The Penn Calc Companion

Part II: Integration and Applications

About this Document

This material is taken from the wiki-format Penn Calc Wiki, created to accompany Robert Ghrist's Single-Variable Calculus class, as presented on Coursera beginning January 2013. Past editors and contributors include: Prof. Robert Ghrist, Mr. David Lonoff, Dr. Subhrajit Bhattachayra, Dr. Alberto Garcia-Raboso, Dr. Vidit Nanda, Ms. Lee Ling Tan, Mr. Brett Bernstein, Prof. Antonella Grassi, and Prof. Dennis DeTurck.

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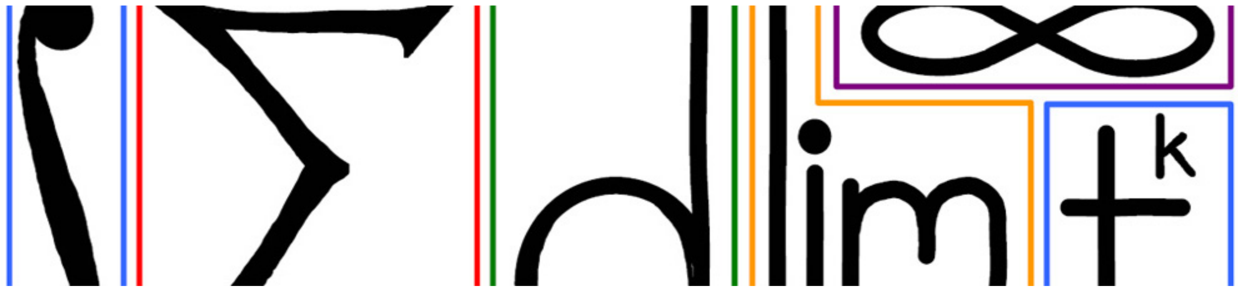


17 Antidifferentiation

This module begins our study of integration. Integration, or *anti-differentiation*, can be thought of as running differentiation in reverse, or undoing the derivative.

This motivates the following definition:

The Indefinite Integral

The *indefinite integral of* $f(t)$, denoted $\int f(t)\,dt$, is the class of functions whose derivative is $f(t)$. $\int f(t)\,dt$ is also referred to as the *anti-derivative of* f. The act of taking the indefinite integral is an operator which is referred to as *anti-differentiation* or *integration*.

Note

The indefinite integral of a function is only defined up to an added constant, called the constant of integration. In other words, if $F(x)$ is an anti-derivative of $f(x)$, then $G(x) = F(x) + C$, C a constant, is also an anti-derivative of f, because C disappears when differentiated. Conversely, any two indefinite integrals of $f(x)$ differ only by some constant.

Any of the known derivatives from the previous chapter can be rephrased as an integral. For example, just as there was a power rule for differentiating monomials, there is a corresponding power rule for integrating monomials. And any anti-derivative can easily be checked by taking the derivative and seeing that the result gives back the original function.

Example

Give the integral of each of the following functions: x^n, $\frac{1}{x}$, $\sin x$, $\cos x$, e^x. (See Answer 1)

There are other functions which are harder to integrate by merely using one of the derivatives we already know. Some of these can be integrated using other techniques from upcoming modules, but there are also functions whose anti-derivative cannot be expressed in terms of simple functions.

17.1 Differential equations

The motivating problem for the study of anti-differentiation is solving a *differential equation*. A differential equation is an equation involving a function and its derivative. In this course, we deal with *ordinary* differential equations, ODEs, which are differential equations involving only functions of one variable and the derivative with respect to that variable (future courses deal with *partial differential equations*, which involve functions of several variables and partial derivatives).

Solving a differential equation means finding the function (or class of functions, usually) which satisfy the differential equation.

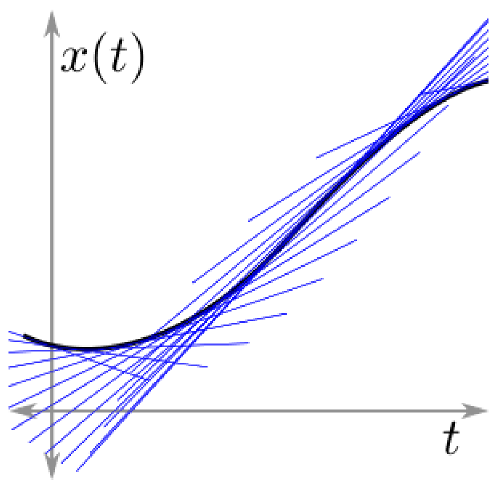
A Simple ODE

The simplest differential equation is of the form

$$\frac{dx}{dt} = f(t).$$

Here, the goal is to find the function $x(t)$ whose derivative with respect to t is $f(t)$. But this is precisely what the integral is. And so, the solution of the differential equation $\frac{dx}{dt} = f(t)$ is given by $x(t) = \int f(t)dt$.

Using the interpretation of the derivative as slope, one can think of the function $f(t)$ as describing the slope of the function $x(t)$:



Thus, $x(t)$ is a function which fits the slopes prescribed by $f(t)$. Note that any constant vertical shift of a solution $x(t)$ will still have the same slope at each point. This is one interpretation of the integration constant: it represents the potential vertical shifts to a solution of the differential equation.

Example

Consider a falling object. Let $x(t)$ be the height of the object at time t, $v(t)$ be the velocity of the object, and assume that acceleration is the constant $-g$ (negative because gravity pulls down). Express the height of the object as a function of t, v_0, and x_0; here, v_0 and x_0 are the velocity and height of the object, respectively, at time $t = 0$. (See Answer 2)

The Next Simplest ODE

Another slightly more complex ODE is of the form

$$\frac{dx}{dt} = f(x).$$

Before we discuss how to solve this in general, we consider a specific example, which is one of the most famous differential equations:

$$\frac{dx}{dt} = ax,$$

where a is a constant. We solve this differential equation in three different ways:

1. (Guess) Solve this differential equation by first observing that $x = Ce^t$ satisfies $\frac{dx}{dt} = x$ and then adjusting the exponent so that an extra factor of a comes out when differentiating. Hint: remember the chain rule. (See Answer 3)

2. (Series) Solve the differential equation by assuming

$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots$$

and then determining what the constants c_i must be to satisfy the differential equation. (See Answer 4)

3. (Integration) Rearrange the differential equation into the form

$$\frac{dx}{x} = a\,dt$$

and integrate both sides to solve the differential equation. (See Answer 5)

The differential equation from this example is sometimes used as a simple model of population growth. In words, the differential equation says that the growth of a population is proportional to the size of the population. As the solution above slows, this model implies the population has exponential growth. This is not a very good model for most populations because of competition for resources and overcrowding. But under certain conditions and for short periods of time, some populations (for instance, bacteria with an abundant food supply) do exhibit exponential growth. For more examples of exponential growth, see the next module.

17.2 Initial value problems

Although a general differential equation's solution often depends on a constant (sometimes several), an additional condition called an *initial value* or *initial condition* can specify a specific solution. This condition is usually of the form $y(t_0) = y_0$. A differential equation with such an initial condition is called an *initial value problem*. To solve such a problem, first find the general solution and then use the initial value to find the specific constant of integration which satisfies the initial condition.

In the context of population growth, the initial value is typically the size of the population at time 0. This is particularly nice in the exponential growth model, because the solution is of the form $P(t) = De^{At}$. So if $P(0) = P_0$ is given, then plugging this in gives $P(t) = P_0 e^{At}$.

17.3 EXERCISES

- $\displaystyle\int (4x^3 + 3x^2 + 2x + 1)\,dx =$
- $\displaystyle\frac{d}{dx}\int \ln\tan x\,dx =$
- $\displaystyle\int \left(\frac{d}{dx}e^{-x}\right)dx =$

- Find the general solution of the differential equation

$$\frac{dx}{dt} = t^2$$

- Find the general solution of the differential equation

$$\frac{dx}{dt} = x^2$$

- There is a large class of differential equations – the so-called "linear" ones – for which we can find solutions using the Taylor series method discussed in the Lecture. One such differential equation is

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + tx = 0$$

  This is a particular case of the more general "Bessel differential equation," and one solution of it is given by the Bessel function $J_0(t)$ that we saw earlier. Notice that this involves not only the first derivative but also the second derivative. For this reason, it is said to be a "second order" differential equation.

  In this problem we will content ourselves with finding a relationship (specifically, a "recurrence relation") on the coefficients of a Taylor series expansion about $t = 0$ of a solution to our equation. Hence consider the Taylor series

$$x(t) = \sum_{k=0}^{\infty} c_k t^k$$

  Substituting this into the differential equation above will give you two conditions. The first one is $c_1 = 0$. What is the other one?

  "Note:" this problem involves some nontrivial manipulation of indices in summation notation. Do not get discouraged if it feels more difficult than other problems: it is!

- Find the general solution of the differential equation

$$\frac{dx}{dt} = t^3 + x^2t^3$$

17.4 Answers to Selected Examples

1.

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C$$
$$\int \frac{1}{x}dx = \ln|x| + C$$
$$\int \sin x dx = -\cos x + C$$
$$\int \cos x dx = \sin x + C$$
$$\int e^x dx = e^x + C$$

(Don't forget the constant!)

(Return)

2. We know from an earlier module that
$$\frac{dv}{dt} = a = -g.$$

   Beginning with the second of these equations, we find that
$$v(t) = \int (-g)\, dt = -gt + C.$$

   We can determine C by plugging in $t = 0$. This cancels that $-gt$ and leaves us with $C = v(0) = v_0$, the initial velocity. Thus,
$$v(t) = -gt + v_0$$

   Now, using the fact that
$$\frac{dx}{dt} = v,$$

   we find that
$$\begin{aligned} x(t) &= \int v(t)\, dt \\ &= \int (-gt + v_0)\, dt \\ &= -\frac{1}{2}gt^2 + v_0 t + C. \end{aligned}$$

   Again, we can find C by plugging in $t = 0$. This leaves us with $x(0) = C$, and so $C = x_0$, the initial height. Thus,
$$x(t) = -\frac{1}{2}gt^2 + v_0 t + x_0.$$

   (Return)

3. Observe that $x = Ce^{at}$ will get an extra factor of a when differentiated by the chain rule. That is,
$$\frac{d}{dt}\left(Ce^{at}\right) = aCe^{at}.$$

   And so $x(t) = Ce^{at}$ is a solution of the differential equation.

   (Return)

4. Assuming that
$$x(t) = c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots ,$$

   and then taking the derivative of this series, term by term, we find
$$\frac{dx}{dt} = 0 + c_1 + 2c_2 t + 3c_3 t^2 + \cdots .$$

   On the other hand, from the original differential equation we have
$$\begin{aligned} \frac{dx}{dt} &= ax \\ &= a\left(c_0 + c_1 t + c_2 t^2 + c_3 t^3 + \cdots\right) \\ &= ac_0 + ac_1 t + ac_2 t^2 + ac_3 t^3 + \cdots . \end{aligned}$$

Because these two series both equal $\frac{dx}{dt}$, they must be equal to each other. But two series are equal if and only if their corresponding coefficients are equal. Therefore,

$$\begin{aligned} c_1 &= ac_0 \\ 2c_2 &= ac_1 \\ 3c_3 &= ac_2, \end{aligned}$$

and so on. Solving these equations one by one gives

$$\begin{aligned} c_1 &= ac_0 \\ c_2 &= \frac{1}{2}ac_1 = \frac{1}{2}a^2c_0 \\ c_3 &= \frac{1}{3}ac_2 = \frac{1}{6}a^3c_0 \end{aligned}$$

And, generally, $c_n = \frac{1}{n!}a^nc_0$ (this can be proven using a method called induction). Doing a little bit of factoring and grouping of factors, we find

$$\begin{aligned} x(t) &= c_0 + ac_0t + \frac{1}{2!}a^2c_0t^2 + \frac{1}{3!}a^3c_0t^3 + \cdots \\ &= c_0\left(1 + (at) + \frac{1}{2!}(at)^2 + \frac{1}{3!}(at)^3 + \cdots\right) \\ &= c_0e^{at}, \end{aligned}$$

which is, again, of the form Ce^{at}.

(Return)

5. Using the chain rule and substituting according to the differential equation, we have

$$\begin{aligned} dx &= \frac{dx}{dt}\,dt \\ dx &= ax\,dt \\ \frac{dx}{x} &= a\,dt. \end{aligned}$$

Integrating both sides of the equation gives

$$\ln x = at + C$$

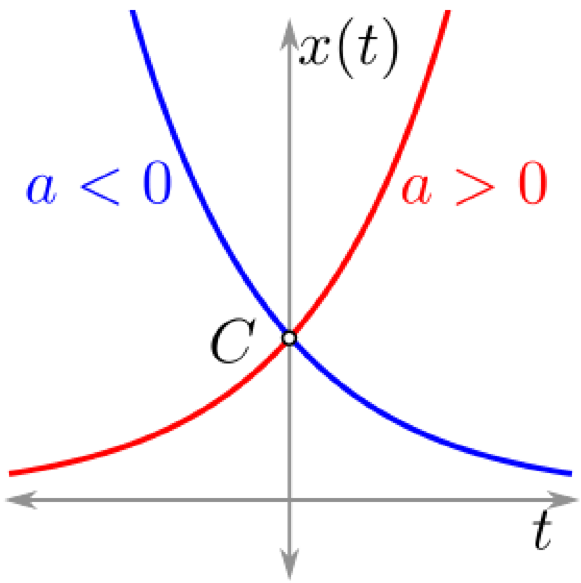
(only one constant of integration is necessary here, because a constant on the left side could be subtracted from both sides and absorbed into C). Now, exponentiating the equation gives $x = e^{at+C}$. By exponential rules, $e^{at+C} = e^{at}e^C$, and the e^C is often rewritten as a new constant, often written C again.

Thus, the solution to $\frac{dx}{dt} = ax$ is $x(t) = Ce^{at}$, where C is any constant.

(Return)

18 Exponential Growth Examples

Recall from the last module that the differential equation $\frac{dx}{dt} = ax$ has solution $x = Ce^{at}$, where C is some constant. The constant C can be thought of as an initial condition, the value of the function at time $t = 0$. When $a > 0$, the function has *exponential growth*. When $a < 0$, the function has *exponential decay*:



This module is devoted to several examples of exponential growth and decay.

18.1 Radioactive decay

Carbon-14 is a radioactive isotope of carbon which exists in organic materials. It is known that the rate at which carbon-14 atoms decay is proportional to the number of carbon-14 atoms present. If I represents the number of atoms, then the differential equation is

$$\frac{dI}{dt} = -\lambda I,$$

where λ is positive (and so the number of atoms is decreasing).

18.2 Population growth

For bacteria with an abundant food supply, the population P satisfies

$$\frac{dP}{dt} = bP,$$

for some positive b. But as the food supply dwindles, or overcrowding occurs, the population growth will necessarily slow (or else the bacteria would eventually consume the earth). Thus, this is not usually an accurate population model.

18.3 Interest accumulation

Consider a bank account with initial deposit (also called the *principal*) P, annual interest rate r, and which is compounded n times a year (so $n = 1$ gives simple interest, $n = 4$ is quarterly interest, etc.). Then the value of the account at the end of k years is $P\left(1 + \frac{r}{n}\right)^{nk}$. What happens as n gets bigger and bigger? Recall that $\lim_{n\to\infty}\left(1 + \frac{\alpha}{n}\right)^n = e^\alpha$. Then it follows that

$$\begin{aligned}\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nk} &= \lim_{n\to\infty} P\left[\left(1+\frac{r}{n}\right)^{n}\right]^k \\ &= Pe^{rk}.\end{aligned}$$

This is called *continuous compounding*. So an account with continuous compounding is worth Pe^{rk} after k years, where P is the initial investment, and r is the annual interest rate.

Rule of 70

The *Rule of 70* is a mental math trick that approximates the number of years it takes for a continuously compounded account to double in value. The rule says that

$$\#\text{ years for the account to double } = \frac{70}{100r}.$$

In other words, the number of years is 70 divided by the annual percentage. Verify the Rule of 70. (See Answer 1)

18.4 Linguistics

Historical linguists study (among other things) word usage and the rate at which words fall out of use. Let $W(t)$ be the number of words which are in active use in English after t years. One model predicts that $\frac{dW}{dt} = -\lambda W$, hence $W = Ce^{-\lambda t}$, and C is the number of common words at time 0.

Example

Suppose the writing of John Milton (from around 1667) consists of 20% words which are unfamiliar to us today. Use the above model to estimate the fraction of words that Shakespeare's audience (from around 1600) recognized of Chaucer's writing (from around 1400). (See Answer 2)

This model ignores the creation of new words as well as the fact that definitions of existing words can evolve over time. So it is probably not the most accurate model. It is worth noting that even if the mathematics are correct, if the underlying model is not very good, then the resulting answer will not be very good either.

18.5 Zombies

Suppose in the zombie apocalypse that the rate of change of the infected population $Z(t)$ is proportional to the uninfected population $U(t)$. Let P be the total population (assumed to be constant). Then

$$\frac{dZ}{dt} = rU = r(P - Z),$$

since $U = P - Z$ (the number of uninfected is the total population less the infected). Taking the derivative of $U = P - Z$ gives

$$\begin{aligned}\frac{dU}{dt} &= \frac{dP}{dt} - \frac{dZ}{dt} \\ &= -\frac{dZ}{dt} \\ &= -rU,\end{aligned}$$

since P was assumed to be constant. Thus, $U(t) = U_0 e^{-rt}$, where U_0 is the initial uninfected population. And so the zombie population is given by

$$Z(t) = P - U_0 e^{-rt}.$$

Note the population is doomed, as $Z(t) \to P$ as $t \to \infty$.

Although a zombie apocalypse is unlikely, this model is still useful for other phenomena involving how quickly something spreads. This includes the spread of disease, propaganda, and technology. Another example is heat transfer, discussed in detail below.

18.6 Newton's law of cooling

Newton's law of cooling says that the rate of change of the temperature of a body is proportional to the difference of the temperatures of the body and ambient environment. Let T be the temperature of the body and A be the ambient temperature. Then

$$\frac{dT}{dt} = k(A - T),$$

for some positive constant k. Separating gives

$$\frac{dT}{T - A} = -k\,dt,$$

and integrating and exponentiating gives $T - A = Ce^{-kt}$. Thus the solution is $T = A + Ce^{-kt}$. Note that k must be positive for this to model to make sense. Indeed, as $t \to \infty$, the temperature of the body should approach the ambient temperature, which will only happen if $k > 0$. Also, note that $C = T_0 - A$ is the initial difference in temperature.

18.7 EXERCISES

- After drinking a cup of coffee, the amount C of caffeine in a person's body obeys the differential equation

  $$\frac{dC}{dt} = -\alpha C$$

  where the constant α has an approximate value of 0.14 hours^{-1}

  How many hours will it take a human body to metabolize half of the initial amount of caffeine?

- The amount I of a radioactive substance in a given sample will decay in time according to the following equation:

$$\frac{dI}{dt} = -\lambda I$$

Nuclear engineers and scientists tend to be concerned with the "half-life" of a substance, that is, the time it takes for the amount of radioactive material to be halved.

Find the half-life of a substance in terms of its decay constant λ.

- In a highly viscous fluid, a falling spherical object of radius r decelerates right before reaching the bottom of the container. A simple model for this behavior is provided by the equation

$$\frac{dh}{dt} = -\frac{\alpha}{r}h,$$

where h is the height of the object measured from the bottom, and α is a constant that depends on the viscosity of the fluid.

Find the time it would take the object to drop from $h = 6r$ to $h = 2r$ in terms of α and r.

- On a cold day you want to brew a nice hot cup of tea. You pour boiling water (at a temperature of 212° F) into a mug and drop a tea bag in it. The water cools down in contact with the cold air according to Newton's law of cooling:

$$\frac{dT}{dt} = \kappa(A - T)$$

where T is the temperature of the water, $A = 32°$F the ambient temperature, and $\kappa = 0.36$ min^{-1}.

The threshold for human beings to feel pain when entering in contact with something hot is around 107° F. How many seconds do you have to wait until you can safely take a sip?

- On the night of April 14, 1912, the British passenger liner RMS Titanic collided with an iceberg and sank in the North Atlantic Ocean. The ship lacked enough lifeboats to accommodate all of the passengers, and many of them died from hypothermia in the cold sea waters. Hypothermia is the condition in which the temperature of a human body drops below normal operating levels (around 36°C). When the core body temperature drops below 28°C, the hypothermia is said to have become severe: major organs shut down and eventually the heart stops.

If the water temperature that night was -2°C, how long did it take for passengers of the Titanic to enter severe hypothermia? Recall from lecture that heat transfer is described by Newton's law of cooling:

$$\frac{dT}{dt} = \kappa(A - T)$$

where T is the body temperature of a passenger, A the water temperature, and $\kappa = 0.016$ min^{-1}.

- The birthrate of a population (number of births per year × 100 / number of population) is 20%, and the mortality rate (number of deaths per year × 100/ number of population) is 5%. If the initial population is 10,000, find the function $P(t)$, the population as a function of time. How long does it take for the population to double?

- Let $y(t)$ denote the number of atoms of a particular radioactive isotope of carbon at year t. We know that the rate at which $y(t)$ decreases is proportional to y itself. (a) What differential equation does $y(t)$ satisfy? (b) If it takes 5 years for the number to decrease to half of its initial number, what is the constant involved in your answer of part (a)?

18.8 Answers to Selected Examples

1. The problem asks for k when the balance has doubled. That is, find k such that

$$2P = Pe^{rk}.$$

Dividing by P and taking logarithms gives $rk = \ln(2)$, so $k = \frac{\ln(2)}{r}$. Since $\ln(2) \approx .7$, this shows that

$$\begin{aligned} k &= \frac{\ln(2)}{r} \\ &\approx \frac{.7}{r} \\ &\approx \frac{70}{100r}, \end{aligned}$$

as desired.

(Return)

2. Let t denote time measured in years. Let $WM(t)$ denote the number of words from Milton's time which are in common use at time t. Then

$$WM(t) = WM(1667)e^{-\lambda(t-1667)}$$

(the $t-1667$ is used in the exponent because we are interested in the number of years since Milton). Let $WC(t)$ denote the number of words from Chaucer's time which are in common use at time t. Then

$$WC(t) = WC(1400)e^{-\lambda(t-1400)}.$$

From the given information, we have that

$$\frac{WM(2013)}{WM(1667)} = \frac{4}{5}.$$

And according to the above formula, we have

$$\frac{WM(1667)e^{-\lambda\cdot 346}}{WM(1667)} = \frac{4}{5}.$$

The factors of $WM(1667)$ cancel, leaving

$$e^{-\lambda\cdot 346} = \frac{4}{5}.$$

Taking the logarithm of both sides and dividing by -346 gives

$$\lambda = \frac{-\ln(4/5)}{346} \approx 6.5 \times 10^{-4}.$$

Knowing λ allows us to compute the fraction of words from Chaucer's time that would be recognized in Shakespeare's time:

$$\begin{aligned} \frac{WC(1600)}{WC(1400)} &= \frac{WC(1400)e^{-\lambda(200)}}{WC(1400)} \\ &= e^{-\lambda(200)} \\ &\approx .88 \end{aligned}$$

So, according to this model, approximately 88% of words from Chaucer's work would have been understood in Shakespeare's time.

(Return)

19 More Differential Equations

Recall that an ordinary differential equation is an equation involving a function and its derivatives. The solution to a differential equation is a function which satisfies the equation. An earlier module introduced a few basic differential equations. This module deals with a few different families of differential equations and the methods of solving them.

19.1 Autonomous differential equations

A differential equation is called *autonomous* if the derivative of the function $x(t)$ is independent of t, i.e. the equation is of the form

$$\frac{dx}{dt} = f(x).$$

Logically, a *nonautonomous* differential equation is one where the derivative equals a function of both x and t:

$$\frac{dx}{dt} = f(x, t).$$

In general, nonautonomous differential equations can be very difficult, but certain types yield to a little algebra and integration. These include separable differential equations and linear first order differential equations, which are covered here.

19.2 Separable differential equations

A *separable differential equation* is one where, with a little algebra, we are able to express the differential equation in the form

$$\frac{dx}{dt} = f(x)g(t).$$

This may involve some algebra. Note in particular that any autonomous equation is separable (think of $g(t) = 1$). Once a differential equation is factored this way, it can be solved by using the chain rule and some algebra:

$$\begin{aligned} dx &= \frac{dx}{dt}\,dt \\ dx &= f(x)g(t)\,dt \\ \frac{dx}{f(x)} &= g(t)\,dt. \end{aligned}$$

(This is why these equations are called separable–the variables can be separated to opposite sides of the equation). To solve this equation, one must find the functions whose derivatives are $\frac{1}{f(x)}$ and $g(t)$, respectively. In other words, one integrates both sides.

Example

Solve the differential equation

$$y' = 3yx^2.$$

(See Answer 1)

Example

Solve the differential equation

$$\frac{dx}{dt} = e^{t-x}.$$

(See Answer 2)

Example

Solve the initial value problem $\frac{dy}{dx} = xy + x$, with $y(0) = 3$. (See Answer 3)

Example

Assume that a falling body with mass m has a drag force proportional to velocity $v(t)$. Then the downward acceleration mg is being counteracted by the upward acceleration κv, for some constant κ. Thus,

$$\begin{aligned} m\frac{dv}{dt} &= mg - \kappa v \\ &= -\kappa\left(v - \frac{mg}{\kappa}\right). \end{aligned}$$

which is separable. Solve this differential equation. (See Answer 4)

19.3 Linear 1st order differential equations

The product rule gives a technique (integration by parts) for seemingly difficult integrals; the product rule also gives a technique for solving a certain class of non-separable differential equations called *linear 1st order differential equations*. This is a differential equation which can be written in the form

$$\frac{dx}{dt} = A(t)x + B(t)$$

(as for separable differential equations, this may involve a little algebra). This form gives the reason for calling these equations *linear*, since dropping the t's gives

$$\frac{dx}{dt} = Ax + B,$$

which is reminiscent of the equation of a line. *1st order* means that the equation only involves the function x and its derivative $\frac{dx}{dt}$ (and no higher derivatives), along with functions of t.

The *standard form* of a linear 1st order differential equation is achieved by bringing all the terms involving x to the left side, which gives

$$\frac{dx}{dt} - A(t)x = B.$$

Example

Identify which of these is a linear 1st order differential equation, and put it in standard form if it is. In the cases that are not, identify which condition is violated. Are all separable equations also linear 1st order?

1. $tx' + x = 0$
2. $x' - e^t x^2 = 0$
3. $x' = x\sin(t)$
4. $x'' - t^2 \frac{dx}{dt} = 0$

(See Answer 5)

Integrating factors

The method for solving linear 1st order differential equations is to use the product rule to factor the sum of two derivatives into the derivative of a product. It is best explained by example.

Example

In part 1 from the previous example, note that $tx' + x = (tx)'$ by the product rule. So that differential equation can be written as $(tx)' = 0$. Integrating both sides gives $tx = C$ for some constant C. Thus the solution is $x = \frac{C}{t}$.

However, not all linear 1st order differential equations are expressed so nicely. For instance, in example 3 above, one cannot rewrite $x' - \sin(t)x$ as the derivative of a product of functions. This is where an *integrating factor* is used.

The integrating factor, denoted by $I(t)$ in this course, is a function which is multiplied through the entire differential equation, giving

$$I\frac{dx}{dt} - IAx = IB.$$

$I(t)$ is chosen so that the left side of this equation can be factored as a derivative of a product using the product rule. Symbolically, the goal is to choose $I(t)$ so that

$$I\frac{dx}{dt} - IAx = \frac{d}{dt}(Ix).$$

To find I, expand the product $\frac{d}{dt}(Ix) = I\frac{dx}{dt} + \frac{dI}{dt}x$. For this to equal the left side of the above equation, it must be that $-IA = \frac{dI}{dt}$. This differential equation is separable, and one finds that $\frac{dI}{I} = -A\,dt$. Integrating and exponentiating gives that

$$I(t) = e^{\int -A(t)\,dt}.$$

One need not work through all this algebra every time but can jump straight to writing down the integrating factor. Multiplying through by the integrating factor allows the left side to be rewritten by the product rule, and integrating both sides finishes the problem.

To summarize the method:

1. Get the differential equation into standard form $\frac{dx}{dt} - A(t)x = B(t)$.
2. Compute the integrating factor $I(t) = e^{-\int A(t)\,dt}$.
3. Multiply the entire equation by $I(t)$, which gives

$$I\frac{dx}{dt} - IAx = IB.$$

1. Rewrite the left side as the derivative of a product (this works because of the way $I(t)$ was chosen): $\frac{d}{dt}(Ix) = IB$.
2. Integrate both sides and then divide by I.
3. The final answer, then, is given by

$$x(t) = e^{\int A} \cdot \int Be^{-\int A}.$$

Example

Solve the differential equation

$$tx' + tx = t^2.$$

Hint: $\int te^t\,dt = te^t - e^t + C$. (See Answer 6)

Example

Suppose a 1000 gallon tank is 90% full. An additive is is pumped into the tank at a rate of 10 gallons per minute. The mixture is well stirred and drained at a rate of 5 gallons per minute.
What is the concentration of the additive when the tank is full? (See Answer 7)

19.4 EXERCISES

- Solve the differential equation $\dfrac{dx}{dt} = \dfrac{x}{t}$.
- Solve the differential equation $\dfrac{dx}{dt} = \dfrac{\sqrt{1-x^2}}{\sqrt{1-t^2}}$.
- Given that $x(0) = 0$ and $\dfrac{dx}{dt} = te^x$, compute $x(1)$.
- What integrating factor should be used to solve the linear differential equation

$$t^2\frac{dx}{dt} = 4t - t^5x$$

- Solve the differential equation $\dfrac{dx}{dt} - 5x = 3$.
- Solve the differential equation $\dfrac{dx}{dt} = \dfrac{x}{1+t} + 2$.

- Suppose that, in order to buy a house, you obtain a mortgage. If the lender advertises an annual interest rate r, your debt $D(t)$ will increase exponentially according to the simple O.D.E.

$$\frac{dD}{dt} = rD$$

If you pay your debt at a rate of P (annual rate, paid continuously), the evolution of your debt will then (under assumptions of continual compounding and payment) obey the linear differential equation

$$\frac{dD}{dt} = rD - P$$

Using this model, answer the following question: if initial amount of the mortgage is for $400,000, the annual interest rate is 5%, and you pay at a rate of $40,000 every year, how many years will it take you to pay off the debt?

- German physician Ernst Heinrich Weber (1795-1878) is considered one of the fathers of experimental psychology. In his study of perception, he noticed that the perceived difference between two almost-equal stimuli is proportional to the *percentual* difference between them. In terms of differentials, we can express Weber's law as

$$dp = k\frac{dS}{S},$$

where p is the perceived intensity of a stimulus and S its actual strength. Observe the relative rate of change on the right hand side. In what way must the magnitude of a stimulus change in time for a human being to perceive a linear growth? Linearly? Logarithmically? Polynomially?

- Some nonlinear differential equations can be reduced to linear ones by a clever change of variables. Bernouilli equations

$$\frac{dx}{dt} + p(t)x = q(t)x^\alpha, \qquad \alpha \in \mathbb{R}$$

constitute the most important case. Notice that for $\alpha = 0$ or $\alpha = 1$ the above equation is already linear. For other values of α, the substitution $u = x^{1-\alpha}$ yields a linear differential equation in the variable u.

Apply the above change of variables in the case

$$\frac{dx}{dt} + 2tx = x^3$$

What differential equation on u do you get?

19.5 Answers to Selected Examples

1. Separating gives

$$\frac{dy}{y} = 3x^2\,dx.$$

Integrating both sides, we have

$$\int \frac{dy}{y} = \int 3x^2\,dx$$
$$\ln y = x^3 + C.$$

Now exponentiating both sides gives

$$\begin{aligned} y &= e^{x^3+C} \\ &= Ce^{x^3}, \end{aligned}$$

for some constant C (remember that the C is not the same in the first and second line above, but we just rewrite it for convenience).

(Return)

2. First, using a law of exponents on the right side, we have

$$e^{t-x} = e^t e^{-x}.$$

Now, separating gives

$$e^x\,dx = e^t\,dt.$$

We might be tempted at this point to say $x = t$ because of the symmetry of this equation. But we must integrate both sides, which introduces an integration constant:

$$e^x = e^t + C.$$

Now taking the logarithm gives

$$x = \ln(e^t + C).$$

Note We must have the integration constant contained within the natural logarithm. In general, it is best to introduce the integration constant as soon as the integration occurs. A common mistake is to forget the constant and then at the very end of the problem add it. This is frequently incorrect, as in this case.

(Return)

3. First, this differential equation does not look like it is of the form of a separable differential equation. However, with a little factoring, one finds that $\frac{dy}{dx} = x(y+1)$. Thus,

$$\frac{dy}{y+1} = x\,dx.$$

Anti-differentiating gives $\ln|y+1| = \frac{1}{2}x^2 + C$. Then, exponentiating gives

$$|y+1| = e^{x^2/2+C} = De^{x^2/2}.$$

Apply the initial condition by plugging in $x = 0$ and $y = 3$, which gives that $D = 4$. Thus, the solution to the initial value problem is $y = 4e^{x^2/2} - 1$. One should double check that this satisfies the differential equation.

(Return)

4. Separating gives

$$\frac{dv}{v - mg/\kappa} = -\frac{\kappa}{m}dt.$$

Integrating both sides gives $\ln(v - mg/\kappa) = -\kappa t/m + C$, and so exponentiating and solving for v gives

$$v = Ce^{-\kappa t/m} + \frac{mg}{\kappa}$$

(here C is replacing the constant e^C from exponentiating). Note that as $t \to \infty$, the exponential term goes to 0, and so $v(t) \to \frac{mg}{\kappa}$, which is the terminal velocity of the falling body (when the force of gravity and drag cancel each other).

(Return)

5. 1. Linear 1st order. Standard form is $x' + \frac{1}{t}x = 0$.
   2. Not linear because of the x^2. Note that this is separable though.
   3. Linear 1st order. Standard form is $x' - \sin(t)x = 0$.
   4. Not 1st order because of the presence of x''.

   Number 2 shows that a differential equation can be separable even though it is not linear 1st order.

   (Return)

6. Divide through by t to get the equation in standard form: $x' + x = t$. Compute the integrating factor

$$I = e^{\int dt} = e^t.$$

   Multiplying through gives

$$e^t x' + e^t x = e^t t.$$

   Rewriting this using the product rule gives $(e^t x)' = te^t$. Integrating both sides and using the hint gives

$$e^t x = te^t - e^t + C.$$

   Finally, dividing by e^t gives

$$x = t - 1 + \frac{C}{e^t}$$

   as a final answer.

   (Return)

7. Begin by setting up a few variables to help make sense of what is happening. Let $V(t)$ be the volume of the total mixture at time t. Let $Q(t)$ be the total amount of additive in the mixture at time t. Let $C(t)$ be the concentration of the mixture, i.e.

$$C(t) = \frac{Q(t)}{V(t)}.$$

   The volume is not too difficult to compute. Since there are 10 gallons per minute entering the tank, and 5 gallons per minute leaving the tank, the net amount of fluid entering the tank is 5 gallons per minute. The tank begins at 90% full, which is 900 gallons. So

$$V(t) = 900 + 5t,$$

   and is full at $t = 20$. Next, consider the rate at which the quantity $Q(t)$ of additive in the tank is changing. There is 10 gallons of pure additive entering per minute and 5 gallons of mixture leaving. Therefore the amount of additive leaving is $5C$. Putting this together gives

$$\frac{d}{dt}Q = 10 - 5C = 10 - \frac{5Q}{900 + 5t}.$$

   Rearranging this gives

$$\frac{dQ}{dt} + \frac{1}{180 + t}Q = 10.$$

This is a 1st order linear differential equation. Computing the integrating factor we find

$$\begin{aligned} I &= \exp\left(\int \frac{1}{180+t}dt\right) \\ &= \exp(\ln(180+t)) \\ &= 180+t. \end{aligned}$$

Multiplying through gives

$$(180+t)\frac{dQ}{dt} + Q = 10(180+t).$$

Now, as always, the left side can be rewritten using the product rule to give

$$\frac{d}{dt}\left[Q(180+t)\right] = 1800 + 10t.$$

Integrating both sides gives

$$Q \cdot (180+t) = 1800t + 5t^2 + K$$

(using K here to avoid confusion with concentration C). We can find K by setting $t = 0$ in this equation. Since $Q(0) = 0$ (there is no additive in the tank initially), we find $K = 0$.

Now, solving for Q and evaluating at $t = 20$ gives

$$\begin{aligned} Q &= \frac{1800t + 5t^2}{180+t} \\ &= 190. \end{aligned}$$

So the final concentration of the full tank is

$$\frac{190}{1000} = 19\%.$$

(Return)

20 ODE Linearization

We have seen techniques for solving two types of differential equations: separable and linear. Unfortunately, there are a lot of differential equations which do not fit into these categories. In some of these cases, we can use linearization to determine the behavior of such differential equations.

20.1 Oscillation

How does one model oscillation? It turns out that a first order differential equation will not work, but a second order (i.e. involving the second derivative) equation will:

$$\frac{d^2x}{dt^2} = -a^2x.$$

Solving such an equation is beyond the scope of this course, but in a course on differential equations one finds the pair of solutions

$$\begin{aligned} x &= C_1 \cos(at) \\ x &= C_2 \sin(at). \end{aligned}$$

For this course we will look at a simpler way to model oscillation.

20.2 Simple Oscillator

Consider a spinner where $\theta(t)$ represents the angle the arrow makes with the positive x-axis at time t. Then θ increases linearly with t and whenever θ gets to 2π, it goes back to 0: (Click Here: Simple Oscillator Animated GIF)

This can be modeled by

$$\begin{aligned} \frac{d\theta}{dt} &= a \\ \theta &= at + \theta_0 \mod 2\pi \end{aligned}$$

where $\mod 2\pi$ means "take the remainder when divided by 2π". Here a can be thought of as the frequency of the spinner (e.g. how many revolutions per minute it makes).

20.3 Coupled Oscillators

Now consider two simple oscillators, θ_1 and θ_2, with the same frequency a, but which are slightly out of phase with each other (i.e. one arrow is slightly ahead of the other): (Click Here: Two Oscillators Animated GIF)

Now suppose these oscillators are *coupled* so that each exerts a small influence on the other (e.g. by connecting their axles with a rod). One way to represent this mathematically is to adjust the rates of change of the oscillators so that they are affected by the difference in angles:

$$\begin{aligned}\frac{d\theta_1}{dt} &= a + \epsilon \sin(\theta_2 - \theta_1)\\ \frac{d\theta_2}{dt} &= a - \epsilon \sin(\theta_2 - \theta_1).\end{aligned}$$

Here, ϵ is some small constant which represents the strength of the effect of the coupling. When θ_2 is bigger than θ_1, the above differential equations speed up θ_1 slightly and slow down θ_2 slightly. One can find by simulation that this coupling effect causes the oscillators to synchronize relatively quickly, depending on how big the phase is between them and how big ϵ is: (Click Here: Coupled Oscillators Animated GIF)

Synchronization

To analyze the synchronization effect mathematically, consider the *phase* φ between the two oscillators:

$$\varphi = \theta_2 - \theta_1.$$

Looking at how the phase φ changes with respect to time gives

$$\begin{aligned}\frac{d\varphi}{dt} &= \frac{d}{dt}(\theta_2 - \theta_1)\\ &= \frac{d\theta_2}{dt} - \frac{d\theta_1}{dt}\\ &= (a - \epsilon\sin(\theta_2 - \theta_1)) - (a + \epsilon\sin(\theta_2 - \theta_1))\\ &= -2\epsilon\sin(\theta_2 - \theta_1)\\ &= -2\epsilon\sin(\varphi).\end{aligned}$$

This is a separable differential equation, but solving it to find φ as an explicit function of t is not so easy, and does not really help us understanding the synchronization phenomenon. But linearization will help us understand the synchronization effect and how quickly it occurs.

Linearization

Going back to the differential equation for the phase, suppose we replace $\sin\varphi$ with its linearization:

$$\begin{aligned}\frac{d\varphi}{dt} &= -2\epsilon\sin\varphi\\ &= -2\epsilon\left(\varphi + O(\varphi^3)\right)\\ &\approx -2\epsilon\varphi.\end{aligned}$$

This will be a good approximation assuming the phase is small (the oscillators are not too far out of sync). This is a familiar differential equation, which gives us the approximate solution

$$\varphi(t) \approx \varphi_0 e^{-2\epsilon t},$$

where φ_0 is the initial phase. This is called the *linearized solution* to the original differential equation. Here, the linearized solution predicts that the phase decays exponentially, which is consistent with the above simulation.

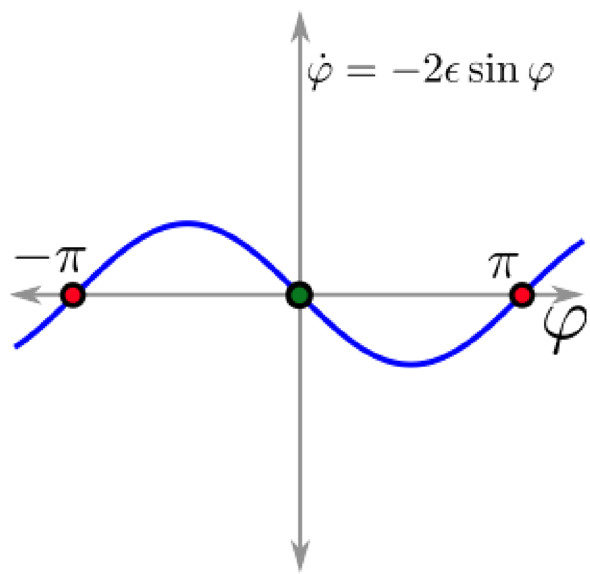
20.4 Equilibria

Another way to study differential equations and predict their behavior, is to study the equilibria of the equation. An *equilibrium* of the equation

$$\dot{x} = f(x)$$

(here, $\dot{x} = \frac{dx}{dt}$), is a solution $x(t) = C$, C a constant, such that $\dot{x} = 0$. In other words, an equilibrium is a root of f. In terms of the differential equation, an equilibrium is a steady state where the quantity x does not change.

One way to find the equilibria of a differential equation is to plot the derivative of a function versus the function itself. From the phase differential equation above, we plot $\dot{\varphi}$ on the y-axis and φ on the x-axis and look for roots:



The roots of this equation are the values of φ for which $\sin\varphi = 0$. For the range of values in which we are interested, the roots are $\varphi = -\pi, 0, \pi$. The equilibrium at 0 is familiar, because that is the state of synchronization to which the above simulation converged. The other two correspond to a phase of π, which means the oscillators are completely opposite one another (it is the same for $-\pi$ since these angles are co-terminal).

Stable and Unstable

A logical question at this point is why did the above coupled oscillator simulation eventually synchronize rather than ending up in opposite directions?

In general, some equilibria are attractive in the sense that if the quantity x gets near such an equilibrium, it will be drawn towards it and stay at it. Some equilibria are repellent in the sense that even if x is very close to such an equilibrium, it will be pushed away from it. Formally,

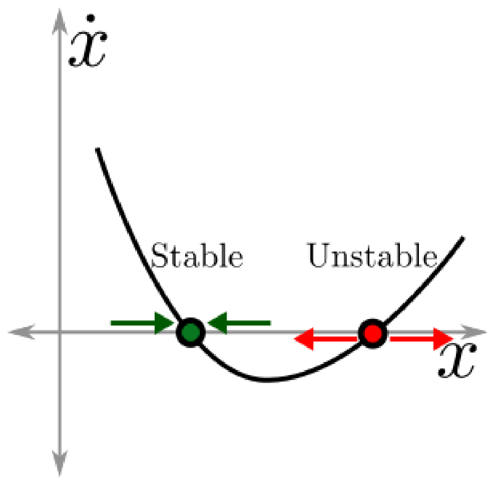
Stable and Unstable Equilibria

An equilibrium C of the differential equation

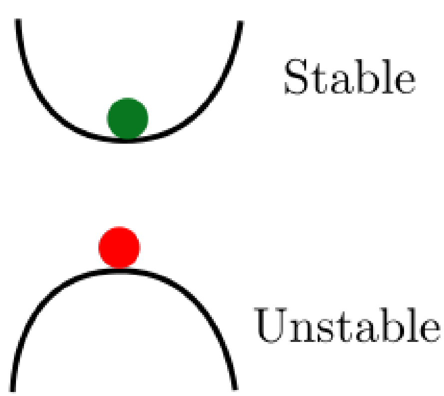
$$\frac{dx}{dt} = f(x)$$

is *stable* if $f'(C) < 0$ and is *unstable* if $f'(C) > 0$.

It is best to make sense of these definitions visually. Plot $\dot{x}$ versus x. Then each root of this equation is an equilibrium. If the graph crosses from positive to negative (going from left to right), then the equilibrium is stable. If the graph crosses from negative to positive (again, from left to right), then the equilibrium is unstable:



Another way to think of stable and unstable equilibria is to visualize one ball sitting in a bowl, and another ball sitting on top of an inverted bowl:



Each of these balls is in equilibrium (it will stay where it is as long as it is not disturbed). But the ball in the bowl is stable because if we nudge it in either direction, it will return to its equilibrium. However, the ball on the inverted bowl is unstable because if it is nudged in either direction it will roll off the bowl.

Example

Find and classify the equilibria of the equation

$$\frac{dx}{dt} = x^2 - 4x + 3.$$

(See Answer 1)

20.5 EXERCISES

- The differential equation

$$\frac{dx}{dt} = (e^x - 1)(x - 1)$$

  has an equilibrium at $x = 0$. What is the linearized equation at this equilibrium? Hint: Taylor-expand the right hand side about zero.

- There is also an equilibrium at $x = 1$ in the equation above. What is the linearized equation at this equilibirum? Hint: let $h = x - 1$ be a local coordinate and compute $\dot{h} = \dot{x} = \cdots$ by Taylor expanding the right hand side at $x = 1$.

- Recall from Lecture 18, Newton's Law of Heat Transfer, which states that

$$\frac{dT}{dt} = \kappa(A - T),$$

  where $\kappa > 0$ is a thermal conductivity constant and A is the (constant) ambient temperature. Find and classify the equilibria in this system (using the derivative of the right hand side at the equilibria, recall...). Wasn't that easy?

