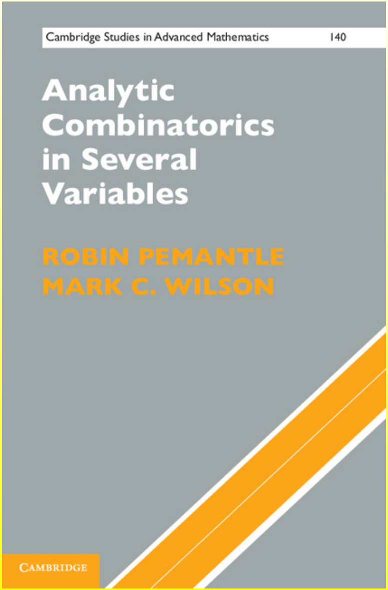
Analytic Combinatorics in Several Variables

Robin Pemantle and Mark Wilson

A of A conference, 30 May, 2013

About the book




Dogs for Peac


Dedication

To the memory of Philippe Flajolet,
on whose shoulders stands all of the work herein.

The book is in four parts

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- General introduction and univariate methods

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I General introduction and univariate methods

II Some complex analysis and some algebra

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III Multivariate asymptotics

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IV Appendices

The Big Question

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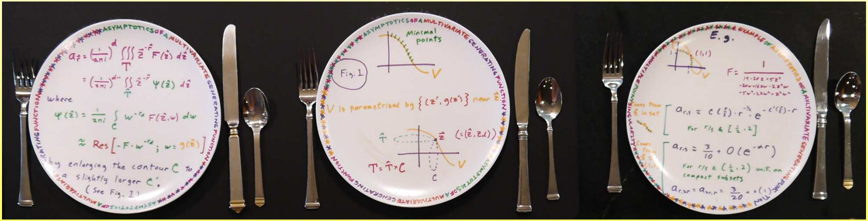
Can I really use these methods without a ridiculous investment of time?


ASYMPTOTICS OF A MULTIVARIATE GENERATING FUNCTION
where
by enlarging the contour C to a slightly larger C'.
(See Fig. I)
Fig. I
Minimal points
V is parametrized by
E.g.
EXAMPLE OF ASYMPTOTICS OF A MULTIVARIATE GENERATING FUNCTION
compact subsets

Scope of method

Structures with recursive nature

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- Analysis of algorithms

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- Random polynomials

Running example

Example

Lattice paths to $(2n, 2n, 2n)$ *with steps* $\{(2,0,0),(0,2,0),(0,0,2),(1,1,0),(1,0,1),(0,1,1)\}$.

$$F(x,y,z) = \frac{1}{1 - x^2 - y^2 - z^2 - xy - xz - yz}$$

Analogy with univariate singularity analysis

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2. Behavior of **f** near the dominant singularity(ies) determines the exact asymptotics

Rational multivariate case

$\mathbf{F}(\mathbf{x}) = \mathbf{P}(\mathbf{x})/\mathbf{Q}(\mathbf{x})$: singularity is the surface $\mathcal{V} := \{\mathbf{Q} = 0\}$.
Carry out same two steps.

STEP 1: Find dominant singularity

1a. Algebra

1b. Geometry

Step 1a: Algebra

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Step 1a: Algebra

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Answer: [1].

Aha, it's smooth.

Table of contents checklist

1. Introduction
2. Enumeration
3. Univariate asymptotics
4. Complex analysis: univariate saddle integrals
5. Complex analysis: multivariate saddle integrals
6. Symbolic algebra
7. Geometry of minimal points (amoebas)

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✓ 1. Introduction

✓ 2. Enumeration

✓ 3. Univariate asymptotics

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5. Complex analysis: multivariate saddle integrals

✓ 6. Symbolic algebra

7. Geometry of minimal points (amoebas)

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Univariate integrals

$$\int_{-\infty}^{\infty} f(t) e^{-\lambda t^2/2} \, dt = \sqrt{2\pi} \; f(0) \; \lambda^{-1/2}$$

Step 1b: Geometry

Next, we use what we know from Step 1a to draw a picture of the singularities "nearest to the origin". In one variable, "nearest" means the least value of $|z|$. In several variables, we mean those points $(x_1, \ldots, x_r) \in \mathcal{V}$ with $(|x_1|, \ldots, |x_d|)$ minimal in the partial order.

Step 1b: Geometry

Chapter 7 is the science of determining this set, which is a portion of the boundary of the *amoeba* of Q. Typically, this set is a real $(d-1)$-dimensional subspace of $\mathcal{V}$. There is a science to this, which you can read about in Chapter 7.

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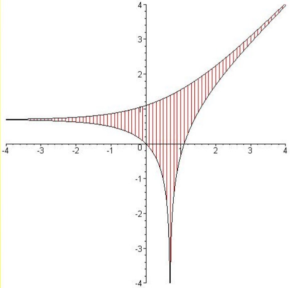
Often we change to logarithmic coordinates, in which case the amoeba looks something like this.



Step 1b: Geometry

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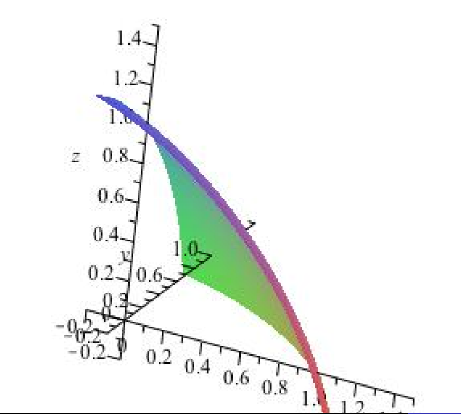

Completing Step 1

Having described the minimal points, we find the dominating point $z \in \mathcal{V}$ (or x in the amoeba boundary) corresponding to the asymptotic direction r of interest. It will be the point on the minimal surface normal to r.

