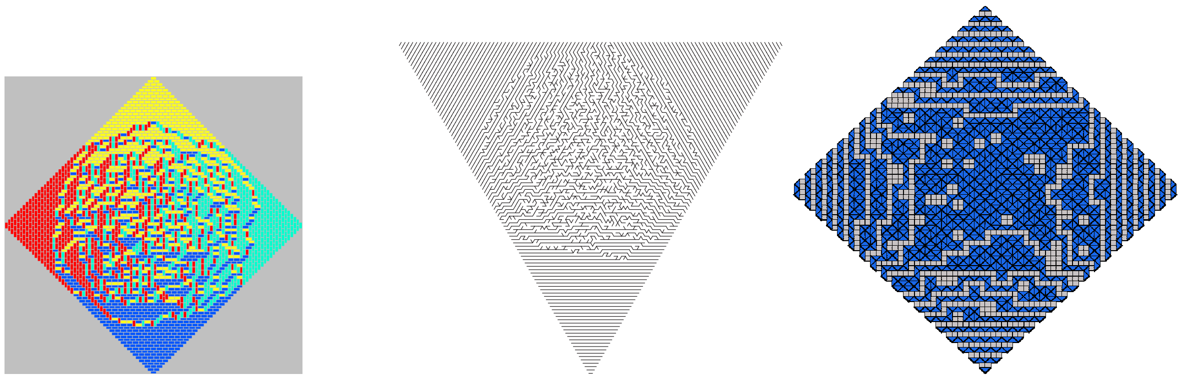

Figure 1: An Aztec tiling, a cube grove and a fortress tiling

The hexahedron recurrence also has a statistical mechanical interpretation. In Section 2 we define the **double-dimer** model on a finite bipartite graph. Terms in the hexahedron recurrence count certain types of double-dimer configurations called *taut* configurations. A randomly generated taut double-dimer configuration is shown in Figure 2.

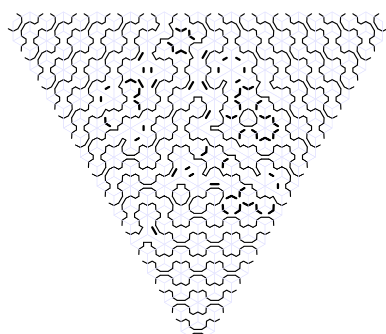

Figure 2: A random taut double-dimer configuration

1.4 Acknowledgements.

We would like to thank Philippe Di Francesco, Sergey Fomin, David Speyer and Dylan Thurston for helpful discussions. Thanks also to two anonymous referees for their useful and detailed reports.

[3]Dubbed the "octic circle" by the fun-loving pioneers of this subject.

2 Dimer model

2.1 Definitions

We will use the following graph theoretic terminology. A **bipartite** graph is a graph $\Gamma = (E, V)$ together with a fixed coloring of the vertices into two colors (black and white) such that edges connect only vertices of different colors. By **planar** graph we mean one with a distinguished planar embedding. The *dual graph* Γ_* to a planar bipartite graph Γ has a vertex f_* for each face f of Γ and edge e_* connecting f_* and g_* where f and g are the two faces containing e.

Let Γ be a finite bipartite graph with positive edge weights $\nu : E \to \mathbb{R}_+$. A **dimer cover** or **perfect matching** is a collection of edges with the property that every vertex is an endpoint of exactly one edge. The "dimers" of a dimer cover are the chosen edges (terminology suggesting a collection of bi-atomic molecules packed into the graph). We let $\Omega_d(\Gamma)$ be the set of dimer covers and we define the probability measure μ_d on Ω_d giving a dimer cover $m \in \Omega_d$ a probability proportional to the product of its edge weights.

A **double-dimer configuration** is a sum of two dimer covers: it is a disjoint union of cycles and doubled edges that contains each vertex precisely once. The **double dimer measure** μ_{dd} is the probability measure defined by taking the sum of two μ_d-independent dimer covers.

2.2 Edge variables, face variables, and cluster variables

2.2.1 Gauge transformations

A gauge transformation consists in multiplying the edge weights of edges incident to a vertex v by a constant. This leaves μ_d unchanged since every dimer cover uses exactly one edge incident to v. The gauge transformations form a group isomorphic to $\mathbb{R}_+^{|V|-1}$ under pointwise multiplication. The -1 in the exponent comes from the fact that there is a single relation: the gauge transformation that multiplies the set of edges incident to v by λ_v for each v is the identity if and only if $\lambda_v \equiv \lambda^{\sigma(v)}$ where λ is any positive real number and σ is the function taking value 1 on white vertices and -1 on black vertices. The quotient space of edge weights modulo gauge transformations is isomorphic to $\mathbb{R}_+^c$, where c is the number of linearly independent cycles of Γ.

For a planar graph (meaning a specific planar embedding has been chosen), whose set of faces is denoted F, it is natural to coordinatize the free abelian group of edge weights modulo gauge transformations with variables $\{X_f\}_{f \in F}$. Here, for a face f, the quantity X_f is the alternating product of the edge weights around that face: orient all edges from white to black, then take the product over edges e in the face of $\nu(e)^{\delta(e)}$ where $\delta = 1$ if the edge e points counterclockwise with respect to the planar embedding and $\delta(e) = -1$ if the edge e points clockwise. These X_f-variables are called *cross-ratio variables*[4]. They have no relations (we do not assign a variable to the outer face).

We will specialize to the case where the edge variables, and hence the face variables, are parametrized by a third set of variables $\{A_f\}_{f\in F}$ called *cluster variables*, also living on faces of Γ. The relationship with the edge variables is as follows. Given $A : F \to \mathbb{C}$, define $\nu_A : E \to \mathbb{C}$ by

$$\nu_A(e) = \frac{1}{A_f A_g}\,, \tag{2.1}$$

where f and g are the two faces containing e. It may be checked that for any face $f \in F$,

$$X_f = \prod_{e\in F} A_f^{\delta(e,f)},$$

where $\delta(e,f) = \pm 1$ according to whether the dual edge e^*f^* sees a white vertex on the right or left. We summarize this in a commuting diagram.

$$\begin{tikzcd}
\{A_f\}_{f\in F} \arrow[rr, "\frac{1}{A(f)A(g)}"] \arrow[dr, "\text{alt. prod.}"', "\rho"] & & \{\nu(e)\}_{e\in E} \arrow[dl, "\pi"', "\text{alt. prod.}"] \\
& \{X_f\}_{f\in F} &
\end{tikzcd}$$

The map π is surjective; its kernel is the set of gauge functions.

Note: the map ρ is not surjective, but contains all the assignments of X variables discussed in this paper.

2.3 Urban renewal

Certain local rearrangements of a bipartite graph Γ preserve the dimer measure μ_d, see [12]. These are: parallel edge reduction, vertex contraction/splitting, and urban renewal. To clarify the terms, suppose graphs G and G' share a subset W of vertices, on which the induced subgraphs are equal. The term "local rearrangement" refers to whatever changes are necessary on the complement of W to turn G into G'. Preservation of the measure means a coupling in the following sense: there is a measure μ'' on pairs (m, m') of configurations on G and G' respectively, such that m has law μ, m' has law μ', and the restrictions of m and m' to W are equal.

The simplest local rearrangement preserving the dimer measure μ_d is *parallel edge reduction.* If there are two parallel edges e and f of Γ between a pair of endpoint, having respective weights a and b, then we can replace these with a single edge g of weight $a + b$. There is

[4]Historically we use notation X for these variables although from the point of view of cluster algebras these are "coefficient variables" and are represented with Ys.

an obvious coupling of the dimer measure in which m' is obtained from m by replacing e or f, if present, by g. Going backward, one could obtain m from m' by replacing g by e with probability $a/(a+b)$ and otherwise replacing g by f.

Vertex contraction is described as follows. Given a vertex v of degree 2, with equal weights on its two edges, one can contract its two edges, erasing v and merging its two neighbors into one vertex. The faces of the new graph are in bijection with the faces of the old graph, the only difference being that two faces have each lost two consecutive equally weighted edges. The dimer measure μ on the original weighted graph Γ may be coupled to the new dimer measure μ' on the new contracted graph Γ' as follows. To sample from μ', sample a matching m from μ and then delete whichever edge of m contains v; the new set of dimers, m' will be a perfect matching on Γ' and it is obvious that the law of m' is μ'. The inverse of this contraction operation, splitting a vertex in two and adding a vertex of degree 2 between them (with equal edge weights on the two edges) is called *vertex splitting*: a vertex w is split into two vertices w_1 and w_2, each incident to a proper subset of the vertices formerly incident to w (this subset being an interval in the cyclic order induced by the planar embedding); a new vertex v is introduced whose neighbors are w_1 and w_2. The new planar embedding is obvious. To sample from the new measure μ', sample from μ and then add either vw_1 or vw_2 depending on which vertex w_1, w_2 is not matched.

The more interesting local rearrangement is called **urban renewal**. It involves taking a quadrilateral face, call it 0, and adding "legs". This is shown in Figure 3, ignoring for the moment the specific values $a_0, \ldots, a_4$ shown for the pre-weights $A(0), \ldots, A(4)$. Let us

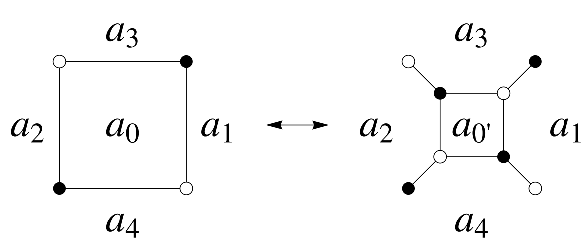

Figure 3: Under urban renewal the central variable a_0 changes from a_0 to $a_{0'} = \frac{a_1a_2+a_3a_4}{a_0}$.

designate the faces around face 0 by the numbers 1, 3, 2 and 4. Each of these faces gains two new edges. In the new graph Γ', there are faces $1', 3', 2'$ and $4'$ each with two more edges than the corresponding face $1, 2, 3, 4$. There is a face $0'$ which is also square. Each other face f of Γ corresponds to a face f' of Γ' with the same number of edges as f. There are four new neighboring relations among faces: $1', 3', 2'$ and $4'$ are neighbors in cyclic order, in addition to any neighboring relations that may have held before. It is not necessary, before or after, that the four faces be distinct. The point of urban renewal is to give a corresponding adjustment of the weights that preserves μ_d. This is most easily done in terms of the A variables. The following proposition is proved in [17].

Proposition 2.1 (urban renewal)**.** *Suppose 0 is a quadrilateral face of* Γ*. Let* Γ' *be con-*

structed from Γ by urban renewal. Define the new pre-weight function $A : F' \to \mathbb{C}$ by $A(f') = A(f)$ if $f \neq 0$ and

$$A(0') := \frac{A(1)A(2) + A(3)A(4)}{A(0)}.$$

The dimer measures μ on Γ and μ' on Γ' can be coupled so as to agree on every edge other than the four edges bounding face 0 in Γ and the eight edges touching face $0'$ in Γ'. $\square$

There are several equivalent versions of urban renewal, all of which are related to each other by vertex contraction and splitting. Two of these are depicted in Figure 4.

Figure 4: Other variants of urban renewal.

Proposition 2.2 ([8]). *The transformation of (Γ, A) to (Γ', A') under urban renewal is a mutation operation in the sense of cluster algebras. Consequently, the final variables after any number of urban renewals are Laurent polynomials in the original variables $\{A_f : f \in F\}$.*

For readers familiar with cluster algebras, the underlying quiver is the dual graph, with edges directed so that white vertices are on the left.

2.4 Superurban renewal transformation for the dimer model

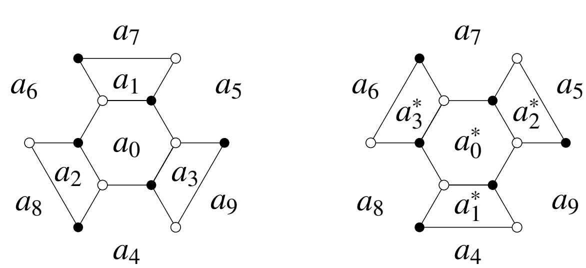

Figure 5: Under superurban renewal the variables a_0, a_1, a_2, a_3 change as indicated in (2.2a)-(2.2d).