- Find and classify all the equilibria of the ODE

$$\frac{dy}{dt} = -2y + y^2 + y^3$$

- Recall from Lecture 19 how we computed the terminal velocity of a falling body with linear drag given by

$$m\frac{dv}{dt} = mg - \kappa v,$$

  where, of course, m is mass, g is gravitation, v is velocity, and $\kappa > 0$ is the drag coefficient. Can you see how easily one can solve for the equilibrium $v_\infty = mg/\kappa$? Do it!

- Very good. Now, let's use a more realistic model of drag that is quadratic as opposed to linear:

$$m\frac{dv}{dt} = mg - \lambda v^2,$$

  where $\lambda > 0$ is a constant drag coefficient. This differential equation is not as easy to solve (but soon you will learn how). Is there is terminal velocity? What is it?

- Recall that with continuous compounding at an interest rate of $r > 0$, an investment $I(t)$ with initial investment $I_0 = I(0)$ is $I(t) = I_0 e^{rt}$. What happens if you wish to withdraw funds from the investment at a rate of spending S, where $S > 0$ is constant? The differential equation is:

$$\frac{dI}{dt} = rI - S.$$

  Your goals are as follows. You have an initial investment I_0, and you cannot change it or the rate r. You want to be able to spend as much as possible but you also don't want to ever spend all your money. What amount of spending rate S can you bear? Hint: if you're not sure what to do, find and classify the equilibria in this model and think about which initial conditions lead to which long-term behaviors.

- In our lesson, we looked at two oscillators with "sinusoidal" coupling. Other types of coupling are possible as well. Consider the system of two oscillators modeled by

$$\frac{d\theta_1}{dt} = 2 + \epsilon(e^{\theta_1-\theta_2} - 1) \quad ; \quad \frac{d\theta_2}{dt} = 2 + \epsilon(1 - e^{\theta_1-\theta_2})$$

  Consider the phase difference $\varphi = \theta_2 - \theta_1$. Note that $\varphi = 0$ (where the oscillators are coupled) is an equilibrium. What is the linearized equation for φ about 0?

  This looks intimidating, but is very straightforward. If you're not sure how to start, compute $\frac{d\varphi}{dt}$. Then linearize this about $\varphi = 0$.

20.6 Answers to Selected Examples

1. Here, $f(x) = x^2 - 4x + 3$. Factoring, one finds

$$\frac{dx}{dt} = (x-1)(x-3).$$

   So the roots (and hence the equilibria) are $x = 1$ and $x = 3$. Taking the derivative, we find

$$\begin{aligned} f'(x) &= 2x - 4 \\ f'(1) &= -2 < 0 \\ f'(3) &= 2 > 0. \end{aligned}$$

   Thus $x = 1$ is a stable equilibrium, and $x = 3$ is an unstable equilibrium.

   (Return)

21 Integration By Substitution

The previous modules gave some of the motivation for integration as a method of solving differential equations. In this and the next few modules, we turn to techniques of integration.

21.1 Integration rules

Since integration is the inverse of differentiation, one can turn differentiation rules into integration rules. For example, by the linearity of the derivative, we have linearity of the integral:

$$\int (u+v)\,dx = \int u\,dx + \int v\,dx$$
$$\int (cu)\,dx = c\int u\,dx$$

where c is a constant. In other words, integration is a linear operator.

The rest of this module deals with turning the chain rule for differentiation into a rule for integration. This rule is called *substitution*, or *u-substitution* traditionally.

21.2 Substitution: the chain rule in reverse

Recall the chain rule, which says that if $u = u(x)$ is a function of x, then

$$du = \frac{du}{dx}dx.$$

Now if $f = f(u)$ is a function of u, then we find

$$\int f(u)\,du = \int f(u(x))\frac{du}{dx}\,dx$$

(To get from the left side to the right, all we have done is replace u and du by $u(x)$ and $\frac{du}{dx}dx$, respectively). This is the formula for substitution, or u-substitution.

Substitution is a useful technique but is not always easy to apply. In a typical problem, one encounters the right side of the above equation, but without knowing what f and u are. If one can find the correct f and u so that the integral can be expressed as above, then one can switch over to the left side of the above equation, which is usually easier to evaluate.

Example

Compute

$$\int e^{\sin x} \cos x \, dx.$$

(See Answer 1)

Example

Compute

$$\int 2xe^{x^2} \, dx.$$

(See Answer 2)

It is not always so easy to see the ideal choice of u, and sometimes it might take a few tries to find the right substitution. Usually, a good strategy is to look for the *inner function* of a composition of functions and let that be u. Another idea is to look for a function whose derivative is also a factor of the integrand.

Example

Compute

$$\int x\sqrt{x-1} \, dx$$

(See Answer 3)

Another general tip for integration by substitution is to try to simplify the integrand as much as possible before integrating.

Example

Compute

$$\int \cot\theta \csc\theta \, d\theta.$$

(See Answer 4)

Example

The Gompertz model for the size $N(t)$ of a tumor at time t is

$$\frac{dN}{dt} = -aN\ln(bN),$$

where $a > 0$ and $0 < b < 1$ are constants. Solve this differential equation. Hint: it is separable. Then find the limit behavior

$$\lim_{t\to\infty} N(t).$$

Finally, find the equilibria of the original differential equation and classify them as stable or unstable. (See Answer 5)

Example

Compute

$$\int 4(2x+5)^4\,dx.$$

(See Answer 6)

21.3 Perspective

The big idea of this module is that a change of variables (a substitution of one variable for a function of another) can change a difficult integral into an easier one. After computing the easier integral, we can change the variables back again. This idea will come up again in this course and in multivariable calculus.

21.4 Additional Examples

Example

Compute

$$\int \frac{(\ln x)^2}{x}\,dx.$$

(See Answer 7)

Example

Compute

$$\int \tan\theta\,d\theta.$$

(See Answer 8)

Example

Compute

$$\int x^5\sqrt{1+x^3}\,dx.$$

(See Answer 9)

21.5 EXERCISES

- Compute the integral $\displaystyle\int 3\cos x\,dx$
- Compute the integral $\displaystyle\int x\sec^2 x^2\,dx$
- Compute the integral $\displaystyle\int \frac{4x}{(x^2-1)^3}\,dx$

- Compute the integral $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\, dx$
- Compute the integral $\int \frac{\ln(15x^5)}{x}\, dx$
- Compute the integral $\frac{x\, dx}{\sqrt{x+3}}\, dx$
- Compute the integral $\frac{dx}{x\sqrt{x^2-1}}\, dx$ using the substitution $u = \sqrt{x^2-1}$
- Now do the same integral using the substitution $u = x^{-1}$ What is going on here?
- Compute the integral $\frac{dx}{x\sqrt{x^2+1}}\, dx$

21.6 Answers to Selected Examples

1. Let $u = \sin x$ and $f(u) = e^u$. Then
$$du = \frac{du}{dx}\, dx = \cos x\, dx,$$
and
$$\begin{aligned}\int e^{\sin x} \cos x\, dx &= \int f(\sin x) d(\sin x) \\ &= \int f(u)\, du \\ &= \int e^u\, du \\ &= e^u + C \\ &= e^{\sin x} + C.\end{aligned}$$
Note that after evaluating the integral in terms of u, we usually replace u with its function of x, since the original integral was with respect to x.

   We can check that this is the correct antiderivative by differentiating (remembering to apply the chain rule) and seeing that we get back the function which we were integrating originally.

   Typically, we need not write out all the details of what f is. It is sufficient to identify u and du and then make the necessary substitutions.

   (Return)

2. The inner function here looks like $u = x^2$. Then $du = 2x\, dx$, which is the remaining factor in the integrand. Thus,
$$\begin{aligned}\int 2xe^{x^2}\, dx &= \int e^u\, du \\ &= e^u + C \\ &= e^{x^2} + C.\end{aligned}$$
(Return)

3. Here, the inner function seems to be $x - 1$. So let $u = x - 1$. Then $du = dx$. This takes care of what is under the square root, and the differential, but what about the extra factor of x? We can take care of this factor by noting that $u = x - 1$ implies $x = u + 1$. So the integral becomes

$$\begin{aligned}\int x\sqrt{x-1}\,dx &= \int (u+1)\sqrt{u}\,du \\ &= \int (u+1)u^{1/2}\,du \\ &= \int u^{3/2} + u^{1/2}\,du \\ &= \frac{2}{5}u^{5/2} + \frac{2}{3}u^{3/2} + C \\ &= \frac{2}{5}(x-1)^{5/2} + \frac{2}{3}(x-1)^{3/2} + C.\end{aligned}$$

From the third to the fourth line above, we used the power rule on each term of the integrand. Again, we can differentiate to make sure we get back to our original integrand (though it might require a little algebra to show that they are in fact equal).

(Return)

4. Since cotangent and cosecant are not very familiar functions, it is helpful to rewrite them in terms of sine and cosine. This gives

$$\begin{aligned}\int \cot\theta\csc\theta\,d\theta &= \int \frac{\cos\theta}{\sin\theta}\cdot\frac{1}{\sin\theta}\,d\theta \\ &= \int \frac{\cos\theta}{\sin^2\theta}\,d\theta.\end{aligned}$$

It is easier now to see that $u = \sin\theta$ is a good choice, since its derivative $du = \cos\theta\,d\theta$ is in the numerator. Making this substitution shows

$$\begin{aligned}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta &= \int \frac{1}{u^2}\,du \\ &= \int u^{-2}\,du \\ &= -u^{-1} + C \\ &= -\frac{1}{\sin\theta} + C \\ &= -\csc\theta + C.\end{aligned}$$

Of course, if one happened to remember the fact that

$$\frac{d}{d\theta}\csc\theta = -\csc\theta\cot\theta,$$

then we would not require a substitution. But substitution allows us to do these integrals (and harder ones) without needing to memorize a lot of information.

(Return)

5. Separating variables and integrating both sides gives

$$\int \frac{dN}{N\ln(bN)} = \int -a\,dt.$$

The right side is easy, but the left side requires some work. Looking for a function whose derivative is a factor in the integrand, we see that $u = \ln(bN)$ is a good choice. In this case

$$du = \frac{1}{bN} \cdot b\, dN = \frac{dN}{N}.$$

And so, the integral on the left above becomes

$$\begin{aligned}\int \frac{dN}{N\ln(bN)} &= \int \frac{1}{u}\, du \\ &= \ln u + C \\ &= \ln\ln(bN) + C.\end{aligned}$$

Putting this together with the integral on the right above (and combining the integration constants to one integration constant on the right), we have

$$\ln\ln(bN) = -at + C$$

Exponentiating twice and then dividing by b gives

$$N = \frac{1}{b}e^{e^{-at+C}} = \frac{1}{b}e^{Ce^{-at}}.$$

By plugging in $t = 0$, we find that $C = \ln(bN_0)$, where N_0 is the initial size of the tumor.

In the long run, the exponential $e^{-at} \to 0$, since $a > 0$ by assumption. Therefore, the entire exponent is going to 0, and so

$$\lim_{t\to\infty} N(t) = \frac{1}{b}.$$

Note that $N(t) = \frac{1}{b}$ is an equilibrium solution to the original differential equation, since it gives $\frac{dN}{dt} = 0$. It is a stable equilibrium since the graph of of $-aN\ln(bN)$ goes from positive to negative as N goes from less than $\frac{1}{b}$ to greater than $\frac{1}{b}$.

Another equilibrium is $N = 0$. This is unstable, since $N > 0$ means $\frac{dN}{dt} > 0$. Intuitively, even if the tumor is very tiny, it will grow according to this model.

(Return)

6. In this case, the inner function is $u = 2x + 5$, and one finds that $du = 2dx$. Thus $dx = \frac{du}{2}$, which gives

$$\begin{aligned}\int 4(2x+5)^4 dx &= \int 4u^4 \frac{du}{2} \\ &= \int 2u^4 du \\ &= \frac{2}{5}u^5 + C \\ &= \frac{2}{5}(2x+5)^5 + C.\end{aligned}$$

(Return)

7. Here, a good inner function is $u = \ln x$, because the derivative $du = \frac{1}{x}\,dx$. Thus

$$\begin{aligned}\int \frac{(\ln x)^2}{x}\,dx &= \int u^2\,du \\ &= \frac{u^3}{3} + C \\ &= \frac{(\ln x)^3}{3} + C.\end{aligned}$$

(Return)

8. First, rewrite tangent in terms of sine and cosine:

$$\int \tan\theta\,d\theta = \int \frac{\sin\theta}{\cos\theta}\,d\theta$$

Now, note that $u = \sin\theta$ would not work, because its derivative, $\cos\theta$ is in the denominator. On the other hand, $u = \cos\theta$ is a good substitution because its derivative (up to a constant) is in in the numerator. That is, $du = -\sin\theta\,d\theta$. Therefore,

$$\begin{aligned}\int \frac{\sin\theta}{\cos\theta}\,d\theta &= \int -\frac{1}{u}\,du \\ &= -\ln(u) + C \\ &= -\ln(\cos\theta) + C.\end{aligned}$$

(Return)

9. The logical choice of inner function is $u = 1 + x^3$, which gives $du = 3x^2\,dx$ and so

$$dx = \frac{du}{3x^2}$$

Substituting in, we find

$$\begin{aligned}\int x^5\sqrt{1+x^3}\,dx &= \int x^5\sqrt{u}\frac{du}{3x^2} \\ &= \frac{1}{3}\int x^3\sqrt{u}\,du.\end{aligned}$$

This seems problematic, because we haven't been able to get everything in terms of u. But we can use our original substitution to help. Since $u = 1 + x^3$, we have that $x^3 = u - 1$, and so

$$\begin{aligned}\frac{1}{3}\int x^3\sqrt{u}\,du &= \frac{1}{3}\int (u-1)\sqrt{u}\,du \\ &= \frac{1}{3}\int u^{3/2} - u^{1/2}\,du \\ &= \frac{1}{3}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\ &= \frac{2}{15}(1+x^3)^{5/2} - \frac{2}{9}(1+x^3)^{3/2} + C.\end{aligned}$$

(Return)

22 Integration By Parts

This module uses the product rule to derive another useful integration technique: integration by parts. Recall the product rule:

$$d(u \cdot v) = u \cdot dv + v \cdot du.$$

Integrating both sides gives

$$\int d(u \cdot v) = \int u\,dv + \int v\,du.$$

Solving for $\int u\,dv$ gives

Integration by parts

If $u = u(x)$ and $v = v(x)$ are two functions of x, then

$$\int u\,dv = uv - \int v\,du.$$

Intuitively, we are given a difficult integral $\int u\,dv$. By breaking the integrand into u and dv and applying the above formula, we are hopefully able to wind up with an easier integral $\int v\,du$. Like with the substitution technique, it requires a little bit of thought to choose suitable u and dv. Once u and dv are picked, it is a fairly mechanical process to apply the formula (assuming a good choice of u and dv).

Note that the selection is constrained by the fact that $u\,dv$ must be the entire integrand. So whatever choice is made for u, whatever factors are left over become dv. Note also that the formula involves finding v, and so dv must be integrable. Ideally, dv should be easy to integrate, which can help guide the selection.

Example

Compute

$$\int xe^x\,dx.$$

(See Answer 1)

Example

Compute

$$\int \ln(x)dx.$$

(See Answer 2)

Example

Try to compute

$$\int \frac{\sin x}{x}\, dx$$

Hint: it cannot be done using integration by parts. (See Answer 3)

22.1 LIPET: A tip for choosing u and dv

It is not always obvious how to choose u and dv. The mnemonic LIPET gives a suggestion for how to select u, and then whatever is left over becomes dv.

1. **L**ogarithm
2. **I**nverse function
3. **P**olynomial
4. **E**xponential
5. **T**rigonometric.

When picking u, go down the list until some factor of the integrand first matches something from the list. So in the first example above, there was no logarithm, no inverse function, but there was a polynomial, x, which was chosen for u. In the second example, there was a logarithm, so that became u.

This will not always work perfectly, because (as the above example showed) some integrals simply cannot be computed using integration by parts. But in most examples where integration by parts works, the above mnemonic will help give the correct selection of u and dv.

Example

Compute

$$\int \frac{\ln x}{x^2}\, dx.$$

(See Answer 4)

22.2 Repeated use

Sometimes integration by parts requires repeated use, if the integral $\int v\, du$ is not easy to compute. It is not always easy to tell when repeating integration by parts will help, but with practice it becomes easier.

Example

Compute

$$\int e^x \cos(x) dx.$$

(See Answer 5)

Example

Compute

$$\int e^{2x} \sin(3x)\, dx.$$

(See Answer 6)

There are integrals that require several applications of integration by parts before they are finished. Unfortunately, it is not always clear when it will work and when it will not. Doing a lot of practice can help develop the intuition to tell the difference.

As the next example shows, sometimes an integral that looks like a perfect candidate for integration by parts does not yield to this method.

Example

Compute

$$\int e^x \cosh x\, dx.$$

(See Answer 7)

22.3 Reduction formulae

A final application of integration by parts is to prove what are known as *reduction formulae*. These formulae express one integral in terms of another slightly simpler integral. One can use a reduction formula to repeatedly simplify an integral, eventually reaching a known integral. These formulae are invariably derived by using integration by parts and some algebra.

Example

For a fixed integer $n \geq 0$, show that

$$\int x^n \cos x\, dx = x^n \sin x + nx^{n-1} \cos x - n(n-1) \int x^{n-2} \cos x\, dx.$$

Use this formula to find

$$\int x^2 \cos x dx.$$

(See Answer 8)

Example

Similar algebra as in the above example shows that for $n \geq 0$

$$\int x^n \sin x\, dx = -x^n \cos x + nx^{n-1} \sin x - n(n-1) \int x^{n-2} \sin x\, dx.$$

Example

Find a reduction formula for

$$\int x^n e^x \, dx.$$

Use it to evaluate

$$\int x^2 e^x \, dx.$$

(See Answer 9)

Example

Show that for $n \geq 2$,

$$\int \sec^n(x)\, dx = \frac{1}{n-1} \sec^{n-2}(x) \tan(x) + \frac{n-2}{n-1} \int \sec^{n-2}(x)\, dx.$$

(See Answer 10)

22.4 Additional examples

Example

Compute

$$\int x \sin(x)\, dx.$$

(See Answer 11)

Example

Compute

$$\int \arctan x \, dx.$$

Hint: recall that

$$\frac{d}{dx} \arctan x = \frac{1}{1+x^2}.$$

(See Answer 12)

22.5 EXERCISES

Compute the following integrals:

- $\int x e^{x/2}\, dx$
- $\int x^2 e^{x/2}\, dx$

- $\int 3x \ln x \, dx$
- $\int 3x^2 \ln x \, dx$
- $\int x^2 \cos \frac{x}{2} \, dx$
- $\int e^{2x} \sin 3x \, dx$
- $\int \ln x \, dx$
- $\int \ln^2 x \, dx$
- $\int \sin(\ln x) \, dx$
- $\int \arcsin(2x) \, dx$

To solve the integral $\int e^x \cos x \, dx$, we used the method of integration by parts twice. Based on how we solved the integral of $e^x \cosh x$, we can try the same with the cosine version, using the fact that $\cos x = \frac{1}{2}(e^{ix} + e^{-ix})$. Try! The integration is the easy part...the hard part is getting the algebra to work out (hello again, Euler's formula...)

- Compute $\int \sin(2x) \cos(3x) \, dx$

22.6 Answers to Selected Examples

1. Letting $u = x$ and $dv = e^x dx$ (we see that these factors together make up our integrand), one finds by differentiating u and integrating dv that $du = dx$ and $v = e^x$. Many students find it helps to organize this information in a grid:

$$\begin{aligned} u &= x & du &= dx \\ dv &= e^x \, dx & v &= e^x. \end{aligned}$$

   Then, from the formula it follows that

$$\begin{aligned} \int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= (x-1)e^x + C. \end{aligned}$$

   One can check that the derivative of this gives back the original integrand, as desired.

   (Return)

2. The selection of u and dv that works is

$$u = \ln(x) \qquad du = \frac{1}{x}\,dx$$
$$dv = dx \qquad v = x.$$

Note that the only other possibility would be $dv = \ln(x)dx$. But this choice would mean that to find v we would need to find the integral of $\ln(x)$, which is the problem at hand.

Applying the formula, we find

$$\begin{aligned}\int \ln(x)dx &= x\ln(x) - \int x \cdot \frac{1}{x}\,dx \\ &= x\ln(x) - \int dx \\ &= x\ln(x) - x + C \\ &= x(\ln(x) - 1) + C.\end{aligned}$$

Again, we can check that the derivative of this function gives back $\ln x$, our original integrand.

(Return)

3. One choice we might try is

$$u = \sin x \qquad du = \cos x\,dx$$
$$dv = \frac{1}{x}\,dx \qquad v = \ln x.$$

However, this requires us to compute the integral

$$\int v\,du = \int \ln x \cos x\,dx,$$

which is no better than the original integral. Another possible choice is

$$u = \frac{1}{x} \qquad du = -\frac{1}{x^2}\,dx$$
$$dv = \sin x\,dx \qquad v = -\cos x.$$

This leads to the integral

$$\int v\,du = \int \frac{\cos x}{x^2}\,dx,$$

which is again no better than the original integral.

It turns out no selection will work. Some integrals cannot be computed using integration by parts. And some integrals (like this one) have no elementary answer (i.e. some combination of trigonometric functions, polynomials, exponentials, etc.).

That said, we could expand $\sin(x)$ as a Taylor series, divide by x and integrate term by term, which gives a series solution. This gives a perfectly suitable solution provided that x is not too far from 0.

(Return)

4. A good choice is

$$u = \ln x \qquad du = \frac{1}{x}\,dx$$
$$dv = \frac{1}{x^2}\,dx \qquad v = -\frac{1}{x}.$$

Then

$$\int \frac{\ln x}{x^2}\,dx = -\frac{1}{x}\ln x - \int -\frac{1}{x^2}\,dx$$
$$= -\frac{1}{x}\ln x - \frac{1}{x} + C.$$

(Return)

5. We can take

$$u = e^x \qquad du = e^x\,dx$$
$$dv = \cos x\,dx \qquad v = \sin x.$$

(it turns out that this would work equally well if we reversed these). Then the formula says that

$$\int e^x \cos(x)dx = e^x \sin(x) - \int \sin(x)e^x dx.$$

This does not seem much better than the original problem. However, with some persistence and algebra, this will work. Let $I = \int e^x \cos(x)dx$ be the original integral, and let $J = \int \sin(x)e^x dx$ be the new integral. So the above calculation shows

$$I = e^x \sin(x) - J$$

Using integration by parts on J, we pick

$$u = e^x \qquad du = e^x\,dx$$
$$dv = \sin x\,dx \qquad v = -\cos x.$$

Then it follows that

$$\begin{aligned} J &= \int \sin(x)e^x dx \\ &= e^x(-\cos(x)) - \int (-\cos(x))e^x dx \\ &= -e^x\cos(x) + \int e^x \cos(x)dx \\ &= -e^x\cos(x) + I. \end{aligned}$$

So the problem has come back to the original integral I. This might seem like cause for despair, but putting together the previous calculations shows

$$\begin{aligned} I &= e^x \sin(x) - J \\ &= e^x \sin(x) - (-e^x\cos(x) + I) \\ &= e^x \sin(x) + e^x\cos(x) - I. \end{aligned}$$

Now, solving for I gives

$$\begin{aligned} 2I &= e^x \sin(x) + e^x \cos x + C \\ I &= \frac{1}{2}(e^x \sin(x) + e^x \cos(x)) + C. \end{aligned}$$

(Return)

6. The algebra is similar to the above example, but care must be taken with the constants. Let

$$I = \int e^{2x} \sin(3x)\, dx.$$

Let

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x}\, dx \\ dv &= \sin(3x)\, dx & v &= -\frac{1}{3}\cos(3x). \end{aligned}$$

Then

$$\begin{aligned} I &= -\frac{1}{3}e^{2x}\cos(3x) - \int \left(-\frac{1}{3}\cos(3x)\right) 2e^{2x}\, dx \\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\int e^{2x}\cos(3x)\, dx. \end{aligned}$$

Now, let

$$J = \int e^{2x}\cos(3x)\, dx.$$

Selecting

$$\begin{aligned} u &= e^{2x} & du &= 2e^{2x}\, dx \\ dv &= \cos(3x)\, dx & v &= \frac{1}{3}\sin(3x), \end{aligned}$$

we have

$$\begin{aligned} J &= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}\int e^{2x}\sin(3x)\, dx \\ &= \frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}I. \end{aligned}$$

Putting this all together, we have

$$\begin{aligned} I &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}J \\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{3}\left(\frac{1}{3}e^{2x}\sin(3x) - \frac{2}{3}I\right) \\ &= -\frac{1}{3}e^{2x}\cos(3x) + \frac{2}{9}e^{2x}\sin(3x) - \frac{4}{9}I. \end{aligned}$$

Solving for I gives

$$\begin{aligned}
\frac{13}{9} I &= -\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) \\
I &= \frac{9}{13} \left(-\frac{1}{3} e^{2x} \cos(3x) + \frac{2}{9} e^{2x} \sin(3x) \right) \\
&= \frac{1}{13} e^{2x} \left(-3 \cos(3x) + 2 \sin(3x) \right).
\end{aligned}$$

Remember that any indefinite integral has an integration constant, so the final answer is

$$\int e^{2x} \sin(3x)\, dx = \frac{1}{13} e^{2x} \left(-3 \cos(3x) + 2 \sin(3x) \right) + C.$$

(Return)

7. This looks so similar to the above examples, that it is reasonable to expect that two applications of integration by parts will allow us to algebraically find this integral. Unfortunately, there is a problem that will soon present itself. Let

$$I = \int e^x \cosh x\, dx.$$

If we set

$$\begin{aligned}
u &= e^x & du &= e^x\, dx \\
dv &= \cosh x\, dx & v &= \sinh x,
\end{aligned}$$

we find

$$\begin{aligned}
I &= e^x \sinh x - \int e^x \sinh x \\
&= e^x \sinh x - J,
\end{aligned}$$

where we have set

$$J = \int e^x \sinh x\, dx.$$

Letting

$$\begin{aligned}
u &= e^x & du &= e^x\, dx \\
dv &= \sinh x\, dx & v &= \cosh x,
\end{aligned}$$

we find

$$\begin{aligned}
J &= e^x \cosh x - \int e^x \cosh x\, dx \\
&= e^x \cosh x - I.
\end{aligned}$$

Putting it all together,

$$\begin{aligned}
I &= e^x \sinh x - J \\
&= e^x \sinh x - (e^x \cosh x - I) \\
&= e^x \sinh x - e^x \cosh x + I.
\end{aligned}$$

Here is where our problem arises. We cannot solve for I because there is a positive I on both sides. This problem is due to the fact that (unlike sine and cosine), the hyperbolic sine and cosine do not introduce negative signs when integrated or differentiated, respectively.

So what do we do? Rewrite our integral using the definition of $\cosh x$ and it becomes easy:

$$\begin{aligned}\int e^x \cosh x\,dx &= \int e^x \cdot \left(\frac{e^x + e^{-x}}{2}\right)\\ &= \frac{1}{2}\int \left(e^{2x}+1\right)\,dx\\ &= \frac{1}{2}\left(\frac{1}{2}e^{2x}+x\right)+C\\ &= \frac{1}{4}e^{2x}+\frac{1}{2}x+C.\end{aligned}$$

(Return)

8. Let

$$\begin{aligned} u &= x^n & du &= nx^{n-1}\,dx\\ dv &= \cos x\,dx & v &= \sin x.\end{aligned}$$

Then according to the formula,

$$\int x^n \cos x\,dx = x^n \sin x - \int nx^{n-1}\sin x\,dx.$$

Now, since we want to get our integral in terms of an integral involving $\cos x$ and a power of x, we can apply integration by parts to

$$\int x^n \sin x\,dx.$$

Here, we let

$$\begin{aligned} u &= x^n & du &= nx^{n-1}\,dx\\ dv &= \sin x\,dx & v &= -\cos x.\end{aligned}$$

This gives

$$\int x^n \sin x\,dx = -x^n\cos x + \int nx^{n-1}\cos x\,dx.$$

Now, using this in our earlier equation (though with n replaced by $n-1$), we find

$$\begin{aligned}\int x^n\cos x\,dx &= x^n\sin x - \int nx^{n-1}\sin x\,dx\\ &= x^n\sin x - n\int x^{n-1}\sin x\,dx\\ &= x^n\sin x - n\left(-x^{n-1}\cos x + \int (n-1)x^{n-2}\cos x\,dx\right)\\ &= x^n\sin x + nx^{n-1}\cos x - n(n-1)\int x^{n-2}\cos x\,dx.\end{aligned}$$

The formula says that

$$\begin{aligned}\int x^2 \cos x\, dx &= x^2 \sin x + 2x \cos x - 2\int \cos x\, dx \\ &= x^2 \sin x + 2x\cos x - 2\sin x + C \\ &= x^2 \sin x + 2x\cos x - 2\sin x + C.\end{aligned}$$

(Return)

9. Letting

$$\begin{aligned} u &= x^n & du &= nx^{n-1}\, dx \\ dv &= e^x\, dx & v &= e^x, \end{aligned}$$

we find that

$$\int x^n e^x = x^n e^x - n\int x^{n-1} e^x\, dx.$$

Applying this when $n = 2$ (then applying it again) gives

$$\begin{aligned}\int x^2 e^x &= x^2 e^x - 2\int xe^x\, dx \\ &= x^2 e^x - 2\left(xe^x - \int e^x\, dx\right) \\ &= x^2 e^x - 2xe^x + 2e^x + C.\end{aligned}$$

(Return)

10. In integrating a power of a trigonometric function, it can be hard to pick how many factors become u and how many become dv. The fact that dv is supposed to be easy to integrate can guide this selection. Since $\int \sec^2 x dx = \tan x$, letting $dv = \sec^2 x dx$ should work well.

Thus, $u = \sec^{n-2}(x)$ and $dv = \sec^2 x dx$, which means $du = (n-2)\sec^{n-3}(x)\sec(x)\tan(x)dx$ (by the chain rule), and $v = \tan x$. Recalling the Pythagorean identity $\tan^2 x = \sec^2 x - 1$, one finds that

$$\begin{aligned}\int \sec^n(x)dx &= \sec^{n-2}(x)\tan(x) - \int (n-2)\sec^{n-2}(x)\tan^2(x)dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^{n-2}(x)(\sec^2(x)-1)dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int (\sec^n(x) - \sec^{n-2}(x))dx \\ &= \sec^{n-2}(x)\tan(x) - (n-2)\int \sec^n(x)dx + (n-2)\int \sec^{n-2}(x)dx\end{aligned}$$

Now, solving for $\int \sec^n(x)dx$ gives

$$\int \sec^n(x)dx = \frac{1}{n-1}\sec^{n-2}(x)\tan(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)dx,$$

as desired.

(Return)

11. The logical choice (either by the LIPET mnemonic, or by picking u to be something which gets simpler when differentiated) for parts is

$$\begin{aligned} u &= x & du &= dx \\ dv &= \sin(x)\,dx & v &= -\cos(x) \end{aligned}$$

Therefore,

$$\begin{aligned} \int x\sin(x)\,dx &= -x\cos(x) - \int -\cos(x)\,dx \\ &= -x\cos(x) + \int \cos(x)\,dx \\ &= -x\cos(x) + \sin(x) + C. \end{aligned}$$

(Return)

12. As in some earlier examples, the only choice we have is to set

$$\begin{aligned} u &= \arctan x & du &= \frac{1}{1+x^2}\,dx \\ dv &= dx & v &= x. \end{aligned}$$

Therefore,

$$\int \arctan x\,dx = x\arctan x - \int \frac{x}{1+x^2}\,dx.$$

This second integral can be solved with a substitution of

$$\begin{aligned} u &= 1+x^2 \\ du &= 2x\,dx. \end{aligned}$$

So $dx = \frac{du}{2x}$. Making the substitution gives

$$\begin{aligned} \int \frac{x}{1+x^2}\,dx &= \int \frac{x}{u}\cdot\frac{du}{2x} \\ &= \frac{1}{2}\int \frac{du}{u} \\ &= \frac{1}{2}\ln u + C \\ &= \frac{1}{2}\ln(1+x^2) + C. \end{aligned}$$

Putting it all together, we find

$$\begin{aligned} \int \arctan x\,dx &= x\arctan x - \int \frac{x}{1+x^2}\,dx \\ &= x\arctan x - \frac{1}{2}\ln(1+x^2) + C. \end{aligned}$$

(Return)

23 Trigonometric Substitution

There is another class of integrals which usually do not involve trigonometric functions, but which can be solved by substituting the variable with a trigonometric function. This can be thought of as using the substitution formula, from Integration By Substitution, in the other direction. That is, going from the left side to right side in the equality

$$\int f(x)\,dx = \int f(x(\theta))\frac{dx}{d\theta}\,d\theta,$$

where we have made the substitution $x = x(\theta)$. We often use θ when making a trigonometric substitution.

Example

Compute

$$\int \frac{dx}{1+x^2}.$$

(See Answer 1)

23.1 Typical substitutions

Trigonometric substitution makes use of the Pythagorean identities. In general, the basic trigonometric substitutions are:

Form	Substitution	Identity used
$1+x^2$	$x=\tan\theta$	$1+\tan^2\theta=\sec^2\theta$
$1-x^2$	$x=\sin\theta$	$1-\sin^2\theta=\cos^2\theta$
x^2-1	$x=\sec\theta$	$\sec^2\theta-1=\tan^2\theta$

Caveat

The form x^2-1 often leads to a messy integral involving $\sec(\theta)$. This can often be avoided using a hyperbolic trigonometric substitution (see below).

After a substitution has been made, the resulting integral will often involve a product of trigonometric functions, possibly raised to powers. These types of integrals are covered in more detail in Trigonometric Integrals. For now, here are a few of the useful identities in evaluating these integrals:

Power reduction
$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

Double angle
$\sin(2\theta) = 2\sin\theta\cos\theta$
$\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Example

Compute

$$\int \sqrt{1-x^2}\, dx.$$

(See Answer 2)

Example

Compute

$$\int \frac{dx}{\sqrt{1-x^2}}.$$

(See Answer 3)

23.2 Forms with other constants

There are other forms which are similar to the above forms but have different constants involved. These are dealt with using similar substitutions which make the constants cancel and factor so that the same identities can be used.

Example

Compute

$$\int \frac{dx}{x^2\sqrt{x^2+4}}$$

(See Answer 4)

Example

Compute

$$\int \frac{dx}{4+9x^2}.$$

(See Answer 5)

The following table summarizes the substitutions to be made when other constants are involved. The identities used are the same Pythagorean identities given in the above table.

Form	Substitution
$a^2x^2+b^2$	$x = \frac{b}{a}\tan\theta$
$b^2-a^2x^2$	$x = \frac{b}{a}\sin\theta$
$a^2x^2-b^2$	$x = \frac{b}{a}\sec\theta$

23.3 Completing the square

Sometimes it is not obvious at first that an integral is of the form where a trigonometric substitution is helpful. It may take a little bit of algebra to see what the right substitution is. This common algebraic tool is known as *completing the square*, which simply rewrites a quadratic expression as the square of a binomial plus a constant. To review the algebra involved in this process, check Wikipedia:Completing the square.

Example

Compute

$$\int \frac{dx}{\sqrt{3+2x-x^2}}$$

(See Answer 6)

23.4 Hyperbolic trigonometric substitutions

Recall that the hyperbolic trigonometric functions $\sinh(x)$ and $\cosh(x)$ are defined by

$$\sinh(\theta) = \frac{e^{\theta} - e^{-\theta}}{2}$$
$$\cosh(\theta) = \frac{e^{\theta} + e^{-\theta}}{2}.$$

These functions satisfy the Pythagorean identity $\cosh^2(\theta) - \sinh^2(\theta) = 1$. Also, note that $\frac{d}{d\theta}\cosh(\theta) = \sinh(\theta)$, and $\frac{d}{d\theta}\sinh(\theta) = \cosh(\theta)$. This means hyperbolic substitutions are another option for dealing with the following forms:

Form	Substitution	Identity used
$1+x^2$	$x = \sinh\theta$	$1+\sinh^2\theta = \cosh^2\theta$
x^2-1	$x = \cosh\theta$	$\cosh^2\theta - 1 = \sinh^2\theta$

This often gives a simpler answer than the $x = \sec\theta$ substitution suggested above, but the trade-off is that the answer will involve hyperbolic functions. Here are some of the other identities for the hyperbolic functions, which are similar to those for regular trigonometric functions:

Double Angle
$\sinh(2\theta) = 2\sinh(\theta)\cosh(\theta)$
$\cosh(2\theta) = \cosh^2(\theta) + \sinh^2(\theta)$
$\cosh(2\theta) = 2\cosh^2(\theta) - 1$
$\cosh(2\theta) = 2\sinh^2(\theta) + 1$

Power reduction
$\sinh^2(\theta) = \frac{\cosh(2\theta)-1}{2}$
$\cosh^2(\theta) = \frac{\cosh(2\theta)+1}{2}$

Example

Compute

$$\int \frac{dx}{\sqrt{1+x^2}}.$$

(See Answer 7)

Example

Compute

$$\int \sqrt{1+x^2}\,dx.$$

(See Answer 8)

23.5 Blow-ups

Sometimes a differential equation can be solved by using a trigonometric substitution. But this can sometimes lead to an unreasonable solution due to blow-ups or singularities, which exist for many trigonometric functions.

Example

Consider a financial model which predicts that marginal profits equal some positive constant plus something which is proportional to the square of net profits. Mathematically,

$$\frac{dP}{dt} = b^2 + a^2P^2,$$

for constants a and b (we square them to ensure that they are positive). Solve this differential equation and find where it has a blow-up. (See Answer 9)

23.6 EXERCISES

Compute the following integrals:

- $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$
- $\displaystyle\int \frac{dx}{\sqrt{x^2-2x}}$
- $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx$
- $\displaystyle\int (1-x^2)^{-3/2}\,dx$
- $\displaystyle\int \frac{x}{\sqrt{1+x^2}}\,dx$
- $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

- $\displaystyle\int \frac{dx}{\sqrt{x^2-6x+10}}$
- $\displaystyle\int \frac{dx}{\sqrt{x^2-2x-8}}$

23.7 Answers to Selected Examples

1. Consider the substitution $x = \tan(\theta)$. Then one finds that $dx = \sec^2\theta\, d\theta$. Making these substitutions and recalling the Pythagorean identity $1+\tan^2\theta = \sec^2\theta$, the integral becomes

$$\begin{aligned}\int \frac{dx}{1+x^2} &= \int \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}\\ &= \int \frac{\sec^2\theta\, d\theta}{\sec^2\theta}\\ &= \int d\theta\\ &= \theta + C\\ &= \arctan(x) + C.\end{aligned}$$

The last line comes from our original substitution:

$$x = \tan\theta \quad \Leftrightarrow \quad \arctan x = \theta.$$

(Return)

2. According to the above guide, the substitution to make is $x = \sin\theta$. Then $dx = \cos\theta\, d\theta$, and it follows that

$$\begin{aligned}\int \sqrt{1-x^2}\, dx &= \int \sqrt{1-\sin^2\theta}\cos\theta\, d\theta\\ &= \int \sqrt{\cos^2\theta}\cos\theta\, d\theta\\ &= \int \cos^2\theta\, d\theta.\end{aligned}$$

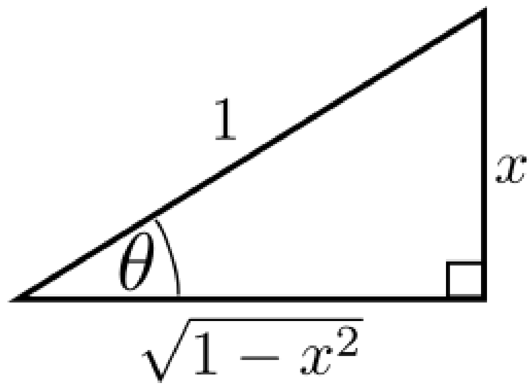
Now using the power reduction identity for cosine, we have

$$\begin{aligned}\int \cos^2\theta\, d\theta &= \int \frac{1}{2}(1+\cos(2\theta))\, d\theta\\ &= \frac{\theta}{2} + \frac{1}{4}\sin(2\theta) + C.\end{aligned}$$

Finally, we must get this back in terms of x. We know that $\theta = \arcsin x$. But to take care of $\sin 2\theta$, we must use the double angle formula from above. This gives

$$\begin{aligned}\sin 2\theta &= 2\sin\theta\cos\theta\\ &= 2x\sqrt{1-x^2}\end{aligned}$$

In the last line above, we knew $\sin\theta = x$ from the original substitution. We found $\cos\theta$ by drawing a right triangle which relates x and θ according to the substitution $\sin\theta = x$:



Putting this all together and doing a little simplification, we find

$$\int \sqrt{1-x^2} = \frac{1}{2}\arcsin x + \frac{1}{2}x\sqrt{1-x^2} + C.$$

(Return)

3. By the above table, the substitution $x = \sin\theta$ should be used (hence $\theta = \arcsin(x)$). Then $dx = \cos\theta\, d\theta$, so the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{1-x^2}} &= \int \frac{\cos\theta\, d\theta}{\sqrt{1-\sin^2\theta}} \\ &= \int \frac{\cos\theta\, d\theta}{\sqrt{\cos^2\theta}} \\ &= \int d\theta \\ &= \theta + C \\ &= \arcsin(x) + C.\end{aligned}$$

(Return)

4. The form x^2+4 in the denominator reminds us of the substitution we made earlier for x^2+1, which was the substitution $x = \tan\theta$. This is the correct impulse, but unfortunately it does not work quite right here since there is no nice simplification for $\tan^2\theta + 4$.

   We can fix this by adjusting the coefficients. The idea is that we could factor out a 4 if we had

$$4\tan^2\theta + 4 = 4(\tan^2\theta + 1).$$

   To get that extra factor of 4, we can make the substitution $x = 2\tan\theta$. Then $dx = 2\sec^2\theta\, d\theta$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{x^2+4}} &= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4\tan^2\theta+4}} \\ &= \int \frac{2\sec^2\theta\, d\theta}{4\tan^2\theta\sqrt{4(\tan^2\theta+1)}} \\ &= \frac{1}{4}\int \frac{\sec^2\theta\, d\theta}{\tan^2\theta\sec\theta}.\end{aligned}$$

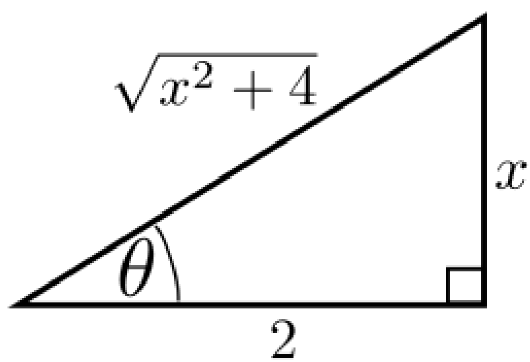
The last equality above comes from again using the identity $\tan^2\theta + 1 = \sec^2\theta$. Doing a little simplification and rewriting in terms of sine and cosine gives

$$\begin{aligned} \frac{1}{4}\int \frac{\sec\theta\, d\theta}{\tan^2\theta} &= \frac{1}{4}\int \frac{1}{\cos\theta}\cdot\frac{\cos^2\theta}{\sin^2\theta}\, d\theta \\ &= \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta}. \end{aligned}$$

This we can handle with a substitution of $u = \sin\theta$ and $du = \cos\theta\, d\theta$, which gives

$$\begin{aligned} \frac{1}{4}\int \frac{\cos\theta\, d\theta}{\sin^2\theta} &= \frac{1}{4}\int \frac{du}{u^2} \\ &= \frac{1}{4}\left(-\frac{1}{u}\right) + C \\ &= -\frac{1}{4u} + C \\ &= -\frac{1}{4\sin\theta} + C \end{aligned}$$

Now, we must do one final bit of right triangle trigonometry to get $\sin\theta$ in terms of x. By the original substitution we have $\tan\theta = \frac{x}{2}$, and this can be expressed by the following triangle:



It follows that $\sin\theta = \frac{x}{\sqrt{x^2+4}}$. Putting it all together, we have

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2+4}} &= -\frac{1}{4\sin\theta} + C \\ &= -\frac{\sqrt{x^2+4}}{4x} + C. \end{aligned}$$

(Return)

5. This is another example which looks like $x = \tan\theta$ is the right type of substitution to make. However, again we need to adjust the coefficient since $4 + 9\tan^2\theta$ does not simplify nicely.

The key is to get the constants to cancel and factor. The substitution $x = \frac{2}{3}\tan\theta$ will work, and in this case $\theta = \arctan(\frac{3}{2}x)$. Then $dx = \frac{2}{3}\sec^2\theta d\theta$, and the integral becomes

$$\begin{aligned}
\int \frac{dx}{4+9x^2} &= \frac{2}{3}\int \frac{\sec^2\theta\, d\theta}{4+9(4/9)\tan^2\theta} \\
&= \frac{2}{3}\int \frac{\sec^2\theta\, d\theta}{4(1+\tan^2\theta)} \\
&= \frac{2}{3}\int \frac{\sec^2\theta\, d\theta}{4\sec^2\theta} \\
&= \frac{2}{3}\cdot\frac{1}{4}\int d\theta \\
&= \frac{1}{6}\theta + C \\
&= \frac{1}{6}\arctan(\frac{3}{2}x) + C.
\end{aligned}$$

(Return)

6. Start by completing the square for the quadratic:

$$\begin{aligned}
3+2x-x^2 &= -x^2+2x+3 \\
&= -(x^2-2x)+3 \\
&= -(x^2-2x+1)+4 \\
&= -(x-1)^2+4 \\
&= 4-(x-1)^2.
\end{aligned}$$

So we can rewrite the integral as

$$\begin{aligned}
\int \frac{dx}{\sqrt{3+2x-x^2}} &= \int \frac{dx}{\sqrt{4-(x-1)^2}} \\
&= \int \frac{du}{\sqrt{4-u^2}},
\end{aligned}$$

where we substituted $u = x-1$ and $du = dx$. This can now be dealt with using a trigonometric substitution of $u = 2\sin\theta$ (remember, the extra factor of 2 is there so that the 4 will factor out). So $du = 2\cos\theta\, d\theta$, and the integral becomes

$$\begin{aligned}
\int \frac{du}{\sqrt{4-u^2}} &= \int \frac{2\cos\theta\, d\theta}{\sqrt{4-4\sin^2\theta}} \\
&= \int \frac{2\cos\theta\, d\theta}{\sqrt{4}\sqrt{1-\sin^2\theta}} \\
&= \int \frac{2\cos\theta\, d\theta}{2\cos\theta} \\
&= \int d\theta \\
&= \theta + C.
\end{aligned}$$

Solving our original substitution for θ, we see that

$$\begin{aligned}\theta &= \arcsin\left(\frac{u}{2}\right)\\ &= \arcsin\left(\frac{x-1}{2}\right).\end{aligned}$$

So the final answer is

$$\int \frac{dx}{\sqrt{3+2x-x^2}} = \arcsin\left(\frac{x-1}{2}\right) + C.$$

(Return)

7. Using a regular trigonometric substitution, we would set $x = \tan\theta$, and $dx = \sec^2\theta\, d\theta$, which, after the usual algebra, gives

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\sec^2\theta\, d\theta}{\sqrt{1+\tan^2\theta}}\\ &= \int \sec\theta\, d\theta.\end{aligned}$$

But the integral of secant is not easy to remember, nor easy to rederive. If instead, we make the hyperbolic trigonometric substitution $x = \sinh u$, so $dx = \cosh u\, du$, then we have

$$\begin{aligned}\int \frac{dx}{\sqrt{1+x^2}} &= \int \frac{\cosh u\, du}{\sqrt{1+\sinh^2 u}}\\ &= \int \frac{\cosh u\, du}{\cosh u}\\ &= \int du\\ &= u + C\\ &= \operatorname{arcsinh} x + C.\end{aligned}$$

So the hyperbolic trigonometric substitution led to a much easier integral to evaluate. The trade-off is that the final result involves the inverse hyperbolic trigonometric functions, as opposed to more familiar functions.