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Example: $Q = 1 - x^2 - y^2 - z^2 - xy - xz - yz$. By symmetry, the point

$$z_* := \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}} \right)$$

is the dominating point for the diagonal direction. The exponential growth rate of a_r is z^{-r}. Thus,

$$a_{2n,2n,2n} = (216 + o(1))^n \, .$$

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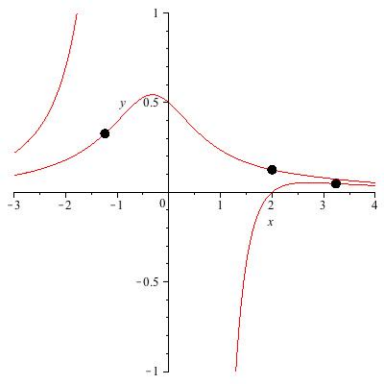
$$\begin{aligned} F &= \frac{P}{Q} \\ Q &= x^5y^2 + 2x^2y - 2x^3y + 4x + y - 2. \end{aligned}$$

The generating function counts certain combinatorial objects but it is only nonnegative in a certain region where the parameters make sense.

The fancy stuff: Morse theory

Finding all possible candidates is an easy algebraic computation, producing three possibilities.



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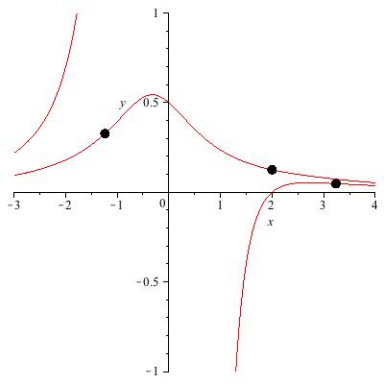

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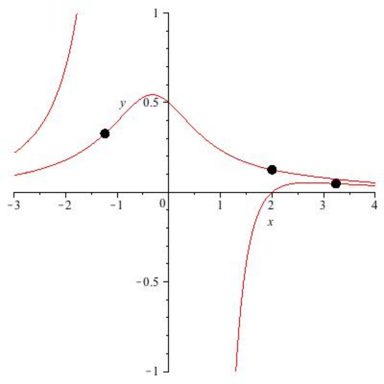

The only minimal point is the rightmost point, but the dominating point is the middle point.

This is difficult to determine but you will not usually need to!

Step 1b: Geometry

Summary: computing the minimal points is not trivial, but in many cases it is not much more than high school geoemtry.

In other words: you don't need Chapter 7 to get started, and it's not so bad anyway.

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- $\gamma = z_*^{-\hat{r}}$
- C is computed in an elementary but tedious way from the partial derivatives of P and Q at z_* (it is the curvature of the minimal surface in logarithmic coordinates).

Step 2, running example

$$\begin{aligned}
\hat{r} &= (1,1,1) \\
Q &= 1 - x^2 - y^2 - z^2 - xy - xz - yz \\
z_* &= \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right) \\
\gamma &= 6^{3/2} \\
C(\hat{r}) &= \frac{\sqrt{3}}{5\pi}
\end{aligned}$$

which leads to

$$a_{2n,2n,2n} \sim 216^n \left[\frac{\sqrt{3}}{5\pi n} + O(n^{-2})\right]$$

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"Easily": SAGE code exists written by A. Raichev + MCW

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Time to update the checklist.

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I will not pretened Chapters 10 and 11 are easy, but you will not need Chapter 11 unless you are lucky enough to run across GF's with an unusual nature.

Interlude: diagonal computation not recommended

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Unextracting the diagonal, unlike diagonal extraction, is not too hard and allows us to extend everything we know about rational functions to the algebraic case. Now that you know about it, we can check off Chapters 12 and 13.

Part III checklist, updated

- ✓ Chapter 9: smooth points
- Chapter 10: interesections of smooth surfaces
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- ✓Chapters 12 and 13: wrapping it up (examples and further speculation)

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Case (*ii*): Self-intersecting smooth surfaces

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Case (*iii*): More complicated local geometry

What's left?

Case (*ii*): Self-intersecting smooth surfaces

Requires theory of iterated residues

Case (*iii*): More complicated local geometry

Requires theory of hyperbolic polynomials,

What's left?

Case (*ii*): Self-intersecting smooth surfaces

Requires theory of iterated residues

Case (*iii*): More complicated local geometry

Requires theory of hyperbolic polynomials, generalized functions,

What's left?

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Case (*iii*): More complicated local geometry

Requires theory of hyperbolic polynomials, generalized functions, Leray and Petrovsky cycles and

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Case (*iii*): More complicated local geometry

Requires theory of hyperbolic polynomials, generalized functions, Leray and Petrovsky cycles and some classical inverse Fourier transform computations.

What's left?

Case (*ii*): Self-intersecting smooth surfaces

Requires theory of iterated residues

Case (*iii*): More complicated local geometry

Requires theory of hyperbolic polynomials, generalized functions, Leray and Petrovsky cycles and some classical inverse Fourier transform computations.

These are difficult and we are not going to check them off today.

You know what to do!