A more complicated local transformation is the *superurban renewal* shown in Figure 5. Figure 6 shows how this can be decomposed into a sequence of six urban renewals. Just as urban renewal is the basis for the octahedron recurrence, we will see that superurban renewal is the basis for the hexahedron recurrence (see Section 3). In section 7 we show how superurban renewal specializes to the Y-Delta transformation for the Ising model.

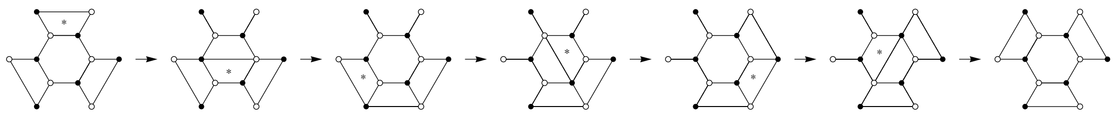

Figure 6: Writing a superurban renewal as a composition of urban renewals. The stars indicate which face undergoes urban renewal

Lemma 2.3. *Under a superurban renewal the A variables transform as in Figure 5, with*

$$a_1^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_4a_7}{a_0a_1} \tag{2.2a}$$

$$a_2^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_5a_8}{a_0a_2} \tag{2.2b}$$

$$a_3^* = \frac{a_1a_2a_3 + a_4a_5a_6 + a_0a_6a_9}{a_0a_3} \tag{2.2c}$$

$$a_0^* = \frac{1}{a_0^2a_1a_2a_3}\Big[a_1^2a_2^2a_3^2 + a_1a_2a_3(2a_4a_5a_6 + a_0a_4a_7 + a_0a_5a_8 + a_0a_6a_9) + (a_5a_6 + a_0a_7)(a_4a_5 + a_0a_9)(a_4a_6 + a_0a_8)\Big] \tag{2.2d}$$

Proof. The proof is a computation: evaluate the superurban renewal as a composition of six urban renewals as in Figure 6, then six times use the urban renewal formula from Figure 3. □

Iterating Proposition 2.2 proves the following result.

Corollary 2.4 (Laurent property for superurban renewal)**.** *Under iterated superurban renewal, all new variables are Laurent polynomials in the original variables.* □

3 Picturing superurban renewal via stepped surfaces

3.1 Graphs associated with stepped surfaces

A **stepped solid** in $\mathbb{R}^3$ is a union U of lattice cubes $[i, i+1] \times [j, j+1] \times [k, k+1]$ which is downwardly closed, meaning that if a cube B is in U then so is any translation of B by negative values in any coordinate. In particular, all our stepped solids are infinite. A **stepped surface** is the topological boundary of a stepped solid. Every stepped surface is the union of lattice squares and every lattice square has vertex set of the form $\{v, v+e_i, v+e_j, v+e_i+e_j\}$ for some $v \in \mathbb{Z}^3$ and some integers $1 \le i < j \le 3$. For each stepped surface ∂U its 1-skeleton ∂U_1 (the topological complex consisiting of the vertices and edges of ∂U

but no faces) is a planar graph. We associate to U another graph, the **associated graph** $\Gamma(U)$, obtained by starting with the dual graph of ∂U_1 and replacing each vertex by a small quadrilateral, as illustrated in Figure 7.

Figure 7 shows the so-called **4-6-12** graph, which is the graph $\Gamma(U)$ when U is the union of all cubes lying entirely within the region $\{(x, y, z) : x + y + z \leq 2\}$. For general stepped surfaces, the faces of $\Gamma(U)$ will be of two types: there is a quadrilateral face centered at the center of each face of ∂U_1 and there is a $(2k)$-gonal face centered at each vertex of ∂U_1, where k is the number of faces of the surface ∂U_1 coming together at the vertex, each contributing one edge and two half-edges ("legs"). The **label** of the face f is the coordinates of the face center or vertex at which f is centered. All face labels are elements of the set

$$\mathbb{Z}^3_{1/2} := \{(x, y, z) \in (1/2)\mathbb{Z}^3 : x + y + z \in \mathbb{Z}\} .$$

The **canonical variables** are the labels of the 4-6-12 graph, namely all elements of $\mathbb{Z}^3$ at levels 0, 1 or 2 and all elements of $(1/2)\mathbb{Z}^3$ at level 1.

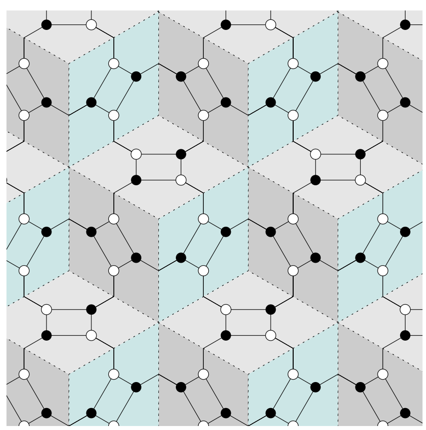

Figure 7: The 4-6-12 graph is drawn on the stepped surface U bounding the union of cubes up to level 2

3.2 Superurban renewal for associated graphs

Let U be a stepped solid with stepped surface ∂U and associated graph $\Gamma(U)$. Suppose that (i, j, k) is a point of ∂U which is a local minimum with respect to the height function $i + j + k$. In other words, $(i - 1, j, k)$, $(i, j - 1, k)$ and $(i, j, k - 1)$ are all in the interior of U. Let U^{+ijk} be the union of U with the cube $[i, i + 1] \times [j, j + 1] \times [k, k + 1]$. The following facts are easily verified by inspection.

Proposition 3.1 (superurban renewal is adding a cube)**.** *Suppose U and (i, j, k) are as above.*

(i) The face in $\Gamma(U)$ *corresponding to* (i,j,k) *is a hexagon.*

(ii) The graph $\Gamma(U^{+ijk})$ *is obtained from the graph* $\Gamma(U)$ *by superurban renewal at this hexagon.*

(iii) The variables associated with each face of $\Gamma(U)$ *transform under superurban renewal according to the hexahedron recurrence* (1.4a)–(1.4d), *provided we interpret*

$$\begin{aligned} h(i,j,k) &= A(i,j,k) \\ h^{(x)}(i,j,k) &= A(i,j+1/2,k+1/2) \\ h^{(y)}(i,j,k) &= A(i+1/2,j,k+1/2) \\ h^{(z)}(i,j,k) &= A(i+1/2,j+1/2,k)\,. \end{aligned} \qquad \square$$

3.3 Cubic corner graph and taut dimer configurations

We now know that adding a cube to a downwardly closed stepped solid corresponds to superurban renewal on the associated graph, which corresponds to the use of the hexahedron recurrence to write the top variable in terms of lower variables. These representations commute. To state this more precisely, let U_0 be the stepped solid coincident with the closed negative orthant. The associated graph $\Gamma(U_0)$ is called the **cubic corner graph** and is shown on the left of Figure 8.

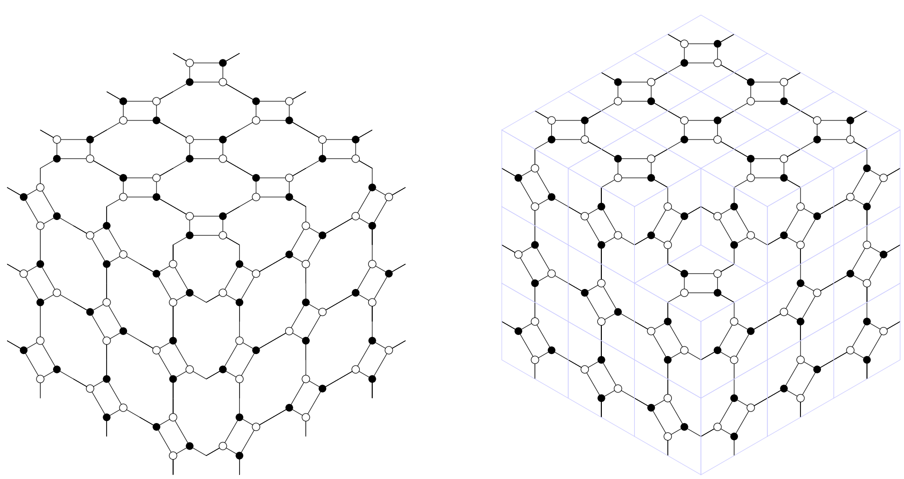

Figure 8: Left: the cubic corner graph. Right: after removing the topmost cube.

Let $\mathcal{L}$ be the lattice poset of all stepped solid subsets of U_0 containing all but finitely many cubes of U_0. For each $U \in \mathcal{L}$, one may add a finite sequence of cubes resulting in U_0. Therefore, a finite sequence of superurban renewals represents $A(0,0,0)$ in terms of the variables in the finite set $\mathcal{I}(U)$, defined to be those associated with faces and vertices of the stepped surface ∂U that are in the union of the removed lattice cubes.

Proposition 3.2. *(i) The rational function F representing $A(0,0,0)$ in terms of the variables in $\mathcal{I}$ is a Laurent polynomial. (ii) If $U' \subset U$ in $\mathcal{L}$ and the representation of each variable $w \in \mathcal{I}(U)$ in terms of variables in $\mathcal{I}(U')$ is substituted into F, the resulting Laurent polynomial is the representation of $A(0,0,0)$ in terms of variables in $\mathcal{I}(U')$.*

Proof. By Proposition 3.1, the expression F is obtained by a sequence of superurban renewals. By definition, each of these is a sequence of six urban renewals, hence part (i) follows from Proposition 2.2. Part (ii) is a consequence of the lack of relations among the variables in any stepped surface. $\square$

Our combinatorial interpretations of these formulae take place on the associated graphs $\Gamma(U)$. An example to keep in mind is U_{-n}, defined to be those cubes of U_0 lying entirely within the halfspace $\{(x,y,z) : x+y+z \leq -n\}$. This solid and its associated graph are illustrated for $n=1$ (only the top cube removed) on the right of Figure 8. This solid is a subset of Z_{-n}, the stepped solid defined by $x+y+z \leq -n$, whose associated graph is isomorphic to the 4-6-12 graph of Figure 7. The labels of Z_{-n} are precisely the points of $\mathbb{Z}^3$ at levels $-n-2, -n-1$ and $-n$ together with the half integer points at level $-n-1$. The hexahedron recurrence imposes no relations on this set of variables, hence from the point of view of determining $A(0,0,0)$ as a function of the variables in $\mathcal{I}(U_{-n})$, we might equally well think of the initial variables as being all of those in $\Gamma(Z_{-n})$.

We define a double-dimer configuration m_0 on the cubic corner graph Γ_0 as on the left of Figure 9. This configuration m_0 plays the role of our initial configuration. This

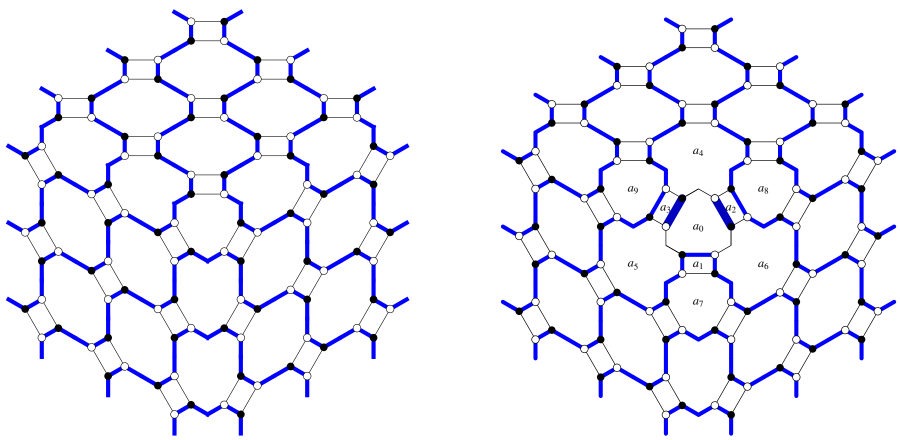

Figure 9: Left: the initial double-dimer configuration m_0 on Γ_0. Right: a taut double-dimer configuration on $\Gamma(U_{-1})$; doubled edges are thicker.

configuration has the following property. If we erase a finite piece of m_0, there is a unique way to complete it to a double-dimer configuration which has the same boundary connections, that is, connections between far-away points. For $U \in \mathcal{L}$, we say that a double-dimer configuration on $\Gamma(U)$ is **taut** if it has the same boundary connections as m_0, that is, it is identical to m_0 far from the origin and there is a bijection between its bi-infinite paths and those of m_0 which is the identity near ∞. There are a finite number of taut configurations (since any taut configuration must agree with m_0 on the domain where ∂U agrees with ∂U_0). See the right of Figure 9 for one such on $\Gamma(U_{-1})$.