(Return)

8. Using the hyperbolic trigonometric substitution $x = \sinh(\theta)$ gives

$$\begin{aligned}\int \sqrt{1+x^2}\, dx &= \int \sqrt{1+\sinh^2\theta}\cosh\theta\, d\theta\\ &= \int \sqrt{\cosh^2\theta}\cosh\theta\, d\theta\\ &= \int \cosh^2\theta\, d\theta\\ &= \frac{1}{2}\int (\cosh(2\theta)+1)\, d\theta\\ &= \frac{1}{2}\left(\theta + \frac{1}{2}\sinh(2\theta)\right) + C\\ &= \frac{1}{2}\theta + \frac{1}{4}2\sinh(\theta)\cosh(\theta) + C\\ &= \frac{1}{2}\sinh^{-1} x + \frac{1}{2}x\sqrt{1+x^2} + C.\end{aligned}$$

(Return)

9. This is a separable equation. Separating the variables and integrating both sides gives

$$\int \frac{dP}{b^2 + a^2P^2} = \int dt.$$

On the left, we can use the trigonometric substitution

$$\begin{aligned} P &= \frac{b}{a}\tan\theta \\ dP &= \frac{b}{a}\sec^2\theta\, d\theta. \end{aligned}$$

Note then that $\theta = \arctan \frac{a}{b}P$. This gives

$$\begin{aligned} \int \frac{dP}{b^2 + a^2P^2} &= \int \frac{\frac{b}{a}\sec^2\theta\, d\theta}{b^2(1+\tan^2\theta)} \\ &= \frac{b}{a}\int \frac{\sec^2\theta\, d\theta}{b^2\sec^2\theta} \\ &= \frac{1}{ab}\int d\theta \\ &= \frac{1}{ab}\theta \\ &= \frac{1}{ab}\arctan\frac{a}{b}P. \end{aligned}$$

(leaving off the constant for now). On the right side we get $t + C$, so

$$\frac{1}{ab}\arctan\frac{a}{b}P = t + C.$$

Solving this for P gives

$$P(t) = \frac{b}{a}\tan(abt + C).$$

If initial profits, at $t = 0$, are 0, then $C = 0$, so the final answer is

$$P(t) = \frac{b}{a}\tan(abt).$$

Since tangent blows up at $\frac{\pi}{2}$, this model implies profit goes to infinity at $t = \frac{\pi}{2ab}$, which is a sign that this model is not perfect.

(Return)

24 Partial Fractions

So far, the techniques of integration covered in this course have all been derived from differentiation rules run in reverse. This module gives an algebraic method for integrating *rational functions*. Recall that a rational function is a function of the form

$$f(x) = \frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials. It turns out that with a little bit of algebraic manipulation, many of these integrals are not too difficult to compute.

Example

Compute

$$\int \frac{3x^2 - 5}{x - 2}\, dx.$$

(See Answer 1)

The rest of this module expands on this method (in particular, when the denominator is of a higher degree), which is known as the method of *partial fractions*.

24.1 Partial fractions

Given a rational function $\frac{P(x)}{Q(x)}$, and P has a lower power than Q, the method of partial fractions uses algebra to rewrite the function as a sum of simpler terms which are easy to integrate. While there are some cases to deal with, the basic outline of the method is:

1. Given the integral $\int \frac{P(x)}{Q(x)} dx$ where P and Q are polynomials.
2. Factor $Q(x) = (x - r_1)(x - r_2) \ldots (x - r_n)$. Assume for now that each of these factors is distinct.
3. We use the following fact, that the rational function can be expressed as

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - r_1} + \frac{A_2}{x - r_2} + \ldots + \frac{A_n}{x - r_n}.$$

4. Use algebra to find what each of the constants A_i is. This step requires the most work.
5. Then

$$\begin{aligned}\int \frac{P(x)}{Q(x)}dx &= \int \left(\frac{A_1}{x-r_1} + \ldots + \frac{A_n}{x-r_n}\right) dx \\ &= A_1 \ln|x-r_1| + \ldots + A_n \ln|x-r_n| + C.\end{aligned}$$

Example

Compute
$$\int \frac{3x-1}{x^2-2x-3}dx.$$
(See Answer 2)

Example

Compute
$$\int \frac{2x^2-6x-2}{x^3-x^2-2x}dx.$$
(See Answer 3)

Example

Compute
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\, dx.$$
(See Answer 4)

Example

A simple model for the deflection $x(t)$ of a thin beam under a load proportional to λ^2 is
$$\frac{dx}{dt} = \lambda^2 x - x^3 = x(\lambda - x)(\lambda + x).$$
Solve this differential equation (but do not solve for $x(t)$ explicitly). Then find the equilibria of the differential equation and classify them as stable or unstable. (See Answer 5)

Example

The *logistic model* for population dynamics says that the rate of change of a population P with respect to time is
$$\frac{dP}{dt} = rP - bP^2$$
where r and b are positive constants which can be thought of as the reproduction rate and death rate, respectively. Factoring and letting $K = \frac{r}{b}$, we have
$$\frac{dP}{dt} = bP(K-P).$$
Solve this differential equation. What is the long run population behavior? (See Answer 6)

24.2 Other technicalities

Higher degree numerator

For the algebra to work out above, the degree of the numerator, $P(x)$, must be lower than that of the denominator, $Q(x)$. However, it is easy to deal with the case when the numerator has equal or higher degree. One can use long division to rewrite the quotient as a divisor plus a remainder, just like writing an improper fraction as a mixed number in middle school.

Repeated factors

If the denominator has one or more repeated factors, i.e.

$$\frac{P(x)}{Q(x)} = \frac{P(x)}{(x-r_1)^{m_1}\ldots(x-r_k)^{m_k}},$$

where one or more of the m_i is greater than 1. Then the way to express the function is

$$\begin{aligned}\frac{P(x)}{(x-r_1)^{m_1}\ldots(x-r_n)^{m_n}} &= \frac{A_1}{x-r_1} + \frac{A_2}{(x-r_1)^2} + \cdots + \frac{A_{m_1}}{(x-r_1)^{m_1}} \\ &+ \frac{B_1}{x-r_2} + \frac{B_2}{(x-r_2)^2} + \cdots + \frac{B_{m_2}}{(x-r_2)^{m_2}} \\ &+ \ldots.\end{aligned}$$

Now, the algebra proceeds as before to find the constants in the numerators. It is easiest to see this through an example.

Example

Compute

$$\int \frac{2x^2-4x-2}{(x+1)(x-1)^2}dx.$$

(See Answer 7)

Quadratic factors

Suppose one of the factors of the denominator is a quadratic which cannot be factored (e.g. x^2+1). Then the numerator of this factor in the expansion should be of the form $Ax+B$. Then the algebra proceeds as before.

Example

Compute

$$\int \frac{3x^2-2x+1}{(x-1)\,(x^2+1)}dx.$$

(See Answer 8)

24.3 EXERCISES

Compute the following integrals:

- $\displaystyle\int \frac{5+x}{x^2+x-6}\,dx$
- $\displaystyle\int \frac{2x+3}{6x^2+5x+1}\,dx$
- $\displaystyle\int \frac{x}{(x+1)(x+2)}\,dx$
- $\displaystyle\int \frac{x^2-x+5}{(x-2)(x-1)(x+3)}\,dx$
- $\displaystyle\int \frac{2x-1}{x^3-x}\,dx$
- $\displaystyle\int \frac{x^2-3}{x^2-4}\,dx$
- $\displaystyle\int \frac{x^3+10x^2+33x+36}{x^2+4x+3}\,dx$
- $\displaystyle\int \frac{x+2}{(x-1)^2}\,dx$
- $\displaystyle\int \frac{dx}{x^4-6x^3+12x^2}$

24.4 Answers to Selected Examples

1. By doing polynomial long division on this ratio, we find

$$\begin{array}{r|l} & \;3x+6 \\ \hline x-2 & 3x^2-5 \\ & -(3x^2-6x) \\ \hline & \quad 6x-5 \\ & \quad -(6x-12) \\ \hline & \qquad\quad 7 \end{array}$$

$$\frac{3x^2-5}{x-2} = 3x+6+\frac{7}{x-2}$$

(For more on polynomial long division, see wikipedia).

The above observation, which is entirely based on algebra, allows us to evaluate the integral as

$$\begin{aligned} \int \frac{3x^2-5}{x-2}\,dx &= \int \left(3x+6+\frac{7}{x-2}\right)dx \\ &= \frac{3}{2}x^2+6x+7\ln|x-2|+C. \end{aligned}$$

(Return)

2. Factoring the denominator gives $x^2 - 2x - 3 = (x+1)(x-3)$. Thus, the goal is to find constants A and B such that

$$\frac{3x-1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}.$$

There are several methods for finding the constants, but one of the simplest is to clear denominators, which gives

$$3x - 1 = A(x-3) + B(x+1).$$

This equation must hold for every value of x. In particular, one can pick convenient values of x which make the algebra easy. In this case, by plugging in $x = 3$, the first term on the right disappears. Thus, the equation becomes $8 = B \cdot 4$, and so $B = 2$. Similarly, picking $x = -1$ makes the second term on the right disappear. Thus, $-4 = A \cdot (-4)$ so $A = 1$. It follows that

$$\begin{aligned}\int \frac{3x-1}{x^2-2x-3}\,dx &= \int \frac{3x-1}{(x+1)(x-3)}\,dx \\ &= \int \left(\frac{1}{x+1} + \frac{2}{x-3}\right) dx \\ &= \ln|x+1| + 2\ln|x-3| + C.\end{aligned}$$

(Return)

3. Factoring the denominator gives

$$x^3 - x^2 - 2x = x(x+1)(x-2)$$

So we are looking for constants A, B, and C such that

$$\frac{2x^2-6x-2}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}.$$

Clearing fractions gives

$$2x^2 - 6x - 2 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1).$$

Now, picking the following convenient values of x allows us to find each constant:

$$\begin{aligned} &x = 0 && -2 = -2A && A = 1 \\ &x = -1 && 6 = 3B && B = 2 \\ &x = 2 && -6 = 6C && C = -1. \end{aligned}$$

So we have that

$$\begin{aligned}\int \frac{2x^2-6x-2}{x(x+1)(x-2)} &= \int \left(\frac{1}{x} + \frac{2}{x+1} - \frac{1}{x-2}\right) dx \\ &= \ln|x| + 2\ln|x+1| - \ln|x-2| + C.\end{aligned}$$

(Return)

4. Factoring gives

$$2x^3 + 3x^2 - 2x = x(2x-1)(x+2)$$

So we are looking for constants A, B, C such that

$$\frac{x^2+2x-1}{2x^3+3x^2-2x} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}.$$

As before, we clear fractions which gives

$$x^2 + 2x - 1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1).$$

Now we pick convenient values of x to make the factors cancel and solve for the constants:

$$\begin{aligned} &x = 0 & &-1 = -2A & &A = \frac{1}{2} \\ &x = \frac{1}{2} & &\frac{1}{4} = \frac{5}{4}B & &B = \frac{1}{5} \\ &x = -2 & &-1 = 10C & &C = -\frac{1}{10}. \end{aligned}$$

So we find

$$\begin{aligned} \int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx &= \int \left(\frac{1/2}{x} + \frac{1/5}{2x-1} + \frac{-1/10}{x+2}\right)\,dx \\ &= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C \\ &= \frac{1}{2}\ln|x| + \frac{1}{10}\ln|2x-1| - \frac{1}{10}\ln|x+2| + C. \end{aligned}$$

Note the extra factor of $\frac{1}{2}$ for the middle term comes from doing a substitution of $u = 2x - 1$, which implies $dx = \frac{1}{2}\,du$.

(Return)

5. Factoring gives

$$\frac{dx}{dt} = x(\lambda - x)(\lambda + x).$$

Separating and integrating gives

$$\int \frac{dx}{x(\lambda-x)(\lambda+x)} = \int dt.$$

Now, we use partial fractions on the left side:

$$\frac{1}{x(\lambda-x)(\lambda+x)} = \frac{A}{x} + \frac{B}{\lambda-x} + \frac{C}{\lambda+x}$$

Clearing fractions gives

$$1 = A(\lambda-x)(\lambda+x) + Bx(\lambda+x) + Cx(\lambda-x).$$

Picking convenient values of x gives

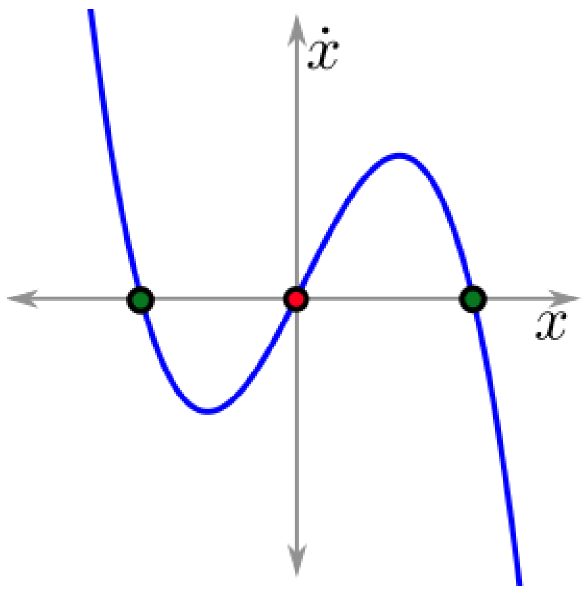
$$\begin{aligned} x &= 0 & 1 &= \lambda^2 A & A &= \frac{1}{\lambda^2} \\ x &= \lambda & 1 &= 2\lambda^2 B & B &= \frac{1}{2\lambda^2} \\ x &= -\lambda & 1 &= -2\lambda^2 C & C &= -\frac{1}{2\lambda^2}. \end{aligned}$$

So we have

$$\begin{aligned} \int \frac{dx}{x(\lambda - x)(\lambda + x)} &= \int \left(\frac{1/\lambda^2}{x} + \frac{1/2\lambda^2}{\lambda - x} - \frac{1/2\lambda^2}{\lambda + x} \right) dx \\ &= \frac{1}{\lambda^2} \ln|x| - \frac{1}{2\lambda^2} \ln|\lambda - x| - \frac{1}{2\lambda^2} \ln|\lambda + x|. \end{aligned}$$

All of this equals $t + C$ on the right.

The equilibria of the differential equation are $x = 0$, $x = \lambda$, and $x = -\lambda$. The equilibrium at 0 is unstable and the other two are stable, as the graph shows:



(Return)

6. Separating gives

$$\frac{dP}{P(K - P)} = b\,dt. \tag{1}$$

Integrating the left side is done using partial fractions, and the denominator is already factored. So the next step is to find A and B such that

$$\frac{1}{P(K - P)} = \frac{A}{P} + \frac{B}{K - P}.$$

Clearing denominators gives $1 = A(K - P) + BP$. Remember, K and A are constants, and this equation must hold for every value of P. Setting $P = K$ cancels the first term and gives $1 = BK$, so $B = \frac{1}{K}$.

Setting $P = 0$ cancels the second term and gives $A = \frac{1}{K}$. Thus,

$$\begin{aligned}
\int \frac{dP}{P(K-P)} &= \int \frac{1}{K}\left(\frac{1}{P} + \frac{1}{K-P}\right) dP \\
&= \frac{1}{K}\left(\ln P - \ln(K-P)\right) \\
&= \frac{1}{K}\ln \frac{P}{K-P},
\end{aligned}$$

by a property of logarithms. Multiplying through by K gives

$$\begin{aligned}
\ln\left(\frac{P}{K-P}\right) &= \int Kb\, dt \\
&= \int r\, dt \\
&= rt + C
\end{aligned}$$

(recall that $Kb = r$ by the definition of K). Now, exponentiating gives

$$\frac{P}{K-P} = \tilde{C}e^{rt},$$

for a new constant $\tilde{C}$. By plugging in $t = 0$, we find that

$$\tilde{C} = \frac{P_0}{K-P_0},$$

where P_0 is the initial population. Multiplying through by $K - P$ and doing a little algebra gives

$$\begin{aligned}
P &= \tilde{C}e^{rt}(K-P) \\
P + P\tilde{C}e^{rt} &= \tilde{C}e^{rt}K \\
P(1+\tilde{C}e^{rt}) &= \tilde{C}e^{rt}K \\
P &= \frac{\tilde{C}e^{rt}K}{1+\tilde{C}e^{rt}}.
\end{aligned}$$

Replacing $\tilde{C} = \frac{P_0}{K-P_0}$ gives

$$\begin{aligned}
P &= \frac{P_0}{K-P_0}e^{rt}K \cdot \frac{1}{1+\frac{P_0}{K-P_0}e^{rt}} \\
&= \frac{KP_0e^{rt}}{K-P_0+P_0e^{rt}} \\
&= \frac{KP_0}{(K-P_0)e^{-rt}+P_0}
\end{aligned}$$

(From the first to the second line, we distributed $(K - P_0)$ in the denominator. From the second to third line, we multiplied the top and bottom by e^{-rt}.).

Note that if $P_0 = 0$ (i.e. there was no population to begin with), then the population will stay at 0. This is consistent with the above equation. On the other hand, if $P_0 > 0$, then as $t \to \infty$, the e^{-rt} in the denominator goes to 0, and so

$$\lim_{t\to\infty} P(t) = \frac{KP_0}{P_0} = K.$$

We can think of K as the carrying capacity for the population, a sort of ideal size for the population. Alternatively, by looking at the original differential equation, we see that K is an equilibrium. It is stable, since populations above K have a negative derivative (hence are decreasing), and populations below K have a positive derivative (hence are increasing).

On the other hand 0 is an unstable equilibrium. The model implies that as long as the population is not extinct to begin with, it will grow and eventually equal K.

(Return)

7. The denominator is already factored, so write

$$\frac{2x^2-4x-2}{(x+1)(x-1)^2} = \frac{A}{x+1} + \frac{B}{x-1} + \frac{C}{(x-1)^2}.$$

Clearing the denominators gives

$$2x^2-4x-2 = A(x-1)^2 + B(x+1)(x-1) + C(x+1).$$

Plugging in $x = 1$ cancels the first two terms on the right, leaving $-4 = 2C$, so $C = -2$. Plugging in $x = -1$ cancels the second two terms and leaves $4 = 4A$, so $A = 1$.

Now, it seems that there are no more nice values of x to help solve for B. But remember that the equation must hold for any value of x. Picking $x = 0$ (which is an easy value to use), gives $-2 = A - B + C$. Knowing $A = 1$ and $C = -2$ gives $B = 1$.

Thus,

$$\begin{aligned}\int \frac{2x^2-4x-2}{(x+1)(x-1)^2}\,dx &= \int \left(\frac{1}{x+1} + \frac{1}{x-1} + \frac{-2}{(x-1)^2}\right) dx \\ &= \ln|x+1| + \ln|x-1| + \frac{2}{x-1} + C\end{aligned}$$

(Return)

8. Write

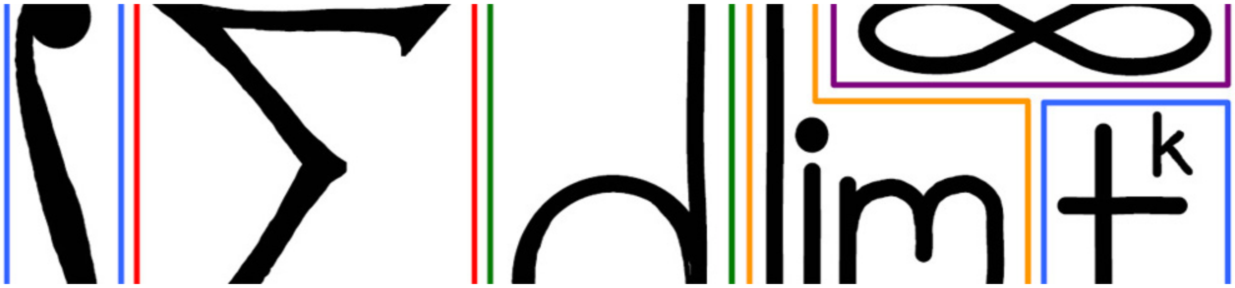
$$\frac{3x^2-2x+1}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+1}.$$

Clearing fractions gives $3x^2-2x+1 = A(x^2+1) + (Bx+C)(x-1)$. Picking $x = 1$ gives $2 = 2A$, so $A = 1$. Now, picking any other two values for x will allow finding B and C. For instance, $x = 0$ gives $1 = A - C$, and so $C = 0$. Finally, picking $x = -1$ gives $6 = 2A + (-B)(-2)$, so $B = 2$.

Thus,

$$\begin{aligned}\int \frac{3x^2-2x+1}{(x-1)(x^2+1)}\,dx &= \int \left(\frac{1}{x-1} + \frac{2x}{x^2+1}\right) dx \\ &= \ln|x-1| + \ln(x^2+1) + C.\end{aligned}$$

(Return)



25 Definite Integrals

This module moves from the indefinite integral, which is a class of functions, to the definite integral, which is a number. The relationship between these seemingly unrelated topics will be revealed in the next module.

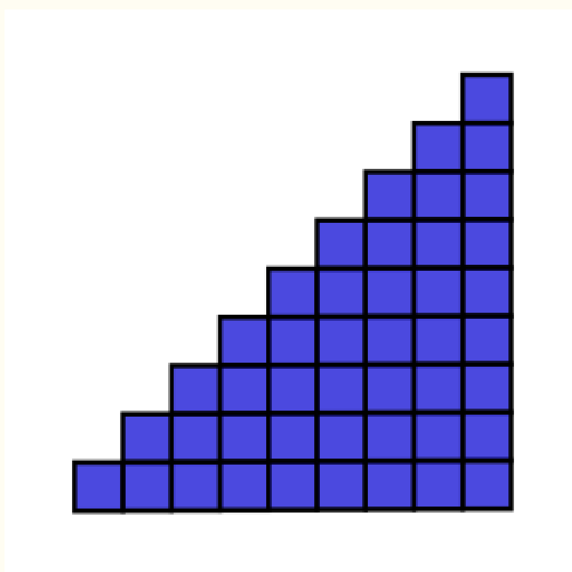
The idea underlying the definite integral is that adding up local increments leads to a global total. Before getting into the details of what this means, consider a simple example.

Example

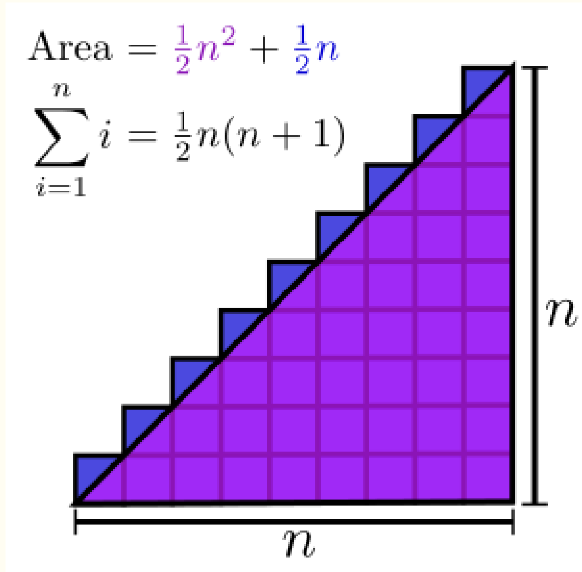
Consider

$$\sum_{i=1}^{n} i = 1 + 2 + 3 + \cdots + n.$$

One can visualize this sum as the area of a triangular stack of 1×1 boxes. The first column has 1 box, the second column has 2 boxes, and so on through the nth column with n boxes:



The area of this roughly triangular region can be found by splitting it into two regions: a right triangle of base and height n, and the half boxes left over:



The total area is therefore $\frac{1}{2}n(n+1)$, and so we find that

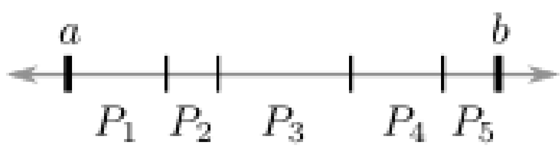
$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1).$$

The point of this example is to compare the amount of computation (e.g. the number of additions) required to do the sum using local information (adding up the terms one by one), verses the global information (evaluating the product on the right above). It is much easier to simply evaluate the product.

The definite integral takes this type of idea and generalizes it to more difficult sums. Before we can define it, we need a few definitions.

25.1 Partitions and Riemann sums

Given an interval $[a, b]$, a *partition* P of $[a, b]$ is a division of the interval $[a, b]$ into subintervals P_i. Visually, think of placing hash marks along the interval $[a, b]$ and then labeling the subintervals $P_1, P_2, \ldots$ from left to right:



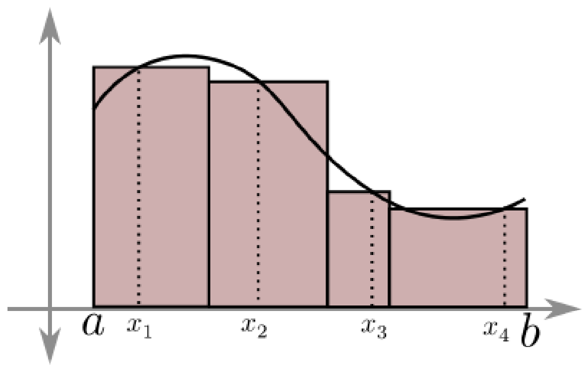
Let $(\Delta x)_i$ be the width of the ith subinterval, P_i.

Choose a *sample point* x_i from the ith subinterval (this can be a point chosen at random from the subinterval or systematically; it does not matter).

Given a function f, a partition P for an interval $[a, b]$, and sample points x_i, the *Riemann sum* of f on P is given by

$$\sum_{i=1}^{N} f(x_i)(\Delta x)_i.$$

The Riemann sum can be interpreted as an approximation of the area under the curve of f from a to b using rectangles. The width and height of the ith rectangle are $(\Delta x)_i$ and $f(x_i)$, respectively. Note that in this area interpretation, a rectangle which is below the x-axis has negative area (since $f(x_i) < 0$ in this case). For an example with $N = 4$ rectangles, consider the following figure:



25.2 The definite integral

The definite integral

The *definite integral* of a function f from a to b, denoted

$$\int_{x=a}^{b} f(x)\,dx,$$

is defined by

$$\int_{x=a}^{b} f(x)\,dx = \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)(\Delta x)_i.$$

The function f being integrated is called the *integrand*.

In other words, the definite integral is the limit of the Riemann sums as the lengths of the subintervals approach 0. In the area interpretation, the widths of all the rectangles are getting arbitrarily small, which ultimately gives the area under the curve:

(Link to Riemann Sum Limit Animated GIF)

Remember that when interpreting the definite integral as the area under the curve, any region which is below the x-axis contributes negative area to the total.

Example

Using the definition of the definite integral, compute

$$\int_{x=0}^{1} x\,dx.$$

(See Answer 1)

Notation

Sums The integral sign $\int$ and the summation sign $\sum$ are both short for sum. The integral sign $\int$ looks like a stylized S, and the summation sign is the Greek sigma, short for sum.

Limits Including the variable in the limits of integration is not strictly necessary, but is a useful habit to develop for future courses where integration will be happening with respect to several variables. It is also fine to suppress the notation and just have $\int_a^b f(x)\,dx$:

$$\int_a^b f(x)\,dx = \int_{x=a}^{b} f(x)\,dx.$$

Variables The variable used in the integrand does not matter; it is sometimes referred to as a dummy variable:

$$\int_{x=a}^{b} f(x)\,dx = \int_{t=a}^{b} f(t)\,dt = \int_{z=a}^{b} f(z)\,dz.$$

However, if there is a variable used in one of the limits of integration (as will happen from time to time), it is important to avoid using that as the dummy variable too. For example,

$$\int_a^x f(t)\,dt \quad \text{instead of} \quad \int_a^x f(x)\,dx.$$

Caveat

Note that, although their notation is similar, definite integrals are not the same as indefinite integrals! The indefinite integral of a function is a class of functions, whereas the definite integral of a function over an interval is a number.

That said, it is no accident that they have similar notations, because of their relationship, which is given by the Fundamental Theorem of Integral Calculus in the next module.

25.3 Properties of definite integrals

Linearity

The definite integral is linear, i.e.

$$\int_{x=a}^{b} \left(f(x)+g(x)\right)\,dx = \int_{x=a}^{b} f(x)\,dx + \int_{x=a}^{b} g(x)\,dx.$$
$$\int_{x=a}^{b} c\cdot f(x)\,dx = c\int_{x=a}^{b} f(x)\,dx.$$

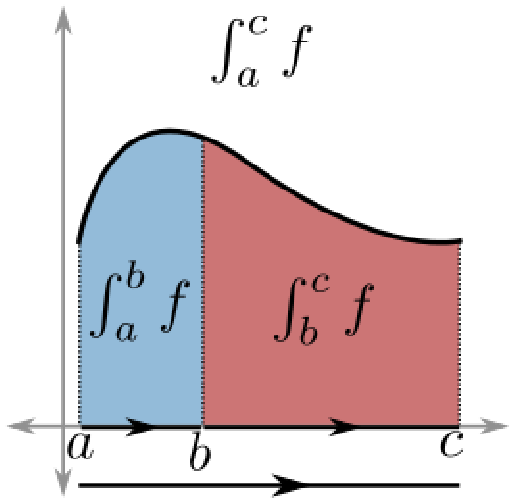
(See Justification 2)

Additivity

When integrating the same function over two adjacent intervals, we have additivity:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

In the area interpretation, this can be thought of as taking the area under the curve from a to b and adding the area under the curve from b to c, which gives the area under the curve from a to c:



Another way of thinking about it is adding the intervals $[a, b]$ and $[b, c]$ together to get $[a, c]$. It is important to note that the orientation of the interval matters, as discussed in the next subsection.

Orientation

The orientation of the interval over which we integrate matters. Integrating from left to right is positive, and integrating from right to left is negative:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

(See Justification 3)

Dominance

This is another intuitive property. If $f(x) \geq 0$ for all x in the interval $[a, b]$, then

$$\int_a^b f(x)\,dx \geq 0.$$

Also, if $f(x) \geq g(x)$ for all x in the interval, then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

(See Justification 4)

25.4 More examples

There are a few definite integrals that we can compute directly from the definition. But for most functions, it is not easy to work directly with the definition.

Example

Compute

$$\int_a^b c\,dx$$

(See Answer 5)

Example

Compute

$$\int_a^b x\,dx$$

(See Answer 6)

25.5 Odd and even functions

There are a few final cases where certain definite integrals can be simplified by using properties of the integrand.

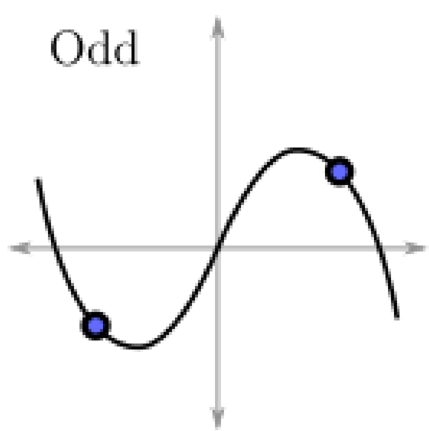
Odd and even functions

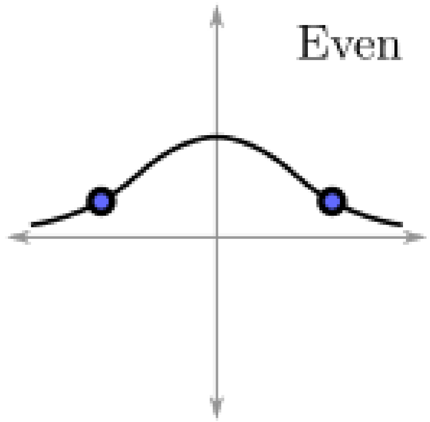
A function $f(x)$ is called *odd* if

$$f(-x) = -f(x).$$

A function $g(x)$ is called *even* if

$$g(-x) = g(x).$$





The reason for the terminology comes from Taylor series. A function is odd if and only if every term in its Taylor series has *odd* power. Similarly, a function is even if and only if every term in its Taylor series has *even* power. (See Justification 7)

Example

Sine and hyperbolic sine are both odd functions because they only have odd powers in their Taylor series. Cosine and hyperbolic cosine are both even functions because they only have even powers in their Taylor series.

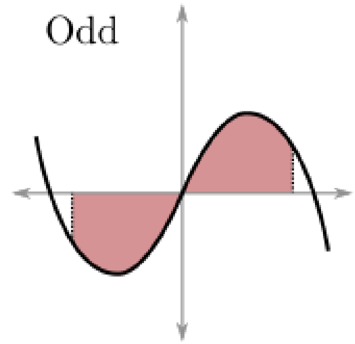
Odd function over a symmetric domain

If an odd function f is integrated over a domain that is symmetric about the origin (i.e., an interval of the form $[-L, L]$, then

$$\int_{x=-L}^{L} f(x)\, dx = 0.$$

Formally, any subinterval's on the left half of the interval will make a contribution to the Riemann sum which is equal and opposite to the contribution of the corresponding subinterval on the right half of the interval. These equal and opposite sums cancel, and so the definite integral over the entire interval is 0.

In terms of the area interpretation, the net area under the curve over the left half of the interval will be equal and opposite in sign to the net area under the curve over the right half of the interval. Therefore, the total area will be 0:



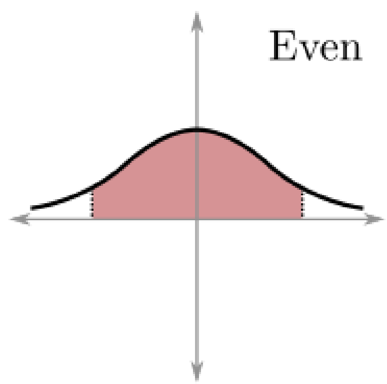
Even function over a symmetric domain.

If an even function g is integrated over a domain that is symmetric about the origin (i.e., an interval of the form $[-L, L]$), then

$$\int_{x=-L}^{L} g(x)\,dx = 2\int_{x=0}^{L} g(x)\,dx.$$

Formally, each subinterval on the left half of the interval has a corresponding subinterval on the right with an equal contribution to the Riemann sum. So one can just take the Riemann sum on the right and double it.

Using the area interpretation, one can see that the region under the curve on the left will be the mirror image of the region under the curve on the right, so the total area is just twice the area on the right:



25.6 EXERCISES

- One particular choice of partition and sampling that can be used to numerically evaluate definite integrals is the following. With n fixed, divide the interval $[a, b]$ into n subintervals P_i of common length $(\Delta x)_i = (b-a)/n$. For the sampling, choose the right endpoint of each P_i; this gives you the formula:

  $$x_i = a + i\frac{b-a}{n}$$

  With these choices of partition and sampling, compute the Riemann sums for the integral

$$\int_{x=1}^{2} \frac{dx}{x}$$

for $n = 1, 2, 3$ subdivisions. Note: in the next Lecture we will learn that

$$\int_{x=1}^{2} \frac{dx}{x} = \ln 2 \simeq 0.693$$

How does this compare to the values you obtained from the Riemann sums?

- With the same choices of partition and sampling as in the previous problem, evaluate the Riemann sum for the integral

$$\int_{x=0}^{3} x^2\, dx$$

for an arbitrary number n of subdivisions. You may need to use the following:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}, \quad \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}, \quad \sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

- The line $y = x$, the x-axis and the vertical line $x = 2$ bound a triangle of area 2. Thus,

$$I = \int_{x=0}^{2} x\, dx = 2$$

Evaluating the Riemann sum for n subdivisions for the above integral with the same choices of partition and sampling as in the previous problem yields an approximation $RS(n)$ for its value I. The error $E(n)$ we commit by using this approximation is defined to be the difference

$$E(n) = RS(n) - I$$

Show that $E(n)$ is in $O(n^{-k})$ for some $k > 0$. What's the best value of k?

- What is the following integral? Think!

$$\int_{x=-\pi/4}^{\pi/4} \left(x^2 + \ln|\cos x|\right) \sin\frac{x}{2}\, dx$$

- Using the definition of definite integrals, compute $\int_0^1 x^3\, dx$. Use a uniform partition and the fact that $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

25.7 Answers to Selected Examples

1. Let the partition P divide the interval $[0,1]$ into N equally sized subintervals. Then the ith subinterval of P is given by $[(i-1)\frac{1}{N}, i\frac{1}{N}]$, and $(\Delta x)_i = \frac{1}{N}$. Choose the right endpoint of each subinterval to be its sample point, i.e. $x_i = \frac{i}{N}$. Finally, note that as $N \to \infty$, $\Delta x \to 0$. It follows that

$$\begin{aligned}
\int_0^1 x\,dx &= \lim_{\Delta x \to 0} \sum_{i=1}^{N} f(x_i)(\Delta x)_i \\
&= \lim_{N\to\infty} \sum_{i=1}^{N} \frac{i}{N} \cdot \frac{1}{N} \\
&= \lim_{N\to\infty} \frac{1}{N^2} \sum_{i=1}^{N} i \\
&= \lim_{N\to\infty} \frac{1}{N^2} \frac{N(N+1)}{2} \\
&= \lim_{N\to\infty} \frac{N^2+N}{2N^2} \\
&= \frac{1}{2}.
\end{aligned}$$

We used the fact from earlier that

$$\sum_{i=1}^{n} i = \frac{1}{2}n(n+1)$$

(Return)

2. The definite integral is defined as the limit of Riemann sums. Note that for any partition P of the interval,

$$\begin{aligned}
\sum_{i=1}^{N} (f+g)(x_i)(\Delta x)_i &= \sum_{i=1}^{N} \left[f(x_i) + g(x_i)\right](\Delta x)_i \\
&= \sum_{i=1}^{N} f(x_i)(\Delta x)_i + g(x_i)(\Delta x)_i \\
&= \sum_{i=1}^{N} f(x_i)(\Delta x)_i + \sum_{i=1}^{N} g(x_i)(\Delta x)_i,
\end{aligned}$$

because of linearity of finite sums. Therefore, as one takes the limit as $\Delta x \to 0$, one finds (by the linearity of limits) that

$$\int_a^b (f(x) + g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

The argument for a constant multiple is almost identical: we can pull a constant out from a sum, and pull a constant out from a limit.

(Return)

3. Consider what happens if one computes

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx.$$

By the additivity property (where c has been replaced by a), this is

$$\int_a^a f(x)\,dx.$$

But this equals 0, which is intuitive in the area interpretation. (More formally, any partition of an interval with 0 width has subintervals of 0 width, so the Riemann sums equal 0). Therefore,

$$\int_a^b f(x)\,dx + \int_b^a f(x)\,dx = 0,$$

and rearranging gives

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx,$$

as desired.

(Return)

4. For the first part, note that regardless of the partition of $[a, b]$, the Riemann sum

$$\sum_{i=1}^{N} f(x_i)(\Delta x)_i \geq 0,$$

because $f(x_i) \geq 0$ by the above assumption. Since each Riemann sum is non-negative, the limit is non-negative.

For the second part, note that

$$f(x) \geq g(x) \Longrightarrow f(x) - g(x) \geq 0.$$

So applying the first part, we have

$$\int_a^b (f(x) - g(x))\ dx \geq 0.$$

Then by linearity of the definite integral (above),

$$\int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0,$$

and rearranging gives

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

(Return)

5. If we use the partition of $[a, b]$ into n equal intervals, then

$$(\Delta x)_i = \frac{b-a}{n}$$

Also, note that $f(x_i) = c$ for all i. So

$$\begin{aligned}\int_a^b c\,dx &= \lim_{\Delta x \to 0} \sum_{i=1}^{n} c\frac{b-a}{n} \\ &= \lim_{\Delta x \to 0} c \cdot n \cdot \frac{b-a}{n} \\ &= c \cdot (b-a).\end{aligned}$$

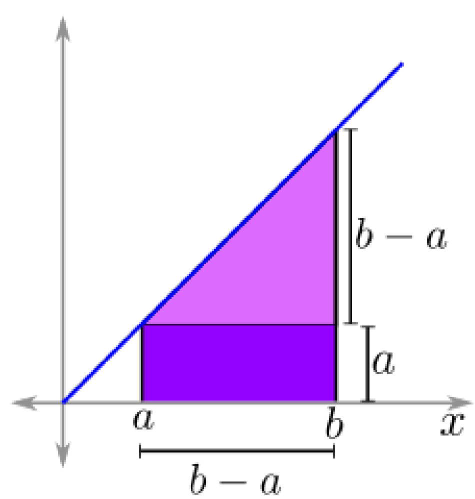
We could also see this by interpreting this definite integral as the area under the curve $y = c$ between $x = a$ and $x = b$, which is simply a rectangle of base $b - a$ and height c.

(Return)

6. Again using a partition into n equal sized subintervals, we have that $(\Delta x)_i = \frac{b-a}{n}$. If we take our sample point to be the right endpoint of each subinterval, then we have $x_i = a + \frac{b-a}{n}i$. So

$$\begin{aligned}\int_{x=a}^{b} x\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} \left(a + \frac{b-a}{n}i\right)\frac{b-a}{n} \\ &= \lim_{n\to\infty} \sum_{i=1}^{n} a\frac{b-a}{n} + \sum_{i=1}^{n} i\left(\frac{b-a}{n}\right)^2 \\ &= \lim_{n\to\infty} n\cdot a\frac{b-a}{n} + \left(\frac{b-a}{n}\right)^2 \frac{n(n+1)}{2} \\ &= a(b-a) + \frac{(b-a)^2}{2} \\ &= \frac{2ab - 2a^2 + b^2 - 2ab + a^2}{2} \\ &= \frac{1}{2}(b^2 - a^2).\end{aligned}$$

This can also be found by interpreting the definite integral as the area under the curve $y = x$, which can be broken into a rectangle with base $b - a$ and height a and a triangle with base and height $b - a$:



(Return)

7. If f only has odd powers in its Taylor series, then

$$f(x) = a_1 x + a_3 x^3 + a_5 x^5 + \cdots$$

for some constants $a_1, a_3, \cdots$. So evaluating $f(-x)$ and doing a little algebra, we find

$$\begin{aligned} f(-x) &= a_1(-x) + a_3(-x)^3 + a_5(-x)^5 + \cdots \\ &= -a_1 x - a_3 x^3 - a_5 x^5 - \cdots \\ &= -\left(a_1 x + a_3 x^3 + a_5 x^5 + \cdots\right) \\ &= -f(x), \end{aligned}$$

as desired. Similarly, if $g(x)$ has even powers, then

$$g(x) = a_0 + a_2 x^2 + a_4 x^4 + \cdots$$

and it follows that

$$\begin{aligned} g(-x) &= a_0 + a_2(-x)^2 + a_4(-x)^4 + \cdots \\ &= a_0 + a_2 x^2 + a_4 x^4 + \cdots \\ &= g(x), \end{aligned}$$

as desired.

(Return)

26 Fundamental Theorem Of Integral Calculus

Computing definite integrals from the definition is difficult, even for fairly simple functions. Fortunately, there is a powerful tool—the *Fundamental Theorem of Integral Calculus*—which connects the definite integral with the indefinite integral and makes most definite integrals easy to compute.

The Fundamental Theorem of Integral Calculus (FTIC)

Given a continuous function f, it follows that

1. $\frac{d}{dx}\left(\int_a^x f(t)dt\right) = f(x)$ and
2. $\int_a^b f(x)dx = \left(\int f(x)dx\right)\Big|_a^b$

(where $F(x)\Big|_a^b = F(b) - F(a)$).

In other words, Part 1 says that the function

$$F(x) = \int_{t=a}^{x} f(t)dt$$

is an anti-derivative of f.

Part 2 says that the definite integral can be computed by finding the indefinite integral of f and subtracting the evaluation at the bottom bound from the evaluation at the top bound. Note that even though the indefinite integral is actually a class of functions that differ by a constant, $F(b)-F(a)$ has the same value for any function F in such a class, so when computing the antiderivative for the purpose of computing a definite integral, it is allowable (and convenient) to forego the constant of integration.

Part 2 can be expressed in a slightly different way which is illustrative. For a differentiable function F we have

$$F\Big|_{x=a}^{b} = \int_{x=a}^{b} dF.$$

This says that the net change in quantity (given on the left side) equals the integral of the rate of change (given on the right side). This interpretation will be used in many applications in the next chapter. (See Rough Proof 1)

Example

Compute
$$\int_{x=1}^{T} \frac{1}{x}\, dx.$$
(See Answer 2)

Example

Compute
$$\int_{1}^{3} x^2 dx.$$
(See Answer 3)

Example

Suppose a publisher prints 12000 books per month with expected revenue of \$60 per book. The marginal cost of each book is given by
$$MC(x) = 10 + \frac{1}{2000}x.$$
What would be the change in profit from a 25% increase in production? (See Answer 4)

Example

Find
$$\frac{d}{dx}\left(\int_0^x \sin(t)dt\right).$$
(See Answer 5)

Example

Find
$$\frac{d}{dx}\left(\int_x^{x^3} \sin(t)dt\right).$$
(See Answer 6)

Caveat

Note that if the integrand $f(t)$ fails to be defined or continuous at a point in the interval $[a, b]$, then the FTIC does not hold. The following example shows this using the singularities of a rational function.

Example

Compute
$$\int_1^4 \frac{dx}{x^2 - 5x + 6}.$$
(See Answer 7)

26.1 Limits of integration and substitution

One must be careful when using Part 2 of the Fundamental Theorem of Integral Calculus along with the method of substitution. The reason this can cause problems is that when a substitution is made, the old limits of integration are still in terms of the original variable. Therefore, one must either get the antiderivative in terms of the original variable before evaluating (this is what we usually did at the end of the substitution anyway), or one can change the limits of integration to reflect the new variable.

Consider the following example which demonstrates both techniques.

Example

Compute

$$\int_{x=0}^{1} x(x-1)^n\,dx,$$

where n is some fixed positive constant. (See Answer 8)

Another case where one must be careful of the limits of integration is with Integration By Parts. One can compute the indefinite integral completely and then apply the limits of integration, or one can apply them as one goes, as in the following:

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

Example

Again compute the definite integral

$$\int_{x=0}^{1} x(x-1)^n\,dx,$$

but this time using integration by parts. (See Answer 9)

26.2 Additional examples

Example

Compute

$$\int_{x=e^3}^{e^5} \frac{\ln x}{x}\,dx.$$

(See Answer 10)

Example

Compute

$$\int_{x=-1}^{1} \frac{1}{1+3x^2}\,dx.$$

(See Answer 11)

26.3 EXERCISES

- Evaluate the following integrals:

$$\int_{x=-1}^{1} \frac{dx}{1+x^2}$$

$$\int_{x=0}^{3} 5x\sqrt{x+1}\,dx$$

$$\int_{x=-\pi}^{\pi} \frac{d}{dx}(x\cos x)\,dx$$

- Compute the following derivatives:

$$\frac{d}{dx}\int_{x=-\pi}^{\pi} x\cos x\,dx$$

$$\frac{d}{dx}\int_{t=0}^{x} \cos t\,dt$$

$$\frac{d}{dx}\int_{t=x^2}^{x^4} e^{-t^2}\,dt$$

$$\frac{d}{dx}\int_{t=1}^{x} \frac{1}{\sqrt{t}}\,dt$$

$$\frac{d}{dx}\int_{t=0}^{\arcsin x} \ln|\sin t+\cos t|\;dt$$

$$\frac{d}{dx}\int_{t=\sin x}^{\tan x} e^{-t^2}\,dt$$

- What is the leading-order term in the Taylor series about $x=0$ of

$$f(x)=\int_{0}^{x} \ln(\cosh(t))dt$$

- We usually use Riemann sums to approximate integrals, but we can go the other way, too, using an antiderivative to approximate a sum. Using only your head (no paper, no calculator), compute an approximation for

$$\sum_{n=0}^{100} n^3$$

  Hint: what integral does this resemble?

- Compute $\dfrac{d}{dx}\displaystyle\int_{3x}^{x^4} e^{-t^2/2}dt$

- Find the critical points of the function $f(x)=\int_{e}^{x^2}\ln(1+t^2)\cos(\sqrt{t})dt$

26.4 Answers to Selected Examples

1. Part 1: By the definition of the derivative, and the definition of the definite integral,

$$\begin{aligned}
\frac{d}{dx}\left(\int_a^x f(t)dt\right) &= \lim_{h\to 0}\frac{\int_a^{x+h} f(t)dt - \int_a^x f(t)dt}{h} \\
&= \lim_{h\to 0}\frac{\int_x^{x+h} f(t)dt}{h} \\
&= \lim_{h\to 0}\frac{f(x^*)\Delta x}{h} \\
&= \lim_{h\to 0}\frac{f(x^*)h}{h} \\
&= \lim_{h\to 0} f(x^*) \\
&= f(x),
\end{aligned}$$

since $x \le x^* \le x+h$.

Part 2: By Part 1, $F(x) = \int_a^x f(t)dt$ is an anti-derivative of f. Furthermore, we have

$$F(b) - F(a) = \int_a^b f(t)dt - \int_a^a f(t)dt = \int_a^b f(t)dt - 0 = \int_a^b f(t)dt.$$

Let $G(x)$ be some anti-derivative of f. Since anti-derivatives of the same function differ only by a constant, $G(x) = F(x) - C$ for some constant C. Then we have

$$G(b) - G(a) = (F(b) - C) - (F(a) - C) = F(b) - F(a)$$

as desired.

(Return)

2. Using Part 2 of the Fundamental Theorem, we find that

$$\begin{aligned}
\int_{x=1}^T \frac{1}{x}\,dx &= \ln x\Big|_{x=1}^T \\
&= \ln T - \ln 1 \\
&= \ln T.
\end{aligned}$$

Note that the above definite integral is sometimes used as the definition of the natural logarithm.

(Return)

3. By Part 2 of FTIC,

$$\begin{aligned}
\int_1^3 x^2 dx &= \frac{1}{3}x^3\Big|_1^3 \\
&= \frac{1}{3}(3^3 - 1^3) \\
&= \frac{26}{3}.
\end{aligned}$$

(Return)

4. The additional 25% means an extra 3000 books. So the goal is to find the change in profit P as x goes from 12000 to 15000. That is,
$$P\Big|_{x=12000}^{15000} = \int_{x=12000}^{15000} dP,$$
according to Part 2 of the Fundamental Theorem. Now, since profit is revenue minus cost, it follows that marginal profit is given by
$$dP = dR - dC.$$
Here,
$$dR = MR(x)\,dx = 60\,dx$$
since the marginal revenue from each book is \$60. And
$$dC = MC(x)\,dx = 10 + \frac{1}{2000}x\,dx.$$
Putting it all together, we find that
$$\begin{aligned} P\Big|_{x=12000}^{15000} &= \int_{x=12000}^{15000} dP \\ &= \int_{x=12000}^{15000} \left(60 - \left(10 + \frac{1}{2000}x\right)\right)\,dx \\ &= \int_{x=12000}^{15000} \left(50 - \frac{1}{2000}x\right)\,dx \\ &= 50x - \frac{1}{4000}x^2\Big|_{x=12000}^{15000} \\ &= \left(50\cdot 15000 - \frac{15000^2}{4000}\right) - \left(50\cdot 12000 - \frac{12000^2}{4000}\right) \\ &= \$129750 \end{aligned}$$
(Return)

5. By Part 1 of FTIC, $\frac{d}{dx}\left(\int_0^x \sin(t)dt\right) = \sin(x)$.
(Return)

6. One must be careful with a function of x in one or both bounds. A good way to break the problem down is to write $F(x) = \int_0^x \sin(t)dt$. By Part 1 of FTIC, $F'(x) = \sin(x)$. Now, note that
$$\begin{aligned} \int_x^{x^3} \sin(t)dt &= \int_0^{x^3} \sin(t)dt - \int_0^x \sin(t)dt \\ &= F(x^3) - F(x). \end{aligned}$$
Next, taking the derivative (and remembering the chain rule) gives
$$\begin{aligned} \frac{d}{dx}\left(\int_x^{x^3} \sin(t)dt\right) &= \frac{d}{dx}\left(F(x^3) - F(x)\right) \\ &= F'(x^3)(3x^2) - F'(x)(1) \\ &= \sin(x^3)(3x^2) - \sin(x). \end{aligned}$$
(Return)

7. This is a rational function, and the denominator factors as $(x-3)(x-2)$, so use partial fractions to express

$$\frac{1}{(x-3)(x-2)} = \frac{A}{x-3} + \frac{B}{x-2}.$$

Clearing denominators gives $1 = A(x-2) + B(x-3)$. Setting $x = 3$ gives $A = 1$. Setting $x = 2$ gives $B = -1$. Thus,

$$\begin{aligned}\int \frac{dx}{x^2 - 5x + 6} &= \int \left(\frac{1}{x-3} - \frac{1}{x-2}\right) dx \\ &= \ln|x-3| - \ln|x-2|.\end{aligned}$$

Then trying to apply FTIC would give

$$\begin{aligned}\int_1^4 \frac{dx}{x^2 - 5x + 6} &= \ln|x-3| - \ln|x-2| \Big|_1^4 \\ &= \ln(1) - \ln(2) - (\ln(2) - \ln(1)) \\ &= -2\ln(2).\end{aligned}$$

However, because $\frac{1}{(x-3)(x-2)}$ has singularities at $x = 2$ and $x = 3$, one must evaluate the improper integral as follows:

$$\int_1^4 \frac{dx}{x^2 - 5x + 6} = \int_1^2 \frac{dx}{x^2 - 5x + 6} + \int_2^3 \frac{dx}{x^2 - 5x + 6} + \int_3^4 \frac{dx}{x^2 - 5x + 6},$$

and none of these integrals exist, as will be shown in the next section on improper integrals, so the entire integral does not exist.

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8. First, we will try the method of substituting back in before evaluating. We make the substitution

$$\begin{aligned}u &= x - 1 \\ du &= dx.\end{aligned}$$

Then the integral becomes

$$\begin{aligned}\int_{x=0}^1 x(x-1)^n \, dx &= \int_{x=0}^1 (u+1)u^n \, du \\ &= \int_{x=0}^1 u^{n+1} + u^n \, du \\ &= \frac{u^{n+2}}{n+2} + \frac{u^{n+1}}{n+1}\Bigg|_{x=0}^1.\end{aligned}$$

This is where a lot of students might make the mistake of plugging in the limits of x when trying to evaluate the integral. This is a reason to include the $x = a$ at the bottom limit of integration: it helps remind us that those limits are in terms of x, even if we have made one or more substitution.

To avoid this pitfall, we now finish getting the antiderivative in terms of x so that we can evaluate:

$$\begin{aligned}
\left.\frac{u^{n+2}}{n+2}+\frac{u^{n+1}}{n+1}\right|_{x=0}^{1} &= \left.\frac{(x-1)^{n+2}}{n+2}+\frac{(x-1)^{n+1}}{n+1}\right|_{x=0}^{1} \\
&= 0+0-\frac{(-1)^{n+2}}{n+2}-\frac{(-1)^{n+1}}{n+1} \\
&= (-1)^{n+2}\left(\frac{1}{n+1}-\frac{1}{n+2}\right) \\
&= \frac{(-1)^{n}}{(n+1)\,(n+2)}.
\end{aligned}$$

On the other hand, as soon as we made the substitution $u=x-1$, we could have changed the limits of integration to reflect our new variable.

Namely, at the lower limit $x=0$, what is the corresponding value of u? Well, $u=x-1$, so when $x=0$, we have $u=-1$. Similarly, when $x=1$, the corresponding value of u is $u=0$. So as we were making our substitution, we could make a corresponding change in the limits of integration so that the new definite integral is entirely in terms of u:

$$\int_{x=0}^{1} x(x-1)^n\,dx = \int_{u=-1}^{0}(u+1)u^n\,du.$$

Now the calculation proceeds as before, and gives the same answer. Changing the limits can sometimes be easier, especially in a complicated integral which may involve (for example) a u-substitution and a trigonometric substitution. Otherwise, the algebra can get messy as one substitutes back in and then substitutes back in again.

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9. The logical choice of parts is

$$\begin{aligned}
u &= x & du &= dx \\
dv &= (x-1)^n\,dx & v &= \frac{(x-1)^{n+1}}{n+1}.
\end{aligned}$$

Then by the formula for parts, we have

$$\begin{aligned}
\int_{x=0}^{1} x(x-1)^n\,dx &= \left.x\frac{(x-1)^{n+1}}{n+1}\right|_{x=0}^{1}-\int_{x=0}^{1}\frac{(x-1)^{n+1}}{n+1}\,dx \\
&= 0-\left.\frac{(x-1)^{n+2}}{(n+1)\,(n+2)}\right|_{x=0}^{1} \\
&= \frac{(-1)^{n+2}}{(n+1)\,(n+2)} \\
&= \frac{(-1)^{n}}{(n+1)\,(n+2)}.
\end{aligned}$$

Note that from the first to second line above, we have $x\frac{(x-1)^{n+1}}{n+1}$ is 0 at both $x=1$ and at $x=0$, so that entire term disappears.

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10. This integral is best computed with a substitution of

$$u = \ln x$$
$$du = \frac{1}{x}\,dx$$

(Integration by parts works too, but it involves a little bit of algebra). Here it is convenient to change the limits of integration as we go, so note that when $x = e^3$, we have $u = 3$. When $x = e^5$ we have $u = 5$. Thus,

$$\begin{aligned}
\int_{x=e^3}^{e^5} \frac{\ln x}{x}\,dx &= \int_{u=3}^{5} u\,du \\
&= \frac{1}{2}u^2\Big|_{u=3}^{5} \\
&= \frac{1}{2}(25-9) \\
&= \frac{1}{2}\cdot 16 \\
&= 8.
\end{aligned}$$

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11. This looks like a good candidate for a trigonometric substitution. Because of the extra factor of 3, the logical substitution is

$$x = \frac{1}{\sqrt{3}}\tan\theta$$
$$dx = \frac{1}{\sqrt{3}}\sec^2\theta\,d\theta.$$

Remember the constant $\frac{1}{\sqrt{3}}$ is there so that when it is squared it will cancel with the factor of 3 and allow us to use the identity $1+\tan^2 = \sec^2$. Again, we will change the bounds as we go. Note that

$$\begin{aligned}
x = -1 \quad &\Rightarrow \quad \frac{1}{\sqrt{3}}\tan\theta = -1 \\
&\Rightarrow \quad \tan\theta = -\sqrt{3}.
\end{aligned}$$

The value of θ for which this holds is $\theta = -\frac{\pi}{3}$. Similarly, when $x = 1$ we have $\theta = \frac{\pi}{3}$. Proceeding, we have

$$
\begin{aligned}
\int_{x=-1}^{1} \frac{1}{1+3x^2} &= \frac{1}{\sqrt{3}} \int_{\theta=-\pi/3}^{\pi/3} \frac{\sec^2\theta \, d\theta}{1+3(\frac{1}{\sqrt{3}}\tan\theta)^2} \\
&= \frac{1}{\sqrt{3}} \int_{\theta=-\pi/3}^{\pi/3} \frac{\sec^2\theta \, d\theta}{1+\tan^2\theta} \\
&= \frac{1}{\sqrt{3}} \int_{\theta=-\pi/3}^{\pi/3} \frac{\sec^2\theta}{\sec^2\theta} \, d\theta \\
&= \frac{1}{\sqrt{3}} \int_{\theta=-\pi/3}^{\pi/3} d\theta \\
&= \frac{1}{\sqrt{3}} \theta \Big|_{-\pi/3}^{\pi/3} \\
&= \frac{1}{\sqrt{3}} \left(\frac{\pi}{3} - \frac{-\pi}{3} \right) \\
&= \frac{2\pi}{3\sqrt{3}}
\end{aligned}
$$

(Return)

27 Improper Integrals

An *improper integral* is a definite integral which cannot be evaluated using the Fundamental Theorem of Integral Calculus (FTIC). This situation arises because the integral either

1. has a point in its interval of integration which is not in the domain of the integrand (the function being integrated) or
2. has ∞ or $-\infty$ as a bound of integration.

As an example of the first type, consider

$$\int_0^2 \frac{dx}{x}.$$

This integral is improper because the left endpoint, 0, is not in the domain of $\frac{1}{x}$.

Another example of the first type is

$$\int_0^4 \frac{dx}{\sqrt[3]{x-2}}.$$

This is improper because the point 2 is in the interval of integration but is not in the domain of $\frac{1}{\sqrt[3]{x-2}}$.

For an example of the second type, consider

$$\int_0^\infty \frac{dx}{1+x^2}.$$

This is improper because the upper bound is ∞.

For an example of the danger of trying to apply the Fundamental Theorem of Integral Calculus when it does not apply, consider

$$\int_{x=-1}^1 \frac{1}{x^2}\,dx.$$

This is an improper integral (hence FTIC does not apply) because the point $x = 0$ is not in the domain of $\frac{1}{x^2}$. If we were to try to apply FTIC anyway, we would find

$$\begin{aligned}\int_{x=-1}^1 \frac{1}{x^2}\,dx &= -\frac{1}{x}\bigg|_{x=-1}^1 \\ &= -1-1 \\ &= -2.\end{aligned}$$

This is problematic since $\frac{1}{x^2} > 0$ for all x (hence should have a positive integral by the dominance property).

27.1 Dealing with improper integrals

To deal with an improper integral of the first type, first consider the integral $\int_a^b f(x)\,dx$, where a is not in the domain of $f(x)$, but f is continuous on the rest of the interval $(a,b]$. In this situation, one replaces the lower bound with a variable T which is slightly larger than a, and then takes the limit as T approaches a from the right (this is denoted by $T \to a^+$).

$$\int_a^b f(x)\,dx = \lim_{T\to a^+} \int_T^b f(x)\,dx.$$

This replacement allows the integral $\int_T^b f(x)\,dx$ to be computed using FTIC, since f is continuous on that interval. After that integral is computed (in terms of T), the limit is computed. If the limit exists, then the original integral exists and equals the result. If the limit does not exist or is infinite, then the original integral does not exist either.

Example

Determine whether the integral

$$\int_0^2 \frac{dx}{x}$$

exists, and if it exists, what its value is. (See Answer 1)

If the right endpoint, b, of the integral $\int_a^b f(x)\,dx$ is not in the domain of f, then b gets replaced with a variable T slightly smaller than b, and again the integral is replaced with a limit, this time as $T \to b^-$ (that is, the limit as T approaches b from the left).

$$\int_a^b f(x)\,dx = \lim_{T\to b^-} \int_a^T f(x)\,dx.$$

Again, the integral equals this limit, if it exists. If the limit does not exist, then the integral does not exist.

Example

Determine whether the integral

$$\int_1^3 \frac{dx}{(x-3)^2}$$

exists. If it exists, find its value. (See Answer 2)

For integrals where a point inside the interval of integration is not in the domain of the integrand, the integral is first split at the bad point, and then each integral is evaluated separately using the above techniques. So, consider $\int_a^b f(x)\,dx$ where the point c is not in the domain of f and $a < c < b$. Then

$$\begin{aligned}\int_a^b f(x)\,dx &= \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \\ &= \lim_{T\to c^-} \int_a^T f(x)\,dx + \lim_{U\to c^+} \int_U^b f(x)\,dx.\end{aligned}$$

Both of the resulting limits must exist for the original integral to exist.

Example

Determine if the integral

$$\int_{-1}^{1} \frac{dx}{x^{4/5}}$$

exists. (See Answer 3)

27.2 Bounds at infinity

For integrals with one bound at infinity, the integral is defined as follows.

$$\int_a^\infty f(x)\,dx = \lim_{T\to\infty} \int_a^T f(x)\,dx.$$

Similarly,

$$\int_{-\infty}^a f(x)\,dx = \lim_{T\to-\infty} \int_T^a f(x)\,dx.$$

In the case of bounds of ∞ and $-\infty$, one can first split the integral at any real number c, and then compute each integral as above:

$$\begin{aligned}\int_{-\infty}^\infty f(x)\,dx &= \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx \\ &= \lim_{T\to-\infty} \int_T^c f(x)\,dx + \lim_{U\to\infty} \int_c^U f(x)\,dx.\end{aligned}$$

As before, both of these limits must exist for the original integral to exist. It is *not* equivalent to computing a single limit such as

$$\lim_{T\to\infty} \int_{-T}^T f(x)\,dx.$$

Example

Determine if the integral

$$\int_0^\infty \frac{dx}{1+x^2}$$

exists. If it exists, find its value. (See Answer 4)

27.3 The p-integral

One class of improper integrals is common enough to get its own name: *the p-integral*. This is the name given to integrals of the form

$$\int \frac{1}{x^p}\,dx = \int x^{-p}\,dx.$$

There are two versions of this integral that are of interest to us. First, with a limit at infinity, and second with a limit at 0.

Example

Show that

$$\int_{x=1}^{\infty} x^{-p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p \leq 1. \end{cases}$$

(See Answer 5)

Example

Consider the p-integral with a limit at 0:

$$\int_{x=0}^{1} x^{-p}\,dx.$$

This integral is improper because 0 is not in the domain of x^{-p}. Show that

$$\int_{x=0}^{1} x^{-p}\,dx = \begin{cases} \infty & \text{if } p \geq 1 \\ \frac{1}{1-p} & \text{if } p < 1. \end{cases}$$

(See Answer 6)

27.4 Converge or diverge

In some contexts, it is enough to know whether an integral *converges* (has a finite answer) or *diverges* (goes to infinity or does not exist). Using the knowledge of the above p-integrals, and some asymptotic tools from earlier in the course, can help quickly determine whether certain improper integrals converge or diverge.

Example

Determine if

$$\int_{x=0}^{1} \frac{dx}{\sqrt{x^2+x}}$$

converges or diverges. (See Answer 7)

Example

Determine if

$$\int_{x=1}^{\infty} \frac{dx}{\sqrt{x^2+x}}$$

converges or diverges. (See Answer 8)

Example

Determine if

$$\int_{x=-\infty}^{\infty} \frac{2x}{1+x^2}\,dx$$

converges or diverges. (See Answer 9)

27.5 EXERCISES

- Use a Talyor expansion of the integrand at $x = 0$ to determine whether the following integrals converge or diverge:

$$\int_{x=0}^{1} \frac{e^{-x}}{x}\, dx$$

$$\int_{x=0}^{1} \frac{\cos^2 x}{\sqrt{x}}\, dx$$

- Use the asymptotics of the integrand as $x \to \infty$ to determine whether the following integrals converge or diverge:

$$\int_{x=1}^{+\infty} \frac{\sqrt[3]{x+3}}{x^3}\, dx$$

$$\int_{x=1}^{+\infty} \frac{1-5^{-x}}{x}\, dx$$

- Compute the following integrals, if they converge, by evaluating a limit.

$$\int_{x=0}^{+\infty} e^{-x} \sin x\, dx$$

$$\int_{x=1}^{2} \frac{dx}{\sqrt{x-1}}$$

$$\int_{x=0}^{4} \frac{2\, dx}{\sqrt{16-x^2}}$$

- The following integral is improper both at $x = 1$ and at $x \to \infty$:

$$\int_{x=1}^{+\infty} \frac{1}{\sqrt{x^3-1}}\, dx$$

  First, as $x \to \infty$, show that the integrand is $x^{-3/2} + O(x^{-9/2})$, so that it converges at this limit.

  Next, as $x \to 1^+$, show that the integrand is $(3(x-1))^{-1/2} + O((x-1)^{1/2})$, so that it converges at this limit also.

- Consider the following two integrals:

$$I_1 = \int_{x=2}^{+\infty} \frac{dx}{\sqrt{x^3-8}}, \qquad I_2 = \int_{x=2}^{+\infty} \frac{1}{\sqrt{(x-2)^3}}\, dx$$

  One converges and one does not. Which is which and why?

- Until now we have used asymptotic analysis to relate an improper integral to a p-integral. But sometimes the leading order term is not a power. Identify the leading order term as $x \to +\infty$ of the integrand of

$$\int_{x=1}^{+\infty} \frac{1}{\sinh x}\, dx$$

and determine whether the integral converges or diverges.

- For $p \geq 0$ an integer, consider the following integral:

$$I_p = \int_{x=1}^{+\infty} \frac{dx}{\ln^p x}$$

Show that this diverges for any value of p.

Hint 1: think about the growth of $\ln^p x$ as $x \to +\infty$ as compared to polynomial growth.

Hint 2: recall from Lecture 25 that if $g(x) \geq f(x)$ for every $x \in [a, b]$, then

$$\int_{x=a}^{b} g(x)\, dx \geq \int_{x=a}^{b} f(x)\, dx$$

This is also true if the domain of integration is unbounded and the integrals are defined...

- Determine whether the following integral converges or diverges.

$$\int_{2}^{\infty} \frac{1}{(x^5 - 4x^3)^{1/4}}\, dx$$

27.6 Answers to Selected Examples

1. Since the left endpoint is not in the domain of $\frac{1}{x}$, the integral becomes

$$\begin{aligned}\int_0^2 \frac{dx}{x} &= \lim_{T\to 0^+} \int_T^2 \frac{dx}{x} \\ &= \lim_{T\to 0^+} \ln x \Big|_T^2 \\ &= \lim_{T\to 0^+} (\ln(2) - \ln(T)).\end{aligned}$$

This limit does not exist because $\lim_{T\to 0^+} \ln(T)$ diverges. Hence, the original integral does not exist.

(Return)

2. The integral becomes

$$\begin{aligned}\int_1^3 \frac{dx}{(x-3)^2} &= \lim_{T\to 3^-} \int_1^T \frac{dx}{(x-3)^2} \\ &= \lim_{T\to 3^-} -\frac{1}{x-3}\Big|_1^T \\ &= \lim_{T\to 3^-} -\frac{1}{T-3} - (-\frac{1}{-2}).\end{aligned}$$

This limit does not exist since $\lim_{T\to 3^-} \frac{1}{T-3}$ diverges. Hence, the integral does not exist.

(Return)

3. The only point not in the domain of the function $\frac{1}{x^{4/5}}$ is 0. Thus, the integral becomes

$$\begin{aligned}
\int_{-1}^{1} \frac{dx}{x^{4/5}} &= \int_{-1}^{0} \frac{dx}{x^{4/5}} + \int_{0}^{1} \frac{dx}{x^{4/5}} \\
&= \lim_{T \to 0^-} \int_{-1}^{T} \frac{dx}{x^{4/5}} + \lim_{U \to 0^+} \int_{u}^{1} \frac{dx}{x^{4/5}} \\
&= \lim_{T \to 0^-} 5x^{1/5}\Big|_{-1}^{T} + \lim_{U \to 0^+} 5x^{1/5}\Big|_{u}^{1} \\
&= \lim_{T \to 0^-} \left(5T^{1/5} - 5(-1)^{1/5}\right) + \lim_{U \to 0^+} \left(5 - 5u^{1/5}\right) \\
&= 0 + 5 + 5 - 0 \\
&= 10.
\end{aligned}$$

So the integral exists and equals 10.

(Return)

4. Because the top bound is ∞, the integral becomes

$$\begin{aligned}
\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{T \to \infty} \int_{0}^{T} \frac{dx}{1+x^2} \\
&= \lim_{T \to \infty} \arctan(x)\Big|_{0}^{T} \\
&= \lim_{T \to \infty} \arctan(T) - \arctan(0) \\
&= \lim_{T \to \infty} \arctan(T) \\
&= \frac{\pi}{2}.
\end{aligned}$$

So the integral exists and equals $\frac{\pi}{2}$.

(Return)

5. First, if $p \neq 1$, then we can use the power rule on x^{-p}. Here we find

$$\begin{aligned}
\int_{x=1}^{\infty} x^{-p}\,dx &= \lim_{T \to \infty} \frac{x^{-p+1}}{-p+1}\Big|_{x=1}^{T} \\
&= \lim_{T \to \infty} \frac{T^{-p+1}}{1-p} - \frac{1}{1-p}.
\end{aligned}$$

Note that T^{-p+1} diverges to infinity if $p < 1$, since in this case we have a positive power of T, which goes to infinity as T goes to infinity. On the other hand, T^{-p+1} goes to 0 if $p > 1$, since in that case it is a negative power of T. Putting this together with the above equations, we have

$$\int_{x=1}^{\infty} x^{-p}\,dx = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \infty & \text{if } p < 1. \end{cases}$$

Finally, consider the case $p = 1$. In this case,

$$\begin{aligned}
\int_{x=1}^{\infty} \frac{1}{x}\,dx &= \lim_{T \to \infty} \ln x\Big|_{1}^{T} \\
&= \lim_{T \to \infty} \ln T \\
&= \infty.
\end{aligned}$$

And so the cases are as claimed above.

(Return)

6. As in the previous example, we first consider when $p \neq 1$ and use the power rule:

$$\begin{aligned}\int_{x=0}^{1} x^{-p}\,dx &= \lim_{T\to 0^+} \left.\frac{x^{1-p}}{1-p}\right|_{x=T}^{1} \\ &= \lim_{T\to 0^+} \frac{1}{1-p} - \frac{T^{1-p}}{1-p}.\end{aligned}$$

Now, note that the limit is as $T \to 0^+$. So we have T^{1-p} diverges if $p > 1$ (since a negative power of T diverges as $T \to 0^+$), and converges to 0 if $p < 1$. Putting this together with the previous computation gives

$$\int_{x=0}^{1} x^{-p}\,dx = \begin{cases} \frac{1}{1-p} & \text{if } p1. \end{cases}$$

Finally, if $p = 1$, then

$$\begin{aligned}\int_{x=0}^{1} \frac{1}{x}\,dx &= \lim_{T\to 0^+} \ln x\Big|_{x=T}^{1} \\ &= \lim_{T\to 0^+} 0 - \ln T\end{aligned}$$

which diverges to infinity. Thus, the original integral diverges for all $p \geq 1$, as claimed.

(Return)

7. This is not a function for which we can easily find an antiderivative. However, since we are interested in the behavior of the function near 0 (that is where the blow-up occurs), we can do a little bit of algebra to see that

$$\begin{aligned}\frac{1}{\sqrt{x^2+x}} &= \frac{1}{\sqrt{x}} \cdot \frac{1}{\sqrt{x+1}} \\ &= \frac{1}{\sqrt{x}} \cdot (1+x)^{-1/2} \\ &= \frac{1}{\sqrt{x}}\left(1 + O(x)\right),\end{aligned}$$

by the binomial expansion. Therefore, the leading order term of this function as $x \to 0$ is $\frac{1}{\sqrt{x}}$. This is a convergent p-integral, because $p = \frac{1}{2}$ and from the above example

$$\int_{x=0}^{1} \frac{dx}{x^p}$$

converges when $p < 1$.

(Return)

8. We must do a slightly different analysis for this integral, since we are interested in the behavior as $x \to \infty$. Because we want to take advantage of the binomial series again (which requires its argument to be less

than 1), we do the following algebra:

$$\begin{aligned}\frac{1}{\sqrt{x^2+x}} &= \frac{1}{\sqrt{x^2}} \cdot \frac{1}{\sqrt{1+x^{-1}}} \\ &= \frac{1}{x}\left(1+\frac{1}{x}\right)^{-1/2} \\ &= \frac{1}{x}\left(1+O\left(\frac{1}{x}\right)\right).\end{aligned}$$

So for this integral, the leading order term is $\frac{1}{x}$, which is the p-integral with $p = 1$. This diverges, as the above example shows, and so the original integral diverges.

(Return)

9. As mentioned above, it is not valid to do this with a single limit such as

$$\lim_{T\to\infty} \int_{x=-T}^{T} \frac{2x}{1+x^2}\,dx.$$

The integral must be split and treated as two separate integrals with limits at infinity:

$$\int_{x=-\infty}^{\infty} \frac{2x}{1+x^2}\,dx = \lim_{S\to-\infty} \int_{x=S}^{0} \frac{2x}{1+x^2}\,dx + \lim_{T\to\infty} \int_{x=0}^{T} \frac{2x}{1+x^2}\,dx.$$

Again, a little bit of asymptotic analysis with the help of the geometric series helps determine the behavior of this function:

$$\begin{aligned}\frac{2x}{1+x^2} &= \frac{2x}{1+x^2} \cdot \frac{x^{-2}}{x^{-2}} \\ &= \frac{2}{x} \cdot \frac{1}{1+x^{-2}} \\ &= \frac{2}{x}\left(1+O(x^{-2})\right).\end{aligned}$$

Here, the geometric series was used to write

$$\frac{1}{1+x^{-2}} = 1 - x^{-2} + x^{-4} - \cdots = 1 + O(x^{-2}),$$

which is justified since $x \to \infty$ and so x^{-2} is very small. So the original integrand behaves like $\frac{2}{x}$, which diverges when integrated to infinity. Therefore

$$\lim_{T\to\infty} \int_{x=0}^{T} \frac{2x}{1+x^2}\,dx$$

diverges, and (but for a sign change) the same reasoning shows

$$\lim_{S\to-\infty} \int_{x=S}^{0} \frac{2x}{1+x^2}\,dx$$

diverges too, so the original integral diverges.

(Return)

28 Trigonometric Integrals

A trigonometric integral is an integral involving products and powers of trigonometric functions: cosine, sine, tangent, secant, cosecant, and cotangent. Many of these integrals can be handled with u-substitution, but there are other methods which are outlined in this module. The three families of integrals discussed in this module are

$$\int \sin^m \theta \cos^n \theta \, d\theta$$
$$\int \tan^m \theta \sec^n \theta \, d\theta$$
$$\int \sin(m\theta) \cos(n\theta) \, d\theta.$$

28.1 Product of sines and cosines

Consider the integral

$$\int \sin^m \theta \cos^n \theta \, d\theta.$$

There are several cases to consider based on whether m and n are odd and even.

m is odd

If m is odd, then one factor of $\sin\theta$ can be set aside. This leaves behind an even power of $\sin\theta$, which can be expressed in terms of $\cos\theta$ using the Pythagorean identity. Then the substitution $u = \cos\theta$ can be made.

Example

Find

$$\int \sin^3(x) \cos(x) \, dx.$$

(See Answer 1)

n is odd

If n is odd, the procedure is very similar. This time, we set aside a factor of $\cos\theta$. This leaves an even power of $\cos\theta$ which can be expressed in terms of $\sin\theta$ using the Pythagorean identities.

Example

Find

$$\int \sin^2(x)\cos^3(x)\,dx.$$

(See Answer 2)

Both m and n are even

If neither m nor n is odd, then both are even. This is a bit more difficult and requires using the power reduction formulas:

Power reduction
$\sin^2(\theta) = \frac{1-\cos(2\theta)}{2}$
$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$

Example

Find

$$\int \sin^2 x\,dx.$$

(See Answer 3)

Example

Find

$$\int \sin^2\theta\cos^2\theta\,d\theta.$$

(See Answer 4)

Example

Find

$$\int \cos^4(x)\,dx.$$

(See Answer 5)

28.2 Product of tangents and secants

Next consider the integral

$$\int \tan^m\theta\sec^n\theta\,d\theta.$$

As with the product of sines and cosines, the method will depend on whether m and n are odd or even.

m is odd

If m is odd, we will set aside a factor of $\tan\theta\sec\theta$. Note that this is the derivative of $\sec\theta$ and so this sets up a substitution of $u=\sec\theta$. After setting aside these factors, we are left with an even power of $\tan\theta$, which can be expressed in terms of $\sec\theta$ using the Pythagorean identity

$$\tan^2\theta=\sec^2\theta-1.$$

Now, the integral can be computed using the substitution $u=\sec\theta$.

Example

Compute

$$\int\tan^3\theta\sec\theta\,d\theta.$$

(See Answer 6)

n is even

If n is even, then we can set aside a factor of $\sec^2\theta$. Note that this is the derivative of $\tan\theta$ and therefore sets up the substitution $u=\tan\theta$. Setting aside $\sec^2\theta$ leaves an even power of $\sec\theta$, which can be expressed in terms of $\tan\theta$ using the Pythagorean identity

$$\sec^2\theta=1+\tan^2\theta.$$

Then the substitution $u=\tan\theta$ allows the computation of the integral.

Example

Compute

$$\int\tan^2\theta\sec^6\theta\,d\theta.$$

(See Answer 7)

m is even, n is odd

If neither of the above cases holds, then the integral is a bit more difficult. It typically requires a bit of algebra and several applications of a reduction formula (or integration by parts). A general method is to rewrite the even power of tangent entirely in terms of secant by using the Pythagorean identity

$$\tan^2\theta=\sec^2\theta-1.$$

This gives an integral which is sums of powers of $\sec\theta$. Each of these can be solved using the reduction formula for secant:

$$\int\sec^n\theta\,d\theta=\frac{1}{n-1}\sec^{n-2}\theta\tan\theta+\frac{n-2}{n-1}\int\sec^{n-2}\theta\,d\theta,$$

along with the fact that

$$\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C.$$

Example

Compute
$$\int \tan^2\theta \sec\theta \, d\theta.$$
(See Answer 8)

28.3 Product of sine and cosine with constants

Finally, consider the integral
$$\int \sin(m\theta)\cos(n\theta)\, d\theta.$$

This integral requires some algebra to simplify the integrand. One can verify using the sum and difference formulas for sine that
$$\sin(m\theta)\cos(n\theta) = \frac{1}{2}\left(\sin((m+n)\theta) + \sin((m-n)\theta)\right).$$

This expression can be integrated term by term to find
$$\begin{aligned}\int \sin(m\theta)\cos(n\theta)\, d\theta &= \int \frac{1}{2}\left(\sin((m+n)\theta) + \sin((m-n)\theta)\right)\, d\theta \\ &= \frac{1}{2}\left(-\frac{\cos((m+n)\theta)}{m+n} - \frac{\cos((m-n)\theta)}{m-n}\right) + C.\end{aligned}$$

There are similar formulas for related integrals:
$$\begin{aligned}\int \sin(m\theta)\sin(n\theta)\, d\theta &= -\frac{\sin((m+n)\theta)}{2(m+n)} + \frac{\sin((m-n)\theta)}{2(m-n)} + C \\ \int \cos(m\theta)\cos(n\theta)\, d\theta &= \frac{\sin((m+n)\theta)}{2(m+n)} + \frac{\sin((m-n)\theta)}{2(m-n)} + C.\end{aligned}$$

These formulas need not be memorized, but be aware they exist and look them up when necessary.

Example

Compute
$$\int \sin(3\theta)\cos(4\theta)\, d\theta.$$
(See Answer 9)

28.4 Additional examples

Example

Compute
$$\int_{\theta=-\pi/2}^{\pi/2} \cos^n\theta \, d\theta.$$

Use the fact (which is proven using integration by parts) that

$$\int \cos^n \theta \, d\theta = \frac{\cos^{n-1}\theta \sin\theta}{n} + \frac{n-1}{n} \int \cos^{n-2}\theta \, d\theta.$$

(See Answer 10)

Example

Compute

$$\int \tan^3 \theta \, d\theta$$

(See Answer 11)

28.5 EXERCISES

Compute the following indefinite integrals. You may need to use reduction formulae or coordinate changes.

- $\int \sin^2 x \cos^2 x \, dx$
- $\int \sin^3 \frac{x}{2} \cos^3 \frac{x}{2} \, dx$
- $\int \frac{x^3 \, dx}{\sqrt{9 - x^2}}$
- $\int 5 \tan^5 x \sec^3 x \, dx$
- $\int 7 \tan^4 x \sec^4 x \, dx$
- $\int 9 \sin^3 3x \, dx$
- $\int \cos^4 x \, dx$
- $\int \sin x \sec x \tan x \, dx$
- $\int \tan^5 2x \sec^4 2x \, dx$
- $\int \cos x \sqrt{1 - \sin x} \, dx$
- $\int \frac{x^2 \, dx}{\sqrt{1 + x^2}}$
- $\int \tan^4(2x) \sec^4(2x) \, dx$

28.6 Answers to Selected Examples

1. Following the above outline, set aside one factor of $\sin x$, which gives

$$\int \sin^3 x \cos x \, dx = \int (\sin^2 x)(\cos x)(\sin x) \, dx.$$

Now, there is an even power of sine remaining, which can be rewritten using the Pythagorean identity

$$\sin^2 x = 1 - \cos^2 x.$$

This gives

$$\begin{aligned} \int \sin^3 x \cos x \, dx &= \int (\sin^2 x)(\cos x) \sin(x) \, dx \\ &= \int (1 - \cos^2 x)(\cos x) \sin(x) \, dx. \end{aligned}$$

Now, the integral can be handled by letting $u = \cos(x)$ (and $du = -\sin(x) dx$).

$$\begin{aligned} \int (1 - \cos^2 x)(\cos x)(\sin x) \, dx &= \int (1 - u^2) u (-du) \\ &= \int (u^3 - u) \, du \\ &= \frac{u^4}{4} - \frac{u^2}{2} + C \\ &= \frac{\cos^4 x}{4} - \frac{\cos^2 x}{2} + C. \end{aligned}$$

(Return)

2. Following the procedure outlined above, we set aside a factor of cosine and use the Pythagorean identity, which gives

$$\begin{aligned} \int \sin^2(x) \cos^3(x) \, dx &= \int \sin^2(x) \cos^2(x) \cos(x) \, dx \\ &= \int \sin^2(x)(1 - \sin^2 x) \cos(x) \, dx. \end{aligned}$$

Now, we are ready to make the substitution $u = \sin(x)$, $du = \cos(x) \, dx$. This gives

$$\begin{aligned} \int \sin^2(x)(1 - \sin^2 x) \cos(x) \, dx &= \int u^2 (1 - u^2) \, du \\ &= \int u^2 - u^4 \, du \\ &= \frac{u^3}{3} - \frac{u^5}{5} + C \\ &= \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C. \end{aligned}$$

(Return)

3. Using the first power reduction formula gives

$$\begin{aligned}\int \sin^2 x\, dx &= \int \frac{1}{2}(1-\cos(2x))\, dx \\ &= \frac{1}{2}\left(x - \frac{\sin(2x)}{2}\right) + C.\end{aligned}$$

(Return)

4. Using both the power reduction formulas and doing some algebra gives

$$\begin{aligned}\int \sin^2\theta\cos^2\theta\, d\theta &= \int \frac{1}{2}\left(1-\cos 2\theta\right)\frac{1}{2}\left(1+\cos 2\theta\right)\, d\theta \\ &= \frac{1}{4}\int (1-\cos 2\theta)(1+\cos 2\theta)\, d\theta \\ &= \frac{1}{4}\int (1-\cos^2 2\theta)\, d\theta \\ &= \frac{1}{4}\int \sin^2 2\theta\, d\theta \\ &= \frac{1}{4}\int \frac{1}{2}\left(1-\cos 4\theta\right)\, d\theta \\ &= \frac{1}{8}\left(\theta - \frac{\sin 4\theta}{4}\right) + C.\end{aligned}$$

(Return)

5. Using the second power reduction formula (and then again in a later step) gives

$$\begin{aligned}\int \cos^4(x)\, dx &= \int (\cos^2(x))^2\, dx \\ &= \int (\frac{1}{2}(1+\cos(2x))^2\, dx \\ &= \frac{1}{4}\int \left(1+2\cos(2x)+\cos^2(2x)\right)\, dx \\ &= \frac{1}{4}\int \left(1+2\cos(2x)+\frac{1}{2}(1+\cos(4x))\right)\, dx \\ &= \frac{1}{4}\left(x+\sin(2x)+\frac{1}{2}x+\frac{1}{8}\sin(4x)\right)+C\end{aligned}$$

(Return)

6. Since the power of tangent is odd, we set aside a factor of $\tan\theta\sec\theta$, and use the Pythagorean identity to find

$$\begin{aligned}\int \tan^3\theta\sec\theta\, d\theta &= \int \tan^2\theta(\tan\theta\sec\theta)\, d\theta \\ &= \int (\sec^2\theta - 1)(\tan\theta\sec\theta) d\theta.\end{aligned}$$

Now, we can make the substitution

$$\begin{aligned}u &= \sec\theta \\ du &= \sec\theta\tan\theta\, d\theta,\end{aligned}$$

which gives

$$\begin{aligned}\int(\sec^2\theta-1)(\tan\theta\sec\theta)d\theta &= \int(u^2-1)\,du\\ &= \frac{1}{3}u^3-u+C\\ &= \frac{1}{3}\sec^3\theta-\sec\theta+C.\end{aligned}$$

(Return)

7. Because the power of secant is even, we set aside $\sec^2\theta$ and use the Pythagorean identity to find

$$\begin{aligned}\int\tan^2\theta\sec^6\theta\,d\theta &= \int\tan^2\theta\sec^4\theta(\sec^2\theta)\,d\theta\\ &= \int\tan^2\theta(1+\tan^2\theta)^2(\sec^2\theta)\,d\theta.\end{aligned}$$

Now, we are prepared for a substitution of

$$\begin{aligned}u &= \tan\theta\\ du &= \sec^2\theta\,d\theta.\end{aligned}$$

Making this substitution and simplifying gives

$$\begin{aligned}\int\tan^2\theta(1+\tan^2\theta)^2(\sec^2\theta)\,d\theta &= \int u^2(1+u^2)^2\,du\\ &= \int u^2(1+2u^2+u^4)\,du\\ &= \int(u^2+2u^4+u^6)\,du\\ &= \frac{1}{3}u^3+\frac{2}{5}u^5+\frac{1}{7}u^7+C\\ &= \frac{1}{3}\tan^3\theta+\frac{2}{5}\tan^5\theta+\frac{1}{7}\tan^7\theta+C.\end{aligned}$$

(Return)

8. Using the Pythagorean identity gives

$$\begin{aligned}\int\tan^2\theta\sec\theta\,d\theta &= \int(\sec^2\theta-1)\sec\theta\,d\theta\\ &= \int(\sec^3\theta-\sec\theta)\,d\theta\\ &= \int\sec^3\theta\,d\theta-\int\sec\theta\,d\theta.\end{aligned}$$

Now, using the reduction formula on the first of these integrals gives

$$\int\sec^3\theta\,d\theta=\frac{1}{2}\sec\theta\tan\theta+\frac{1}{2}\int\sec\theta\,d\theta.$$

Combining this with the above expression and using the integral of secant, we find

$$\begin{aligned}
\int \tan^2\theta \sec\theta \, d\theta &= \int \sec^3\theta \, d\theta - \int \sec\theta \, d\theta \\
&= \left(\frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\int \sec\theta \, d\theta\right) - \int \sec\theta \, d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\int \sec\theta \, d\theta \\
&= \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.
\end{aligned}$$

(Return)

9. Using the formula given, we have

$$\begin{aligned}
\int \sin(3\theta)\cos(4\theta) \, d\theta &= -\frac{\cos(7\theta)}{14} - \frac{\cos(-\theta)}{-2} + C \\
&= -\frac{\cos(7\theta)}{14} + \frac{\cos\theta}{2} + C,
\end{aligned}$$

where we have used the fact that cosine is even to simplify the final expression.
(Return)

10. Applying the limits of integration in the above reduction formula gives

$$\int_{\theta=-\pi/2}^{\pi/2} \cos^n\theta \, d\theta = \left.\frac{\cos^{n-1}\theta\sin\theta}{n}\right|_{-\pi/2}^{\pi/2} + \frac{n-1}{n}\int_{\theta=-\pi/2}^{\pi/2} \cos^{n-2}\theta \, d\theta.$$

Now, notice that

$$\left.\frac{\cos^{n-1}\theta\sin\theta}{n}\right|_{-\pi/2}^{\pi/2} = 0$$

because cosine is 0 at $\pm\pi/2$. Therefore,

$$\int_{\theta=-\pi/2}^{\pi/2} \cos^n\theta \, d\theta = \frac{n-1}{n}\int_{\theta=-\pi/2}^{\pi/2} \cos^{n-2}\theta \, d\theta.$$

This is itself a reduction formula. By computing the base cases $n = 0$ and $n = 1$, respectively, we find

$$\begin{aligned}
\int_{\theta=-\pi/2}^{\pi/2} d\theta &= \theta\Big|_{-\pi/2}^{\pi/2} \\
&= \pi,
\end{aligned}$$

and

$$\begin{aligned}
\int_{\theta=-\pi/2}^{\pi/2} \cos\theta \, d\theta &= \sin\theta\Big|_{-\pi/2}^{\pi/2} \\
&= 2.
\end{aligned}$$

Now the value of the integral for any higher value of n can be found by repeatedly using the above formula until the integral reduces to one of the base cases above. Using induction, one finds

$$\int_{\theta=-\pi/2}^{\pi/2} \cos^n \theta \, d\theta = \begin{cases} \frac{1\cdot 3\cdot 5\cdots(n-1)}{2\cdot 4\cdot 6\cdots n}\pi & \text{if } n \text{ is even} \\ \frac{2\cdot 4\cdot 6\cdots(n-1)}{3\cdot 5\cdot 7\cdots n}\cdot 2 & \text{if } n \text{ is odd.} \end{cases}$$

(Return)

11. Here the power of tangent is odd, so the method calls for setting aside a factor of $\sec\theta\tan\theta$. However, there is no factor of secant in this integral! It turns out that this is not a problem; we can multiply the top and the bottom by secant to introduce a factor of secant, and the algebra works out:

$$\begin{aligned} \int \tan^3\theta \, d\theta &= \int \frac{\tan^2\theta}{\sec\theta}(\sec\theta\tan\theta)\, d\theta \\ &= \int \frac{\sec^2\theta - 1}{\sec\theta}(\sec\theta\tan\theta)\, d\theta \\ &= \int \left(\sec\theta - \frac{1}{\sec\theta}\right)(\sec\theta\tan\theta)\, d\theta. \end{aligned}$$

Now, proceed as usual with the substitution

$$\begin{aligned} u &= \sec\theta \\ du &= \sec\theta\tan\theta\, d\theta. \end{aligned}$$

Thus,

$$\begin{aligned} \int \left(\sec\theta - \frac{1}{\sec\theta}\right)(\sec\theta\tan\theta)\, d\theta &= \int \left(u - \frac{1}{u}\right)\, du \\ &= \frac{1}{2}u^2 - \ln|u| + C \\ &= \frac{1}{2}\sec^2\theta - \ln|\sec\theta| + C. \end{aligned}$$

(Return)

29 Tables And Computers

This module discusses some of the shortcuts available by using tables of integrals and mathematical software. Many integrals can be easily computed by a computer algebra system, but it is still important to know the underlying concepts so as to be able to use these tools efficiently and accurately.