4 Main formula

Given a taut double-dimer configuration m, let $c(m)$ denote the number of loops in m and define $c(m; i, j, k) := L(i, j, k) - 2 - d(m; i, j, k)$ where $L(i, j, k)$ is the number of edges in the face (i, j, k) and $d(m; i, j, k)$ is the number of dimers lying along the face (i, j, k) in the matching m. Define the *weight* of a taut configuration m to be

$$2^{c(m)} \prod_{(i,j,k) \in \mathcal{I}(U)} A(i, j, k)^{c(m;i,j,k)} .$$

In the configuration m_0, all quadrilateral faces have 2 dimers and all octagonal faces have 6 dimers, so the only face (i, j, k) with $c(m_0; i, j, k) \neq 0$ is the hexagonal face which has 3 dimers and $c(m_0; 0, 0, 0) = 6 - 2 - 3 = 1$. Any taut configuration differs from m_0 in finitely many places, hence its weight has finitely many variables appearing in it. For example, the configuration on the right of Figure 9 has weight $\dfrac{a_4^2 a_5 a_6 a_7}{a_0 a_1 a_2 a_3}$.

Theorem 4.1. *Fix any $U \in \mathcal{L}$ and let $\mathcal{I}(U)$ be the labels of $\Gamma(U)$. Use the notation $m \preceq U$ to signify that m is a taut double-dimer configuration on $\Gamma(U)$. Then the representation of $A(0, 0, 0)$ as a Laurent polynomial in the variables in $\mathcal{I}(U)$ is given by*

$$A(0, 0, 0) = \sum_{m \preceq U} 2^{c(m)} \prod_{(i,j,k) \in \mathcal{I}(U)} A(i, j, k)^{c(m;i,j,k)} . \tag{4.1}$$

Specializing to U_{-n} and $A(i, j, k) = 1$ for all i, j, k with $-n - 2 \leq i + j + k \leq -n$ gives the formula

$$A(0, 0, 0) = \sum_{m \preceq U_{-n}} 2^{c(m)} .$$

Remark. These formulae are translation-invariant, once one accounts for the distinguished role played by $(0, 0, 0)$ in Γ_0. Let (r, s, t) be any point of $\mathbb{Z}^3$ and U be any lattice solid obtained by removing finitely many cubes from the top of the orthant $\{x \leq r, y \leq s, z \leq t\}$. Then $A(r, s, t)$ is a Laurent polynomial in the set $\mathcal{I}(U)$ of labels in $\Gamma(U)$ and (4.1) holds for $A(r, s, t)$ in place of $(0, 0, 0)$.

Proof. We induct on U. It is true for $U = U_0$: there is one configuration, m_0, with $c(m_0; i, j, k) = 1$ if $i = j = k = 0$ and zero otherwise. The sum is therefore equal to $A(0, 0, 0)$, yielding the identity $A(0, 0, 0) = A(0, 0, 0)$ which is the correct representation.

For the induction to run, we need to see that the conclusion remains true if we remove a maximal cube. This corresponds to a superurban renewal, which is a composition of ordinary urban renewals with additional vertex splittings and contractions, depending on which version of ordinary urban renewal is used. Under a vertex splitting, two faces increase in length by 2 and get two additional dimers on them. Thus their contribution does not change. A similar argument holds for vertex contraction.

Consider an urban renewal of type shown in Figure 3. There are several cases to consider, see Figure 10, depending on the various possible boundary connections.

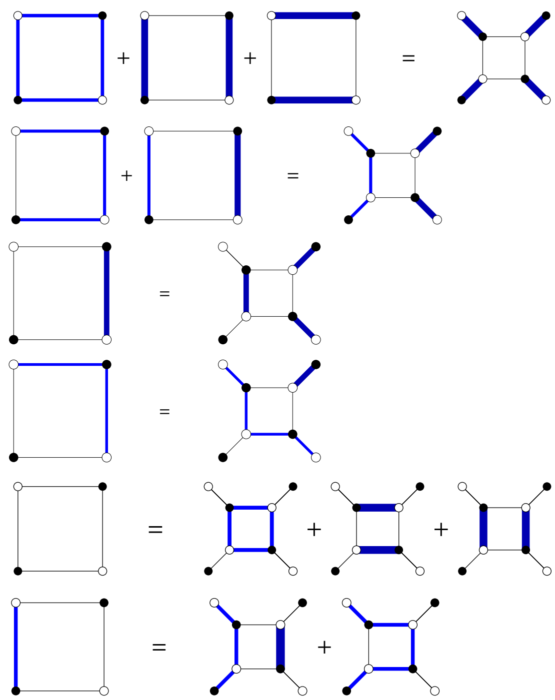

Figure 10: Checking consistency of urban renewal formulas and A monomials.

In the first case, the ratio of monomials on the right side and left side of the equation is

$$\frac{(a_0')^2 a_1^{2-4} a_2^{2-4} a_3^{2-4} a_4^{2-4}}{2a_0^{-2} a_1^{-1} a_2^{-1} a_3^{-1} a_4^{-1} + a_0^{-2} a_1^{-2} a_2^{-2} a_3^0 a_4^0 + a_0^{-2} a_1^0 a_2^0 a_3^{-2} a_4^{-2}} = \frac{(a_0')^2 a_0^2}{a_1 a_2 a_3 a_4 (2 + \frac{a_3 a_4}{a_1 a_2} + \frac{a_1 a_2}{a_3 a_4})} = 1\,.$$

In the second case, the ratio is

$$\frac{(a_0')^1 a_1^{2-4} a_2^{2-3} a_3^{2-3} a_4^{2-3}}{a_0^{-1}(a_1^{-1} a_2^0 a_3^{-1} a_4^{-1} + a_1^{-2} a_2^{-1} a_3^0 a_4^0)} = \frac{a_0 a_0'}{a_1 a_2 + a_3 a_4} = 1\,.$$

The remaining four cases are similar. These cover all possible cases (up to rotations). $\square$

5 Limit shapes

In this section we discuss general recurrences on $\mathbb{Z}^d$. In the following section we specialize to the hexahedron recurrence.

Let $\ell : \mathbb{Z}^d \to \mathbb{Z}$ be a linear function which will be the direction of propagation (hence "time" or "height"). A function $f : \mathbb{Z}^d \to \mathbb{Z}$ is said to be (spatially) isotropic if $f(v)$ depends only on $\ell(v)$. Solutions to recurrences that are isotropic and have the particularly simple form $f(v) = \gamma^{\ell(v)^k}$ lead to linear recurrences for the formal logarithmic derivatives $(\partial/\partial t) \log f(v,t)$; in the examples which interest us these have probabilistic interpretations and satisfy limit shape theorems. We begin by identifying the exponent δ for which such a solution exists.

5.1 Isotropic solutions and homogeneity

In order to discuss general $\mathbb{Z}^d$-invariant algebraic recurrences some notation is required. Let $\mathcal{A}$ be a finite index set, let $\{E_\alpha : \alpha \in \mathcal{A}\}$ be a finite collection of multisets of elements of $\mathbb{Z}^d$, and let $\{c_\alpha : \alpha \in \mathcal{A}\}$ be constants. Denote the multiset union by $E := \bigcup_{\alpha \in \mathcal{A}} E_\alpha$. For each $v \in \mathbb{Z}^d$, define a polynomial $P_{(v)}$ on indeterminates $\{x_w : w \in \mathbb{Z}^d\}$ by the equation

$$P_{(v)} = \sum_{\alpha \in \mathcal{A}} c_\alpha \prod_{w \in E_\alpha} x_{v+w} \,. \tag{5.1}$$

A lattice function $f : \mathbb{Z}^d \to \mathbb{C}$ is said to satisfy the algebraic recurrence $P = \{P_{(v)} : v \in \mathbb{Z}^d\}$ if at all $v \in \mathbb{Z}^d$, $P_{(v)}$ vanishes upon setting $x_w = f(w)$ for all $w \in \mathbb{Z}^d$. Say that the recurrence is k-homogeneous with respect to the linear functional $\ell : \mathbb{Z}^d \to \mathbb{Z}$ if there are constants $\beta_0, \ldots, \beta_k$ such that for each $\alpha \in \mathcal{A}$ and each $0 \le j \le k$,

$$\sum_{w \in E_\alpha} \ell(w)^j = \beta_j \,.$$

For instance, the recurrence is 0-homogeneous if $|E_\alpha| = \beta_0$, that is, if the polynomials $P_{(v)}$ are homogeneous of degree β_0. The **degree of homogeneity** of the recurrence P is the maximum δ such that P is δ-homogeneous.

Example 5.1. The octahedron recurrence (1.1) is 1-homogeneous with respect to the height function $\ell(i,j,k) = j+k$. To see this, put the recurrence in the form of (5.1) by taking $E_1 = \{(1,0,0),(0,1,1)\}$, $E_2 = \{(0,1,0),(1,0,1)\}$ and $E_3 = \{(0,0,1),(1,1,0)\}$. The corresponding multisets of heights are $\{0,2\}, \{1,1\}$ and $\{1,1\}$. The sums of the zero powers of the heights are the constant $\beta_0 = 2$. The sums of the first powers of the heights are the constant $\beta_1 = 2$. The sums of the squares of the heights are not constant (4, 2 and 2) therefore the degree of homogeneity of the octahedron recurrence with respect to $\ell(i,j,k) = j+k$ is 1.

The degree of homogeneity of a recurrence tells us for which power δ we can expect a solution to the recurrence of the form $f(v) = \gamma^{\ell(v)^\delta}$. These solutions are the ones for which we can most easily compute corresponding linear recurrences for the the derivatives $(\partial/\partial t)f(v,t)$.

Proposition 5.2. *Suppose the degree of homogeneity of the recurrence P is $\delta - 1$. Suppose further that $\sum_{\alpha \in S} c_\alpha$ is nonvanishing for at least two equivalence classes $S = S_m = \{\alpha : \sum_{w \in E_\alpha} \ell(w)^\delta = m\}$. Then there is a number $\gamma \neq 0$ such that $f(v) = \gamma^{\ell(v)^\delta}$ satisfies P at every $v \in \mathbb{Z}^d$. In fact $f(v) = \gamma^{q(\ell(v))}$ satisfies P for every monic polynomial q of degree δ.*

Proof. Let q be a monic polynomial of degree δ and define $f(v) = f_q(v) = \gamma^{q(\ell(v))}$. This satisfies P at v if and only if

$$\sum_{\alpha \in \mathcal{A}} c_\alpha \gamma^{\sum_{w \in E_\alpha} q(\ell(v+w))} = 0\,. \tag{5.2}$$

For $j < \delta$,

$$\begin{aligned}
\sum_{w \in E_\alpha} \ell(v+w)^j &= \sum_{w \in E_\alpha} (\ell(v) + \ell(w))^j \\
&= \sum_{w \in E_\alpha} \sum_{i \le j} \binom{j}{i} \ell(v)^i \ell(w)^{j-i} \\
&= \sum_{i \le j} \binom{j}{i} \beta_{j-i} \ell(v)^i \\
&=: C(v)
\end{aligned}$$

independent of j. Factoring out $\gamma^{C(v)}$ from (5.2) leaves only the leading term of q, thus,

$$\sum_{\alpha \in \mathcal{A}} c_\alpha \gamma^{\sum_{w \in E_\alpha} (\ell(v+w))^\delta} = 0\,. \tag{5.3}$$

Again we may decompose the power $\ell(v+w)^\delta = (\ell(v) + \ell(w))^\delta$, this time arriving at

$$\sum_{w \in E_\alpha} \ell(v+w)^\delta = \sum_{w \in E_\alpha} \ell(w)^\delta + \sum_{1 \le i \le \delta} \binom{\delta}{i} \beta_{\delta - i} \ell(v)^i := \ell(w)^\delta + D(v)\,.$$

Factoring out $\gamma^{D(v)}$ from (5.3) leaves

$$\sum_{\alpha \in \mathcal{A}} c_\alpha \gamma^{\sum_{w \in E_\alpha} (\ell(w))^\delta} = 0\,. \tag{5.4}$$

This no longer depends on v, and is not a monomial because we have assumed that P is not δ-homogeneous. Therefore, there is at least one nonzero value of γ for which (5.4) holds, and this finishes the proof. $\square$

5.2 Recurrence for the derivative

Suppose that each set E_α resides in the ℓ-nonnegative halfspace $\mathbb{Z}^d_\ell := \{v \in \mathbb{Z}^d : \ell(v) \geq 0\}$, and suppose further that the values $\{f(v) : v \in \mathbb{Z}^d_\ell\}$ all depend smoothly on some parameter t. Consider the logarithmic derivative $g(v) = (d/dt) \log f(v) = f'(v)/f(v)$, which is defined whenever $f(v) \neq 0$. We claim that g satisfies a linear recurrence with constant coefficients.