29.1 Tables of integrals

Most calculus textbooks have an appendix containing one hundred or more integral formulas. All of these formulas can be derived using the techniques of the previous modules (possibly with some additional technical algebra), and it is a good exercise to try to derive some of them.

Using the table is sometimes difficult, because finding the correct form can be tricky. And even with the correct form, an integral may not match the form precisely. It may take some algebra and a u-substitution to match the form.

Example

Use the formula

$$\int \frac{dx}{x^2\sqrt{x^2-a^2}} = \frac{\sqrt{x^2-a^2}}{a^2x} + C$$

to evaluate the integral

$$\int \frac{dx}{(4x^2+4x+1)\sqrt{4x^2+4x-3}}.$$

(See Answer 1)

Other table entries are more inductive in nature, like the reduction formulas mentioned in the integration by parts module.

Example

Use the formulas

$$\int \sec ax\,dx = \frac{1}{a}\ln|\sec ax + \tan ax| + C,$$

and (for $n \geq 2$)

$$\int \sec^n ax\,dx = \frac{\sec^{n-2} ax \tan ax}{a(n-1)} + \frac{n-2}{n-1}\int \sec^{n-2} ax\,dx + C$$

to find

$$\int \sec^3(x)\, dx.$$

(See Answer 2)

29.2 Mathematical software

Expensive computer algebra systems such as *Maple* and *Mathematica* can quickly and accurately dispense with most integrals that can be done by hand. One free alternative, from the makers of Mathematica, is Wolfram Alpha, which for most purposes is as good as its more costly relatives, and in many cases it can explain the intermediate steps of longer computations (though it now only provides three such explanations per day for a user without a paid subscription).

Note that the form of the answer given by computer systems may look different from what one gets by hand or by a table, so care should be taken when comparing answers.

Example

Compute $\int \sec^3(x)\, dx$ using Wolfram Alpha, or other computer algebra system. Note the syntax of the entry (though it is pretty good at parsing other forms of entry). Also note that the answer given is in a different form than that found in the earlier example.

Answer

Example

There are limits to what a computer algebra system can do. Consider the integral

$$\int_{x=0}^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx.$$

It turns out that the value of this integral is $\frac{\pi}{4}$ for all n, although Wolfram Alpha is not able to compute it.

But if we enter small particular values of n into Wolfram Alpha, then it does give the answer, although sometimes only in decimal form.

29.3 Answers to Selected Examples

1. Although it is not exactly in the correct form, completing the square should get it closer. Indeed, factoring and completing the square gives

$$\int \frac{dx}{(4x^2+4x+1)\sqrt{4x^2+4x-3}} = \int \frac{dx}{(2x+1)^2\sqrt{(2x+1)^2-4}}.$$

Now, making the u-substitution $u = 2x+1$ (hence $dx = \frac{1}{2}\, du$) gives

$$\begin{aligned}\int \frac{dx}{(2x+1)^2\sqrt{(2x+1)^2-4}} &= \frac{1}{2}\int \frac{du}{u^2\sqrt{u^2-4}} \\ &= \frac{1}{2}\frac{\sqrt{u^2-4}}{4u} + C \\ &= \frac{1}{2} \cdot \frac{\sqrt{(2x+1)^2-4}}{4\,(2x+1)} + C.\end{aligned}$$

(Return)

2. Reducing using the second formula, and then using the first formula gives

$$\begin{aligned}\int sec^3 x &= \frac{\sec x \tan x}{2} + \frac{1}{2}\int \sec x \, dx \\ &= \frac{\sec x \tan x}{2} + \frac{1}{2}\ln|\sec x + \tan x| + C.\end{aligned}$$

(Return)

30 Simple Areas

We know the basic standard formulae for the area of basic shapes, but why are they true? From the point of view of calculus, area A is the integral of dA, the *area element*.

In this chapter, we will use the following procedure to determine a quantity U:
1. Determine the differential element dU.
2. Integrate to compute $U = \int dU$.

30.1 Length of an interval

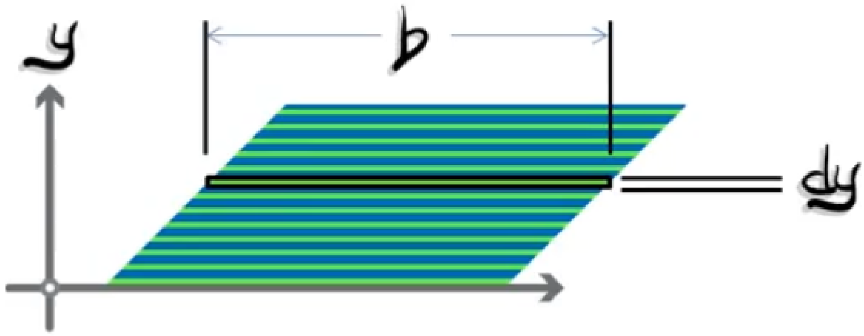
Before getting to areas, first consider how this method works for computing the length L of the interval from a to b. If the length is denoted L, then the length element will be denoted dL, and $L = \int dL$. In this context, the appropriate length element would be dx if we're working along the x-axis.

So, we want to integrate dx as x goes from a to b.
The length,

$$\begin{aligned} L &= \int dL \\ &= \int_{x=a}^{b} dx \\ &= x\Big|_{x=a}^{b} \\ &= b - a \end{aligned}$$

30.2 Parallelogram

The formula for the area of a parallelogram is base $\times$ height (bh). Consider the following rearrangement into differential elements, where we carve the parallelogram into parallel horizontal strips of width b and height dy, where y is the y-axis.



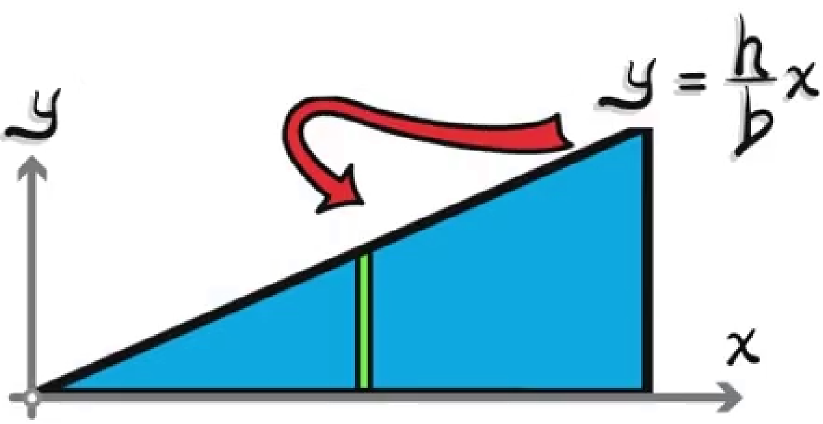
In this case, the area element, $dA = b\,dy$, is the area of this infinitesimal rectangle. The limits on y should go from 0 to the height, h of the parallelogram.
The area,

$$\begin{aligned} A &= \int dA \\ &= \int_{y=0}^{h} b\,dy \\ &= by\Big|_{y=0}^{h} \\ &= bh \end{aligned}$$

We have our familiar answer bh. This means that we've done a rearrangement in terms of infinitesimal strips. Shearing that parallelogram preserves the area element and hence, the area. That is why a parallelogram has the same area as the corresponding rectangle.

30.3 Triangle

The formula for the area of a triangle is $\frac{1}{2}\times$ base $\times$ height ($\frac{1}{2}bh$). Let's think in terms of a differential area element. Given the fact that we can shear and preserve the area element, and thus the area, let's present our triangle as having a hypotenuse modeled by the line $y = \frac{h}{b}x$.



To compute the area element, let's use a vertical strip.

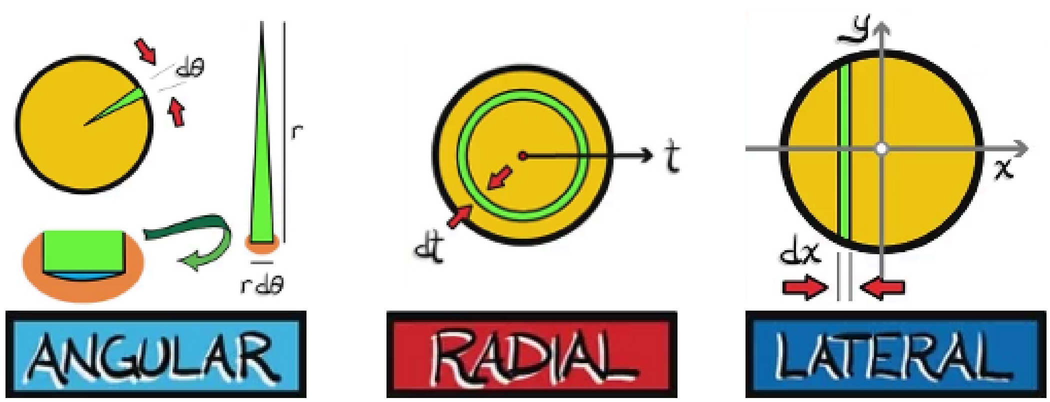
$$dA = \frac{h}{b}x\, dx$$

where the height of that vertical strip is $\frac{h}{b}x$ and the width is the length element dx.
The area,

$$\begin{aligned} A &= \int dA \\ &= \int_{x=0}^{b} \frac{h}{b}x\, dx \\ &= \frac{h}{b}\frac{x^2}{2}\Big|_{x=0}^{b} \\ &= \frac{hb^2}{2b} \\ &= \frac{1}{2}bh \end{aligned}$$

30.4 Disc

We will use three ways to find the area of a circular disc of radius r:
1. Using an angular area element.
2. Using a radial variable.
3. Using a lateral, or a vertical rectangular strip.



1. **Angular** In this case, we'll use an angular area element. We will take a wedge with angle $d\theta$. If we look at that close up, it's modeled fairly well as a triangle. It's not a perfect triangle, there's a bit of curvature at the end. This is a triangle with two sides of length r whose included angle is $d\theta$. Such a triangle has area $\frac{1}{2}r^2\sin(d\theta) \approx \frac{1}{2}r^2 d\theta$, since $d\theta$ is very small. If we model that as a triangle with height r, and width $r\,d\theta$, we can ignore the higher order terms in the Taylor expansion of that area. We obtain an area element $dA = \frac{1}{2}r(r\,d\theta)$.

   Integrating to get the area, θ has to spin all the way around the circle from 0 to 2π.

The area,

$$
\begin{aligned}
A &= \int dA \\
&= \int_{\theta=0}^{2\pi} \frac{1}{2} r^2 \, d\theta \\
&= \frac{1}{2} r^2 \int_{\theta=0}^{2\pi} d\theta \\
&= \frac{1}{2} r^2 \theta \Big|_{\theta=0}^{2\pi} \\
&= \frac{1}{2} r^2 (2\pi) \\
&= \pi r^2.
\end{aligned}
$$

2. **Radial** Let's consider a radial variable. We can sweep out the area of the circular disk using annuli with a radial coordinate t. Then, we're looking at an annular strip of width dt. The corresponding area element is the circumference $(2\pi t)$ $\times$ thickness (dt).

   $$dA = 2\pi t \, dt$$

   Integrating this from 0 to the radius r gives us the area.

   $$
   \begin{aligned}
   A &= \int dA \\
   &= \int_{t=0}^{r} 2\pi t \, dt \\
   &= \pi t^2 \Big|_{t=0}^{r} \\
   &= \pi r^2.
   \end{aligned}
   $$

3. **Lateral** We will use a vertical rectangular strip. Again, it is not a perfect rectangle and there's a little bit of curvature at the end. But, these are higher order terms, and we just care about the differential element. So, using a vertical strip with width dx, and knowing that the formula for the boundary circle is $x^2 + y^2 = r^2$, we solve for y along the upper and lower branches.

   $$y = \pm\sqrt{r^2 - x^2}$$

   We then obtain an area element that is the area of this rectangular strip.

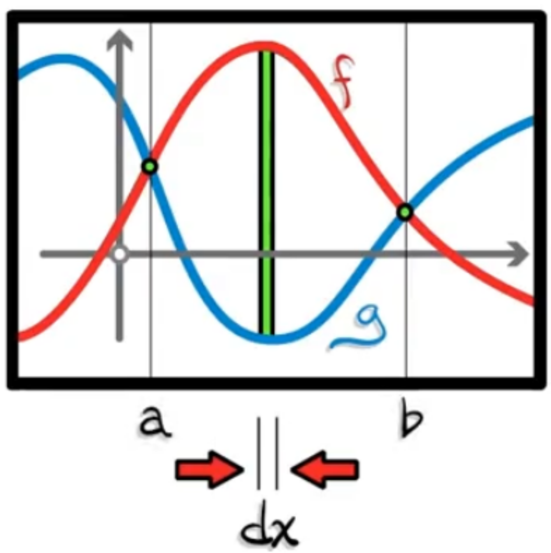
   $$dA = 2\sqrt{r^2 - x^2} \, dx$$

   In the case of strips, assume the circle is centered at the origin, and let x keep track of where the strip intersects the x-axis. Thus, x ranges from $-r$ to r. Integrating, and using a trigonometric substitution

$x = r \sin u$ gives

$$\begin{aligned} A &= \int dA \\ &= \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx \\ &= 2\int_{-\pi/2}^{\pi/2} \sqrt{r^2(1-\sin^2 u)}\; r\cos u\, du \\ &= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2 u\, du \\ &= 2r^2 \int_{-\pi/2}^{\pi/2} \frac{1}{2}(1+\cos(2u))du \\ &= r^2(u + \frac{1}{2}\sin\ 2u)\Big|_{-\pi/2}^{\pi/2} \\ &= \pi r^2 \end{aligned}$$

30.5 The area between two curves



Let's say the f is on top and the g is below. Then as we sweep a vertical strip from left to right, we obtain the area. In this case, the area element is a vertical rectangle of width dx and of height $f(x) - g(x)$, the length of the interval between the two.

$$dA = (f(x) - g(x))dx$$

The general formula for the area between two curves $f(x)$ and $g(x)$,

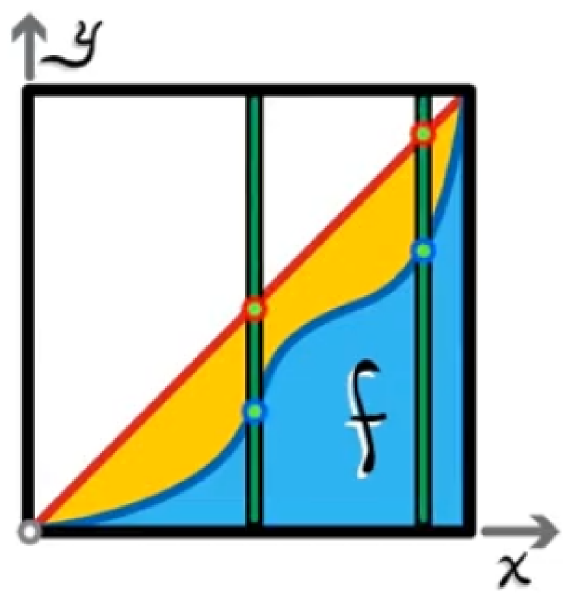
$$
\begin{aligned}
A &= \int dA \\
&= \int_{x=a}^{b} (f(x) - g(x)) dx
\end{aligned}
$$

Example

Find the area of the region bounded above by $f(x) = 4 + x - x^2$ and below by $g(x) = 1 - x$. (See Answer 1)

30.6 Gini Index (An application of area formula)

In economics, this ratio is used to quantify income inequality in a population.



Let $f(x)$ = Fraction of total income earned by the lowest x fraction of the populace, $0 < x < 1$.

The Gini index quantifies how far f is from a "flat" distribution. This means that f(0) = 0, f(1) = 1.

$$f$$

is probably going to be below the flat distribution where $y = x$, the lowest x fraction earns the lowest x fraction.

The Gini index, $G(f)$ is measuring the difference between these two distributions, in terms of area. It's the ratio of the area between the flat distribution $y = x$ and the given population's income distribution $y = f(x)$. One normalizes that by the area between the flat distribution $y = x$ and $y = 0$, namely the area of that triangle, or

$\frac{1}{2}$.

$$G(f) = \frac{\text{Area between the } y = x \text{ and } y = f(x)}{\text{Area between } y = x \text{ and } y = 0}$$
$$= 2\int_{x=0}^{1} (x - f(x))\, dx.$$

Example

Compute G for a power law distribution $f(x) = x^n$. (See Answer 2)

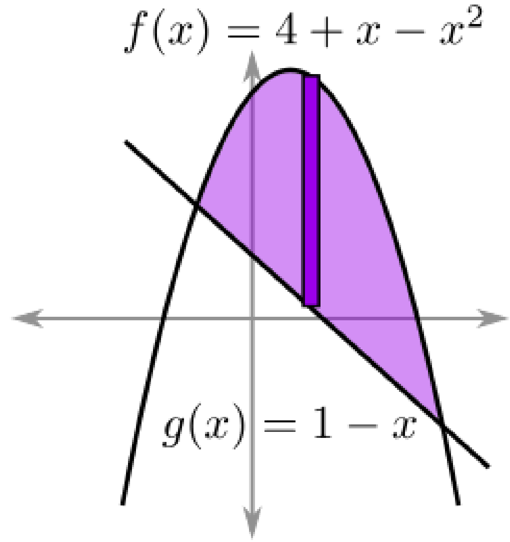
The Gini Index doesn't tell you the income distribution, but we could approximate it in the assumption of a power law. For example, in the year 2010, in the state of New York in USA, the Gini Index was very close to $\frac{1}{2}$. If we assume that it went by a power law distribution, that would imply a cubic distribution of income.

30.7 EXERCISES

- What is the area between the curve $f(x) = \sin^3 x$ and the x-axis from $x = 0$ to $x = \dfrac{\pi}{3}$?
- Find the area of the bounded region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$.
- What is the area between the curve $y = \sin x$ and the x-axis for $0 \leq x \leq \pi$?
- Calculate the Gini index of a country where the fraction of total income earned by the lowest x fraction of the populace is given by
$$f(x) = \frac{2}{5}x^2 + \frac{3}{5}x^3.$$
- Compute the area between the curves $f(x) = e^x \sec^2 x$ and $g(x) = e^x \tan^2 x$ for $0 \leq x \leq \pi$.
- Consider a cone of height h with base a circular disc of radius r. Let's compute the "surface area" – the area of the "outside" of the cone, not including the bottom. Following how we computed the area of a circular disc (which is, indeed, such a cone with $h = 0$), we can decompose its area into infinitesimal triangles with base $r d\theta$ and height the slant length $L = \sqrt{h^2 + r^2}$. The area element dA is then the area of this infinitesimal triangle. Integrating dA from $\theta = 0$ to $\theta = 2\pi$ gives the "surface area" of the cone. What is its value?
- Compute the area between the curves $\sin(x)$ and $\cos(x)$ from $x = 0$ to $x = \pi/2$.
- Compute the area of a triangle with vertices at (0,0), (2,1), (3,6)

30.8 Answers to Selected Examples

1. The logical choice for area element is a vertical strip:



The height of this strip is $f(x) - g(x) = 3 + 2x - x^2$, and the width of the strip is dx. So the area element is $dA = (3 + 2x - x^2)dx$. To find the intersection points, set the curves equal, which gives $1 - x = 4 + x - x^2$. This implies $x^2 - 2x - 3 = 0$, which factors to $(x + 1)(x - 3) = 0$. Thus, the intersections are $x = -1$ and $x = 3$. It follows that

$$\begin{aligned} A &= \int dA \\ &= \int_{-1}^{3} (3 + 2x - x^2) dx \\ &= \left(3x + x^2 - \frac{1}{3}x^3\right)\Bigg|_{-1}^{3} \\ &= (9 + 9 - 9) - \left(-3 + 1 + \frac{1}{3}\right) \\ &= \frac{32}{3}. \end{aligned}$$

(Return)

2.

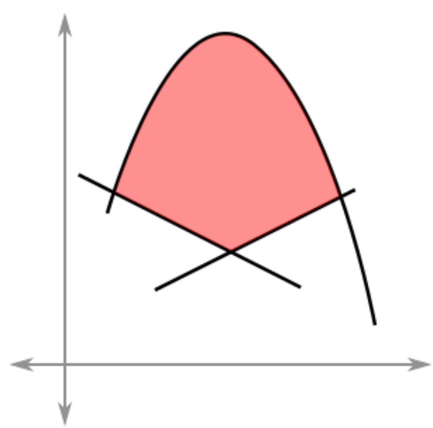
$$\begin{aligned} G(f) &= 2\int_{x=0}^{1} (x - f(x))\, dx \\ &= 2\int_{x=0}^{1} (x - x^n)\, dx \\ &= 2\left(\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right)\Bigg|_{x=0}^{1} \\ &= 1 - \frac{2}{n+1} \\ &= \frac{n-1}{n+1} \end{aligned}$$

(Return)

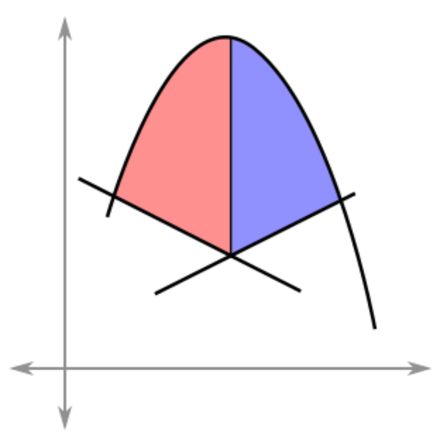
31 Complex Areas

31.1 Complex regions

Some regions in the plane are more complicated and cannot be evaluated with a single integral. This happens when the area element is not bounded by the same curves throughout the region. For instance, consider the region bounded by a parabola and two lines:

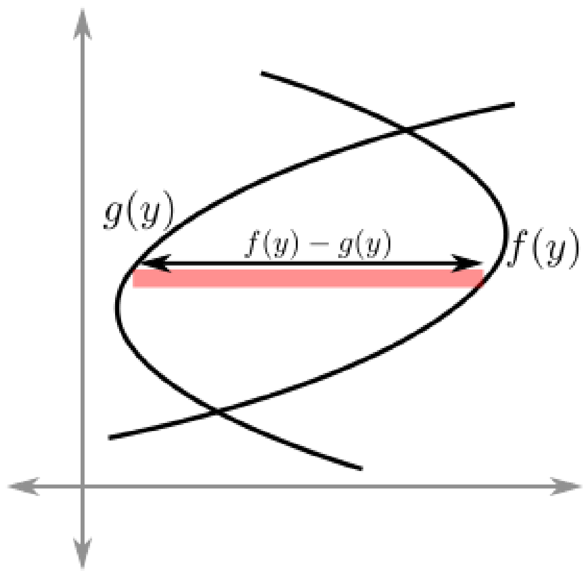


In this case, the only way to find the area of the region is to divide it into regions which can be integrated separately:



31.2 Horizontal strips

Other regions are difficult to integrate using vertical strips as the area element, but work well with horizontal strips as the area element. For example, consider the following region bounded on the left by $x = g(y)$ and on the right by $x = f(y)$:



In this case, the area of a horizontal strip is a function of y, namely $(f(y) - g(y))dy$, where $x = f(y)$ is the curve on the right and $x = g(y)$ is the curve on the left.

Example

Find the area between the curves

$$y - x = 0$$
$$y^2 + x = 2.$$

(See Answer 1)

Example

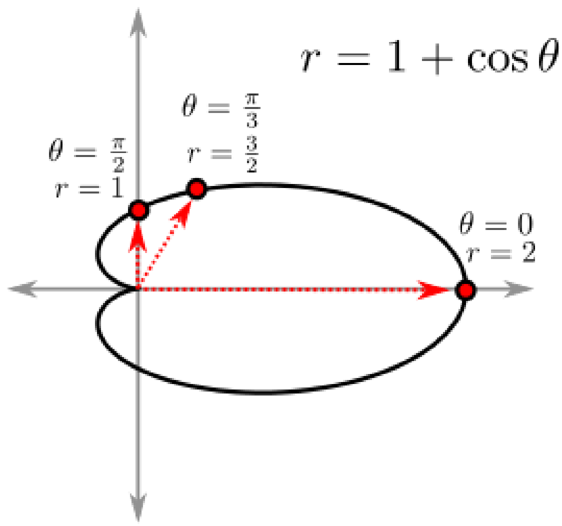
Find the area of the region bounded by $x = 3y$ and $x = y^2 - 4$. (See Answer 2)

Example

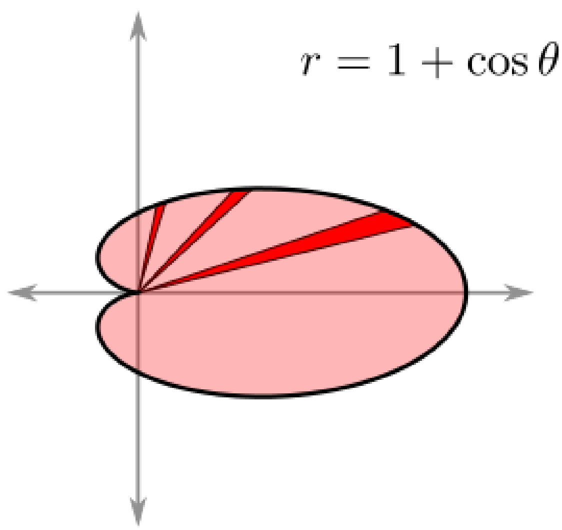
Find the area of the region bounded by $y = \ln(x)$ and the lines $y = 0$, $y = 1$, and $x = 0$. (See Answer 3)

31.3 Polar shapes

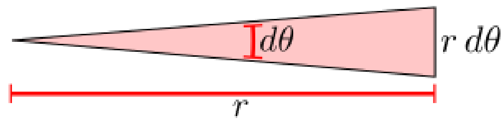
A polar shape is the graph of a polar function $r = f(\theta)$. Here, the input to the function is θ, which is the angle formed with the positive x-axis (known as the *pole*). The output r is the distance from the origin (or *radial* *distance*). For example, the following shows the graph of the polar function $r = 1 + \cos\theta$, which is known as a *cardioid*:



The area of such a region is not usually easy to compute by integrating with respect to x or y (for one thing, the polar equation would need to be expressed in terms of x and y first!). Instead, the way to integrate over such regions is to use a *polar area element*, which is a wedge shaped region. Here are several examples of the polar area element for various values of θ:



To compute what the polar area element is in terms of θ, r, and $d\theta$, note that the region is roughly triangular (the curved portion at the base of the triangle can be ignored since it is a higher order term). The angle at the tip of the triangle is $d\theta$, the height of the triangle is r, and the base of the triangle is $r\,d\theta$:



Thus, the polar area element is

$$dA = \frac{1}{2}(r)(r\,d\theta) = \frac{1}{2}(f(\theta))^2\,d\theta,$$

since $r = f(\theta)$. Thus, the area of a polar region defined by $r = f(\theta)$, where $a \le \theta \le b$, is

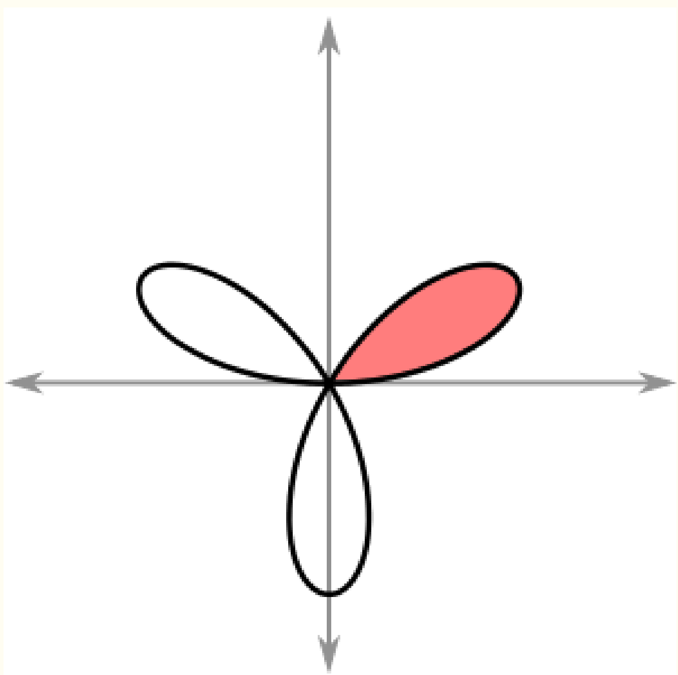
$$A = \int_{\theta=a}^{b} \frac{1}{2}(f(\theta))^2\,d\theta.$$

Example

Compute the area of the cardioid $r = 1 + \cos\theta$. (See Answer 4)

Example

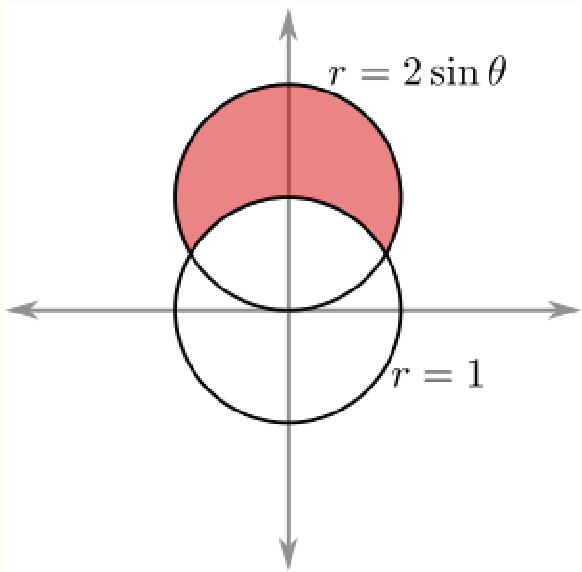
Find the area of a single petal of the polar curve $r = \sin(3\theta)$:



Hint: To find the bounds on θ, compute when $r = 0$. (See Answer 5)

Example

Find the area inside the circle $r = 2\sin\theta$ and outside the circle $r = 1$:



(See Answer 6)

31.4 EXERCISES

- Find the area enclosed by the curves $y = 1$, $x = 1$, and $y = \ln x$.
- Find the area of the bounded region enclosed by the x-axis, the lines $x = 1$ and $x = 2$ and the hyperbola $xy = 1$.
- Compute the area in the bounded (i.e., finite) regions between $y = x(x-1)(x-2)$ and the x-axis.
- Find the area of the sector of a circular disc of radius r (centered at the origin) given by $1 \leq \theta \leq 3$ (as usual, θ is in radians).
- Use polar coordinates to find an area within $r = 3 - 2cos(\theta)$ and outside $r = 3$.
- Find the area of the overlap between two circles of radius 2 that pass through each others' centers. You can do so with either cartesian or polar coordinates (though one might be easier than the other!).
- Find the area bound by the curves $y = \cos^2 x$ and $y = \dfrac{8x^2}{\pi^2}$.
- Kepler's First Law states that the orbit of every planet is an ellipse with the Sun at one of its two foci. If we think of the Sun as being situated at the origin, we can describe the orbit with the equation:

$$r = \frac{p}{1 + \varepsilon \cos\theta}$$

  The point at which the planet is closest to the Sun (the perihelion) corresponds to $\theta = 0$, while the planet is furthest away from the Sun at $\theta = \pi$ (the aphelion). Knowing the distance between the Sun and the planet at these two points would allow you to fix the values of the constants p and ε. Notice that $\varepsilon = 0$ describes a perfect circle, so that the "eccentricity" ε measures how far the orbit is from being a circle.

Kepler's Second Law states that the line joining a planet and the Sun sweeps out equal areas during equal intervals of time. Another way of expressing this fact is by saying that the "areal velocity"

$$v_A = \frac{dA}{dt}$$

of that line is constant in time.

Express the area element dA in terms of the angle element $d\theta$ and use Kepler's Second Law to deduce the differential equation governing the time evolution of θ.

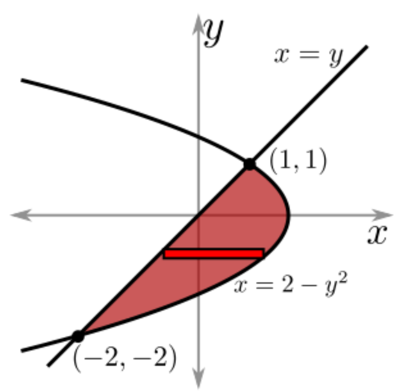
- Let C_1 be the circle given by $r = \sin(\theta)$. Let C_2 be the circle given by $r = \cos(\theta)$. Find the area of region in C_2 that is not in C_1.

31.5 Answers to Selected Exercises

1. Expressing these curves as functions of y, we find

$$\begin{aligned} x &= y \\ x &= 2 - y^2. \end{aligned}$$

Graphing these curves, one finds the bounded region:



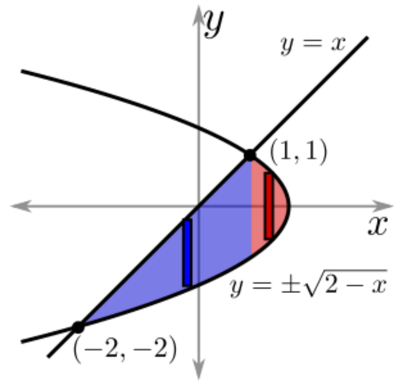
To find the intersections, set the curves equal to one another. This gives

$$y = 2 - y^2.$$

A rearranging and factoring gives

$$y^2 + y - 2 = (y-1)(y+2) = 0,$$

and so we find that the intersection points are at $y = 1$ and $y = -2$ (the x-coordinates are the same, since they are on the line $x = y$). Note that using a vertical rectangle as the area element here would not be so easy, because the area element depends on the value of x. Sometimes the strip goes from the parabola below to the line above, as shown in blue, and sometimes the strip goes from parabola to parabola, shown in red:



In particular, the area element for a vertical strip is

$$dA = \begin{cases} (x + \sqrt{2-x})\,dx & \text{if } -2 \le x \le 1 \\ 2\sqrt{2-x}\,dx & \text{if } 1 \le x \le 2 \end{cases}$$

But using a horizontal strip as the area element works much better because throughout the region the strip is always going from the line on the left to the parabola on the right. So using a horizontal strip gives the area element

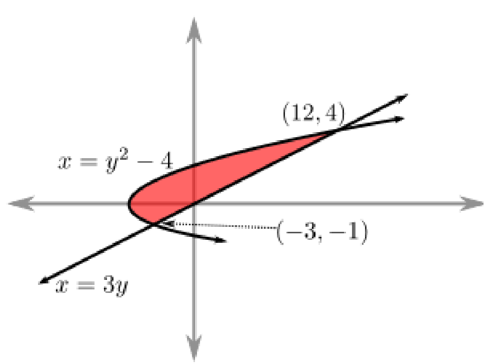
$$dA = ((2 - y^2) - y)\,dy.$$

Integrating this over the range of $-2 \le y \le 1$ gives the area:

$$\begin{aligned} A &= \int dA \\ &= \int_{y=-2}^{1} 2 - y^2 - y\,dy \\ &= 2y - \frac{y^3}{3} - \frac{y^2}{2}\Big|_{y=-2}^{1} \\ &= \left(2 - \frac{1}{3} - \frac{1}{2}\right) - \left(-4 + \frac{8}{3} - 2\right) \\ &= \frac{9}{2}. \end{aligned}$$

(Return)

2. The region looks roughly as in the following:



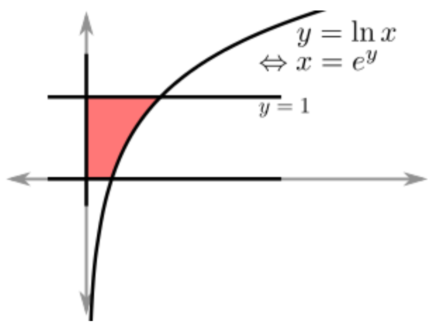
By setting $3y = y^2 - 4$, collecting like terms, and factoring, one finds the intersection points at $y = -1$ and $y = 4$, as indicated in the figure. The area element is a horizontal rectangle, which has area $dA = (3y - (y^2 - 4))dy$.

Thus, the area between the curves is

$$\begin{aligned} A &= \int dA \\ &= \int_{-1}^{4} (3y - y^2 + 4)dy \\ &= \frac{3}{2}y^2 - \frac{1}{3}y^3 + 4y\Big|_{-1}^{4} \\ &= \frac{125}{6} \end{aligned}$$

(Return)

3. The region looks roughly like the following:



Note that using vertical rectangles would not be ideal because this would require two integrals (for x from 0 to 1 and from 1 to e). Instead, one can express the curve $y = \ln x$ as $x = e^y$. Now, using horizontal rectangles gives an area element of $dA = e^y dy$. Thus

$$\begin{aligned} A &= \int dA \\ &= \int_0^1 e^y dy \\ &= e^y\Big|_0^1 \\ &= e - 1. \end{aligned}$$

(Return)

4. In this case, $f(\theta) = 1 + \cos\theta$, and so the area element is

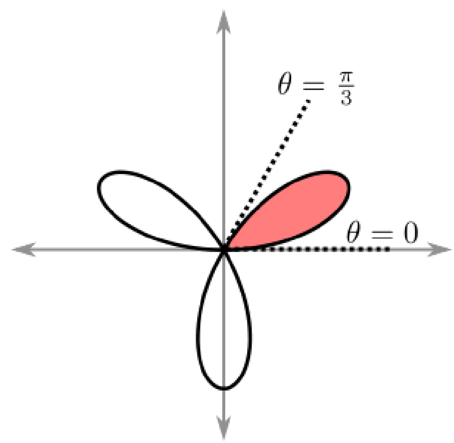
$$\begin{aligned} dA &= \frac{1}{2}(1 + \cos\theta)^2 \, d\theta \\ &= \frac{1}{2}(1 + 2\cos\theta + \cos^2\theta) \, d\theta \end{aligned}$$

Because θ ranges from 0 to 2π to trace out the entire cardioid, it follows that the area is

$$\begin{aligned}
A &= \int dA \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta \\
&= \frac{1}{2}\int_{\theta=0}^{2\pi} \left(1 + 2\cos\theta + \frac{1}{2}(1+\cos(2\theta))\right) d\theta \\
&= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\right)\Bigg|_{\theta=0}^{2\pi} \\
&= \frac{3\pi}{2}.
\end{aligned}$$

(Return)

5. The area element is $dA = \frac{1}{2}\sin^2(3\theta)\, d\theta$. To find the bounds on θ, set $r = 0$, which gives $\sin(3\theta) = 0$. The smallest values of θ for which this occurs is $\theta = 0$ and $\theta = \frac{\pi}{3}$:



Thus, the area of a single petal is

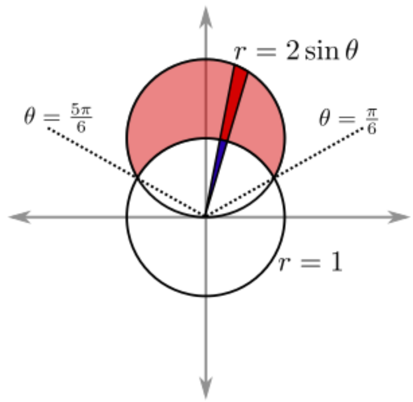
$$\begin{aligned}
A &= \int dA \\
&= \int_{\theta=0}^{\pi/3} \frac{1}{2}\sin^2(3\theta)\, d\theta \\
&= \frac{1}{2}\int_{\theta=0}^{\pi/3} \frac{1}{2}(1 - \cos(6\theta))\, d\theta \\
&= \frac{1}{4}\left(\theta - \frac{1}{6}\sin(6\theta)\right)\Bigg|_{\theta=0}^{\pi/3} \\
&= \frac{\pi}{12}
\end{aligned}$$

(Return)

6. First, we find the intersections by setting the curves equal, which gives

$$2\sin\theta = 1 \quad \Rightarrow \quad \sin\theta = \frac{1}{2},$$

and so the intersections are at $\frac{\pi}{6}$ and $\frac{5\pi}{6}$. The area element of the region is the polar area element of the circle $r = 2\sin\theta$ minus the polar area element of the circle $r = 1$:



So we have that

$$dA = \left(\frac{1}{2}(2\sin\theta)^2 - \frac{1}{2}(1)^2\right)\, d\theta.$$

Thus, the area is

$$\begin{aligned}
A &= \int dA \\
&= \frac{1}{2}\int_{\theta=\pi/6}^{5\pi/6} (4\sin^2\theta - 1)\, d\theta \\
&= \frac{1}{2}\int_{\theta=\pi/6}^{5\pi/6} 2(1-\cos(2\theta)) - 1\, d\theta \\
&= \frac{1}{2}\left(\theta - \sin(2\theta)\right)\Big|_{\pi/6}^{5\pi/6} \\
&= \frac{\pi}{3} + \frac{\sqrt{3}}{2}.
\end{aligned}$$

From the second to the third line above, we used the power reduction formula for sine:

$$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta)).$$

(Return)

32 Volumes

32.1 Finding the volume element

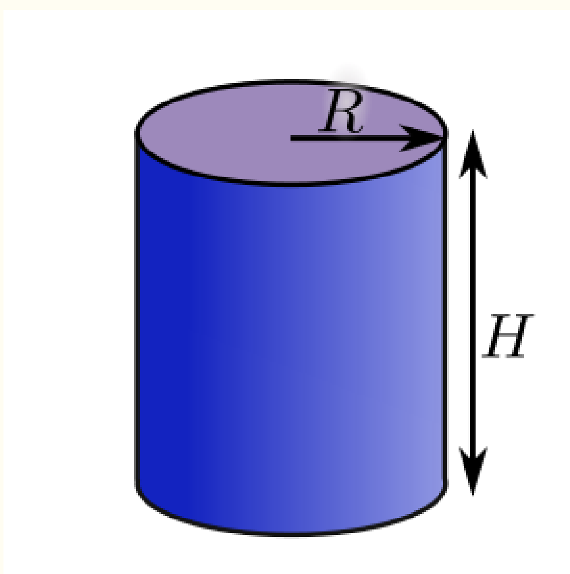
Just as area was computed by finding the area element and integrating, volume is computed by determining the volume element (i.e. the volume of a slice) and then integrating:

$$V = \int dV.$$

The difficulty is in finding a suitable volume element dV. Once that is chosen, the rest is a matter of evaluating the integral.

Example

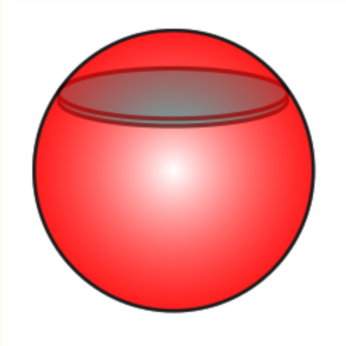
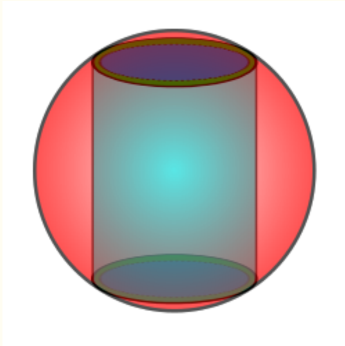
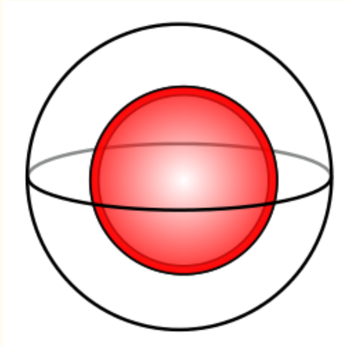
Compute the volume of a cylinder of radius R and height H using several different volume elements dV:



(See Answer 1)

Example

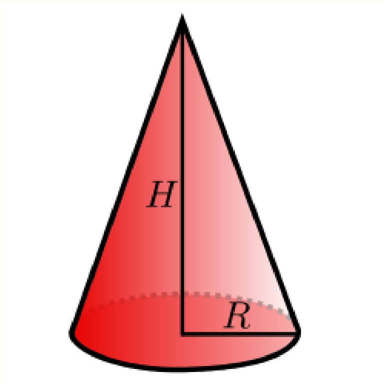
Find the volume of a sphere of radius R. First, by using discs as the volume element (shown on left below). Then use cylindrical shells as the volume element (shown in the middle below). Finally, use a spherical shell for the volume element, as shown in the third diagram.