Proposition 5.3. *Suppose that P is $(\delta - 1)$-homogeneous and choose γ according to Proposition 5.2 so that $f_0(v) = \gamma^{\ell(v)^\delta}$ solves P. Let $f(t, v)$ be smooth in t with $f(0, v) = f_0(v)$ and let $g(v) = (d/dt) \log f(t, v)|_{t=0}$. Then*

$$\sum_{\alpha \in \mathcal{A}} c'_\alpha \sum_{w \in E_\alpha} g(v + w) = 0 \tag{5.5}$$

for all $v \in \mathbb{Z}^d_\ell$, where

$$L(\alpha) := \sum_{w \in E_\alpha} \ell(w)^\delta .$$

and

$$c'_\alpha = c_\alpha \gamma^{L(\alpha)}$$

Proof. Differentiating the recurrence at v gives

$$\begin{aligned} 0 &= \left(\frac{d}{dt}\right) \sum_{\alpha \in \mathcal{A}} c_\alpha \prod_{w \in E_\alpha} x_{v+w} \Bigg|_{t=0} \\ &= \sum_{\alpha \in \mathcal{A}} c_\alpha \left(\sum_{w \in E_\alpha} g(v + w) \right) \prod_{w \in E_\alpha} f(v + w) \\ &= \sum_{\alpha \in \mathcal{A}} c_\alpha \sum_{w \in E_\alpha} g(v + w) \gamma^{\sum_{w \in E_\alpha} \ell(v+w)^\delta} . \end{aligned}$$

The sum $\sum_{w \in E_\alpha} \ell(v + w)^\delta$ is equal to $D(v) + L(\alpha)$, whence, factoring out $\gamma^{D(v)}$ from the last equation proves the proposition. $\square$

The **characteristic polynomial** of a recurrence $\sum_{w \in E} b_w g(v + w) = 0$ is the Laurent polynomial $\sum_{w \in E} b_w x^w$, where x^w denotes the monomial $x_1^{w_1} \cdots x_d^{w_d}$. In particular, the characteristic polynomial of (5.5) is

$$H = H_P = \sum_{\alpha \in \mathcal{A}} c'_\alpha \sum_{w \in E_\alpha} x^w .$$

Example 5.4. For the octahedron recurrence, let $E_1 := \{(1, 0, 0), (0, 1, 1)\}$, $E_2 := \{(0, 1, 0), (1, 0, 1)\}$ and $E_3 := \{(0, 0, 1), (1, 1, 0)\}$ and denote these six vectors by $w_1, \ldots, w_6$ respectively. Then

$L(1) = 4, L(2) = 2, L(3) = 2$. The equation for γ is $\sum_\alpha c'_\alpha |E_\alpha| = 0$, which is $2\gamma^4 - 4\gamma^2 = 0$, so $\gamma = \sqrt{2}$. The linear recurrence on derivatives is then given by

$$2[g(v + w_1) + g(v + w_2)] - g(v + w_3) - g(v + w_4) - g(v + w_5) - g(v + w_6) = 0\,.$$

Dividing by 2, we see that the sum of the x and yz points is equal to one half the sum of the other four points in any elementary octahedron. This recurrence has characteristic polynomial $2(x + yz) - (y + z + xz + xy)$.

Example 5.5 (cube recurrence). Let $w_1, \ldots, w_8$ denote

$$(0, 0, 0), (1, 1, 1), (1, 0, 0), (0, 1, 1), (0, 1, 0), (1, 0, 1), (0, 0, 1), (1, 1, 0)$$

respectively and let $E_j = \{w_{2j-1}, w_{2j}\}$ for $j = 1, 2, 3, 4$. This puts the cube recurrence (1.2) in standard form with $c_1 = 1$ and $c_2 = c_3 = c_4 = -1$. With $\ell(i, j, k) = i + j + k$, the cube recurrence has degree of homogeneity equal to 1. The values of $L(\alpha) = \sum_{w\in E_\alpha} \ell(w)^2$ for $\alpha = 1, 2, 3, 4$ are $9, 5, 5, 5$ and the resulting equation for γ is $\gamma^9 - 3\gamma^5$ which has one positive solution $\gamma = 3^{1/4}$. This leads to values for c'_α (we may divide everything by $3^{5/4}$) of $3, 1, 1$ and 1 respectively. Thus the recurrence for the derivatives is given by

$$g(v) + g(v + w_2) = \frac{1}{3} \sum_{j=3}^{8} g(v + w_j)$$

and the characteristic polynomial for the cube recurrence with respect to $\ell = i + j + k$ is

$$3(xyz + 1) - (x + y + z + xy + xz + yz)\,.$$

5.3 Behavior of linear recurrences

5.3.1 Boundary conditions

With the right boundary conditions, the linear recurrence (5.5) will have tractable asymptotics and a limiting shape. This is assured when the boundary conditions are such that (5.5) holds for all but finitely many $v \in \mathbb{Z}^d$. The most common way this arises is as follows. Re-indexing if necessary, suppose that $0 \in E$ (recall $E = \bigcup_\alpha E_\alpha$), suppose that ℓ attains its unique maximum there. The recurrence (5.5) determines $g(v)$ as a linear function of $\{g(v+w) : 0 \neq w \in E\}$. A canonical boundary condition is to take $g(w) = 0$ when $\ell(w) < 0$, to take $g(0) = 1$ and to define g everywhere else by the recurrence. In this case (5.5) holds everywhere except at the origin.

Define a d-variable generating function

$$F(x) := \sum_v g(v) x^v$$

where x^v denotes the monomial $x_1^{v_1} \cdots x_d^{v_d}$. Let H denote the characteristic polynomial of the recurrence. The fact that (5.5) holds except at finitely many points implies that the generating function F satisfies $HF = G$ where G is a Laurent polynomial.

Example 5.6 (octahedron recurrence, continued). If we impose the octahedron recurrence everywhere except at the origin, setting the right-hand side of (5.5) equal to 1 at the origin and setting $g(v) = 0$ when $\ell(v) \leq 2$ except for $g(0,1,1) = 1$, then $HF = x_2 x_3$, whence

$$F = \frac{yz}{H} = \frac{yz}{2x + 2yz - y - z - xz - xy}\,.$$

The usual form of the octahedron recurrence differs from ours by an affine transformation, whence one usually sees (e.g. for the Aztec Diamond creation rate generating function)

$$F(x,y,z) = \frac{z}{2 - (x + x^{-1} + y + y^{-1})z + 2z^2}\,;$$

see [4] or [**?**, Section 4.1] for the generating function in this form.

5.3.2 Coefficient asymptotics

Laurent series for rational functions obey limit laws. A proper statement of this, even without proofs, is possible only in a very abbreviated manner. The purpose of this subsection is to give a description allowing interested non-experts to grasp the relevant results, viewing certain terms as black boxes, while also laying the groundwork for more sophisticated users who know what's inside the black box to follow the application in Section 6. The sketch of proof of Theorem 5.7 is included only so that the transfer of results from [15] can be checked. For further details, see, [15, Chapter 7].

A one-paragraph outline is as follows. Let G/H be a rational function with Laurent expansion $\sum_v a_v \mathbf{z}^v$. The asymptotic behavior of the coefficients $\{a_v\}$ as $v \to \infty$ while $v/|v| \to \hat{v}$ is determined by the behavior of the denominator H at one or more dominating points $\mathbf{z}^*(\hat{v})$ depending on the direction $\hat{v}$. The most interesting behavior ensues when the dominating point is a singular point, that is, when $\nabla H(\mathbf{z}^*) = 0$. In this case, the coefficient asymptotics are determined by the leading homogeneous part $\overline{H}$ of the Taylor expansion of H at the point $\mathbf{z}^*$. Hyperbolicity of $\overline{H}$ (see the next paragraph) guarantees that its real zero set bounds a convex cone K, known as the **cone of hyperbolicity**. The inverse Fourier transform of $1/\overline{H}$, call it Ψ, is supported on K^*, the convex dual to K. The continuous function Ψ gives the asymptotics of the discrete parameter function $v \mapsto a_v$. In cases where the support of Ψ is all of K^*, it follows that the phase boundary for the statistical mechanical ensemble enumerated by G/H is the projective curve dual to $\overline{H}$. This will be the case for the double-dimer model.

The following highlighted terms may be thought of as black boxes, which need not be understood in order to follow the remainder of this subsection and its application in Section 6. The function G/H has many Laurent expansions, corresponding to components of the complement of the **amoeba** of H. Locally at any point $\mathbf{z}$ on the boundary of one of these components, C, the polynomial H satisfies a condition known as **hyperbolicity**. This guarantees that its real zero set bounds a convex cone K; the **tangent cone** K_1 to C at $\mathbf{z}$ is a convex subcone of K. The **dominating points** $\mathbf{z}^*$ project to $\mathbf{z}$ under the **log-modulus map** and satisfy certain equations concerning the **logarithmic gradient** of H. The **inverse Fourier transform** or **IFT** of $1/\overline{H}$ is defined by taking a limit of integrals in the domain $\mathbf{x} + i\mathbb{R}^n$ as $\mathbf{x}$ approaches $\mathbf{z}$ from withing the complement of the amoeba. Every convex cone has a **convex dual**; the convex dual K^* to K is a subcone of the convex dual K_1^* to K_1. When K^* has nonempty interior, the **cone point hypotheses** of [**?**] and [15, Chapter 11] are satisfied. The results stated below specialize to the case where $\mathbf{z}^* = (1, \ldots, 1)$. In this case the projection $\mathbf{z}$ of $\mathbf{z}^*$ under the log-modulus map is the origin, meaning that the non-frozen region is characterized by sub-exponential decay: $\lim |v|^{-1} \log |a_v| = 0$.

Theorem 5.7. *Let $F = G/H$ be a quotient of polynomials in n variables. Assume that the zero set of H touches the closed unit polydisk only at $(1, \ldots, 1)$. Let $\overline{H}$ denote the homogeneous part of H at the origin, let K denote its cone of hyperbolicity, and let K^* denote the convex dual to K. Assume that K^* has nonempty interior.*

1. *The coefficients a_v decay exponentially in $|v|$ when $v \notin K^*$, the exponential rate being uniform if $v/|v|$ is contained in a compact set disjoint from K^*. Here $|v|$ can be any norm.*

2a. *If $G(1, \ldots, 1) = 1$ then $a_v \sim \Psi(v)$ on K^* (the ratio tends to 1), where Ψ is the IFT of $1/\overline{H}$.*

2b. *The IFT Ψ is homogeneous of degree* $\deg(\overline{H}) - d$. ***[What is** d**?]***

2c. *If G vanishes to degree δ at $(1, \ldots, 1)$ then on K^*, $a_v \sim \Psi_G$, where Ψ_G is a linear combination of partial derivatives of order δ of Ψ.*

3. *If instead the zero set of H intersects the unit torus in finitely many points then K^* is the union of the dual cones at each zero of H, the coefficients decay exponentially away from K^*, and the leading asymptotic on K^* is obtained by summing Ψ over the finitely many contact points of the zero set with the unit torus.*

Sketch of proof: The first part is essentially the Paley-Wiener Theorem. It was proved in the special case of cube groves (the cube recurrence) in [16]. The general proof may be found in [15, Chapter 8]. The second part is proved as [**?**, Lemma 6.3]. There, hypotheses are assumed to restrict the vanishing degree of H at $(1, \ldots, 1)$ to 2, but in fact the proof is valid for any degree. The third part follows from the general theory of inverse Fourier transforms, and the fourth part is [**?**, Proposition 6.2], again removing the restriction on the degree of vanishing of H at $(1, \ldots, 1)$. □

The above hypotheses alone do not imply that the support of Ψ is all of the cone K^*. Also, the assumption of finite intersection of the zero set of H with the closed unit polydisk is difficult to check. Therefore, we state the following version of Theorem 5.7. Again, the proof may be taken from [**?**]: hypotheses (i) below implies that the unit torus is a minimal torus, while hypothesis (ii) below satisfies the hypotheses of Theorem 5.8 of [**?**]; the third conclusion merely states an algorithm for computing the dual to the projective algebraic curve defined by $\overline{H}$.

Corollary 5.8. *Let F, G, H, K and K^* be as in Theorem 5.7. For each v in the interior of K^*, let $W(v)$ denote the set of z in the unit torus such that v is in the dual cone to H at z as defined in [**?**, Definition 2.21]. In particular when $\nabla H(z) \neq 0$, the point z is in $W(v)$ exactly when v is a scalar multiple of the logarithmic gradient $(z_1 \partial H/\partial z_1, \ldots, z_d \partial H/\partial z_d)(z)$. Replace the hypothesis of Theorem 5.7 that H vanishes finitely often on the closed polydisk by the following hypotheses:*

(i) The coefficients a_v of F satisfy $\limsup_v |v|^{-1} \log |a_v| \leq 0$.

(ii) For each $v \in K^*$, the set $W(v)$ is finite and non-empty.

Then the following conclusions hold.

1. *The coefficients a_v tend to zero exponentially rapidly when v goes to infinity varying over a closed subcone disjoint from the closure of K^*.*

2. *If $v \to \infty$ over a closed subcone of the interior of K^* then $a_v \sim \sum_{\mathbf{z}^* \in W(v)} \Psi_{\mathbf{z}^*}(v)$ where $\Psi_{\mathbf{z}^*}$ is the IFT of $1/\overline{H_{\mathbf{z}^*}}$, the homogeneous part of $\overline{H}$ at $\mathbf{z}^*$.* ***[Do you mean the homogeneous part of H at $\mathbf{z}^*$?]***

3. *In three variables, the boundary of K^* is an algebraic curve in $\mathbb{RP}^2$ and may be computed from $H(x, y, z)$ be setting $z = -ax - by$ and eliminating x and y from the simultaneous polynomial equations*
$$H = 0 \ ; \ \frac{\partial H}{\partial x} = 0 \ ; \ \frac{\partial H}{\partial y} = 0 \, . \tag{5.6}$$

□

6 Application to the hexahedron recurrence

We now apply the results of the last section to the hexahedron recurrence. We have developed these results independently of the specific recurrence for several reasons. First, the methods are more general and it is good to see them in the abstract. Secondly, the hexahedron recurrence is not $\mathbb{Z}^d$-invariant but invariant under a sublattice of finite index. Because of this, it is not easy to see what is going on if one begins with the hexahedron recurrence. The exposition is clearest for general $\mathbb{Z}^d$-invariant recurrences, after which we may work the hexahedron recurrence along similar lines. We do not develop a general theory of periodic recurrences for finite index sublattices because the notation is even messier. We first compute the isotropic solutions, then compute the linear recurrences for the derivative, then prove limit shape results.

6.1 Isotropic solution

Let $\ell(i,j,k) = i + j + k$. An isotropic set of initial variables $\mathcal{I}$ is given by the 4-6-12 graph: those (i,j,k) with $0 \leq i+j+k \leq 2$. The hexahedron recurrence, beginning with these variables, preserves the isotropy. Therefore, the solution will be described by constants $A_n, B_n : n \geq 0$ such that $f(i,j,k) = A_n$ for integer points with $i+j+k = n$ and $f(i,j,k) = B_n$ for half-integer points with $i+j+k = n+1$. We will solve this general recursion, then specialize to solutions of the form $A_n = \gamma^n, B_n = \kappa\gamma^n$. The initial conditions A_0, A_1, A_2 and B_0 and the hexahedron recurrence determine A_n and B_n for all positive n. The recurrence becomes

$$\begin{aligned}
A_n &= \frac{2A_{n-2}^3B_{n-3}^3 + 3A_{n-3}A_{n-1}A_{n-2}B_{n-3}^3 + A_{n-2}^6 + 3A_{n-3}A_{n-1}A_{n-2}^4 + 3A_{n-3}^2A_{n-1}^2A_{n-2}^2 + A_{n-3}^3A_{n-1}^3 + B_{n-3}^6}{A_{n-3}^2B_{n-3}^3} \\
B_n &= \frac{A_n^3 + A_{n-1}A_nA_{n+1} + B_{n-1}^3}{A_{n-1}B_{n-1}}.
\end{aligned}$$

The values of A_3, B_1 and B_2, are determined by the initial conditions, but we may still use them in formulae for the remaining A and B values, leading to a mysteriously simple solution to the recurrence.

$$A_4 = \frac{A_0A_3^2}{A_1^2}, \quad A_5 = \frac{A_0^2A_3^4}{A_1^3A_2^2}, \quad A_6 = \frac{A_0^4A_3^6}{A_1^6A_2^3}, \quad \ldots$$

$$B_3 = \frac{A_0B_1B_2^2}{A_2B_0^2}, \quad B_4 = \frac{A_0^2B_2^4}{A_2^2B_0^3}, \quad B_6 = \frac{A_0^4B_1B_2^6}{A_2^4B_0^6}, \quad \ldots$$

and generally one can verify by induction that

$$A_n = \frac{A_0^{\lfloor (n-2)^2/4 \rfloor} A_3^{\lfloor (n-1)^2/4 \rfloor}}{A_1^{\lfloor ((n-1)^2-1)/4 \rfloor} A_2^{\lfloor ((n-2)^2-1)/4 \rfloor}}.$$

$$B_n = \frac{A_0^{\lfloor (n-1)^2/4 \rfloor} B_1^{\frac{1}{2}(1-(-1)^n)} B_2^{\lfloor n^2/4 \rfloor}}{A_2^{\lfloor (n-1)^2/4 \rfloor} B_0^{\lfloor (n^2-1)/4 \rfloor}}.$$

This can be written as

$$A_{2n} = \frac{A_0^{(n-1)^2}A_3^{n^2-n}}{A_1^{n^2-n}A_2^{n^2-2n}} \tag{6.1a}$$

$$A_{2n+1} = \frac{A_0^{n^2-n}A_3^{n^2}}{A_1^{n^2-1}A_2^{n^2-n}} \tag{6.1b}$$

$$B_{2n} = \frac{A_0^{n^2-n}B_2^{n^2}}{A_2^{n^2-n}B_0^{n^2-1}} \tag{6.1c}$$

$$B_{2n+1} = \frac{A_0^{n^2}B_1B_2^{n^2+n}}{A_2^{n^2}B_0^{n^2+n}}. \tag{6.1d}$$

The simplest nontrivial solution to this recursion and the one in the form of which we spoke earlier is

$$A_n = 3^{n^2/2}, \quad B_n = 2 \cdot 3^{(n+1)^2/2} . \tag{6.2}$$

There is another reasonably simple solution with a more direct combinatorial meaning. This is obtained by setting the initial variables A_0, A_1, A_2, B_0 all equal to 1. This implies $A_3 = 14, B_1 = 3$ and $B_2 = 14$ and produces the result

$$\begin{aligned} A_{2n} &= 14^{n(n-1)} \\ A_{2n+1} &= 14^{n^2} \\ B_{2n} &= 14^{n^2} \\ B_{2n+1} &= 3 \times 14^{n(n+1)} . \end{aligned} \tag{6.3}$$

Setting the initial A variables equal to 1 amounts to setting all the edge weights $\nu(e)$ equal to 1 in $\Gamma(U)$, as dictated by change of variables (2.1). By Theorem 4.1, $A(0,0,0)$ counts taut double-dimer configurations of $\Gamma(U_{-n})$, with the weight of m counted as $2^{c(m)}$ when the canonical initial variables are set to 1. By translation invariance, if $i+j+k = n+2$ and variables at levels 0, 1 and 2 are set to 1, then $A(i,j,k)$ counts taut double-dimer configurations in a graph isomorphic to $\Gamma(U_{-n})$, again with weights $2^{c(m)}$. Evaluating $A(i,j,k) = A_{n+2} = 14^{(n/2)(n/2+1)}$ if n is even and $14^{(n/2)(n/2+1)+1/4}$ if n is odd. Thus we have proved:

Corollary 6.1. *The number of taut double-dimer configurations of $\Gamma(U_{-n})$, weighted by $2^{c(m)}$, is equal to*

$$14^{\frac{n}{2}\left(\frac{n}{2}+1\right)+\frac{1}{4}\delta_{\mathrm{odd}}(n)} .$$

6.2 Recurrence for the derivative

Let us interpret Proposition 5.3 in the context of statistical mechanical ensembles. Suppose that over the set $E = \bigcup_\alpha E_\alpha$, the function ℓ has a minimum value of 0 and a maximum value of J. Suppose further that P is an algebraic recurrence that determines the values of $\{f(v) : \ell(v) \geq J\}$ in terms of initial values $\{f(v) : 0 \leq \ell(v) < J\}$. Consider $t = f(0,\ldots,0)$ to be variable while all other initial conditions remain fixed at $f(v) = \gamma^{\ell(v)^\delta}$. Applying Proposition 5.3 gives the constant coefficient linear recurrence (5.5) for the logarithmic derivatives of $f(v)$. Specializing further to the case $f(v) = A(v)$ for one of the Laurent recurrences we have studied, the monomials in the expression of $A(v)$ in terms of initial variables correspond to configurations, the value $A(v)$ is the partition function for all configurations. The logarithmic derivative $\dfrac{1}{A(v)}\,\dfrac{\partial A(v)}{\partial A(0,0,0)}$ at the initial conditions $f(v) = \gamma^{\ell(v)^\delta}$ may be interpreted as the expected value of the exponent on the term $A(0,0,0)$ in the statistical mechanical ensemble in which the probability of the configuration ξ is $M_\xi(v)/A(v)$ where M_ξ is the monomial corresponding to ξ.

We now apply this to the hexahedron recurrence and the double-dimer ensemble. We choose initial conditions (6.2) rather than (6.3) because these correspond to the solution $f(v) = \gamma^{\ell(v)^2}$ of Proposition 5.2 with $\gamma(i,j,k) = i+j+k$. The logarithmic derivative $g(i,j,k) = A(i,j,k)^{-1}\partial A(i,j,k)/\partial A(0,0,0)$ is the expected number of dimers lying along the face at the origin in a double-dimer configuration picked from all taut configurations on Γ_{ijk} according to the double-dimer measure μ_{dd} corresponding to the initial conditions (6.2). By translation invariance, this is the same as the expected number of dimers lying along the face centered at $(-i,-j,-k)$ on the graph $\Gamma(U_{-i-j-k})$. We may then ask about the limiting shape function, that is, about the values of $g(i,j,k)$ as $n = i+j+k \to \infty$ with $(i/n, j/n, k/n) \to (\alpha_1, \alpha_2, \alpha_3)$ in the 2-simplex.