(See Answer 2)

Example

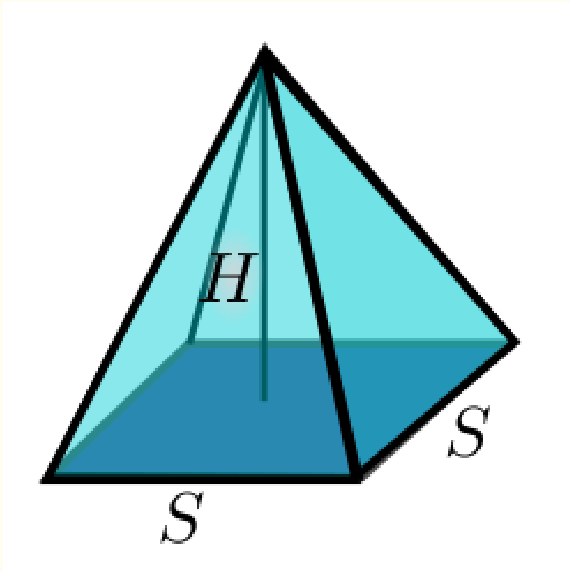
Find the volume of a cone of base radius R and height H.



(See Answer 3)

Example

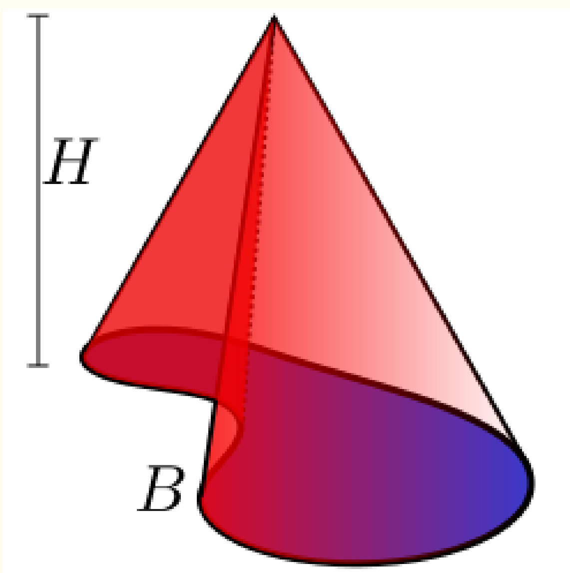
Find the volume of a square pyramid of base edge S and height H.



(See Answer 4)

Example

Show that the volume of a generalized cone of base area B and height H is $\frac{1}{3}BH$. Explain the reason for the factor of $\frac{1}{3}$.



(See Answer 5)

32.2 EXERCISES

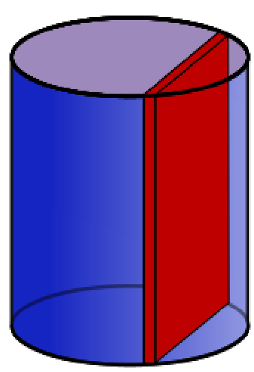
- Find the volume of the following solid: for $1 \leq x < \infty$, the intersection of the this solid with the plane perpendicular to the x-axis is a circular disc of radius e^{-x}.
- The base of a solid is given by the region lying between the y-axis, the parabola $y = x^2$, and the line $y = 16$ in the first quadrant. Its cross-sections perpendicular to the y-axis are equilateral triangles. Find the volume of this solid.
- The base of a solid is given by the region lying between the y-axis, the parabola $y = x^2$, and the line $y = 4$. Its cross-sections perpendicular to the y-axis are squares. Find the volume of this solid.
- Find the volume of the solid whose base is the region enclosed by the curve $y = \sin x$ and the x-axis from $x = 0$ to $x = \pi$ and whose cross-sections perpendicular to the x-axis are semicircles.
- Consider a cone of height h over a circular base of radius r. We computed the volume by slicing parallel to the base. What happens if instead we slice orthogonal to the base? What is the volume element obtained by taking a wedge at angle θ of thickness $d\theta$?
- Consider the following solid, defined in terms of polar coordinates: $0 \leq r \leq R$; $0 \leq \theta \leq 2\pi$; $0 \leq z \leq r$. Can you describe this shape? Compute its volume.
- Consider the following solid, defined in terms of polar coordinates: $0 \leq r \leq R$; $0 \leq \theta \leq 2\pi$; $0 \leq z \leq \theta$. Can you describe this shape? Compute its volume.
- Challenge: compute the volume intersection of the (infinite) cylinders of radius R centered along the x and y axes in 3-d. That is, compute the volume of intersection of

$$x^2 + z^2 \leq R^2$$
$$x^2 + z^2 \leq R^2$$

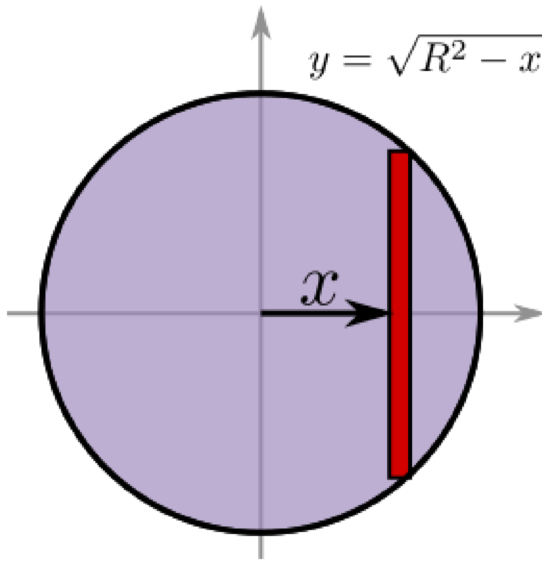
  in the 3-dimensional (x, y, z) space.

32.3 Answers to Selected Examples

1. First, consider making a slice perpendicular to the base of the cylinder:



This gives a rectangular slice whose height is H, the same as the cylinder. The width of the rectangle can be determined by looking at an overhead view of the cylinder. Let x be the distance of the slice from the center of the cylinder (so x ranges from $-R$ to R as the slice sweeps across the cylinder):

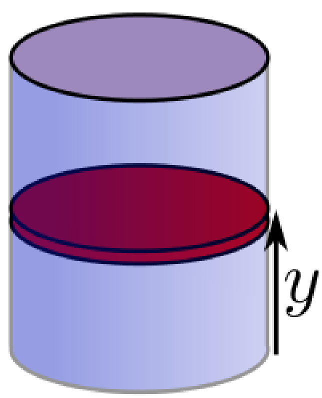


Doing a little algebra, we find that the width of the rectangle is $2\sqrt{R^2 - x^2}$. Finally, the thickness of the slice is dx, and so the volume element in this case is

$$dV = 2H\sqrt{R^2 - x^2}\,dx.$$

Integrating this requires the trigonometric substitution $x = R\sin\theta$. There are easier volume elements we could choose, as we shall see.

Another way to slice is to make cuts parallel to the base of the cylinder. Let y denote the distance of the slice from the base of the cylinder:



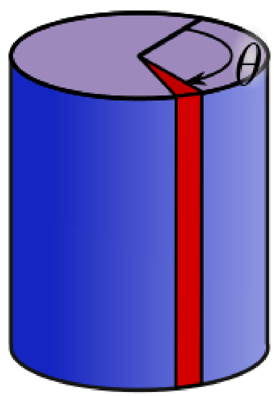
Then each slice is a circle of radius R and thickness dy. Thus

$$dV = \pi R^2\,dy,$$

and y ranges from 0 to H, so the volume is

$$\begin{aligned}
V &= \int dV \\
&= \int_{y=0}^{H} \pi R^2 \, dy \\
&= \pi R^2 y \Big|_{y=0}^{H} \\
&= \pi R^2 H.
\end{aligned}$$

Another possible choice is a wedge shaped volume element. Let θ be the angle that the wedge forms with a fixed axis (so θ ranges from 0 to 2π):



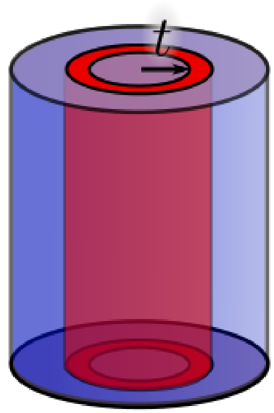
Here, the area of the sector of the circle is $\frac{1}{2}R^2 \, d\theta$. Thus the volume of the wedge is

$$dV = \frac{1}{2} R^2 H \, d\theta.$$

Thus the volume is

$$\begin{aligned}
V &= \int dV \\
&= \int_{\theta=0}^{2\pi} \frac{1}{2} R^2 H \, d\theta \\
&= \frac{1}{2} R^2 H \theta \Big|_{\theta=0}^{2\pi} \\
&= \pi R^2 H.
\end{aligned}$$

One final option is to use cylindrical shells. Let t be the radius of the shell, so that t ranges from 0 to R as the shells sweep through the cylinder.



The height of the cylindrical shell is H and the thickness of the shell is dt. Recalling that the lateral surface area of a cylinder is $2\pi RH$, we have

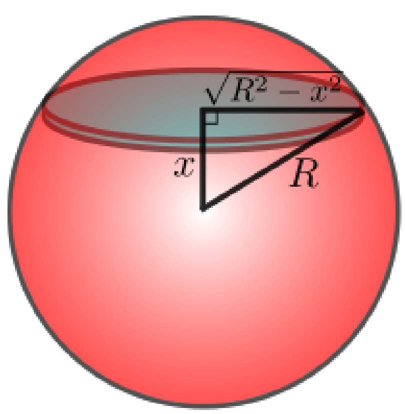
$$dV = 2\pi tH\,dt.$$

Integrating gives that the volumes is

$$\begin{aligned} V &= \int dV \\ &= \int_{t=0}^{R} 2\pi tH\,dt \\ &= 2\pi H \frac{t^2}{2}\bigg|_{t=0}^{R} \\ &= \pi R^2 H. \end{aligned}$$

(Return)

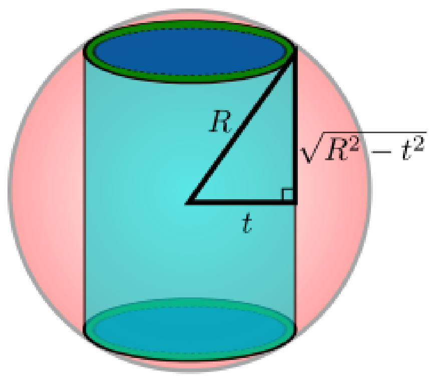
2. Let x be the distance from the center of the disc to the center of the sphere (so x ranges from $-R$ to R as the discs sweep across the sphere). Then drawing a right triangle shows that the radius of the disc is $\sqrt{R^2 - x^2}$ (since the radius of the sphere is R). See the diagram below:



The thickness of the disc is dx, and so the volume of the disc is $\pi(\sqrt{R^2 - x^2})^2 dx$ (the area of the disc times its thickness), and so the volume of the sphere is

$$
\begin{aligned}
V &= \int dV \\
&= \int_{x=-R}^{R} \pi(R^2 - x^2)dx \\
&= \pi \left(R^2 x - \frac{x^3}{3} \right) \Big|_{x=-R}^{R} \\
&= \pi \left(\left(R^3 - \frac{R^3}{3} \right) - \left(-R^3 + \frac{R^3}{3} \right) \right) \\
&= \frac{4}{3}\pi R^3.
\end{aligned}
$$

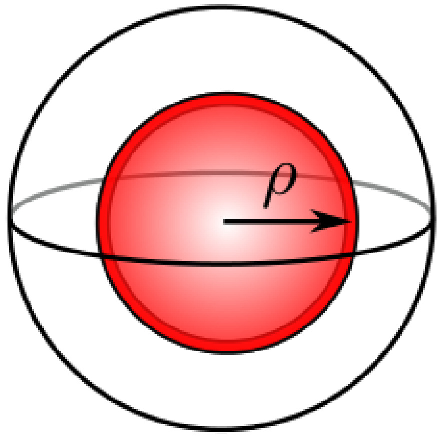
For the cylindrical shell, let t be the radius of the cylinder (so t ranges from 0 to R as the cylinders sweep out the sphere). Then by drawing in a right triangle, one finds that the height of the cylinder is $2\sqrt{R^2 - t^2}$:



Recall that the lateral surface area of a cylinder with radius r and height h is $2\pi r h$. Thus, the lateral surface area of the cylinder is $4\pi t\sqrt{R^2 - t^2}$. The thickness of the shell is dt, and so the volume element is $4\pi t\sqrt{R^2 - t^2}dt$. It follows (after making the u-substitution $u = R^2 - t^2$) that the volume of the sphere is

$$
\begin{aligned}
V &= \int dV \\
&= \int_{t=0}^{R} 4\pi t\sqrt{R^2 - t^2}dt \\
&= 4\pi \int_{u=R^2}^{0} \frac{-1}{2}\sqrt{u}du \\
&= -2\pi \left(\frac{2}{3}u^{3/2} \Big|_{u=R^2}^{0} \right) \\
&= -2\pi \left(0 - \frac{2}{3}R^3 \right) \\
&= \frac{4}{3}\pi R^3.
\end{aligned}
$$

Finally, for the spherical shell, let ρ denote the radius of the spherical shell:



Recall that the surface area of a sphere of radius ρ is $4\pi\rho^2$. Therefore, the volume of the spherical shell (i.e. our volume element) is

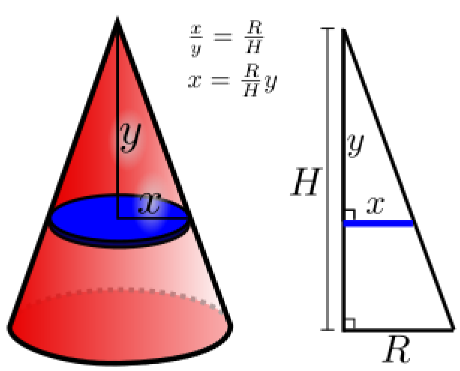
$$dV = 4\pi\rho^2\, d\rho.$$

Note that to sweep over the entire sphere, ρ must range from 0 to R. Therefore,

$$\begin{aligned} V &= \int dV \\ &= \int_{\rho=0}^{R} 4\pi\rho^2\, d\rho \\ &= 4\pi \frac{1}{3}\rho^3 \Big|_{\rho=0}^{R} \\ &= \frac{4}{3}\pi R^3. \end{aligned}$$

(Return)

3. The easiest choice for volume element is a slice parallel to the base of the cone, which gives a disc. Let y be the distance from the tip of the cone to the center of the disc (so y ranges from 0 to H as the disc sweeps across the cone), and x be the radius of the disc:



The volume element is the area of the disc, πx^2, times the thickness of the disc, dy. It remains to find x in terms of y. In the cutaway in the figure on the right above, one sees that by similar triangles, $\frac{x}{y} = \frac{R}{H}$, and so $x = \frac{R}{H}y$. Thus, the volume element is

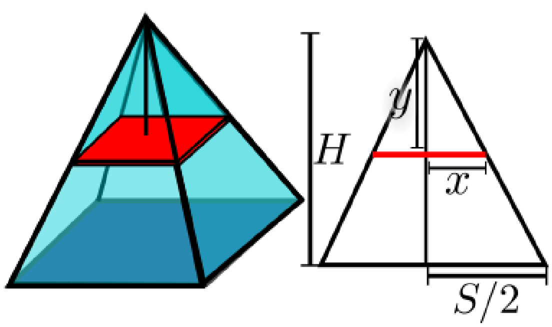
$$\begin{aligned} dV &= \pi x^2 dy \\ &= \pi \left(\frac{R}{H}y\right)^2 dy. \end{aligned}$$

Thus, the volume of the cone is

$$\begin{aligned} V &= \int dV \\ &= \int_{y=0}^{H} \pi \frac{R^2}{H^2} y^2 dy \\ &= \frac{\pi R^2}{H^2} \left(\frac{y^3}{3} \Big|_{y=0}^{H} \right) \\ &= \pi \frac{R^2}{H^2} \cdot \frac{H^3}{3} \\ &= \frac{1}{3} \pi R^2 H. \end{aligned}$$

(Return)

4. Again, use slices parallel to the base. Let y be the distance from the tip of the cone to the center of the slice (so y ranges from 0 to H), and let x be half of the side length of the slice.



As shown in the above cutaway, one finds by similar triangles that $\frac{x}{y} = \frac{S/2}{H}$, and so $x = \frac{S}{2H}y$. Therefore, the area of a slice is $(2x)^2 = \frac{S^2}{H^2}y^2$, and the thickness of a slice is dy, so the volume element is

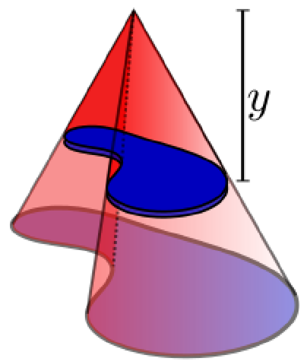
$$dV = \frac{S^2}{H^2} y^2 dy.$$

And so the volume of the pyramid is

$$
\begin{aligned}
V &= \int dV \\
&= \int_{y=0}^{H} \frac{S^2}{H^2} y^2 \, dy \\
&= \frac{S^2}{H^2} \left(\left. \frac{y^3}{3} \right|_{y=0}^{H} \right) \\
&= \frac{S^2}{H^2} \cdot \frac{H^3}{3} \\
&= \frac{S^2 H}{3}.
\end{aligned}
$$

(Return)

5. Let y be the distance from the tip of the cone to the slice.



Because the linear dimensions of the slice grow proportionally with y (e.g. the length of the slice is $\frac{y}{H}$ times the length of the base), the area of the slice will grow proportionally with the square of y. This means that

$$\text{Area of the slice } = \left(\frac{y}{H}\right)^2 B$$

Thus, the volume element is $dV = B\frac{y^2}{H^2}dy$, and it follows that the volume of the cone is

$$
\begin{aligned}
V &= \int dV \\
&= \int_{y=0}^{H} B \frac{y^2}{H^2} dy \\
&= \frac{B}{H^2} \left(\left. \frac{y^3}{3} \right|_{y=0}^{H} \right) \\
&= \frac{B}{H^2} \cdot \frac{H^3}{3} \\
&= \frac{1}{3} BH.
\end{aligned}
$$

The factor of $\frac{1}{3}$ comes from the fact that the volume element is proportional to the square of y, hence the integral has a y^2, which produces a factor of $\frac{1}{3}$ by the power rule.

(Return)

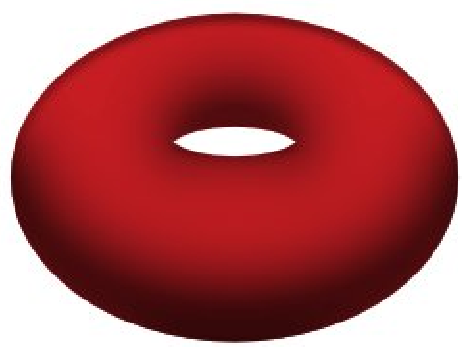
33 Volumes Of Revolution

33.1 Volume element for solid of revolution

Consider a region R in the plane and a line L. The *solid of revolution* of R about the axis L is the solid which results from taking the region R and revolving it around the line L:

(Solid of Revolution Animated GIF)

The result is typically something doughnut shaped. The question of this module is to find the volume of the solid:



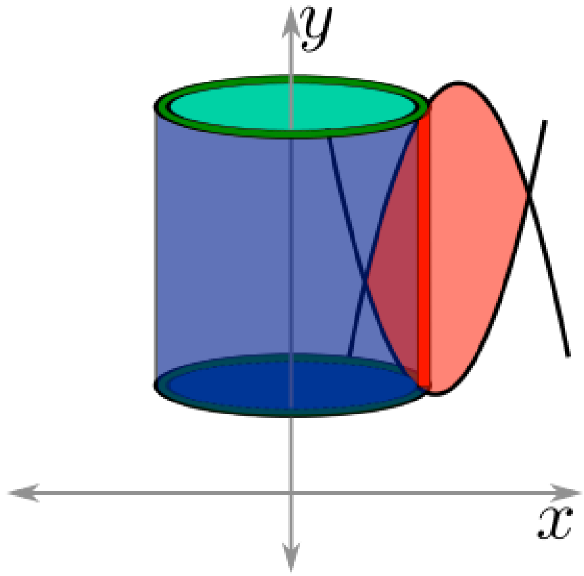
The method is the same as the previous modules: find the volume element (the contribution of a small slice of the region to the total volume) and integrate. Just as area can be computed using vertical or horizontal slices, volume can be computed using corresponding methods: shells or washers, respectively.

The basic outline of finding the volume element for a solid of revolution is

1. Find a convenient area element for the region R in the plane
2. Determine the volume as that area element gets revolved around the axis L.

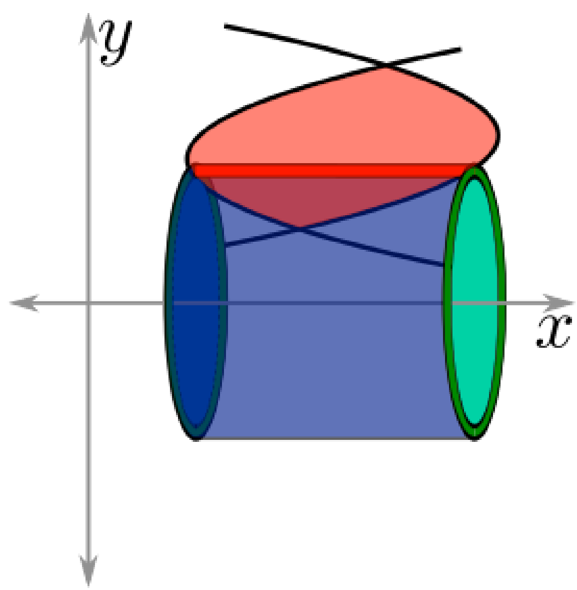
33.2 Cylindrical shells

When the area element is parallel to the axis of rotation, the volume element is a cylindrical shell. Here, the region is bounded by two parabolas. The natural area element for such a region is a vertical rectangle (shown in red). As the region is revolved about the y-axis, the volume element traces out a cylindrical shell, whose volume becomes the volume element of the solid of revolution.



Recall that a cylinder has lateral surface area $2\pi rh$. The thickness of the cylindrical shell is dx (if the axis of rotation is a vertical line) or dy (if the axis of rotation is a horizontal line). Here r and h will generally be functions of x or y (again, depending on whether the axis of rotation is vertical or horizontal).

If a horizontal rectangle is the natural area element (for instance, the region between two horizontal parabolas), and the axis of revolution is the x-axis, then cylindrical shells again arise naturally as the volume element:

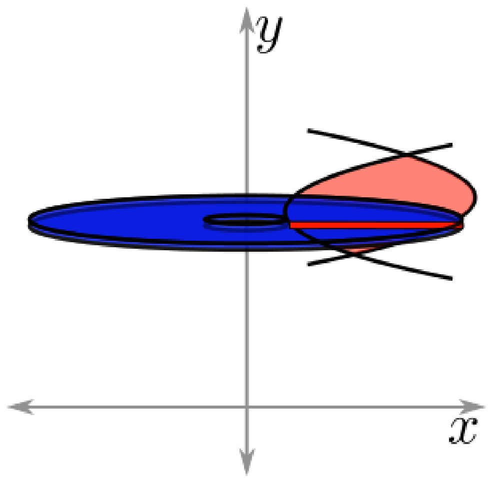


Example

Suppose the region bounded by $y = 3x - x^2$ and $y = x$ is revolved about the y-axis. What is the volume of the resulting solid? (See Answer 1)

33.3 Washers

When the area element is perpendicular to the axis of rotation, the volume element is a washer. So when the area element is a horizontal rectangle (as in a region bounded by horizontal parabolas) and the axis of revolution is vertical, the region traced out by the rectangle is a washer:



A washer is just an annulus (a circle with a circle cut out of it) which has been thickened. The volume of the washer is the area of the annulus times the thickness of the washer. The area of the annulus is $\pi R^2 - \pi r^2$, where R is the radius of the outer circle and r is the radius of the inner circle. The thickness of the washer is dx or dy (depending on the orientation of the washer. Thus, the volume element when using washers is

$$dV = \pi(R^2 - r^2)\, dx \text{ or } dy.$$

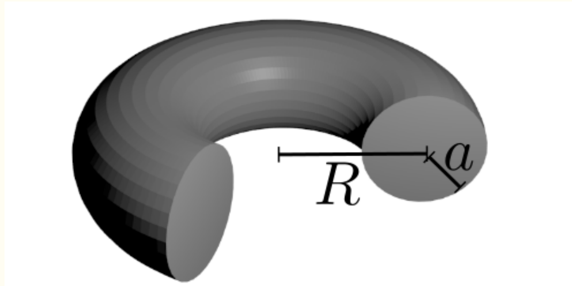
Example

Given the region bounded by $y = 3x - x^2$ and $y = x$, find the volume of the solid resulting from revolving the region about the x-axis. (See Answer 2)

33.4 Additional Examples

Example

Find the volume of a doughnut formed by rotating a disc of radius a about the y-axis. Let the radius of the doughnut be R, as shown in this cutaway:



Use a vertical area element (which leads to a cylindrical shell). (See Answer 3)

Example

Compute the volume of the doughnut again, this time using a horizontal area element (which leads to a washer). (See Answer 4)

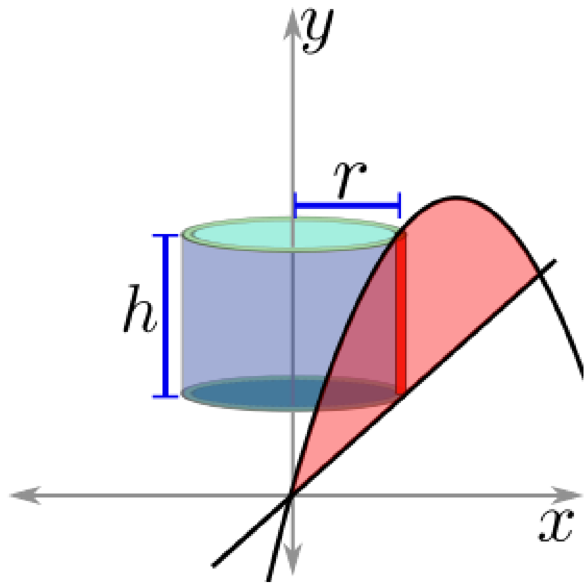
33.5 EXERCISES

- Let D be the region bounded by the curve $y = x^3$, the x-axis, the line $x = 0$ and the line $x = 2$. Find the volume of the region obtained by revolving D about the x-axis.
- Let D be the same region as above. What is the volume of the region formed by rotating this D about the line $x = 3$?
- Let D be the region between the curve $y = -(x-2)^2 + 1$ and the x-axis. Find the volume of the region obtained by revolving D about the y-axis.
- Find the volume obtained by revolving the region between the curves $y = x^3$ and $y = \sqrt[3]{x}$ in the first quadrant about the x-axis.
- Let D be the region under the curve $y = \ln \sqrt{x}$ and above the x-axis from $x = 1$ to $x = e$. Find the volume of the region obtained by revolving D about the x-axis.
- Let D be the region bounded by the graph of $y = 1 - x^4$, the x-axis and the y-axis in the first quadrant. Which of the following integrals can be used to compute the volume of the region obtained by revolving D around the line $x = 5$?
- Challenge: compute the volume of the region obtained by rotating the disc $x^2 + y^2 \leq \epsilon^2$ about the axis given by $y = 1 - x$ for $\epsilon \leq \frac{1}{2}$.
- Let D be the region under the curve $\sqrt{x-1}$ above the x-axis from $x = 1$ to $x = 2$. Compute the volume of solid obtained by rotating D about the x-axis. Compute the volume twice, using both methods.

33.6 Answers to Selected Examples

1. The first step in such a calculation is to draw a decent picture of the region. Then determine whether a vertical or horizontal rectangle would make the best area element. In this case, a vertical rectangle is the best choice.

   Since a vertical rectangle is being revolved about a vertical axis, the result is a cylindrical shell:



The radius of the shell is x (the distance from the y-axis), and the height of the shell is the distance from the top curve to the bottom curve: $h = (3x - x^2) - x = 2x - x^2$. The thickness of the shell is dx. Recalling that the surface area of a cylinder is $2\pi r h$, it follows that the volume element is just the surface area multiplied by the thickness dx:

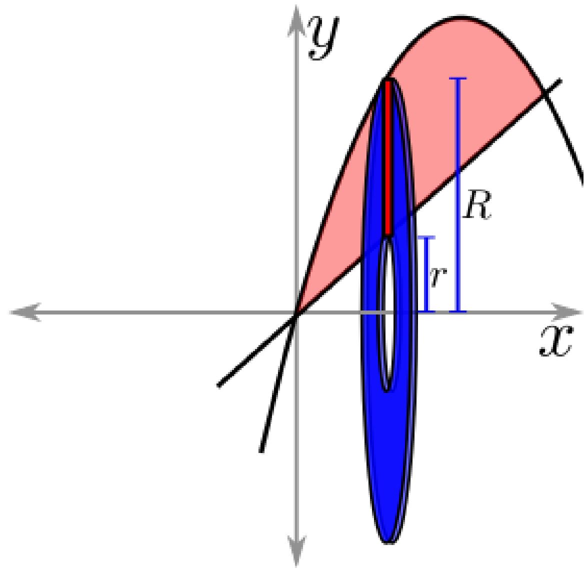
$$\begin{aligned} dV &= 2\pi r h \, dx \\ &= 2\pi x(2x - x^2) \, dx. \end{aligned}$$

A little algebra shows that the intersections occur at $x = 0$ and $x = 2$, so the volume is

$$\begin{aligned} V &= \int dV \\ &= \int_{x=0}^{2} 2\pi x(2x - x^2) \, dx \\ &= 2\pi \int_{x=0}^{2} (2x^2 - x^3) \, dx \\ &= 2\pi \left(\frac{2}{3}x^3 - \frac{1}{4}x^4 \right) \bigg|_{x=0}^{2} \\ &= \frac{8}{3}\pi. \end{aligned}$$

(Return)

2. As in the previous example, the optimal area element is a vertical rectangle. A vertical rectangle revolved about a horizontal axis results in a washer:



The outer radius R is the upper curve: $R = 3x - x^2$, and the inner radius r is the inner curve: $r = x$. The thickness of the washer is dx, and so

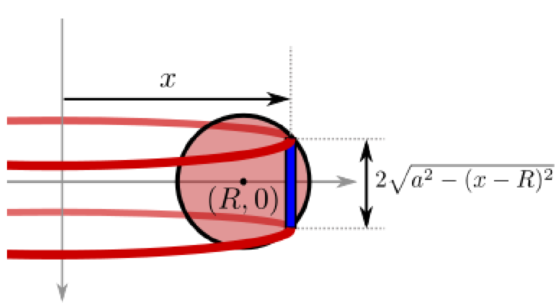
$$\begin{aligned} dV &= \pi(R^2 - r^2)\, dx \\ &= \pi((3x - x^2)^2 - x^2)\, dx. \end{aligned}$$

As in the last example, the intersections are at $x = 0$ and $x = 2$, so

$$\begin{aligned} V &= \int dV \\ &= \int_{x=0}^{2} \pi(9x^2 - 6x^3 + x^4 - x^2)\, dx \\ &= \pi\left(\frac{8}{3}x^3 - \frac{3}{2}x^4 + \frac{1}{5}x^5\right)\Bigg|_{x=0}^{2} \\ &= \frac{56}{15}\pi. \end{aligned}$$

(Return)

3. First, suppose we use a vertical area element. Since we are rotating about a vertical axis, the area element and axis of rotation are parallel, and so the resulting volume element is a cylindrical shell. Let x be the distance of the area element (the rectangle) from the y-axis. This also happens to be the radius of the cylindrical shell:



The equation of the circle is

$$(x - R)^2 + y^2 = a^2,$$

and solving for y gives

$$y = \pm\sqrt{a^2 - (x - R)^2}.$$

Therefore, the height of the area element (and hence the height of the cylindrical shell) is

$$2\sqrt{a^2 - (x - R)^2}.$$

Now, the volume of the cylindrical shell (our volume element) is

$$\begin{aligned} dV &= 2\pi r h\, dx \\ &= 2\pi x(2\sqrt{a^2 - (x - R)^2})\, dx \\ &= 4\pi x\sqrt{a^2 - (x - R)^2}\, dx. \end{aligned}$$

Note that x ranges from $R - a$ to $R + a$ as it sweeps across the circle. Therefore the volume is

$$\begin{aligned} V &= \int dV \\ &= \int_{x=R-a}^{R+a} 4\pi x\sqrt{a^2 - (x - R)^2}\, dx. \end{aligned}$$

This is a bit messy, but with a substitution of

$$\begin{aligned} u &= x - R \\ du &= dx, \end{aligned}$$

we find

$$\begin{aligned} \int_{x=R-a}^{R+a} 4\pi x\sqrt{a^2 - (x - R)^2}\, dx &= \int_{u=-a}^{a} 4\pi(u + R)\sqrt{a^2 - u^2}\, du \\ &= \int_{u=-a}^{a} 4\pi u\sqrt{a^2 - u^2}\, du + \int_{u=-a}^{a} 4\pi R\sqrt{a^2 - u^2}\, du. \end{aligned}$$

Here, we have used linearity to split the integral into two integrals. Notice that the first integrand is an odd function of u, and since it is integrated over a symmetric interval, the first integral is 0:

$$\int_{u=-a}^{a} 4\pi u\sqrt{a^2 - u^2}\, du = 0.$$

The second integral can be found by noting that

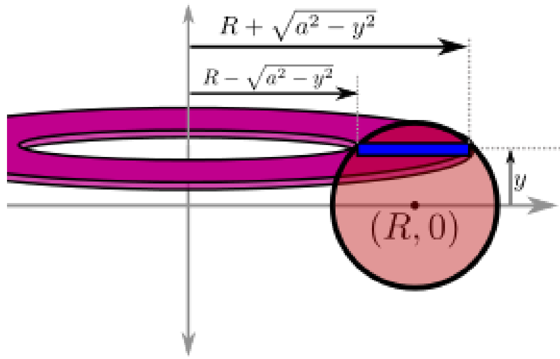
$$\int_{u=-a}^{a} 2\sqrt{a^2 - u^2}\, du$$

gives the area of a disc of radius a (this was an integral done in the simple areas module). Therefore, the volume is

$$\begin{aligned} \int_{u=-a}^{a} 4\pi R\sqrt{a^2 - u^2}\, du &= 2\pi R \int_{u=-a}^{a} 2\sqrt{a^2 - u^2}\, du \\ &= 2\pi R(\pi a^2) \\ &= 2\pi^2 R a^2. \end{aligned}$$

(Return)

4. Carefully drawing and labeling the outer and inner radii of the washer gives the following diagram:



The outer and inner radii can be found by solving the equation of the circle for x:

$$(x - R)^2 + y^2 = a^2 \quad \Rightarrow \quad x = R \pm \sqrt{a^2 - y^2}.$$

Thus, the volume element is the area of the washer times its thickness, dy. Computing this and doing a little algebra gives

$$\begin{aligned} dV &= \left[\pi(R + \sqrt{a^2 - y^2})^2 - \pi(R - \sqrt{a^2 - y^2})^2\right] dy \\ &= 4\pi R\sqrt{a^2 - y^2}\, dy. \end{aligned}$$

Note that y ranges from $-a$ to a, and so the volume integral is the same one arrived at above:

$$\begin{aligned} V &= \int dV \\ &= \int_{y=-a}^{a} 4\pi R\sqrt{a^2 - y^2}\, dy \\ &= 2\pi R(\pi a^2) \\ &= 2\pi^2 R a^2. \end{aligned}$$

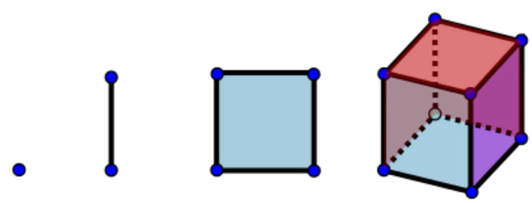
(Return)

34 Volumes In Arbitrary Dimension

The motivation for this module is to find the volume (often referred to as *hypervolume*) of an object in dimension n. This has physical meaning for $n \leq 3$, but what happens for $n \geq 4$?

34.1 The cube in dimension n

Consider the unit cube (i.e. the cube of side length 1) in n dimensions, sometimes called the n-hypercube or just the n-cube. Formally, this is defined to be the set of n-tuples (i.e. lists of length n) $(x_1, x_2, \ldots, x_n)$ such that $0 \leq x_i \leq 1$ for all $1 \leq i \leq n$. For $n = 0, 1, 2, 3$, these are familiar figures: the point, line segment, square, and cube, respectively.



Now, consider some of the various measurements for each of these cubes.

Volume of the cube

For $n = 0$, the cube is just a point, and volume is defined to just be the number of points. So a single point has volume 1.

For $n = 1$, the cube is a line segment. The volume in one dimension is just length, so the one dimension cube has volume 1.

For $n = 2$, the cube is a square of side length 1. In two dimensions, volume is area, so the cube in two dimensions has volume $w \times h = 1 \times 1 = 1$.

For $n = 3$, the cube is a (traditional) cube of side length 1, which has (traditional) volume $l \times w \times h = 1 \times 1 \times 1 = 1$.

For higher values of n, this pattern continues. The intuition is that each additional dimension adds an extra factor of 1, thus the volume of each unit n-cube is 1.

Surface area of cubes

Consider the *surface area* of the cube in dimension n. As with volume, this has physical meaning for $n = 2$ and $n = 3$.

For $n = 2$, the surface area of a square is really its perimeter, which is 4.

For $n = 3$, the surface area is the total area of the faces which bound the cube. There are 6 faces each with area 1, so the surface area is 6.

In general, the n dimension cube will have $2n$ boundary faces, and each face is a cube of dimension $n - 1$, so the surface area (really the hypervolume of the boundary) is $2n$.

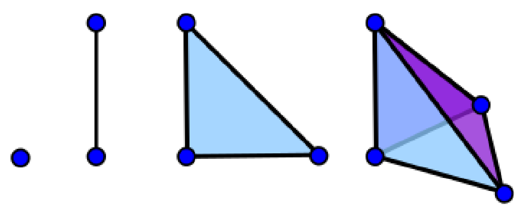
Other features

The *diagonal* of the n-cube can be defined to be the distance from $(0, 0, \ldots, 0)$ to $(1, 1, \ldots, 1)$. Using the distance formula, one finds that the diagonal of the n-cube is $\sqrt{n}$.

The number of corners is fairly easy to count. For $n = 0, 1, 2, 3$, the number of corners is 1, 2, 4, and 8 respectively. Since the n-cube can be thought of as two copies of the $(n-1)$-cube, one can show by induction that there are 2^n corners in the n-cube.

34.2 Simplex

A *simplex* is a generalization of a triangle or a pyramid. In dimension n, the simplex is defined to be the set of n-tuples $(x_1, x_2, \ldots, x_n)$ such that $0 \leq x_i \leq 1$ and $\sum x_i \leq 1$. This can be thought of as the corner of the n dimension cube where the sum of the coordinates is less than 1. Here are the simplices of dimension $n = 0, 1, 2, 3$:



34.3 Volume of spheres in arbitrary dimension

Now, consider a sphere of radius r in n dimensions. This is the set of points $(x_1, x_2, \ldots, x_n)$ such that $x_1^2 + x_2^2 + \ldots + x_n^2 \leq r^2$. Let $V_n(r)$ be the volume of the sphere of radius r in n dimensions (as above, volume means length, area, volume, hypervolume for $n = 1, 2, 3, \ldots$, respectively). With some careful integration and induction, one finds that

$$V_n(r) = \begin{cases} \frac{\pi^k}{k!} r^n & \text{if } n = 2k \\ \frac{2^n \pi^k k!}{n!} r^n & \text{if } n = 2k+1 \end{cases}.$$

Now, note that as $n \to \infty$ (and r stays fixed), the volume goes to 0 (since factorial grows faster than exponentials).

34.4 EXERCISES

- Consider a four-dimensional box (or "rectangular prism") with side-lengths 1, 1/2, 1/3, and 1/4. What is the 4-dimensional volume of this box?
- What is the "diameter" – i.e., the farthest distance between two points – in this 4-d box? Hint: think in terms of diagonals.
- High-dimensional objects are everywhere and all about. Let's consider a very simple model of the space of digital images. Assume a planar digital image (such as that captured by a digital camera), where each pixel is given values that encode color and intensity of light. Let's assume that this is done via an RGB (red/green/blue) model. Though there are many RGB model specifications, let us use one well-suited for mathematics: to each pixel on associates three numbers (R, G, B), each taking a value in $[0, 1]$.

  Since the red/green/blue values are independent, each pixel has associated to it a 3-d cube of possible color values. Consider a (fairly standard) 10-megapixel camera. If I were to consider the "space of all images" that my camera can capture, what does the space look like? How many dimensions does it have? Note: there's no calculus in this problem...just counting!
- Consider an n-dimensional "hypercube" C of all side-lengths equal to 1. Its n-dimensional volume is, clearly, 1. Now consider what happens when you shrink the hypercube's side-lengths by 1 percent (concentrically, so that the shrunken cube has the same center as the original) and remove it from the original cube. By subtracting the n-dimensional volume of this slightly smaller hypercube, conclude how much volume remains in the 1-percent outer "shell."
- In the previous question, what happens to the volume of the 1-percent shell as $n \to \infty$?
- We have seen that the n-dimensional volume of a unit radius ball in dimension n converges to zero as $n \to \infty$. But what about a really large ball? For a ball of radius $R = 10^{10}$ meters in dimension n, what is the limit as $n \to \infty$ of its volume? (in unit of meters-to-the-n^{th})
- For the brave: so, as $n \to \infty$, the volume of the n-ball all concentrates near the surface shell. OK, you've got that. Now answer this: what proportion of the volume is concentrated along the "equatorial plane"? Let's make that specific. Recall, we computed the volume V_n as $I_n \cdot V_{n-1}$, where

  $$I_n = \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^n \theta \, d\theta.$$

  We can compute the volume of the equatorial slice of thickness 2ϵ (for some small but fixed $\epsilon > 0$) as

  $$V_{n,2\epsilon} = V_{n-1} \int_{-\epsilon}^{\epsilon} \cos^n \theta \, d\theta.$$

  So, here is the (hard!) problem. Compute the limit as $n \to \infty$ of the ratio of $V_{n,2\epsilon}$ to V_n:

  $$\lim_{n\to\infty} \frac{V_{n,2\epsilon}}{V_n} = \lim_{n\to\infty} \frac{1}{I_n} \int_{-\epsilon}^{\epsilon} \cos^n \theta \, d\theta.$$

  If you can do this (a very big if...) you will get a surprise...

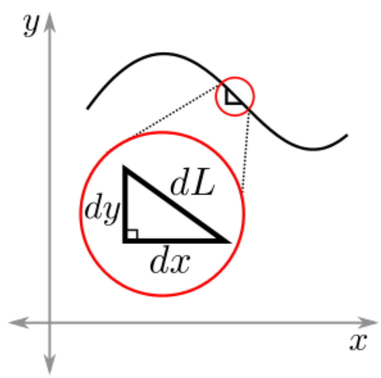
35 Arclength

Consider the graph of a function $y = f(x)$ for $a \leq x \leq b$. The purpose of this module is to find the length of this piece of the curve, known as the *arclength* of the function f from a to b.

As in previous modules, the basic method is to find the arclength element dL and then integrate it:

$$L = \int dL.$$

By zooming in on a portion of the curve, it begins to look like a straight line. Then one can express dL in terms of the infinitesimal horizontal change dx and vertical change dy:



Now, by the Pythagorean theorem one finds that $dL = \sqrt{dx^2 + dy^2}$. A little algebra and the chain rule gives that

$$\begin{aligned} dL &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{dx^2 + \left(\frac{dy}{dx}dx\right)^2} \\ &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \\ &= \sqrt{1 + (f'(x))^2}\,dx. \end{aligned}$$

So the arclength of the function f from a to b is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx.$$

Example

Find the arclength of the curve

$$y = \ln \sin x; \quad \frac{\pi}{4} \le x \le \frac{\pi}{2}.$$

Hint: recall the facts that

$$1 + \cot^2 x = \csc^2 x$$
$$\int \csc x \, dx = -\ln|\csc x + \cot x| + C.$$

(See Answer 1)

Example

Find the arclength of the curve

$$y = x^2 - \frac{1}{8}\ln(x); \quad 1 \le x \le 4.$$

(See Answer 2)

35.1 Parametric curves

If a curve is defined parametrically, i.e. $x = x(t)$ and $y = y(t)$ for $a \le t \le b$, then the arclength element can be written as

$$\begin{aligned} dL &= \sqrt{dx^2 + dy^2} \\ &= \sqrt{\left(\frac{dx}{dt}dt\right)^2 + \left(\frac{dy}{dt}dt\right)^2} \\ &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \sqrt{x'(t)^2 + y'(t)^2}\,dt. \end{aligned}$$

So the arclength of a parametric curve $(x(t), y(t))$ for $a \le t \le b$ is given by

$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\,dt.$$

Example

Find the arclength for a circle of radius r. (See Answer 3)

Example

Find the arclength for the spiral $x(t) = t\cos(t)$, $y(t) = t\sin(t)$ for $0 \le t \le 2\pi$. (See Answer 4)

35.2 Additional Examples

Example

Compute the arclength of the curve

$$y = \frac{2}{3}x^{3/2}; \quad 0 \leq x \leq 3$$

(See Answer 5)

Example

A *catenary* is the curve that is formed by hanging a cable between two towers. It is a fact that the rate of change of the slope of a hanging cable is proportional to the rate of change of arclength with respect to x. Mathematically,

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \kappa \cdot \frac{dL}{dx},$$

for some constant κ. Use this fact to find the equation of the catenary. Then find the length of the catenary for $-l \leq x \leq l$. (See Answer 6)

Example

Show that the spiral

$$x = \frac{1}{t}\cos t$$
$$y = \frac{1}{t}\sin t$$

for $2\pi \leq t$ has infinite arclength. (See Answer 7)

35.3 EXERCISES

- Compute the arclength of $y = \frac{x^3}{3} + \frac{1}{4x}$, from $x = 1$ to $x = 2$.