Taking the logarithmic derivatives of the recurrence relations (1.4a)–(1.4d), where h is now A and g is the logarithmic derivative, and plugging in the initial conditions (6.2), gives the linear system

$$\begin{aligned}
g_{(123)} &= -g + \frac{1}{3}(g_{(1)} + g_{(2)} + g_{(3)} + g_{(23)} + g_{(13)} + g_{(12)}) && \text{(6.4a)}\\
g^{(x)}_{(1)} &= \frac{1}{12}\left(-9g + 4g_{(1)} + g_{(2)} + g_{(3)} + 3g_{(23)} - 4g^{(x)} + 8g^{(y)} + 8g^{(z)}\right) && \text{(6.4b)}\\
g^{(y)}_{(2)} &= \frac{1}{12}\left(-9g + g_{(1)} + 4g_{(2)} + g_{(3)} + 3g_{(23)} + 8g^{(x)} - 4g^{(y)} + 8g^{(z)}\right) && \text{(6.4c)}\\
g^{(z)}_{(3)} &= \frac{1}{12}\left(-9g + g_{(1)} + g_{(2)} + 4g_{(3)} + 3g_{(12)} + 8g^{(x)} + 8g^{(y)} - 4g^{(z)}\right). && \text{(6.4d)}
\end{aligned}$$

As it happens, the first equation gives a self-contained recurrence for the logarithmic derivatives at the integer points. Not only that, but the recurrence is recognizable as that arising in the cube recurrence (1.2). In other words, letting $F(x,y,z) = \sum g_{i,j,k} x^i y^j z^k$, we see that the solution to the first recurrence above with boundary conditions $g(0,0,0) = 1$, $g(i,j,k) = 0$ for other points (i,j,k) with $i+j+k \le 0$ and satisfying the recurrence everywhere except at $(-1-1-1)$, is

$$F(x,y,z) = \frac{G(x,y,z)}{H(x,y,z)} = \frac{1}{1 + xyz - \frac{1}{3}(x+y+z+xy+xz+yz)}.$$

6.3 Limit shape

This is the same as that satisfied by the cube grove placement probabilities [16]. The boundary of the dual cone is known as the "arctic circle", which is the inscribed circle in the triangular region $\{x+y+z = n, x,y,z \ge 0\}$. Outside of this, the placement probabilities decay exponentially while inside the arctic circle they do not. Inside, the limit function is homogeneous of degree -1 and is asymptotically equal to the inverse of the distance to the arctic circle in the plane normal to the $(1,1,1)$ direction [?]. We can conclude from this that with high probability, a random configuration from Γ_n is equal to m_0 outside a neighborhood of size $o(n)$ of the arctic circle and that there is positive local entropy everywhere inside the arctic circle.

6.4 General double-dimer shape theorems

Different periodic initial conditions lead to different limiting shapes. This section describes the possible limit shapes arising from periodic initial conditions for the double-dimer ensemble. Because the odd and even vertices play a different role in the hexahedron recurrence, instead of four variables in (6.4a)-(6.4d) described in terms of ten old variables, we will have eight new variables in terms of twenty old ones. We differentiate (1.4a)-(1.4d) and use (6.1a)-(6.1d) to get 8 linear equations, four for $i+j+k$ odd and four for $i+j+k$ even. To name the variables, we denote the logarithmic derivatives $A(i,j,k)^{-1}\partial A(i,j,k)/\partial A(0,0,0)$ by $g(i,j,k)$ if $i+j+k$ is odd, and by $h(i,j,k)$ if $i+j+k$ is even. Define $g^{(x)}, h^{(x)}$, and so forth, as before. This allows us to compute the general solution, still assuming isotropy, but no longer assuming that $\{A_n\}$ and $\{B_n\}$ are two geometric sequences.

The 8×8 matrices that arise and their characteristic polynomials are already somewhat unwieldy when specific real numbers are used for the periodic initial conditions. As rational functions of indeterminates, these expressions are completely impenetrable. Therefore, we present this computation first in a numerical case where one can see more or less what is going on, then give the general result.

A generic example

The sequence $\{A_n, B_n\}$ is determined by a_0, a_1, a_2 and b_0. Let $a_0 = 1, b_0 = 1, a_1 = 2, a_2 = 3$. Then $b_1 = 15, b_2 = 189, a_3 = 378$. The linear system is

$$\begin{aligned}
g_{(123)} &= \frac{1}{105}\left(84h + 4(g_{(1)} + g_{(2)} + g_{(3)}) + 98(h_{(12)} + h_{(13)} + h_{(23)}) - 95(h^{(x)} + h^{(y)} + h^{(z)})\right)\\
h_{(123)} &= \frac{1}{42}\left(-33g + 46(h_{(1)} + h_{(2)} + h_{(3)}) + 17(g_{(12)} + g_{(13)} + g_{(23)}) - 38(g^{(x)} + g^{(y)} + g^{(z)})\right)\\
g^{(x)}_{(1)} &= \frac{1}{210}\left(-126h + 85g_{(1)} + g_{(2)} + g_{(3)} + 84h_{(23)} - 85h^{(x)} + 125h^{(y)} + 125h^{(z)}\right)\\
g^{(y)}_{(2)} &= \frac{1}{210}\left(-126h + g_{(1)} + 85g_{(2)} + g_{(3)} + 84h_{(13)} + 125h^{(x)} - 85h^{(y)} + 125h^{(z)}\right)\\
g^{(z)}_{(3)} &= \frac{1}{210}\left(-126h + g_{(1)} + g_{(2)} + 85g_{(3)} + 84h_{(12)} + 125h^{(x)} + 125h^{(y)} - 85h^{(z)}\right)\\
h^{(x)}_{(1)} &= \frac{1}{15}\left(-9g + 6g_{(23)} + 14h_{(1)} + 8h_{(2)} + 8h_{(3)} - 14g^{(x)} + g^{(y)} + g^{(z)}\right)\\
h^{(y)}_{(2)} &= \frac{1}{15}\left(-9g + 6g_{(13)} + 8h_{(1)} + 14h_{(2)} + 8h_{(3)} + g^{(x)} - 14g^{(y)} + g^{(z)}\right)\\
h^{(z)}_{(3)} &= \frac{1}{15}\left(-9g + 6g_{(12)} + 8h_{(1)} + 8h_{(2)} + 14h_{(3)} + g^{(x)} + g^{(y)} - 14g^{(z)}\right).
\end{aligned}$$

In terms of the generating functions, this is

$$\begin{pmatrix} G \\ H \\ G^{(x)} \\ G^{(y)} \\ G^{(z)} \\ H^{(x)} \\ H^{(y)} \\ H^{(z)} \end{pmatrix} = \begin{pmatrix} \frac{4}{105}(xy+zy+xz) & \frac{4xyz}{5}+\frac{14}{15}(x+y+z) & 0 & 0 & 0 & -\frac{19}{21}xyz & -\frac{19}{21}xyz & -\frac{19}{21}xyz \\ \frac{17}{42}(x+y+z)-\frac{11xyz}{14} & \frac{23}{21}(xy+zy+xz) & -\frac{19}{21}xyz & -\frac{19}{21}xyz & -\frac{19}{21}xyz & 0 & 0 & 0 \\ \frac{1}{210}x\left(\frac{1}{y}+\frac{1}{z}+\frac{85}{x}\right) & \frac{1}{210}x\left(\frac{84}{yz}-\frac{126}{x}\right) & 0 & 0 & 0 & -\frac{17x}{42} & \frac{25x}{42} & \frac{25x}{42} \\ \frac{1}{210}y\left(\frac{85}{y}+\frac{1}{z}+\frac{1}{x}\right) & \frac{1}{210}y\left(\frac{84}{xz}-\frac{126}{y}\right) & 0 & 0 & 0 & \frac{25y}{42} & -\frac{17y}{42} & \frac{25y}{42} \\ \frac{1}{210}\left(\frac{1}{y}+\frac{85}{z}+\frac{1}{x}\right)z & \frac{1}{210}\left(\frac{84}{xy}-\frac{126}{z}\right)z & 0 & 0 & 0 & \frac{25z}{42} & \frac{25z}{42} & -\frac{17z}{42} \\ \frac{1}{15}x\left(\frac{6}{yz}-9\right) & \frac{1}{15}x\left(\frac{8}{y}+\frac{8}{z}+\frac{14}{x}\right) & -\frac{14x}{15} & \frac{x}{15} & \frac{x}{15} & 0 & 0 & 0 \\ \frac{1}{15}y\left(\frac{6}{xz}-9\right) & \frac{1}{15}y\left(\frac{14}{y}+\frac{8}{z}+\frac{8}{x}\right) & \frac{y}{15} & -\frac{14y}{15} & \frac{y}{15} & 0 & 0 & 0 \\ \frac{1}{15}\left(\frac{6}{xy}-9\right)z & \frac{1}{15}\left(\frac{8}{y}+\frac{14}{z}+\frac{8}{x}\right)z & \frac{z}{15} & \frac{z}{15} & -\frac{14z}{15} & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} G \\ H \\ G^{(x)} \\ G^{(y)} \\ G^{(z)} \\ H^{(x)} \\ H^{(y)} \\ H^{(z)} \end{pmatrix} + I_0 ,$$

where I_0 represents the initial conditions.

The denominator of the generating functions is given by the determinant of $I - M$ where M is the matrix on the RHS above. This polynomial factors and the zero set has two components with equations

$$P_1 = 63x^2y^2z^2 - 62(x^2yz+xy^2z+xyz^2) - (x^2y^2+x^2z^2+y^2z^2) + 62(xy+xz+yz) + (x^2+y^2+z^2) - 63$$

and

$$\begin{aligned} P_2 = {} & 198x^2y^2z^2 - 171(x^2y^2z + x^2yz^2 + xy^2z^2) + 5(x^2y^2 + x^2z^2 + y^2z^2) \\ & + 481(x^2yz + xy^2z + xyz^2) - 513xyz - 310(xy + xz + yz) - 5(x^2 + y^2 + z^2) + 315. \end{aligned}$$

Evidently this is not irreducible. However, writing the generating function as $N/(P_1P_2)$, there is a soft argument that the polynomial N in the numerator contains a factor of P_2 and therefore that the generating function takes the form $F = G/P_1$. To see this, observe by direct computation that P_2 has nontrivial intersection with $(-1, 1)^3$. Suppose that N does not vanish on the intersection of the zero set of P_2 with the open unit polydisk in $\mathbb{C}^3$. Then the Taylor series for F fails to converge at some point in the open unit polydisk which means that the limsup growth of the coefficients is exponential. The probabilistic interpretation contradicts this. We conclude that N vanishes on the intersection of P_2 with the open unit polydisk, which is a variety of complex codimension 1. By irreducibility of P_2, we see that N vanishes on the whole zero set of P_2. The upshot of all this is that we may write F in reduced form as G/P_1.

Before checking the hypotheses we compute the dual curve to get a picture of what we expect to find. Translating by taking $x = 1 + X, y = 1 + Y$ and $z = 1 + Z$ and then taking the leading homogeneous (cubic) part gives

$$\overline{H} = 62(X^2Y + XY^2 + X^2Z + Y^2Z + XZ^2 + YZ^2) + 132XYZ\,.$$

The arctic boundary is the dual of this cubic curve. Computing it as in (5.6), we arrive at a polynomial $P^*(a, b)$ defining an algebraic curve in $\mathbb{CP}^2$:

$$\begin{aligned} P^*(a,b) \quad = \quad & 923521 + 5125974\,ba - 3044572\,ab^2 - 2085370\,ab^5 - 3044572\,b^3a - 3044572\,a^2b + 45167\,a^2b^4 \\ & +5125974\,b^4a + 6191514\,a^2b^2 + 2233364\,b^3a^3 + 45167\,a^4b^2 - 3044572\,a^2b^3 - 2085370\,a^5b \\ & -3044572\,a^3b + 5125974\,a^4b - 3044572\,b^2a^3 - 2085370\,a - 2085370\,b + 45167\,a^2 + 45167\,b^2 \\ & +45167\,b^4 + 2233364\,b^3 + 2233364\,a^3 - 2085370\,b^5 + 45167\,a^4 - 2085370\,a^5 + 923521\,b^6 + 923521\,a^6\,. \end{aligned}$$

The zero set of P^* contains two components in $\mathbb{RP}^2$. These are shown in Figure 11. The parametrization of the curve P^* above is via the representation of points in $\mathbb{RP}^2$ as $(a : b : 1)$. The picture is more symmetric in barycentric coordinates $(\alpha, \beta, 1-\alpha-\beta)$ where $a = \alpha/(1-\alpha-\beta)$ and $b = \beta/(1-\alpha-\beta)$. Referring to figure 11, we call the region inside the inner curve the "facet" and the region between the two curves the "annular region". The set K^* is the union of these two regions.

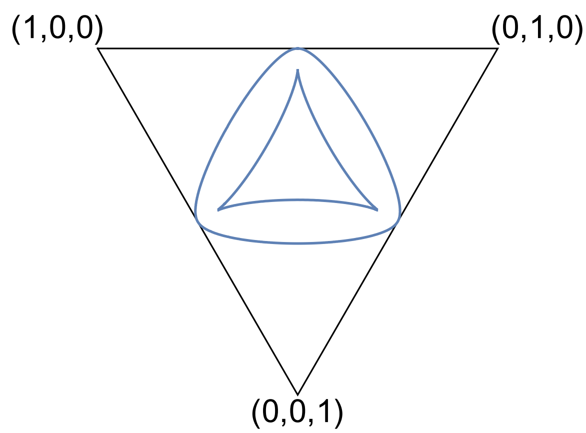

Figure 11: The arctic boundary for the example. The curve is a homogeneous degree-6 curve.

Next, we compute the IFT at the points of the intersection of the zero set of H with the unit torus. It is easily seen that these consist of $\pm(1,1,1)$ along with a two-dimensional set of smooth points, where ∇H is nonvanishing; we denote the smooth set by $\mathcal{V}$. Because the degree of $\overline{H}$ is 3 at $\pm(1,1,1)$ and 1 at any point of $\mathcal{V}$, the IFT will have homogeneous degree 0 at $\pm(1,1,1)$ and -2 at smooth points.

The IFT at $(1,1,1)$ is computed by an elliptic integral. It is nonvanishing on the entire interior of K^*, varying over the annular region and remaining constant on the facet. While everything else in this example can easily be verified, this computation is not routine and will be detailed in forthcoming work [1]. The role of the contribution from $(-1,-1,-1)$ is to double the contribution to a_v from $(1,1,1)$ when the parity of the integer vector v is even and kill it when the parity is odd.

Finally, to put this all together, we check the hypotheses of Corollary 5.8. Boundeness of the coefficients implies hypothesis (i). To check hypothesis (ii), we need only check that only finitely many points of $\mathcal{V}$ have a given logarithmic gradient. This is true with the exception of the projective directions $(1,1,0), (1,0,1)$ and $(0,1,1)$, which are the midpoints of the sides of the triangle and are on the outer boundary of the annular region. Therefore, the hypotheses are satisfied over the interior of K^*.

We conclude that there is exponential decay outside K^*. In the annular region, as $v \to \infty$ with $v/|v|$ tending to $\hat{v}$ and the parity of v remaining even, the coefficient a_v tends to a function $\Phi(\hat{v})$ given by an elliptic integral. If $\hat{v}$ is in the facet, a_v tends to a constant; due to the three-fold symmetry, the constant must be 1/3.

The general case

In general there is a one-parameter family of curves, the foregoing example being a generic instance.

We now replace the specification $(a_0, a_1, a_2, b_1) = (1, 2, 3, 1)$ by any four positive real numbers: $(a_0, a_1, a_2, b_1) = (a, b, c, d)$. Instead of the 8×8 matrix M of Laurent polynomials in x, y and z, we will have a matrix $\tilde{M}$ of rational functions of a, b, c and d that specializes to M when $(a, b, c, d) = (1, 2, 3, 1)$. Computing the characteristic polynomial and factoring yields $P_1 P_2$ with

$$\begin{aligned}
P_1 &= (C_1 + C_2)(x^2y^2z^2 - 1) - C_1(x^2y^2 + x^2z^2 + y^2z^2 - x^2 - y^2 - z^2) \\
&\quad -C_2(x^2y^2 + x^2z^2 + x^2y^2 + y^2z^2 + x^2z^2 + y^2z^2 - xy - xz - yz) \\
C_1 &:= ab^3cd \\
C_2 &:= b^6 + c^6 + 3ac^4d + 3a^2c^2d^2 + 2ab^3cd + 2b^3c^3 + a^3d^3 .
\end{aligned}$$

In a neighborhood of the values $a = b = 1, c = 2, d = 3$ from the worked example, the same argument shows there must be a factor of P_2 in the numerator, so that the reduced generating function is rational with denominator P_1; by analytic continuation, this is true for all parameter values.

Recentering at $(1, 1, 1)$ via the substitution $x = 1 + X, y = 1 + Y, z = 1 + Z$ and taking the lowest degree homogeneous term at the origin yields

$$\overline{H} = (1 - \theta)(X^2Y + X^2Z + Y^2X + Y^2Z + Z^2X + Z^2Y) + (2 + 6\theta)XYZ$$

where $\theta := C_1/(C_1 + C_2)$. Up to a constant factor, this is equal to

$$X^2Y + X^2Z + Y^2X + Y^2Z + Z^2X + Z^2Y + \lambda XYZ \tag{6.5}$$

where

$$\begin{aligned}
\lambda &= \frac{2 + 6\theta}{1 - \theta} \\
&= 2\,\frac{2\,c^3b^3 + 6\,acdb^3 + c^6 + 3\,ac^4d + 3\,a^2c^2d^2 + a^3d^3 + b^6}{a^3d^3 + b^6 + 3\,a^2c^2d^2 + 3\,ac^4d + 2\,acdb^3 + c^6 + 2\,c^3b^3} .
\end{aligned}$$

As a, b, c, d vary over positive reals, the quantity λ varies over the half-open interval $(2, 3]$. It reaches its maximum value when $a = 1, b = 2\sqrt{3}, c = \sqrt{3}$ and $d = 9$ (or at any scalar multiple of this 4-tuple of values) and corresponds to the initial conditions (6.2).

When $\lambda = 3$, the polynomial $\overline{H}$ factors as $(X+Y+Z)(XY+XZ+YZ)$ and when $\lambda = 2$ it factors as $(X + Y)(X + Z)(Y + Z)$. However, for $2 < \lambda < 3$ this polynomial is irreducible with the zero set looking like a cone together with a ruffled collar. Figure 12 shows two examples: on the left $\lambda = 5/2$ and on the right $\lambda = 66/31$, a value much nearer to 2 which is the value from the example with $a = b = 1, c = 2$ and $d = 3$. At $\lambda = 3$ the dual shape is, as we have seen, the inscribed circle; the facet region in this case is degenerate, having shrunk to a point. As λ decreases from 3 to 2, the circle deforms to look more like an inscribed triangle and the facet grows, approaching the outer curve.

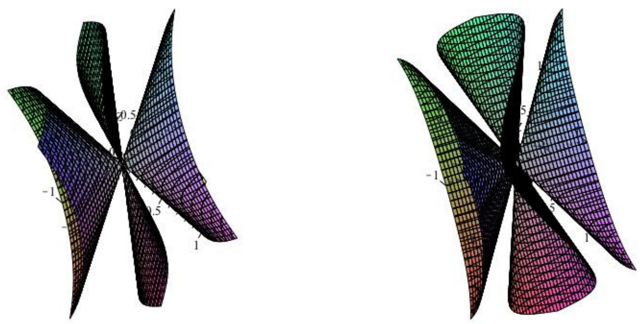

Figure 12: As $\lambda \to 2$ the collar becomes more ruffled (further from a plane)

7 Ising model

In this section we will show how the Y-Delta transformation for the Ising model is a special case of the hexahedron recurrence. We begin by recalling the definition of the Ising model. Let $G = (V, E)$ be a finite graph with $c : E \to \mathbb{R}_+$ a positive weight function on edges. The Ising model is a probability measure μ on the configuration space $\Omega = \{\pm 1\}^V$. A configuration of spins $\sigma \in \Omega$ has probability

$$\mu(\sigma) = \frac{1}{Z} \prod_{e=\{v,v'\} \in E} c(e)^{(1+\sigma(v)\sigma(v'))/2}, \tag{7.1}$$

where the product is over all edges in E and the partition function Z is the sum of the unweighted probabilities $\prod c(e)^{(1+\sigma(v)\sigma(v'))/2}$ over all configurations σ. In other words, the probability of a configuration is proportional to the product of the edge weights of those edges where the spins are equal. The Ising model originated as a thermodynamical ensemble with energy function $H(\sigma) = -\sum_e \sigma(v)\sigma(v')J(e)$: take $J(e) = (T/2)\log c(e)$ where T is Boltzmann's constant times the temperature.

7.1 Ising-Y-Delta move

The Ising-Y-Delta move on the weighted graph $G = (V, E, c)$ transforms the graph the same way as does the Y-Delta move for electrical networks but transforms the edge weights

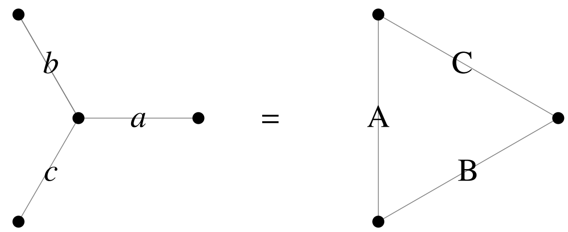

Figure 13: The Y-Delta move.

differently. The transformation is depicted in Figure 13. As is apparent, it converts a Y-shape to a triangle shape, or vice versa. The old weights (a, b, c) are replaced by weights (A, B, C) defined by

$$A = \sqrt{\frac{(abc+1)(a+bc)}{(b+ac)(c+ab)}} \tag{7.2}$$

$$B = \sqrt{\frac{(abc+1)(b+ac)}{(a+bc)(c+ab)}} \tag{7.3}$$

$$C = \sqrt{\frac{(abc+1)(c+ab)}{(a+bc)(b+ac)}}. \tag{7.4}$$

Lemma 7.1. *Equations* (7.2)–(7.4) *are the unique positive solution to*

$$[abc+1 : a+bc : b+ac : c+ab] = [ABC : A : B : C] \tag{7.5}$$

Referring to Figure 13, if the Ising edge weights satisfy (7.5) *then there is a coupling of the configurations before and after a Y-Delta move, agreeing at all vertices other than the one removed when going from Y to Delta.*

Proof. The first statement is a calculation. For the second statement, observe that up to a global sign there are four possible assignments of spins to the three vertices of the triangle (the three outer vertices of the Y). For example if the three spins are $+++$ then the edges of the triangle contribute weight ABC to the weight of the configuration; in this case for the Y graph the central spin is either $+$, in which case the contribution is abc, or $-$, in which case the contribution is 1. Similarly the contributions for the $++-, +-+,$ and $-++$ configuration are C, B, A and $c+ab, b+ac, a+bc$, respectively. As long as the quadruple of local contributions from the Y is proportional to that of the Δ the measures will be the same for the two graphs. ☐

7.2 Kashaev's relation

Suppose that $\mathcal{G} = (V, E)$ is any planar graph with positive weight function $c : E \to \mathbb{R}^+$. Kashaev [9] showed how the space of edge weights for the Ising model on $\mathcal{G}$ can be parametrized differently using weights on the vertices and faces, rather than the edges. This parametrization has the advantage that the Y-Delta move has a simpler form in these new coordinates. Let f be any positive function on vertices and faces of $\mathcal{G}$. On each edge $e = vv'$ with adjacent faces F, F', let $b(e)$ be the ratio $b(e) = \frac{f(v)f(v')}{f(F)f(F')}$. Kashaev associated the weight function w with f where $w(e)$ is the solution greater than 1 of $(w - 1/w)^2/4 = b(e)$.