35.4 Answers to Selected Exercises

1. Computing the arclength element from the above formula gives

$$\begin{aligned} dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \sqrt{1 + \left(\frac{1}{\sin x}\cos x\right)^2}\, dx \\ &= \sqrt{1 + \cot^2 x}\, dx \\ &= \sqrt{\csc^2 x}\, dx \\ &= \csc x\, dx. \end{aligned}$$

Therefore, we find that the arclength is

$$\begin{aligned}
L &= \int dL \\
&= \int_{x=\pi/4}^{\pi/2} \csc x \, dx \\
&= -\ln|\csc x + \cot x| \Big|_{x=\pi/4}^{\pi/2} \\
&= -\ln(1+0) + \ln(\sqrt{2}+1) \\
&= \ln(1+\sqrt{2}).
\end{aligned}$$

(Return)

2. First, one finds $\frac{dy}{dx} = 2x - \frac{1}{8x}$. So, with some careful algebra one sees that

$$\begin{aligned}
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{1+\left(2x-\frac{1}{8x}\right)^2} \\
&= \sqrt{1+(2x)^2 - 2\frac{2x}{8x} + \frac{1}{(8x)^2}} \\
&= \sqrt{1+(2x)^2 - \frac{1}{2} + \frac{1}{(8x)^2}} \\
&= \sqrt{(2x)^2 + \frac{1}{2} + \frac{1}{(8x)^2}}
\end{aligned}$$

Now note that by reversing the cancellation done in an earlier step when simplifying $-2\frac{2x}{8x} = -\frac{1}{2}$, one finds that $\frac{1}{2} = 2\frac{2x}{8x}$. And so, continuing the computation, one finds

$$\begin{aligned}
\sqrt{1+\left(\frac{dy}{dx}\right)^2} &= \sqrt{(2x)^2 + 2\frac{2x}{8x} + \frac{1}{(8x)^2}} \\
&= \sqrt{\left(2x+\frac{1}{8x}\right)^2} \\
&= 2x + \frac{1}{8x}.
\end{aligned}$$

Thus, $dL = \left(2x + \frac{1}{8x}\right) dx$, and it follows that

$$\begin{aligned}
L &= \int dL \\
&= \int_{x=1}^{4} \left(2x + \frac{1}{8x}\right) dx \\
&= \left(x^2 + \frac{1}{8}\ln(x)\right) \Bigg|_{x=1}^{4} \\
&= (16 + \frac{1}{8}\ln(4)) - (1 + \frac{1}{8}\ln(1)) \\
&= 15 + \frac{\ln(4)}{8}.
\end{aligned}$$

(Return)

3. A simple parametrization for the circle of radius r is

$$\begin{aligned} x &= r\cos t \\ y &= r\sin t. \end{aligned}$$

Note that t ranges from 0 to 2π. Using the above formula, we find that the arclength element is

$$\begin{aligned} dL &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\ &= \sqrt{(-r\sin t)^2 + (r\cos t)^2}\, dt \\ &= \sqrt{r^2(\sin^2 t + \cos^2 t)}\, dt \\ &= \sqrt{r^2}\, dt \\ &= r\, dt. \end{aligned}$$

(we used the Pythagorean identity $\sin^2 t + \cos^2 t = 1$ from line three to line four). Therefore,

$$\begin{aligned} L &= \int dL \\ &= \int_{t=0}^{2\pi} r\, dt \\ &= r \cdot t\Big|_{t=0}^{2\pi} \\ &= 2\pi r, \end{aligned}$$

as desired.

(Return)

4. First, compute $x'(t) = \cos(t) - t\sin(t)$ and $y'(t) = \sin(t) + t\cos(t)$. Then

$$\begin{aligned} \sqrt{x'(t)^2 + y'(t)^2} &= \sqrt{(\cos(t) - t\sin(t))^2 + (\sin(t) + t\cos(t))^2} \\ &= \sqrt{\cos^2(t) - 2t\cos(t)\sin(t) + t^2\sin^2(t) + \sin^2(t) + 2t\cos(t)\sin(t) + t^2\cos^2(t)} \\ &= \sqrt{1 + t^2}. \end{aligned}$$

Thus, $dL = \sqrt{1+t^2}dt$. So one finds that

$$L = \int_0^{2\pi} \sqrt{1+t^2}dt.$$

This integral was computed in the Trigonometric Substitution module. The answer becomes

$$\begin{aligned} L &= \left(\frac{1}{2}\sinh^{-1}(t) + \frac{1}{2}t\sqrt{1+t^2}\right)\Big|_0^{2\pi} \\ &= \frac{1}{2}\sinh^{-1}(2\pi) + \pi\sqrt{1+4\pi^2}. \end{aligned}$$

(Return)

5. Computing the arclength element, we find

$$\begin{aligned} dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \sqrt{1 + \sqrt{x}^2}\, dx \\ &= \sqrt{1 + x}\, dx. \end{aligned}$$

Therefore, the arclength is

$$\begin{aligned} L &= \int dL \\ &= \int_{x=0}^{3} \sqrt{1+x}\, dx \\ &= \frac{2}{3}(1+x)^{3/2}\Big|_{x=0}^{3} \\ &= \frac{16}{3} - \frac{2}{3} \\ &= \frac{14}{3}. \end{aligned}$$

(Return)

6. Using the formula for the arclength element, the fact tells us that

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \kappa\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Now, making a substitution of

$$u = \frac{dy}{dx}$$

simplifies the equation to become

$$\frac{du}{dx} = \kappa\sqrt{1+u^2}.$$

This is a separable differential equation. Separating and integrating gives

$$\int \frac{du}{\sqrt{1+u^2}} = \int \kappa\, dx.$$

The left side can be handled with either a trigonometric or hyperbolic trigonometric substitution. We take the latter approach, and let

$$\begin{aligned} u &= \sinh t \\ du &= \cosh t\, dt. \end{aligned}$$

So we have (remembering the Pythagorean identity for hyperbolic trigonometric functions from the trigonometric substitution module)

$$\begin{aligned}
\int \frac{du}{\sqrt{1+u^2}} &= \int \frac{\cosh t}{\sqrt{1+\sinh^2 t}}\,dt \\
&= \int \frac{\cosh t}{\sqrt{\cosh^2 t}}\,dt \\
&= \int \frac{\cosh t}{\cosh t}\,dt \\
&= \int dt \\
&= t \\
&= \operatorname{arcsinh} u
\end{aligned}$$

(we leave the constant of integration off for now since we will be integrating on the right side as well). On the right side, we have

$$\int \kappa\,dx = \kappa x + C.$$

Putting it together, we have

$$u = \sinh(\kappa x + C).$$

If we pick our coordinates so that $x = 0$ occurs at the low point of the catenary, then note that at this point, we have

$$u = \frac{dy}{dx} = 0,$$

since the slope of the catenary is 0 at the low point. Using this fact and plugging in $x = 0$ into the earlier equation gives

$$u = \sinh(C) = 0$$

and so $C = 0$. This gives

$$u = \frac{dy}{dx} = \sinh(\kappa x).$$

Now integrating both sides gives

$$y = \frac{1}{\kappa}\cosh(\kappa x) + C,$$

where $C = y_0$ is the y value of the low point of the catenary.

To find the length of the catenary, we have

$$\begin{aligned}
L &= \int dL \\
&= \int_{x=-l}^{l} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_{x=-l}^{l} \sqrt{1 + \sinh^2 \kappa x}\, dx \\
&= \int_{x=-l}^{l} \cosh \kappa x\, dx \\
&= \frac{1}{\kappa} \sinh \kappa x \Big|_{x=-l}^{l} \\
&= \frac{1}{\kappa} \left(\sinh \kappa l - \sinh \kappa(-l)\right) \\
&= \frac{2}{\kappa} \sinh \kappa l.
\end{aligned}$$

since hyperbolic sine is an odd function. This grows very quickly as l increases, because

$$\frac{2}{\kappa} \sinh \kappa l \approx \frac{1}{\kappa} e^{\kappa l}.$$

(Return)

7. Computing

$$\begin{aligned}
\frac{dx}{dt} &= \frac{-t \sin t - \cos t}{t^2} \\
\frac{dy}{dt} &= \frac{t \cos t - \sin t}{t^2}.
\end{aligned}$$

Plugging these into the formula for the arclength of a parametric curve and noting the cancellation of cross terms, we have

$$\begin{aligned}
dL &= \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \sqrt{\left(\frac{-t \sin t - \cos t}{t^2}\right)^2 + \left(\frac{t \cos t - \sin t}{t^2}\right)^2}\, dt \\
&= \sqrt{t^2 \cdot \frac{\sin^2 t + \cos^2 t}{t^4} + \frac{\cos^2 t + \sin^2 t}{t^4}}\, dt \\
&= \sqrt{\frac{1}{t^2} + \frac{1}{t^4}}\, dt \\
&= \frac{\sqrt{t^2+1}}{t^2}\, dt.
\end{aligned}$$

Therefore, the arclength is

$$\int_{t=2\pi}^{\infty} \frac{\sqrt{t^2+1}}{t^2}\, dt.$$

This integral is difficult to compute exactly, but we only want to show it diverges, which is not as difficult. Note that

$$\frac{\sqrt{t^2+1}}{t^2} \geq \frac{\sqrt{t^2}}{t^2} = \frac{t}{t^2} = \frac{1}{t}.$$

And so by the dominance of definite integrals,

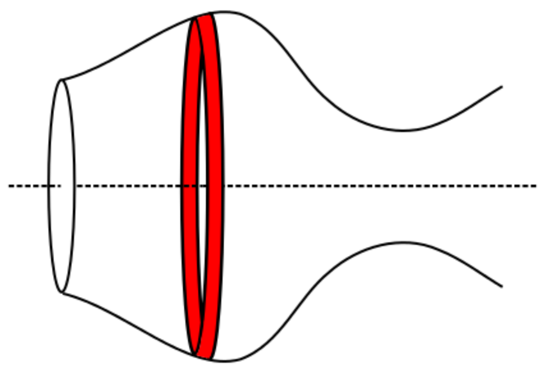
$$\int_{t=2\pi}^{\infty} \frac{\sqrt{t^2+1}}{t^2}\,dt \geq \int_{t=2\pi}^{\infty} \frac{1}{t}\,dt$$

but the integral on the right diverges to infinity by our earlier discussions of p-integrals. Thus, our integral on the left, being larger, also diverges to infinity.

(Return)

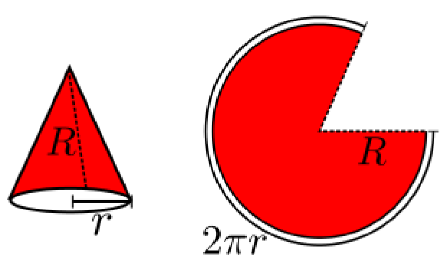
36 Surface Area

This module deals with the surface area of solids of revolution. Consider the portion of a curve $y = f(x)$ for $a \le x \le b$ revolved about a horizontal axis to create a solid. In earlier modules the goal was to find the volume of such a solid, but now the focus is on finding the surface area. As always, the method will be to find the surface area element and integrate it. The surface area element which works well is the thin band shown here:



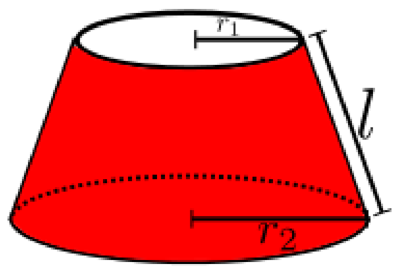
36.1 Surface area of a cone

The first step towards finding the surface area element is to find the lateral surface area of a more simple solid: the cone. Consider a cone whose base has radius r and lateral height R (the lateral height is the distance from the tip of the cone to a point on the circumference of the base; see the left diagram below).



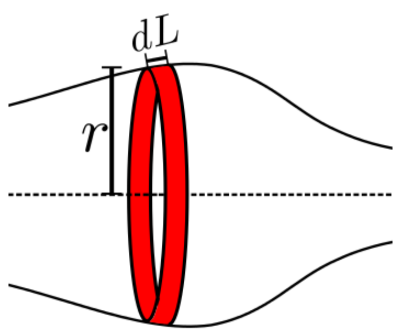
To find the area, consider cutting the cone along the straight dotted line from base circumference to tip and unrolling the cone. The result is a portion of a circle whose radius is R, as shown on the right in the diagram above. Note that the circumference of the base of the cone, $2\pi r$, becomes the length of arc of the unrolled cone. This means that the unrolled cone is a fraction of the full circle of radius R, and that fraction is $\frac{2\pi r}{2\pi R}$ (the ratio of the circumference of the partial circle to the circumference of the whole circle). Thus the surface area of the cone is $\frac{2\pi r}{2\pi R}\pi R^2 = \pi r R$.

The surface area of a cone can be used to find the area of a frustum of a cone whose top radius is r_1, bottom radius is r_2, and lateral height l (as in the below diagram). The area of this frustum is $\pi(r_1 + r_2)l$. Expressed another way, the area is $2\pi r l$, where $r = \frac{r_1+r_2}{2}$ is the average of the two radii of the frustum.



36.2 Surface area element

Now, the surface area element can be found. When the curve is partitioned into sufficiently small pieces, the surface area element is just the area of the frustum formed by rotating the arclength element about the axis (see the diagram):



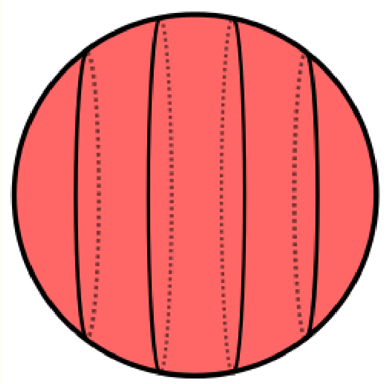
Thus, the surface area element is $dS = 2\pi r dL$, where r is the distance from the curve to the axis of rotation, and dL is the arclength element (i.e. $dL = \sqrt{1 + (f'(x))^2}dx$). In the (common) case where the axis of rotation is the x-axis, one finds that $r = f(x)$.

Thus, the surface area resulting from revolving the curve $y = f(x)$ for $a \le x \le b$ about the x-axis is given by

$$\begin{aligned} S &= \int_a^b 2\pi r \, dL \\ &= 2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2} dx. \end{aligned}$$

Example

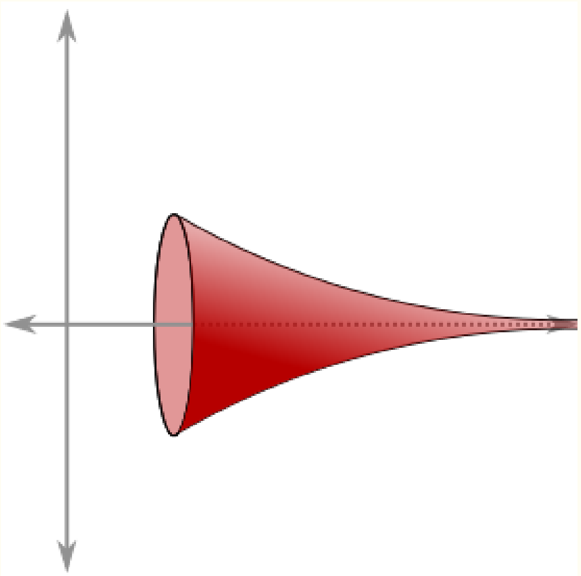
Consider the sphere of radius r. If the sphere is cut into slices of equal width, which slice has the most surface area?



(See Answer 1)

Example

Consider the surface generated by revolving the curve $y = \frac{1}{x^p}$ for $x \ge 1$ about the x-axis.



Find the values of p for which the surface has finite surface area. Then find the values of p for which the solid of revolution has finite volume. (See Answer 2)

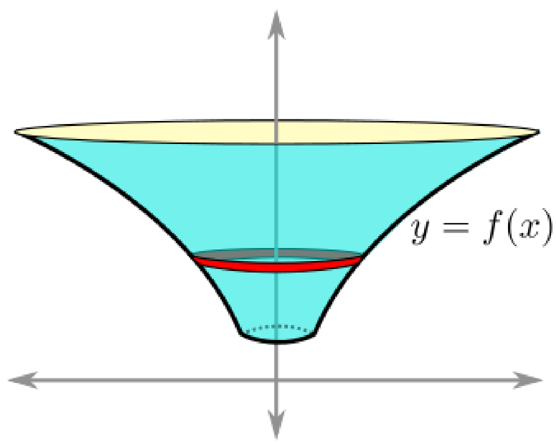
36.3 Rotations about the y-axis

Suppose we want to know the surface area which results from revolving the curve

$$y = f(x); a \leq x \leq b$$

about the y-axis. There are two main ways one can go about finding this surface area:

1. Express everything as a function of y (including range of inputs), and then use the above formula but with the roles of x and y switched.
2. Leave things in terms of x, but adjust the formula slightly.



The first method expresses the curve as

$$x = f^{-1}(y); c \leq y \leq d,$$

where $c = f^{-1}(a)$ and $d = f^{-1}(b)$. Then express the surface area element as

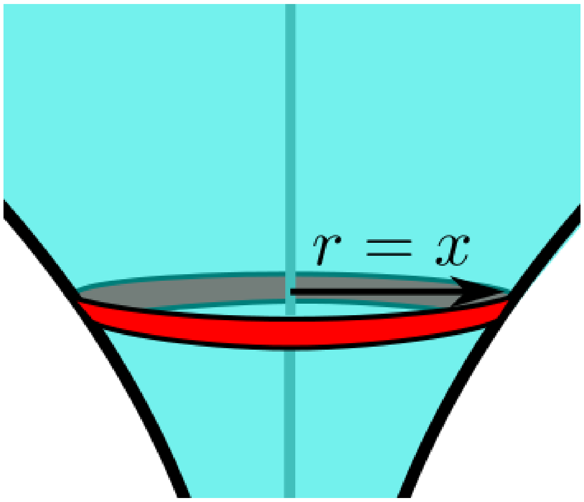
$$\begin{aligned} dS &= 2\pi r\, dL \\ &= 2\pi f^{-1}(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \end{aligned}$$

Putting it together, the surface area can be expressed as

$$\begin{aligned} S &= \int dS \\ &= 2\pi \int_{y=c}^{d} f^{-1}(y)\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \end{aligned}$$

Again, this is really just a reuse of the original formula, with the roles of x and y flipped.

The second method is sometimes simpler to apply because it involves less algebra. The main observation to make is that the radius in the surface area element is simply x when the curve is revolved around the y-axis:



So the surface area element can be written

$$\begin{aligned} dS &= 2\pi r\, dL \\ &= 2\pi x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx. \end{aligned}$$

This integral is with respect to x, and so it should be integrated over the original range of x:

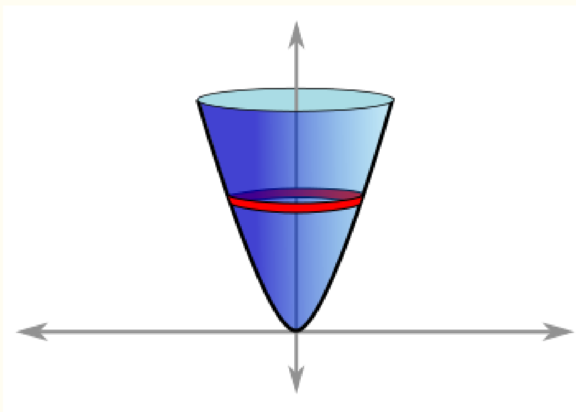
$$S = 2\pi \int_{x=a}^{b} x\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx.$$

Example

Compute the surface area of the surface resulting from revolving the curve

$$y = \frac{1}{2}x^2; \quad 0 \leq x \leq 4.$$

about the y-axis:



(See Answer 3)

36.4 EXERCISES

- Compute the surface area resulting from revolving the curve $f(x) = \cosh(x)$, $0 \le x \le 2$ about the x-axis.

36.5 Answers to Selected Examples

1. If we center the sphere at the origin, we can think of the sphere as the surface of revolution obtained by revolving the curve

$$y = \sqrt{r^2 - x^2}; \quad -r \le x \le r$$

about the x-axis. First, we compute the arclength element:

$$\begin{aligned} dL &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\ &= \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx \\ &= \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx \\ &= \sqrt{\frac{r^2}{r^2 - x^2}}\, dx \\ &= \frac{r}{\sqrt{r^2 - x^2}}\, dx. \end{aligned}$$

Plugging this into the surface area element, we find

$$\begin{aligned} dS &= 2\pi y\, dL \\ &= 2\pi \sqrt{r^2 - x^2} \cdot \frac{r}{\sqrt{r^2 - x^2}}\, dx \\ &= 2\pi r\, dx. \end{aligned}$$

Note that this is independent of x! This means that every slice of the sphere has equal surface area.

For example, if we were to slice the sphere into four slices of equal thickness, then a middle slice goes from $x = 0$ to $x = \frac{r}{2}$, and its surface area

$$\begin{aligned} \int_{x=0}^{r/2} 2\pi r\, dx &= 2\pi r x \Big|_{x=0}^{r/2} \\ &= 2\pi r \cdot \frac{r}{2} \\ &= \pi r^2. \end{aligned}$$

The end-cap slice, on the other hand, goes from $x = \frac{r}{2}$ to $x = r$, so its surface area is

$$\begin{aligned} \int_{x=r/2}^{r} 2\pi r\, dx &= 2\pi r x \Big|_{x=r/2}^{r} \\ &= 2\pi r \left(r - \frac{r}{2}\right) \\ &= 2\pi r \cdot \frac{r}{2} \\ &= \pi r^2. \end{aligned}$$

So we see that the pieces have equal surface area.

(Return)

2. The surface area, in terms of p, is

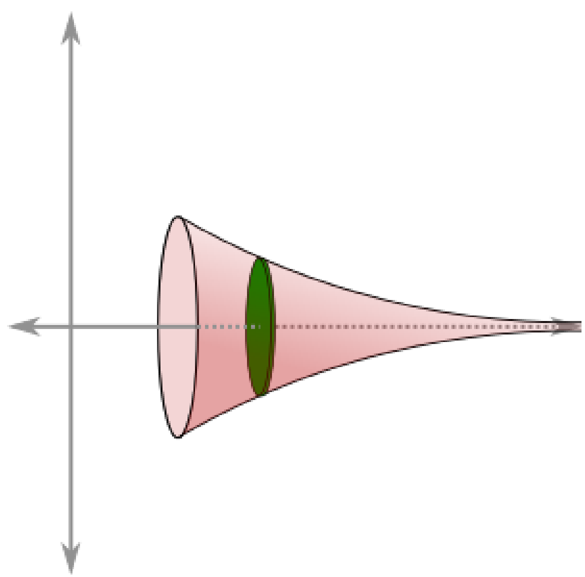
$$S = 2\pi \int_1^\infty \frac{1}{x^p}\sqrt{1+(-px^{-p-1})^2}dx$$
$$= 2\pi \int_1^\infty \frac{1}{x^p}\sqrt{1+\frac{p^2}{x^{2p+2}}}dx$$

Unfortunately, this integral is not computable using standard methods, but we can use a binomial expansion to determine the leading order term of the integrand, which will tell us whether the integral converges or not. We see that

$$\frac{1}{x^p}\sqrt{1+\frac{p^2}{x^{2p+2}}} = \frac{1}{x^p}\left(1+\frac{p^2}{x^{2p+2}}\right)^{1/2}$$
$$= \frac{1}{x^p}\left(1+\frac{1}{2}\cdot\frac{p^2}{x^{2p+2}}+O\left(\frac{1}{x^{4p+4}}\right)\right)$$
$$= \frac{1}{x^p}+O\left(\frac{1}{x^{3p+2}}\right).$$

Therefore, the leading order term in this integral is $\frac{1}{x^p}$, which we know converges for $p > 1$ and diverges for $p \leq 1$ (from our study of p-integrals). So this surface of revolution has finite area if and only if $p > 1$.

Turning to the volume of this solid, it is best to use slices perpendicular to the x-axis, which leads to discs whose radius is y:



The volume element is therefore

$$dV = \pi y^2\, dx$$
$$= \pi\left(\frac{1}{x^p}\right)^2\, dx$$
$$= \pi\cdot\frac{1}{x^{2p}}\, dx.$$

Thus, the volume is

$$\begin{aligned} V &= \int dV \\ &= \pi \int_{x=1}^{\infty} \frac{1}{x^{2p}}\, dx. \end{aligned}$$

We know this is convergent if $2p > 1$, i.e. $p > \frac{1}{2}$. So the volume of the solid is finite if $p > \frac{1}{2}$. This leads to the surprising fact that for

$$\frac{1}{2} < p \leq 1,$$

the volume of the solid is finite, but the surface area is infinite.

(Return)

3. Using the first method requires some algebra. The curve becomes

$$x = \sqrt{2y}; \quad 0 \leq y \leq 8.$$

So the area element is

$$\begin{aligned} dS &= 2\pi r\, dL \\ &= 2\pi r \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy \\ &= 2\pi\sqrt{2y}\sqrt{1 + \left(\frac{1}{\sqrt{2y}}\right)^2}\, dy \\ &= 2\pi\sqrt{2y+1}\, dy. \end{aligned}$$

So the surface area is

$$\begin{aligned} S &= \int dS \\ &= 2\pi \int_{y=0}^{8} \sqrt{2y+1}\, dy \\ &= 2\pi(2y+1)^{3/2} \cdot \frac{1}{3}\Big|_{y=0}^{8} \\ &= \frac{2}{3}\pi\left(17^{3/2} - 1\right). \end{aligned}$$

Using the second method, we have

$$\begin{aligned} dS &= 2\pi r\, dL \\ &= 2\pi x\sqrt{1 + \left(\frac{dy}{dx}\right)}\, dx \\ &= 2\pi x\sqrt{1 + x^2}\, dx. \end{aligned}$$

So the surface area is

$$
\begin{aligned}
S &= \int dS \\
&= \pi \int_{x=0}^{4} 2x\sqrt{1+x^2}\, dx \\
&= \pi (1+x^2)^{3/2} \cdot \frac{2}{3} \Big|_{x=0}^{4} \\
&= \frac{2}{3}\pi \left(17^{3/2} - 1\right).
\end{aligned}
$$

So we get the answer with (perhaps) slightly less algebra involved.

(Return)

37 Work

Recall that *work* is the amount of energy required to perform some action. When the amount of force is constant, work is simply

$$\text{work} = \text{force} \times \text{distance}.$$

For example, if a book weighing 22 Newtons (about 5 pounds) is lifted 2 meters, the total work done is $22N \times 2m = 44J$ (J is the Joule, which equals one Newton-meter).

Consider a situation where the force is not constant. For instance, if one were to lift a weight using a non-negligible rope, there is less rope being pulled up (and hence less force) as the weight goes further up. It is in situations like these that we need a better formula to compute work.

37.1 Work element

Computing work when the force is not constant requires integration. As in previous sections, the first step is to determine the *work element* dW, and then integrate:

$$W = \int dW.$$

Because work arises in a variety of situations, there is not one simple formula for the work element. For different applications the work element will look different. In some situations, it is best to consider a small movement dx, where the force can be thought of as constant for that small movement, which allows the work element to be expressed as $dW = F \cdot dx$.

Springs

The force required to displace a spring varies with the displacement. The further the spring is stretched, the more resistant it becomes to being stretched further. Consider three types of springs:

- **Linear**. A spring is linear if the force of resistance grows linearly with the displacement. That is,

$$F(x) = \kappa x.$$

for some constant κ, which represents the stiffness of the spring.

- **Hard**. A spring is hard if the force of resistance grows faster than linearly with the displacement:

$$F(x) = \kappa x + O(x^2).$$

- **Soft**. A spring is soft if the force of resistance grows slower than linearly with the displacement:

$$F(x) = \kappa x - O(x^2).$$

Consider for any of these springs what the work element dW is. When the spring is stretched to x, the force of resistance is $F(x)$. For the next infinitesimal amount of stretching dx, the force can be presumed to be constant:

(Stretching Spring Animated GIF)

Therefore, the work element (i.e. the amount of work to stretch the spring the additional amount dx) is

$$dW = F(x)\,dx.$$

Example

Compute the amount of work it takes to stretch a linear spring from rest (when $x = 0$) to $x = a$. (See Answer 1)

Example

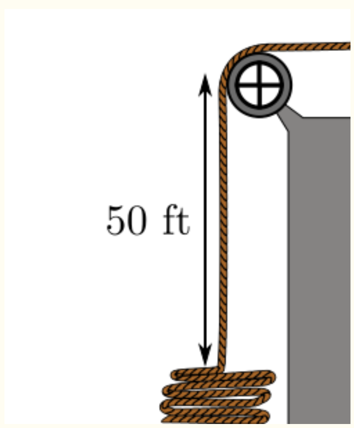
Consider a nonlinear, soft spring which exerts a force of $F(x) = 3x - x^2$ Newtons when the spring is stretched to x meters. Determine how much work is required to stretch the spring from 1 meter to 3 meters. (See Answer 2)

Pulling up a rope

In some situations, one must do a little work to determine what $F(x)$ is, and then one can integrate, as in the above examples.

Example

Consider a rope which is 100 feet long and density 1 pound/foot. It hangs from a wall which is 50 feet high (so 50 feet of rope runs down the length of the wall and the remaining 50 feet is coiled at the bottom of the wall). How much work (in foot-pounds) is required to pull the rope to the top of the wall?



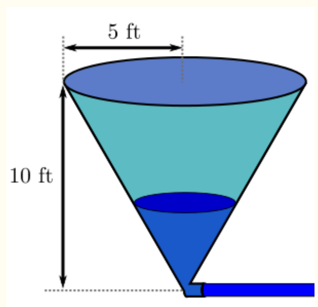
(See Answer 3)

37.2 Work element by slices

In other situations, such as pumping liquid, digging a hole, or piling gravel, a fruitful method for determining the work involved is to consider a slice of the material which is being moved. Determining the weight of the slice, and multiplying by the distance the slice has to be lifted gives the amount of work required for that slice. That is precisely the work element. Integrating over all the slices in the object gives the total amount of work to move that object.

Example

Pumping Liquid Consider an inverted conical tank (so the tip of the cone points downward) with base radius 5 feet and height 10 feet. Water is pumped into the tank through a valve at the tip of the cone:



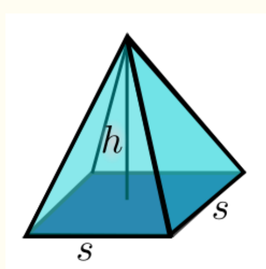
How much work is required to fill the tank with water? Leave the weight density of water as the constant ρ. (See Answer 4)

Example

Digging a Hole Consider two workers digging a hole. How deep should the first worker dig so that each does the same amount of work? Let the weight density of the dirt be the constant ρ, the depth of the hole is D, and the cross-sectional area of the hole is the constant A (so we assume that the hole does not get any wider or narrower as the workers dig). (See Answer 5)

Example

Gravel Pyramid Compute the amount of work required to build a pyramid of gravel. Assume the gravel is infinitesimal with weight density ρ, and that the pyramid has a square base of side length s, and height h:



(See Answer 6)

Example

Rope Revisited Consider the rope example from above, but this time suppose l total feet of rope are hanging from a h foot building, where $l \geq h$, and let ρ be the weight density of the rope. Compute the work required to lift the rope to the top of the building.
For a different perspective, this time, use a work element which equals the amount of work required to lift an infinitesimal length of rope to the top of the building (this will depend on whether the infinitesimal length of rope is hanging at the beginning or is part of the coil at the bottom of the building). Then integrate along the entire length of rope. (See Answer 7)

37.3 EXERCISES

- Consider a conical tank of height 10m. The vertex of the cone is at the bottom, and the base of cone (which is at height 10m) has radius 2m. Let ρ denote the weight density of water. The water inside the tank has height 4m. How much work would it take to pull all the water to the top of the tank?

37.4 Answers to Selected Examples

1. As shown above, the work element (i.e. the amount of work to stretch the spring a short distance dx) is

$$dW = F(x)\,dx = \kappa x\,dx.$$

It follows that

$$\begin{aligned} W &= \int dW \\ &= \int_{x=0}^{a} \kappa x\,dx \\ &= \kappa \frac{x^2}{2}\Big|_{x=0}^{a} \\ &= \kappa \frac{a^2}{2}. \end{aligned}$$

(Return)

2. The work element is

$$\begin{aligned} dW &= F(x)\,dx \\ &= (3x - x^2)\,dx. \end{aligned}$$

Thus, the total work to stretch the spring from 1 meter to 3 meters is

$$\begin{aligned} W &= \int dW \\ &= \int_{x=1}^{3} (3x - x^2)\, dx \\ &= \frac{3}{2}x^2 - \frac{1}{3}x^3 \Big|_{x=1}^{3} \\ &= \left(\frac{27}{2} - 9\right) - \left(\frac{3}{2} - \frac{1}{3}\right) \\ &= \frac{10}{3} \text{ Joules.} \end{aligned}$$

(Return)

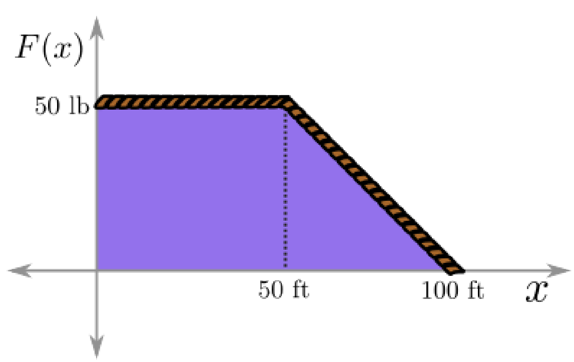
3. As the first 50 feet of rope are brought up, there is always precisely 50 feet of rope hanging from the building (because every foot of rope brought onto the top of the building is replaced by a rope which is coiled below). These 50 feet of rope weigh 50 lbs, so that is the force required to support them. If x denotes the amount of rope which has been taken onto the roof, then

$$F(x) = 50; \quad 0 \leq x \leq 50.$$

After the first 50 feet of rope have been brought to the roof, there is now 50 feet of rope dangling with nothing left coiled below. Therefore, as we bring up these last 50 feet, there is less and less rope hanging, and so the weight of the rope (and hence the force we exert) is decreasing. It decreases linearly, since the rope has constant density. Each foot of rope we bring up decreases the weight by 1 lb, and so

$$F(x) = 100 - x; \quad 50 \leq x \leq 100$$

(to see that this is right, note that it is linear and matches at the endpoints). We can graph the force as a function of the amount of rope we have brought up:



Now, we can find the work by integrating the work element

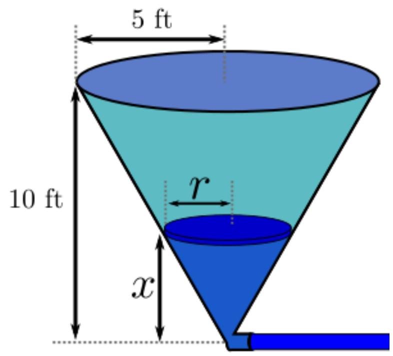
$$\begin{aligned} W &= \int dW \\ &= \int_{x=0}^{100} F(x)\, dx. \end{aligned}$$

Note that this is the area under the graph of the force (highlighted above), which is easier to compute than to do it algebraically. Splitting it into a square and a triangle, the area (and hence the work) is

$$50 \text{ lb} \times 50 \text{ ft} + \frac{1}{2} 50 \text{ lb} \times 50 \text{ ft} = 3750 \text{ ft-lb}.$$

(Return)

4. Consider a slice of the water in the tank. Let x be the distance of the slice from the tip of the tank. That is, x is the distance that the slice of water has to be lifted. Let r be the radius of the slice:



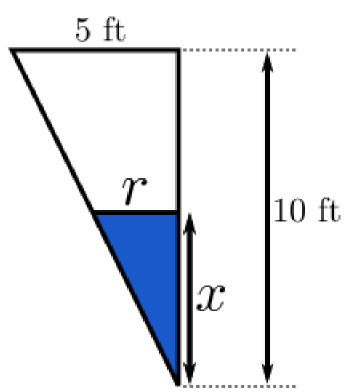
Above, we said that it is the weight of the slice multiplied by the distance the slice had to be moved. But the weight of a slice is just the volume of the slice times the density of the slice. Letting ρ denote the weight density of the substance (in this case water), we have

$$\begin{aligned} dW &= \text{ weight of slice } \times distanceslicetravels \\ &= \rho \cdot dV \cdot \text{ distance slice travels }. \end{aligned}$$

In the problem at hand, we have

$$dV = \pi r^2 \, dx,$$

and the distance the slice is lifted is x, by the way we labeled our diagram. To finish, we must get r in terms of x, which requires a little bit of geometry. If we flatten our cone and look at it from the side, we get similar triangles:



Therefore,

$$\frac{r}{x} = \frac{5}{10} = \frac{1}{2},$$

and so $r = \frac{x}{2}$. Putting this together, we have

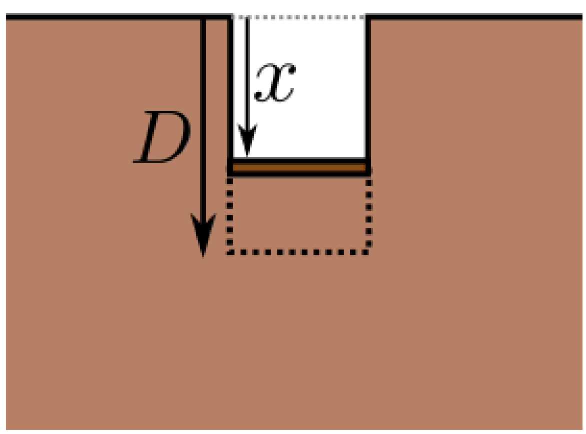
$$\begin{aligned} dW &= \rho \cdot dV \cdot \text{ distance slice travels} \\ &= \rho(\pi r^2 \, dx)x \\ &= \pi\rho\frac{1}{4}x^3 \, dx. \end{aligned}$$

Note that x ranges from 0 to 10, so the work required to fill the tank is

$$\begin{aligned} W &= \int dW \\ &= \int_{x=0}^{10} \pi\rho\frac{1}{4}x^3 \, dx \\ &= \frac{\pi\rho}{4}\frac{x^4}{4}\Big|_{x=0}^{10} \\ &= 625\pi\rho. \end{aligned}$$

(Return)

5. Let x be the distance down to to the layer of dirt currently being dug:



This is convenient because this is the distance that the current slice of dirt has to be lifted to get out of the hole. The area of the slice of dirt is A, its thickness is dx, and the density is ρ, so we have

$$\begin{aligned} dW &= \text{ weight of slice } \times \text{ distance slice moves} \\ &= (\rho \cdot dV) \cdot x \\ &= (\rho A \, dx) \cdot x \\ &= \rho A x \, dx. \end{aligned}$$

Note that x varies from 0 to D as the hole gets dug. Thus, the total work required to dig the hole is

$$\begin{aligned} W &= \int dW \\ &= \int_{x=0}^{D} \rho A x \, dx \\ &= \frac{1}{2}\rho A D^2. \end{aligned}$$

To find the depth $\tilde{D}$ where the work done is half, we set

$$\int_{x=0}^{\tilde{D}} \rho A x \, dx = \frac{1}{2}W = \frac{1}{4}\rho A D^2.$$

Computing the integral on the left, we find

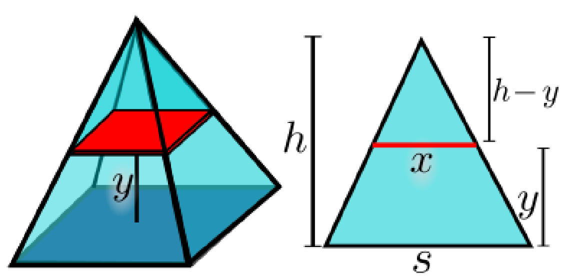
$$\frac{1}{2}\rho A \tilde{D}^2 = \frac{1}{4}\rho A D^2.$$

Solving for $\tilde{D}$ gives

$$\tilde{D} = \frac{1}{\sqrt{2}} D.$$

(Return)

6. If we think of building the pyramid slice by slice, let y be the distance from the base of the pyramid to the slice. This is convenient because this is the distance that the slice must be lifted to be put in place. Also, let x be the side length of the slice:



Then using similar triangles, as shown on the right above, we find that

$$\frac{x}{s} = \frac{h-y}{h}.$$

So we find that

$$x = \frac{h-y}{h} s.$$

Thus, the volume of a slice is just the area x^2 multiplied by the thickness dy, and so we have

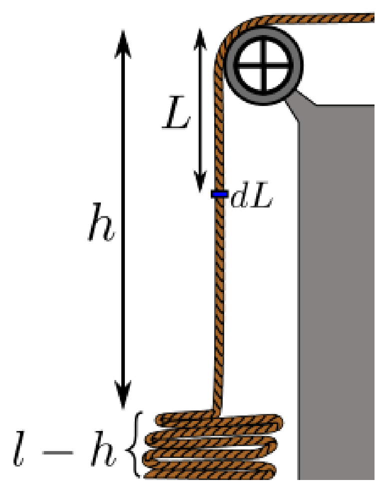
$$\begin{aligned} dW &= \rho dV\, y \\ &= \rho(x^2\, dy)\, y \\ &= \rho\left(\frac{h-y}{h}s\right)^2 y\, dy \\ &= \frac{\rho s^2}{h^2}(h-y)^2 y\, dy. \end{aligned}$$

Because y ranges from 0 to h, we have

$$\begin{aligned} W &= \int dW \\ &= \frac{\rho s^2}{h^2}\int_{y=0}^{h}(h-y)^2 y\, dy \\ &= \frac{\rho s^2}{h^2}\int_{y=0}^{h}(h^2 y - 2hy^2 + y^3)\, dy \\ &= \frac{\rho s^2}{h^2}\left(\frac{1}{2}h^2y^2 - \frac{2}{3}hy^3 + \frac{1}{4}y^4\right)\Bigg|_{y=0}^{h} \\ &= \frac{\rho s^2}{h^2}\left(\frac{1}{2}h^4 - \frac{2}{3}h^4 + \frac{1}{4}h^4\right) \\ &= \frac{\rho s^2}{h^2}\cdot\frac{1}{12}h^4 \\ &= \frac{\rho s^2 h^2}{12}. \end{aligned}$$

(Return)

7. Let L be the distance along the rope of the infinitesimal piece being considered:



So L is the distance that the infinitesimal piece must be lifted to get to the top of the building. The weight of the infinitesimal piece is density multiplied by length, and so the work element for a piece of rope which is hanging is

$$dW = \rho L\, dL; \quad 0 \le L \le h.$$

For a piece of rope which is part of the coil at the bottom, the distance it must be lifted is always h, so the work element there is

$$dW = \rho h\, dL; \quad h \leq L \leq l.$$

So the work can be computed by integrating these work elements over their respective ranges and then adding:

$$\begin{aligned}
W &= \int dW \\
&= \int_{L=0}^{h} \rho L\, dL + \int_{L=h}^{l} \rho h\, dL \\
&= \rho \frac{1}{2} L^2 \Big|_{L=0}^{h} + \rho h L \Big|_{L=h}^{l} \\
&= \frac{\rho h^2}{2} + \rho h(l - h).
\end{aligned}$$

Another way to think about this is to treat the coiled rope at the bottom of the wall as a single solid object. The rope in the coil has length $l - h$, and so its weight is $\rho(l - h)$. The distance the coil (as a unit) must be lifted is h. It follows that the work to lift the coiled portion of the rope is $\rho h(l - h)$, the result of the second integral above.

(Return)

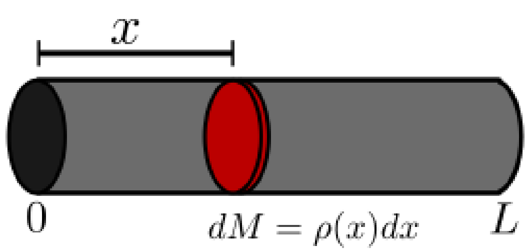
38 Elements

This module deals with various problems that can be modeled using integral calculus. As in the previous sections, the problem will be to find the total accumulation of some quantity U, and the method will be to determine a slice of the quantity, the U element dU, and integrate.

38.1 Mass

Mass of a rod

Consider the problem of determining the mass of a rod. Suppose the rod's density varies along the length of the rod (but the rod is uniform in cross section). Let $\rho(x)$ denote the linear density (i.e. the mass per unit of length) of the rod at position x:

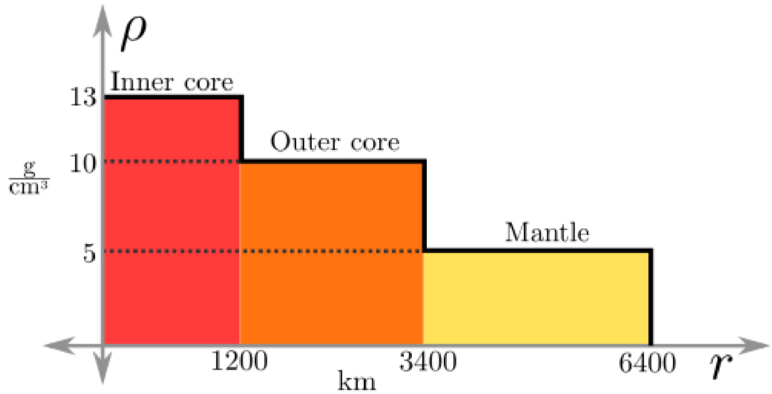


Then the mass element dM is the density $\rho(x)$ times the thickness of the slice dx, as shown above, and it follows that the mass of the rod is

$$\begin{aligned} M &= \int dM \\ &= \int_{x=0}^{L} \rho(x)dx. \end{aligned}$$

Mass of the earth

Consider the problem of finding the mass of the earth. Suppose the density of the earth $\rho(r)$ is given as a function of the distance from the center of the earth. Assume that there are just three layers (inner core, outer core, and mantle) and that the density is constant within each layer.



What is the mass element in this case? It is important to note that in this example we are measuring the contribution of a spherical shell to the mass of the earth. This contribution is the volume of the spherical shell multiplied by the density of the shell. Mathematically,

$$dM = \rho(r) \cdot dV.$$

Recalling that the surface area of a sphere of radius r is $4\pi r^2$, we have that the volume element is

$$dV = 4\pi r^2\, dr,$$

and so the mass element is

$$dM = 4\pi \rho(r) r^2\, dr.$$

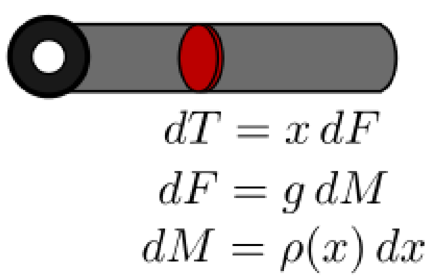
Example

Using the approximate graph of density above, estimate the mass of the earth. (See Answer 1)

38.2 Torque

Imagine a rod of variable density which is attached to a hinge. The torque at the hinge depends not just on the weight of the rod but on the distribution of the weight.