Lemma 7.2 ([9])**.** *Let $f_0, \ldots, f_7$ be the values at the faces and vertices involved in a Y-Delta transformation, as in Figure 14. Then we have the identity*

$$f_0^2 f_7^2 + f_1^2 f_4^2 + f_2^2 f_5^2 + f_3^2 f_6^2 - 2(f_1 f_2 f_4 f_5 + f_1 f_4 f_3 f_6 + f_2 f_3 f_5 f_6) \\ - 2 f_0 f_7 (f_1 f_4 + f_2 f_5 + f_3 f_6) - 4(f_0 f_4 f_5 f_6 + f_7 f_1 f_2 f_3) = 0\,. \quad (7.6)$$

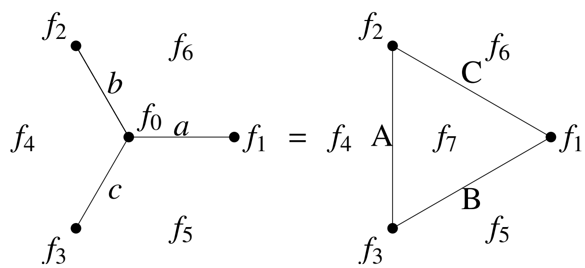

Figure 14: The f variables in the Y-Delta move.

Proof. This is easy enough to check from (7.2)–(7.4), with $\frac{(a-1/a)^2}{4} = \frac{f_0 f_1}{f_5 f_6}$ and so forth. $\square$

We note that the remarkable formula (7.6) has another origin: it is the algebraic identity relating the principal minors of a symmetric matrix, as follows.

Lemma 7.3. *Let M be a symmetric $n \times n$ matrix and for $S \subset \{1, 2, 3\}$ let M_S be the principal minor of M which is the determinant of the matrix obtained from M by removing rows and columns indexed by S. Then for the 8 subsets of S the identity (7.6) holds with*

$$f_0 = M_\phi, f_1 = M_1, f_2 = M_2, f_3 = M_3, f_4 = -M_{23}, f_5 = -M_{13}, f_6 = -M_{12}, f_7 = -M_{123}\,.$$

For an explanation of this, as well as the analogous fact for the hexahedron relation, see [**?**].

Proof. One checks this easily for a 3×3 matrix. For an $n \times n$ matrix M, recall that Jacobi's identity relates minors of M with complementary minors of M^{-1}:

$$\frac{M_S}{M_\phi} = (M^{-1})_{S^c}.$$

(In general there is a sign involved but for principal minors this sign is $+1$.) The equation (7.6) holds for the 3×3 submatrix of M^{-1} indexed by S; this implies that it holds for M for the complementary minors. $\square$

When placed on a lattice, the relation (7.6) has an interpretation as a recurrence for stepped surfaces. Previously we associated a graph $\Gamma(U)$ with each stepped surface ∂U; now we associate another graph $\Upsilon(U)$. The vertices of $\Upsilon(U)$ are taken to be the even vertices of ∂U and the edges of $\Upsilon(U)$ are the diagonals of the faces of ∂U whose endpoints are even. Because every face of ∂U is a quadrilateral, the graph ΥU is planar. If $f : \mathbb{Z}^d \to \mathbb{R}^+$ is a positive function, define edge weight $w(e)$ on an edge e of $\Upsilon(U)$ to be the positive solution to $(w - 1/w)^2/4 = b$ where $b = f(v)f(v')/(f(u)f(u'))$, where $e = \{v, v'\}$ and where u and u' are the other two vertices of the face of ∂U on which e lies. The previous lemma results in the following lattice relation, known as *Kashaev's difference equation.*

Lemma 7.4. *Let $U \subseteq U'$ be stepped solids differing by a single cube.*

1. *The graph $\Upsilon(U')$ differs from $\Upsilon(U)$ by a Y-Delta move: Y to Delta if the bottom vertex of the added cube was even and Delta to Y otherwise.*

2. *If e is a weight function on the edges of $\Upsilon(U)$, extended by the Ising-Y-Delta relations to the edges of $\Upsilon(U')$, and if f is a function on the vertices of $\mathbb{Z}^d$ inducing e on the edges of $\Upsilon(U)$ and $\Upsilon(U')$ then at the eight vertices of the added cube, f satisfies the relations (1.3).*

$\square$

Kashaev's relation is almost a recurrence: $f_{(123)}$ is determined from the other seven values up to the choice of root of a quadratic equation. It turns out there is a canonical choice.

Proposition 7.5. *Let*

$$X = \sqrt{ff_{(23)} + f_{(2)}f_{(3)}}, \quad Y = \sqrt{ff_{(13)} + f_{(1)}f_{(3)}}, \quad Z = \sqrt{ff_{(12)} + f_{(1)}f_{(2)}}.$$

Then the recurrence (1.3) *can be written*

$$X_{(1)} = \frac{f_{(1)}X + YZ}{f} \tag{7.7}$$

$$Y_{(2)} = \frac{f_{(2)}Y + XZ}{f} \tag{7.8}$$

$$Z_{(3)} = \frac{f_{(3)}Z + XY}{f} \tag{7.9}$$

$$f_{(123)} = \frac{2f_{(1)}f_{(2)}f_{(3)} + ff_{(1)}f_{(23)} + ff_{(2)}f_{(13)} + ff_{(3)}f_{(12)} + 2XYZ}{f^2}. \tag{7.10}$$

$\square$

The proof is a simple verification.

7.3 Embedding Kashaev's recurrence in the hexahedron recurrence

The proof of all results in this section are straightforward substitutions and are omitted.

Proposition 7.6. *Suppose $f : \mathbb{Z}^3_{1/2} \to \mathbb{C}$ satisfies the following relation for integer (i,j,k):*

$$
\begin{aligned}
f(i+1/2, j+1/2, k)^2 &= f(i,j,k)f(i+1,j+1,k) + f(i,j+1,k)f(i+1,j,k) \\
f(i+1/2, j, k+1/2)^2 &= f(i,j,k)f(i+1,j,k+1) + f(i,j,k+1)f(i+1,j,k) \\
f(i, j+1/2, k+1/2)^2 &= f(i,j,k)f(i,j+1,k+1) + f(i,j,k+1)f(i,j+1,k)\,.
\end{aligned}
$$

Then f satisfies the Kashaev relation (1.3) *at integer points if f satisfies the hexahedron relations* (1.4a)–(1.4d), *where as usual we interpret $h = f$, $h^{(x)} = f_{(0,1/2,1/2)}$, $h^{(y)} = f_{(1/2,0,1/2)}$ and $h^{(z)} = f_{(1/2,1/2,0)}$.* □

We obtain the Ising-Y-Delta recurrence as a corollary.

Corollary 7.7. *Suppose the initial conditions for f at the vertices of a stepped surface ∂U are real and positive. Define f on the z-faces of the stepped surface by*

$$f(i+1/2, j+1/2, k) := \sqrt{f(i,j,k)f(i+1,j+1,k) + f(i,j+1,k)f(i+1,j,k)}$$

and similarly for the x- and y-faces, always taking the positive square root. Then the values produced by the hexahedron recurrence at all points above the stepped surface, restricted to integer points, yield the Ising-Y-Delta recurrence. □

Remark. Another way to say this is that the equation (7.6) is a special case of (2.2d). Setting $a_1 = \sqrt{a_5a_6 + a_0a_7}$, $a_2 = \sqrt{a_4a_6 + a_0a_8}$ and $a_3 = \sqrt{a_4a_5 + a_0a_9}$, the recurrence (2.2d) becomes (7.6) after relabelling variables to correspond to the same geometric positions: $f_0 = a_0, f_1 = a_4, f_4 = a_7$ etc.

As in the dimer case let us consider initial conditions on the stepped surface defined by $0 \le i+j+k \le 2$ (take U to be the lattices cubes lying entirely within $\{(x,y,z) : x+y+z \le 2\}$). Recall that $\Gamma(U)$ was the 4-6-12 graph. It is easy to see that $\Upsilon(U)$ is the regular triangulation. Indeed, there is a vertex of $\Upsilon(U)$ at the center of each dodecagon of $\Gamma(U)$ and an edge of $\Upsilon(U)$ connecting dodecagons that share a quadrilateral neighbor. The function f now takes values on the vertices and faces of this triangulation and the X, Y, Z values lie on the three directions of edges. Starting with initial data of the f values on $\{0 \le i+j+k \le 2\}$, one can determine $X_{i,j,k}, Y_{i,j,k}, Z_{i,j,k}$ on the set $i+j+k = 0$. From (7.7)–(7.9) one can then determine the values of X, Y, Z on $i+j+k = 1$; then from (7.10) one can determine the values of f on $i+j+k = 3$, and so on; in this way one determines X, Y, Z, f on all planes $i+j+k \ge 0$. Theorem 4.1 has the following consequence for the $f_{i,j,k}$.

Theorem 7.8. *$f_{i,j,k}$ is a Laurent polynomial in the initial variables $\{f_{i,j,k}\}_{0\le i+j+k\le 2}$ and $\{X_{i,j,k}, Y_{i,j,k}, Z_{i,j,k}\}_{i+j+k=0}$. The X, Y, Z variables only appear with power 1.*

Proof. Let $\{a_{ijk}\}$ denote the initial conditions for the hexahedron recurrence (before specialization), thus $f_{ijk} = a_{ijk}$ for $0 \leq i+j+k \leq 2$. Take a monomial M in the Laurent expansion of $a_{i,j,k}$. Let a_0 be a quadrilateral variable; it occurs in M with exponent in $[-2, 2]$.

There are several cases to consider. If a_0 occurs with power 2, we can replace it with $a_1a_3 + a_2a_4$ where a_1, a_2, a_3, a_4 are the four faces in cyclic order adjacent to a_0. Then the monomial M becomes a sum of two monomials not involving a_0.

If a_0 appears with degree -1, there is another monomial M' of $a_{i,j,k}$ which pairs with it, in the sense that $M/M' = \frac{a_1a_3}{a_2a_4}$ (these two monomials correspond to the two possible configurations of double-dimers which have three edges lying along the quad face at a_0, and use the same two edges joining the quad face to adjacent faces: as in line 2 or Figure 10). The sum of these two monomials is, up to monomial factors M^* not involving a_0, $M + M' = M^*(a_1a_3 + a_2a_4)/a_0 = M^*a_0$. Upon replacing a_0 by $\sqrt{a_1a_3 + a_2a_4}$ this now becomes a monomial in which $a_0 = \sqrt{a_1a_3 + a_2a_4}$ occurs with power $+1$.

If a_0 which appears in M with degree -2, there are two other monomials M', M'', which in the appropriate order have the ratios $[2 : \dfrac{a_1a_3}{a_2a_4} : \dfrac{a_2a_4}{a_1a_3}]$ (these correspond to the three possible configurations of double-dimers which have four edges lying along the quad face at a_0, as in the first line of Figure 10). The sum of these is

$$M + M' + M'' = M^* \frac{2 + \dfrac{a_1a_3}{a_2a_4} + \dfrac{a_2a_4}{a_1a_3}}{a_0^2} = M^* \frac{a_0^2}{a_1a_2a_3a_4}$$

and upon the substitution $a_0^2 = a_1a_3 + a_2a_4$ this is a sum of two monomials not involving a_0.

Once all these substitutions (and groupings) are done, a_0 only appears in the numerator and has degree 1 or 0. We can similarly regroup terms for all the other quad variables: since no two quad faces are adjacent the groupings "commute" in the sense that they can be done in any order.

After regrouping all quad variables, we see that $f_{i,j,k}$ (the specialization of $a_{i,j,k}$) is a Laurent polynomial, with positive coefficients, in the initial variables f, X, Y, Z, and with the X, Y, Z (the quad variables) appearing in the numerator only and of degree 0 or 1. $\square$

7.4 Open question

What are the natural combinatorial structures counted by the monomials in $f_{i,j,k}$? Using Proposition 7.6 it appears possible (although we have not succeeded) to get an interpretation of the monomials in the expansion of $f_{i,j,k}$ in terms of collections of double-dimer covers of Γ_{i+j+k}.

Note however that when we apply the substitutions of the proof of that proposition, it is possible that the new monomials are not distinct, and combine to make monomials of $f_{i,j,k}$ in other ways. This is already true of $a_{1,1,1}$, in which 9 monomials collapse into a single monomial of $f_{1,1,1}$.

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