If there were just a mass-less rod with a single point mass, the torque would be Force $\times$ Distance . This can be used to determine the torque element dT by thinking of each slice of the rod as a point mass. What is the torque on such a slice?



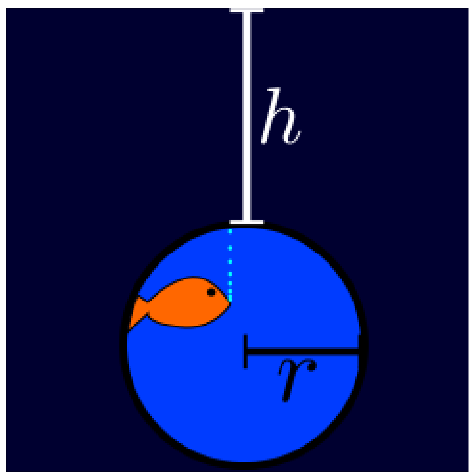
First, the torque element is the distance from the hinge, x, times the force element dF (the force on the slice). The force element is the mass of the slice dM times the gravitational constant g. Finally, as in the previous example, the mass element $dM = \rho(x)\,dx$. Putting it all together, one finds

$$dT = x \cdot g \cdot \rho(x)\,dx.$$

Integrating this over the length of the rod gives the torque.

38.3 Hydrostatic force

The next application is to compute the total force exerted by a tank of fluid on a surface submerged in the tank, often called the *hydrostatic force*. For a tangible example, consider a large aquarium with a circular glass viewing window (see the diagram below). If the viewing window has radius r, and the top of the viewing window is at depth h, then the problem is to find the total force of the water on the viewing window.



As always, the method will be to find the force element dF (the force on a small strip of the window), and then use integration to find the total force.

Recall that if pressure is constant across a surface, the force on the surface is area $\times$ pressure. Hydrostatic pressure is given by

$$P = \text{weight density of fluid} \times \text{depth}.$$

Note the units: $\frac{N}{m^3} \cdot m = \frac{N}{m^2}$, which is the correct unit for pressure (force per unit of area). Since the density of the fluid is assumed to be constant, the pressure only depends on the depth.Therefore, the most logical choice for the force element is a horizontal strip, since the depth, and hence the pressure, will be constant across the strip. Letting dA denote the area of the strip, we find that the force element is given by

$$dF = P\,dA = \rho x\,dA,$$

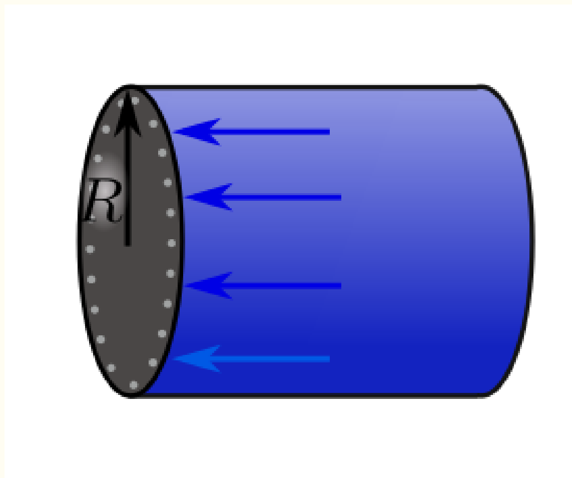
where ρ is the weight density of the fluid and x is the depth of the strip.

Example

Compute the total force exerted on the circular viewing window in the aquarium shown above. (See Answer 2)

Example

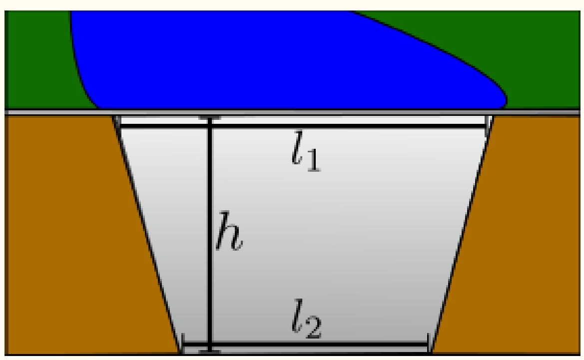
Compute the force on the endcap of a full cylindrical tank of radius R on its side.



(See Answer 3)

Example

Consider a dam in the shape of a trapezoid with height h, top edge l_1 and bottom edge l_2. Find the total force exerted on the dam by the water:



(See Answer 4)

38.4 Present value

Consider the problem of determining the present value of some amount of money at a future time. Turning the problem around, first consider the value of an initial amount of money P_0 at a future time t. Assuming a constant annual nominal interest rate r and continuous compounding, this problem was an example of exponential growth, and had solution

$$P(t) = P_0 e^{rt},$$

where t is the time in years. Given some amount of money, P, at time t, finding its present value is a matter of solving $P = P_0 e^{rt}$ for P_0. In other words, solving for present value in this simple case is the same as finding the initial investment P_0 which yields P after t years of continuous compounding interest. Solving this equation gives that the present value of a future amount P at time t is given by

$$P_0 = Pe^{-rt}.$$

Example

Find the present value of \$1000000 in 30 years, assuming an interest rate of $r = .08$. (See Answer 5)

Now consider an income stream, say from a job. If $I(t)$ is the rate of income at time t, what is the present value of that income stream? Let PV be the present value. Then consider the income earned over a small amount of time t years in the future:

$$dI = I(t)\,dt$$

(the income element). This small bit of income at time t contributes $e^{-rt}I(t)dt$ to the present value of the income stream. Thus the present value element is given by

$$dPV = e^{-rt}I(t)dt.$$

Integrating this over the range of values of t (the time period of the income stream) gives the present value of that income stream.

Example

The Bigbucks lottery has an option of either a single lump sum payment today or an annuity which pays a constant amount each year for 20 years. Suppose the annuity pays \$3 million a year (for 20 years), and that the interest rate will remain steady at $r = .05$. What is the fair lump sum payout today? (See Answer 6)

38.5 EXERCISES

- Consider a dam with the shape of an isosceles triangle. The base of the triangle, which is parallel to the ground, is 5m long, and the height of the triangle is 10m. The weight density of water is given by ρ. Compute the force exerted on the dam by water.

38.6 Answers to Selected Examples

1. Note that our volume is being measured in cubic kilometers, but the density $\rho(r)$ is in grams per cubic centimeter. We need a conversion factor C to make sure the units come out correctly. A little unit conversion gives us that

$$C = 1\frac{\text{g}}{\text{cm}^3} = 10^{12}\frac{\text{kg}}{\text{km}^3}.$$

   So we need to multiply by this so that the units are correct (and the final answer will be in kilograms).

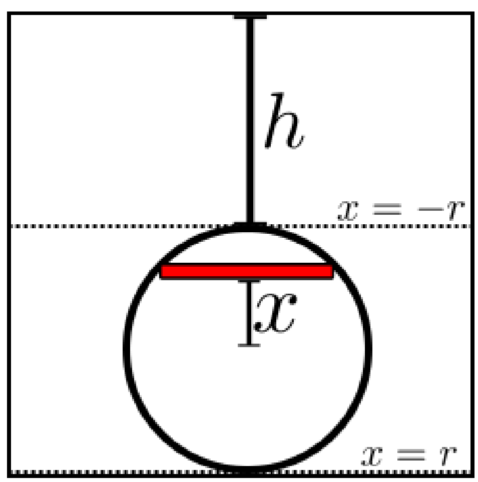
Splitting the integral based on the values of r for which $\rho(r)$ is constant, we find

$$\begin{aligned}
M &= \int dM \\
&= C \int_{r=0}^{6400} 4\pi r^2 \rho(r)\, dr \\
&= 4\pi C \left(\int_{r=0}^{1200} r^2 \rho(r)\, dr + \int_{r=1200}^{3400} r^2 \rho(r)\, dr + \int_{r=3400}^{6400} r^2 \rho(r)\, dr \right) \\
&= 4\pi C \left(13 \cdot \frac{r^3}{3}\bigg|_0^{1200} + 10 \cdot \frac{r^3}{3}\bigg|_{1200}^{3400} + 5 \cdot \frac{r^3}{3}\bigg|_{3400}^{6400} \right) \\
&\approx 6.3 \cdot 10^{24} \text{ kilograms}
\end{aligned}$$

According to Wolfram Alpha, the mass of the earth is approximately $5.97 \cdot 10^{24}$ kilograms, so our rough estimate is not too far off.

(Return)

2. As mentioned above, we will use horizontal strips of the window as the area element. The force element is the amount of force on that strip of the window. Let x be the distance from the center of the window to the horizontal strip. Let up be negative, down be positive (so the top of the window is $x = -r$ and the bottom of the window is $x = r$:



Then the depth of the strip is $h + r + x$, and the area of the strip is $2\sqrt{r^2 - x^2}\, dx$. Thus the force element in this example is

$$dF = (h + r + x)\rho \cdot 2\sqrt{r^2 - x^2}\, dx$$

where ρ is the weight density of water. So

$$
\begin{aligned}
F &= \int dF \\
&= 2\rho \int_{-r}^{r} (h + r + x)\sqrt{r^2 - x^2}\, dx \\
&= 2\rho \int_{-r}^{r} (h + r)\sqrt{r^2 - x^2}\, dx + 2\rho \int_{-r}^{r} x\sqrt{r^2 - x^2}\, dx.
\end{aligned}
$$

Now, notice that $x\sqrt{r^2 - x^2}\, dx$ is an odd function, so its integral from $-r$ to r is 0. Thus

$$
\begin{aligned}
F &= 2\rho(h + r) \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx \\
&= 2\rho(h + r)\frac{\pi r^2}{2} \\
&= \rho(h + r)\pi r^2,
\end{aligned}
$$

since $\int \sqrt{r^2 - x^2}\, dx$ gives half the area of a circle of radius r.

It is worth observing that with a very symmetric window such as the circle in this example, one can take the area of the window, πr^2, and multiply by the pressure at the center of the window $\rho(h + r)$, to find the hydrostatic force:

$$F = \rho\pi r^2(h + r).$$

The reason this works is that the pressure on a horizontal strip above the center of the window averages with the pressure on the strip's mirror image below the center to give the pressure at the center of the window.

(Return)

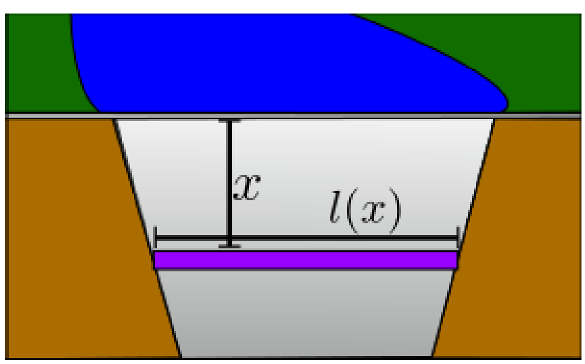
3. Using the knowledge gleaned from the previous example, we can take the area of the endcap, πR^2, and multiply by the hydrostatic pressure at the center of the endcap, which is ρR, to find that the force is

$$F = \pi R^2 \cdot \rho R = \rho\pi R^3.$$

We could also note that this is really a special case of the aquarium window example above, by setting $h = 0$ in that example.

(Return)

4. As above, the force element dF is the force exerted on a horizontal strip. Let x be the distance of the horizontal strip from the top of the dam, and $l(x)$ be the length of the strip



Since the shape is a trapezoid, $l(x)$ is a linear function of x, and from the top and the bottom of the dam, one finds that $l(0) = l_1$ and $l(h) = l_2$. It follows from the slope intercept form of a line that $l(x) = l_1 + \frac{l_2 - l_1}{h} x$.

So the force acting on the strip is $dF = \rho x \, dA$, where ρ is the weight density of the water, x is the depth of the strip, and $dA = (l_1 + \frac{l_2 - l_1}{h} x) \, dx$ is the area of the strip. Putting it all together, one finds

$$\begin{aligned} F &= \int dF \\ &= \int_0^h \rho x \left(l_1 + \frac{l_2 - l_1}{h} x \right) \, dx \\ &= \rho \left(\frac{l_1 x^2}{2} + \frac{l_2 - l_1}{3h} x^3 \right) \Big|_0^h \\ &= \rho \left(\frac{l_1 h^2}{2} + \frac{l_2 - l_1}{3h} h^3 \right) \\ &= \frac{\rho h^2}{6} (l_1 + 2 l_2). \end{aligned}$$

(Return)

5. From the above equation one finds that

$$\begin{aligned} P_0 &= P e^{-rt} \\ &= 1000000 e^{(-.08) \cdot 30} \\ &\approx 90717. \end{aligned}$$

(Return)

6. The income stream $I(t)$ is constant at $3 \cdot 10^6$. Thus,

$$\begin{aligned} PV &= \int dPV \\ &= \int_{t=0}^{20} e^{-rt} I(t) dt \\ &= 3 \cdot 10^6 \int_{t=0}^{20} e^{-.05t} dt \\ &= 3 \cdot 10^6 \left(\frac{1}{-.05} e^{-.05t} \Big|_{t=0}^{20} \right) \\ &= 3 \cdot 10^6 \cdot (-20)(e^{-1} - 1), \end{aligned}$$

which is approximately \$38 million.

(Return)

39 Averages

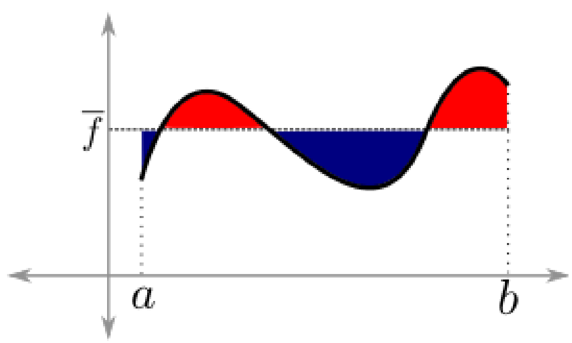
Consider the problem of finding the average test score in a class of 100 students. The answer is to add up all the scores and divide by 100. But what would happen if there were infinitely many students? This module deals with the problem of finding the average value of a function.

39.1 Average value of a function

The definition of the *average value* of a function $f(x)$ over the interval $[a, b]$, denoted $\overline{f}$, is

$$\overline{f} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

One way to interpret the average value is to find the rectangle of length $b-a$ whose area equals the area under the curve f over the interval $[a, b]$. The height of this rectangle is $\overline{f}$. Put another way, $\overline{f}$ is the height of the horizontal line such that the area above the line and below $f(x)$ equals the area which is below the line and above $f(x)$. These areas are shown in red and blue, respectively, in the following diagram:



A better formulation of the average value, which will be useful in other situations and higher dimensions, is

$$\overline{f} = \frac{\int_{x=a}^b f\,dx}{\int_{x=a}^b\,dx}.$$

This emphasizes that the average value over a region is the integral of the function over the region divided by the volume of that region (in this case, the 1-dimensional volume is just the length of the interval). This generalizes nicely to higher dimensions.

If we go down a dimension to the discrete average, if f_i denotes the ith data point out of n, then the average value of the data is

$$\overline{f} = \frac{\sum_{i=1}^{n} f_i}{n} = \frac{\sum_{i=1}^{n} f_i}{\sum_{i=1}^{n} 1}.$$

This shares a common feature with the earlier formula for average value. Namely, it is the sum (integral) of the function values over a range of inputs divided by the sum (integral) of 1 over that range of inputs.

Example

Compute the average value of $\sin^2 x$ over the interval $[0, 2\pi]$. (See Answer 1)

Example

Compute the average of x^n and e^x over $0 \le x \le T$. Compute the average of $\ln x$ over $1 \le x \le T$. (See Answer 2)

Example

Suppose we are given the density function $\rho(r)$ for the density of the earth at a distance r from the center. Find a formula for the average density of the earth, but do not try to evaluate the integral. (See Answer 3)

39.2 Root mean square

There is another type of average of a function called the *root mean square*. The root mean square of f, denoted f_{RMS} is defined by

$$f_{RMS} = \sqrt{\overline{f^2}}.$$

So the root mean square is the square root of the average value of the square of the function. This is a useful metric when the average value of f is uninteresting.

Example

Compute and compare the average value and the root mean square of $f(x) = \sin x$ on the interval $[0, 2\pi]$. (See Answer 4)

39.3 EXERCISES

- Consider the polar function $f(\theta) = \cos^2(\theta)$. Compute the average value of f from $\theta = 0$ to $\theta = 2\pi$.

39.4 Answers to Selected Exercises

1. From the definition,

$$\begin{aligned}
\overline{f} &= \frac{\int_0^{2\pi} \sin^2 x dx}{2\pi} \\
&= \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{2} \left(1 - \cos(2x)\right) dx \\
&= \frac{1}{2\pi} \left(\frac{x}{2} - \frac{1}{4} \sin(2x) \right) \Bigg|_0^{2\pi} \\
&= \frac{1}{2\pi} \cdot \frac{2\pi}{2} \\
&= \frac{1}{2}.
\end{aligned}$$

   (Return)

2. For $f(x) = x^n$, one finds

$$\begin{aligned}
\overline{f} &= \frac{1}{T} \int_0^T x^n \, dx \\
&= \frac{1}{T} \frac{x^{n+1}}{n+1} \Bigg|_0^T \\
&= \frac{T^n}{n+1}.
\end{aligned}$$

   For $f(x) = e^x$, the average value is

$$\begin{aligned}
\overline{f} &= \frac{1}{T} \int_0^T e^x \, dx \\
&= \frac{e^T - 1}{T}
\end{aligned}$$

   For $f(x) = \ln x$, recalling the integral using integration by parts, one finds

$$\begin{aligned}
\overline{f} &= \frac{1}{T-1} \int_1^T \ln x \, dx \\
&= \frac{1}{T-1} (x \ln x - x) \Bigg|_1^T \\
&= \frac{1}{T-1} (T \ln T - T + 1).
\end{aligned}$$

   (Return)

3. Note that we cannot simply integrate the density function and divide by the radius of the earth, for the same reason that we could not integrate the density function to find the mass of the earth in the previous module.

   One way to logically think about it is to note that the average density times the volume of the earth should give the mass of the earth. That is,

$$\overline{\rho} \cdot V = M.$$

Remember that when we found the mass of the earth, we had $dM = \rho\, dV$, where the volume element dV is a spherical shell. So we can write

$$\overline{\rho} = \frac{M}{V} = \frac{\int \rho\, dV}{\int dV}.$$

Then, remembering that the volume of the spherical shell (i.e. the volume element) is $4\pi r^2\, dr$, we have

$$\overline{\rho} = \frac{\displaystyle\int_{r=0}^{R} 4\pi r^2 \rho(r)\, dr}{\displaystyle\int_{r=0}^{R} 4\pi r^2\, dr}.$$

(Return)

4. The average value of $\sin x$ is

$$\begin{aligned}
\overline{f} &= \frac{\displaystyle\int_{x=0}^{2\pi} \sin x\, dx}{2\pi} \\
&= \frac{1}{2\pi}(-\cos x)\Big|_{x=0}^{2\pi} \\
&= \frac{1}{2\pi}(-1-(-1)) \\
&= 0.
\end{aligned}$$

Using the result of an example from above, the root mean square of $\sin x$ is

$$\begin{aligned}
f_{RMS} &= \sqrt{\frac{\displaystyle\int_{x=0}^{2\pi} \sin^2 x\, dx}{2\pi}} \\
&= \sqrt{\frac{1}{2}} \\
&= \frac{1}{\sqrt{2}}.
\end{aligned}$$

(Return)

40 Centroids And Centers Of Mass

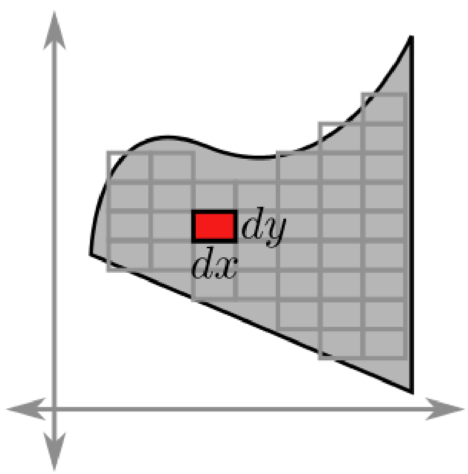
The motivation for this module are the questions:

- what is the average of several locations (e.g. cities on a map)?
- what is the average of an entire region?

The centroid and center of mass give answers to these questions. The formulas for the centroid and the center of mass of a region in the plane seem somewhat mysterious for their apparent lack of symmetry. So before giving the formulas, a brief aside is helpful.

40.1 The area element revisited

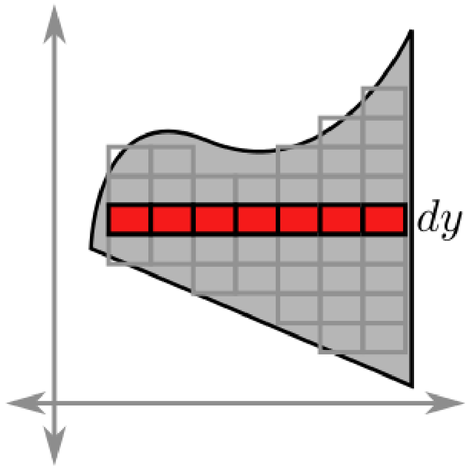
In future courses, the area element of a region will not be a strip of area but a small rectangle with width dx and height dy:



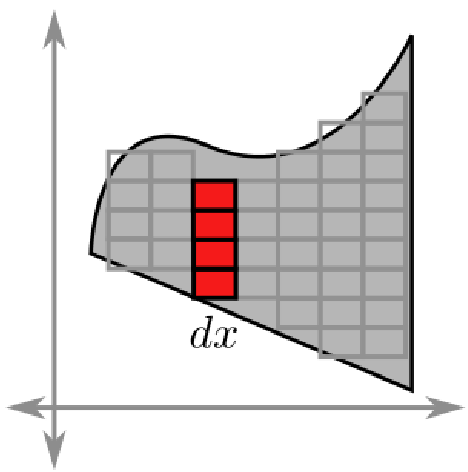
The area of the region, then, is the limit of the sum of the areas of all these small rectangles as the rectangles get infinitely small. The notation used to express this is called a double integral, written

$$\text{Area} = \iint_R dx\, dy.$$

Think of the double integral as a nested integral: $\iint dx\, dy = \int (\int dx)\, dy$. The inner integral is performed first, with respect to x (since the dx is left of the dy). Then the result is integrated with respect to y. Conceptually, the inner integral is adding up the contribution of a row of boxes, and then the outer integral is adding up the rows:



Double integrals can be computed in the other order too: $\iint dy\,dx$. First the inner integral is performed with respect to y, which adds up the contribution of a column of boxes. Then the outer integral adds up the contribution of the columns:



Example

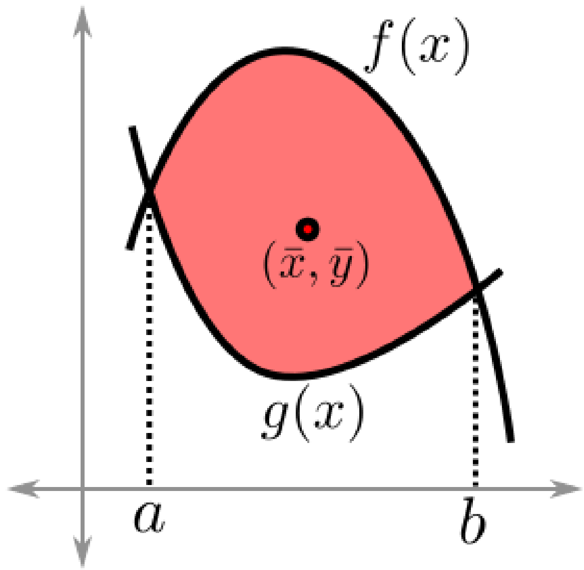
Express the area of the region bounded by the curves $y = x^2 - 4x + 5$ and $y = x + 1$ as a double integral and evaluate the integral. (See Answer 1)

40.2 Centroid

The *centroid* of a region R in the plane is defined to be the point $(\overline{x}, \overline{y})$, where $\overline{x}$ is the average x-coordinate of R and $\overline{y}$ is the average y-coordinate of R. One interpretation is that if the region were cut out of a sheet of uniform density metal and a pin were placed at its centroid, the region would balance on the pin.

The centroid is best expressed mathematically using double integrals:

$$\overline{x} = \frac{\iint_R x\, dx\, dy}{\iint_R\, dx\, dy}$$
$$\overline{y} = \frac{\iint_R y\, dx\, dy}{\iint_R\, dx\, dy}.$$



Suppose the region R is bounded above by the curve $y = f(x)$ and below by the curve $y = g(x)$, and the intersection points are at $x = a$ and $x = b$. Then integration is easier in the $dy\, dx$ order, and the centroid can be written more explicitly as

Centroid of a region

The centroid of the region bounded above by $y = f(x)$ and below by $y = g(x)$ is given by

$$\begin{aligned}\overline{x} &= \frac{\int_a^b \int_{g(x)}^{f(x)} x dy dx}{\int_a^b \int_{g(x)}^{f(x)} dy dx} \\ &= \frac{\int_a^b x(f(x) - g(x))dx}{\int_a^b (f(x) - g(x))dx}.\end{aligned}$$

Similarly,

$$\begin{aligned}\overline{y} &= \frac{\int_a^b \int_{g(x)}^{f(x)} y\, dy\, dx}{\int_a^b \int_{g(x)}^{f(x)}\, dy\, dx} \\ &= \frac{\int_a^b \frac{1}{2}(f(x)^2 - g(x)^2)\, dx}{\int_a^b (f(x) - g(x))\, dx}.\end{aligned}$$

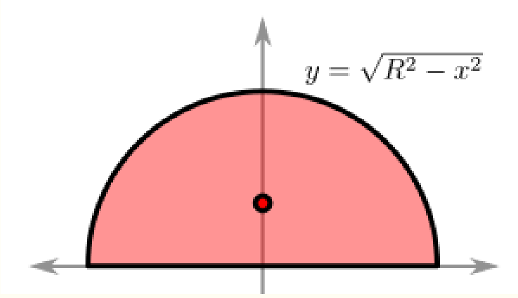
Note that the denominator in each case is the area of the region.

Example

Find the centroid of a triangle with vertices at $(a, 0), (b, 0)$, and $(0, c)$. (See Answer 2)

Example

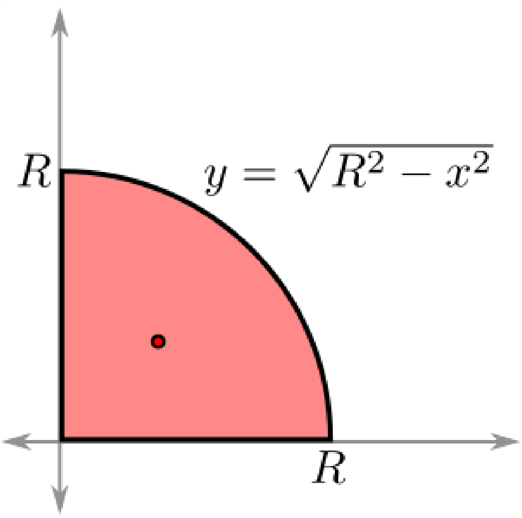
Compute the centroid of the upper half circle of radius R.



(See Answer 3)

Example

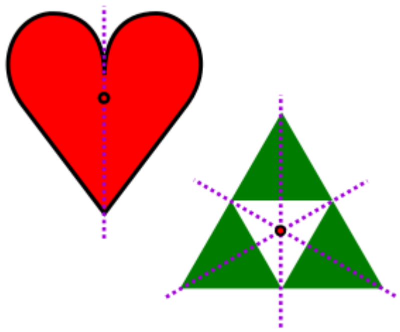
Compute the centroid of the quarter circle of radius R:



(See Answer 4)

Symmetry

It is important to note that centroids respect symmetry. What that means is that if there is an axis of symmetry (i.e. a line where if we reflect the region about the line we get the same region back), then the centroid must lie on the axis of symmetry. If there is more than one axis of symmetry, then the centroid will lie at the intersection of these axes:



40.3 Center of mass

Now consider a region R of the plane cut from a sheet of metal of variable density $\rho(x, y)$. Again, the problem is to find the balancing point $(\overline{x}, \overline{y})$, but in this context it is called the *center of mass*. Again, it is expressed as a double integral:

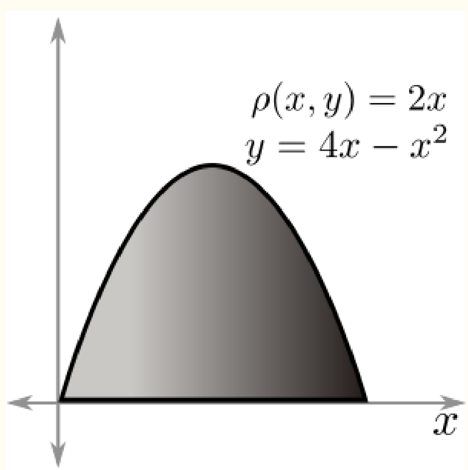
$$\overline{x} = \frac{\iint_R \rho(x, y)x\,dx\,dy}{\iint_R \rho(x, y)\,dx\,dy}$$
$$\overline{y} = \frac{\iint_R \rho(x, y)y\,dx\,dy}{\iint_R \rho(x, y)\,dx\,dy}.$$

The only difference between these and the centroid formulas is that instead of the area element $dA = dx\,dy$, the mass element $dM = \rho(x, y)\,dx\,dy$ is used (multiplying the area element by the density of that point gives the mass contributed by that small rectangle). Indeed, if the density is constant, then $\rho(x, y) = \rho$ factors out of both the numerator and denominator and cancel, leaving the formula for centroid.

Note that the denominator for both $\overline{x}$ and $\overline{y}$ is the mass of the region.

Example

Compute the center of mass of the region bounded above by $y = 4x - x^2$ and below by the x-axis, where the density function is given by $\rho(x, y) = 2x$:



(See Answer 5)

40.4 Centroids using point masses

Given a complex region which consists of the union of simpler regions, there is a method for finding the centroid:

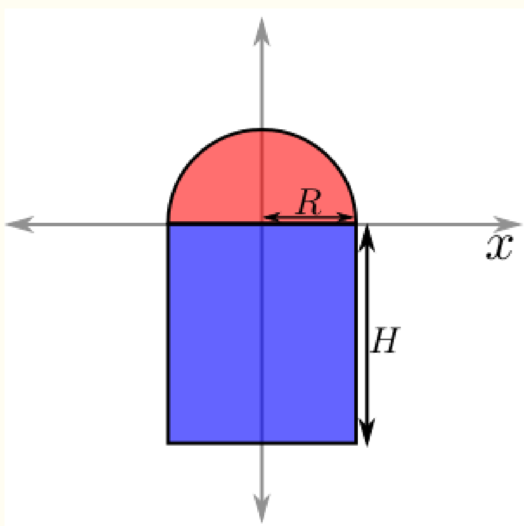
1. Find the centroid of each simple region.
2. Replace each region with a point mass at its centroid, where the mass is the area of the region.
3. Find the centroid of these point masses (this is done by taking a weighted average of their x and y coordinates).

(Centroids and Point Masses Animated GIF)

This is easiest to see with an example:

Example

Find the centroid of a region consisting of a rectangle of width $2R$ and height H which has a semicircle of radius R on one end:



(See Answer 6)

40.5 Application: Pappus' theorem

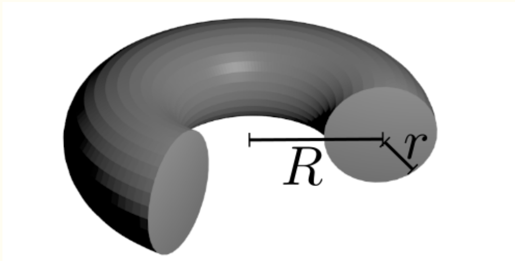
One application of the centroid is known as *Pappus' theorem*, after the Greek mathematician Pappus of Alexandria. It uses the centroid to find the volume and surface area of a solid of revolution.

Pappus' theorem

Consider the solid which results from rotating the plane region R about the axis L.
The volume of this solid is equal to the area of R times the distance the centroid travels (as it gets revolved around the axis).
The surface area of the solid is equal to the perimeter of R times the distance the centroid travels.

Example

Find the volume and surface area of a torus (i.e. a doughnut) with cross sectional radius r and main radius R:

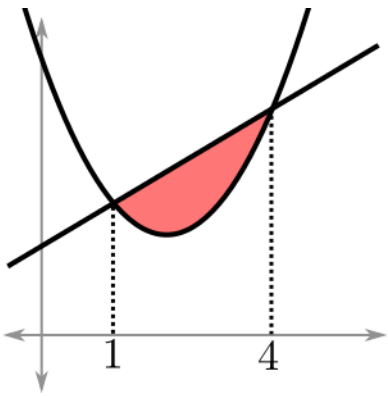


(See Answer 7)

40.6 EXERCISES

- Compute the area of region bounded by curves $x = (y-2)^2 + 2$ and $y = x - 2$ using double integrals.
- Consider the region under the graph $y = x^2$, above the x-axis, from $x = 0$ to $x = 1$. Let S be the solid obtained by revolving this region about the y-axis. Compute the average height (average y-coordinate) of S.
- Let $R1$ denote the region inside the triangle with vertices at (0,1), (-2,0), (0,-1). Given a unit circle centered at the origin, let $R2$ denote the region inside the semicircle for $x \geq 0$. Let R denote the union $R1$ and $R2$. compute the centroid of R.

40.7 Answers to Selected Exercises

1. 

The easier order of integration is $dy\,dx$ because every vertical strip is bounded on top by $y = x + 1$ and bounded below by $y = x^2 - 4x + 5$; whereas a horizontal strip would sometimes be bounded on the left by $y = x + 1$, and other times be bounded by $y = x^2 - 4x + 5$.

Setting the curves equal gives the intersections at $x = 1$ and $x = 4$. So the area can be found by computing

$$\begin{aligned}
\int_{x=1}^{x=4} \int_{y=x^2-4x+5}^{y=x+1} dy\,dx &= \int_{x=1}^{x=4} \left(y \Big|_{x^2-4x+5}^{x+1} \right) dx \\
&= \int_{x=1}^{x=4} (x + 1 - (x^2 - 4x + 5))\,dx \\
&= \int_{x=1}^{x=4} (-x^2 + 5x - 4)\,dx \\
&= -\frac{x^3}{3} + \frac{5}{2}x^2 - 4x \Big|_1^4 \\
&= \frac{9}{2}.
\end{aligned}$$

(Return)

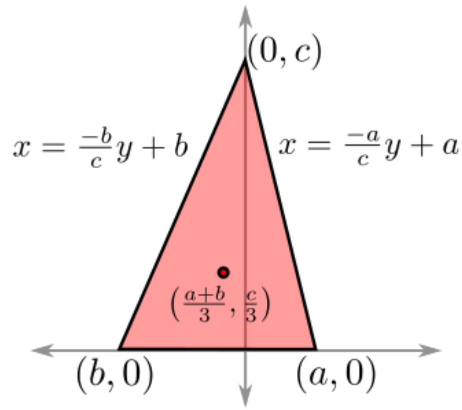
2. The easier order of integration is $dx\,dy$ because a horizontal strip is always bounded on the left by $x = \frac{-b}{c}y + b$ and on the right by $x = \frac{-a}{c}y + a$ (see the diagram below). So one finds that

$$\begin{aligned}
\overline{x} &= \frac{\int_{y=0}^{y=c} \int_{x=\frac{-b}{c}y+b}^{x=\frac{-a}{c}y+a} x\,dx\,dy}{\int_{y=0}^{y=c} \int_{x=\frac{-b}{c}y+b}^{x=\frac{-a}{c}y+a} dx\,dy} \\
&= \frac{\int_{y=0}^{y=c} \int_{x=\frac{-b}{c}y+b}^{x=\frac{-a}{c}y+a} x\,dx\,dy}{\text{Area}}
\end{aligned}$$

Noting that the area of the triangle is $\frac{1}{2}(a - b)c$, one finds

$$\begin{aligned}
\overline{x} &= \frac{2}{(a-b)c} \int_{y=0}^{y=c} \int_{x=\frac{-b}{c}y+b}^{x=\frac{-a}{c}y+a} x\,dx\,dy \\
&= \frac{2}{(a-b)c} \int_{y=0}^{y=c} \frac{1}{2} \left((\frac{-a}{c}y + a)^2 - (\frac{-b}{c}y + b)^2 \right) dy \\
&= \frac{1}{(a-b)c} \cdot \frac{1}{3} \left((\frac{-a}{c}y + a)^3 \frac{-c}{a} - (\frac{-b}{c}y + b)^3 \frac{-c}{b} \right) \Big|_0^c \\
&= \frac{1}{3(a-b)c}(a^2c - b^2c) \\
&= \frac{1}{3(a-b)c} c(a+b)(a-b) \\
&= \frac{1}{3}(a+b).
\end{aligned}$$

A similar computation gives that $\overline{y} = \frac{c}{3}$.



More generally, the centroid of a triangle with coordinates (x_0, y_0), (x_1, y_1), and (x_2, y_2) is

$$(\overline{x}, \overline{y}) = \left(\frac{x_0 + x_1 + x_2}{3}, \frac{y_0 + y_1 + y_2}{3}\right).$$

In other words, the centroid of a triangle is the average of the x coordinates and the average of the y coordinates.

(Return)

3. By the symmetry about the y-axis, the x-coordinate of the centroid is 0.

   To find the y-coordinate, note that the equation of the curve is $y = \sqrt{R^2 - x^2}$. Also, note that the area of the region is $\frac{1}{2}\pi R^2$. Thus,

$$\begin{aligned}\overline{y} &= \frac{2}{\pi R^2}\int_{x=-R}^{R} \frac{1}{2}(\sqrt{R^2 - x^2})^2\,dx \\ &= \frac{1}{\pi R^2}\left(R^2 x - \frac{1}{3}x^3\right)\Big|_{x=-R}^{R} \\ &= \frac{1}{\pi R^2}\cdot\frac{4R^3}{3} \\ &= \frac{4R}{3\pi}.\end{aligned}$$

   (Return)

4. We know that the area of the region is $\frac{1}{4}\pi R^2$. So we have that

$$\begin{aligned}\overline{x} &= \frac{1}{A}\int x(f(x) - g(x))\,dx \\ &= \frac{4}{\pi R^2}\int_{x=0}^{R} x(\sqrt{R^2 - x^2} - 0)\,dx.\end{aligned}$$

   Making a substitution of

$$\begin{aligned}u &= R^2 - x^2 \\ du &= -2x\,dx\end{aligned}$$

gives

$$\frac{4}{\pi R^2}\int_{x=0}^{R} x\sqrt{R^2-x^2}\,dx = \frac{4}{\pi R^2}\int_{u=R^2}^{0} -\frac{1}{2}\sqrt{u}\,du$$
$$= \frac{-2}{\pi R^2}\frac{2}{3}u^{3/2}\Big|_{u=R^2}^{0}$$
$$= \frac{2}{\pi R^2}\cdot\frac{2}{3}R^3$$
$$= \frac{4R}{3\pi}.$$

Because the region is symmetric about the line $y = x$, we predict that $\overline{y} = \frac{4R}{3\pi}$ as well. We can verify this by integrating:

$$\overline{y} = \frac{1}{A}\int \frac{1}{2}(f(x)^2 - g(x)^2)\,dx$$
$$= \frac{2}{\pi R^2}\int_{x=0}^{R}(R^2 - x^2)\,dx$$
$$= \frac{2}{\pi R^2}\left(R^2x - \frac{1}{3}x^3\right)\Big|_{x=0}^{R}$$
$$= \frac{2}{\pi R^2}\left(R^3 - \frac{1}{3}R^3\right)$$
$$= \frac{2}{\pi R^2}\cdot\frac{2}{3}R^3$$
$$= \frac{4R}{3\pi},$$

as claimed.

(Return)

5. Setting $y = 0$, we find that the curve intersects the x-axis at $x = 0$ and $x = 4$. First, we compute the mass of the region, which is the denominator for both $\overline{x}$ and $\overline{y}$. It is easier to integrate in the $dy\,dx$ order, so we will do that here, and in the integrals that follow.

$$M = \iint_R \rho(x, y)\,dy\,dx$$
$$= \int_{x=0}^{4}\int_{y=0}^{4x-x^2} 2x\,dy\,dx$$
$$= \int_{x=0}^{4}\left(2xy\Big|_{y=0}^{4x-x^2}\right)dx$$
$$= \int_{x=0}^{4} 2x(4x - x^2)\,dx$$
$$= \int_{x=0}^{4}(8x^2 - 2x^3)\,dx$$
$$= \frac{8}{3}x^3 - \frac{1}{2}x^4\Big|_{x=0}^{4}$$
$$= \frac{512}{3} - 128 = \frac{128}{3}.$$

So to compute $\overline{x}$, we find

$$\begin{aligned}
\overline{x} &= \frac{1}{M}\iint_R \rho(x,y)x\,dy\,dx \\
&= \frac{3}{128}\int_{x=0}^{4}\int_{y=0}^{4x-x^2}(2x)x\,dy\,dx \\
&= \frac{3}{128}\int_{x=0}^{4}\left(2x^2y\Big|_{y=0}^{4x-x^2}\right)dx \\
&= \frac{3}{128}\int_{x=0}^{4}2x^2(4x-x^2)\,dx \\
&= \frac{3}{128}\left(2x^4-\frac{2}{5}x^5\right)\Big|_{x=0}^{4} \\
&= \frac{3}{128}512-\frac{2048}{5} \\
&= \frac{3}{128}\cdot\frac{512}{5}=\frac{12}{5}.
\end{aligned}$$

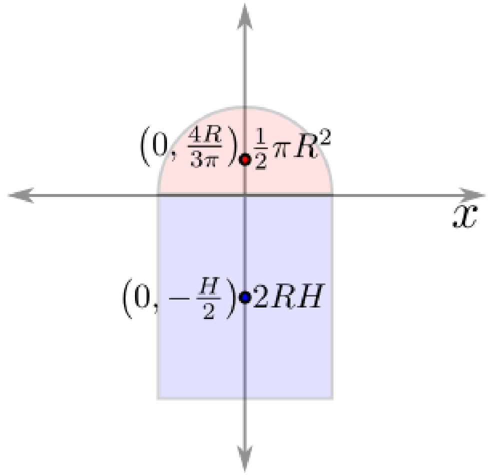
Similarly,

$$\begin{aligned}
\overline{y} &= \frac{1}{M}\iint_R \rho(x,y)y\,dy\,dx \\
&= \frac{3}{128}\int_{x=0}^{4}\int_{y=0}^{4x-x^2}(2x)y\,dy\,dx \\
&= \frac{3}{128}\int_{x=0}^{4}xy^2\Big|_{y=0}^{4x-x^2}dx \\
&= \frac{3}{128}\int_{x=0}^{4}x(4x-x^2)^2\,dx \\
&= \frac{3}{128}\int_{x=0}^{4}(16x^3-8x^4+x^5)\,dx \\
&= \frac{3}{128}\left(4x^4-\frac{8}{5}x^5+\frac{1}{6}x^6\right)\Big|_{x=0}^{4} \\
&= \frac{3}{128}\cdot 1024\left(1-\frac{8}{5}+\frac{2}{3}\right)=\frac{8}{5}.
\end{aligned}$$

(Return)

6. From an earlier example, the centroid of the semicircle is $(0,\frac{4R}{3\pi})$, and the weight (the area of the semicircle) is $\frac{1}{2}\pi R^2$.

   The rectangle is symmetric, so its centroid (as it is drawn in the coordinate plane) is $(0,-\frac{H}{2})$, and its weight is $2RH$:



By symmetry,

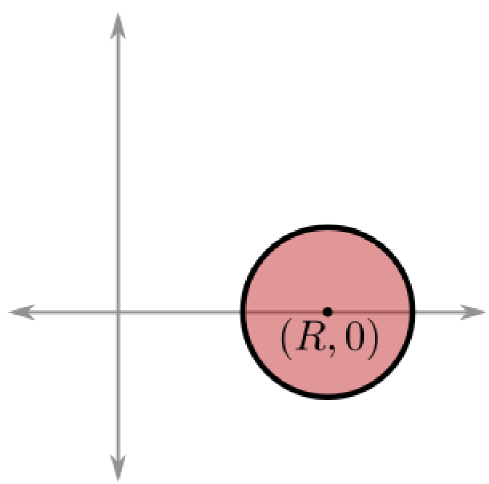
$$\overline{x} = 0.$$

Taking the weighted average of the y-coordinates of the points gives

$$\begin{aligned}\overline{y} &= \frac{\frac{1}{2}\pi R^2 \cdot \frac{4R}{3\pi} + 2RH \cdot \left(-\frac{H}{2}\right)}{\frac{1}{2}\pi R^2 + 2RH} \\ &= \frac{\frac{2}{3}R^3 - H^2 R}{\frac{1}{2}\pi R^2 + 2RH} \\ &= \frac{4R^2 - 6H^2}{3\pi R + 12H}.\end{aligned}$$

We can check that this is reasonable by noting that if $H = 0$ we get the y-coordinate of the centroid of the semicircle, and when $R = 0$ we get the y-coordinate of the centroid of the line segment from $(0, 0)$ to $(0, -H)$.

(Return)



7.

Here, the region being rotated is a circle, which is easy to work with because a circle's centroid is just its center. For convenience, center the circle at $(R, 0)$ and revolve around the y-axis. Then the distance which the centroid travels is $2\pi R$ (the path of the centroid is just a circle of radius R).

Therefore the surface area of the torus is

$$\begin{aligned}\text{Surface area} &= \text{Perimeter} \ \cdot \ \text{Centroid travel distance} \\ &= (2\pi r) \cdot (2\pi R) \\ &= 4\pi^2 r \cdot R.\end{aligned}$$

And the volume of the torus is

$$\begin{aligned}\text{Volume} &= \text{Area} \ \cdot \ \text{Centroid travel distance} \\ &= (\pi r^2) \cdot (2\pi R) \\ &= 2\pi^2 r^2 R.\end{aligned}$$

(Return)

41 Moments And Gyrations

This module deals with the moment of inertia and the radius of gyration, which are two properties of an object with physical interpretations.

41.1 Moment of inertia

The *moment of inertia* of an object, usually denoted I, measures the object's resistance to rotation about an axis. To get an intuitive understanding of moment of inertia consider swinging a hammer by its handle (higher moment of inertia, harder to swing) versus swinging a hammer by its head (lower moment of inertia, easier to swing). So moment of inertia depends on both the object being rotated and the axis about which it is being rotated.

(Hammer Animated GIF)

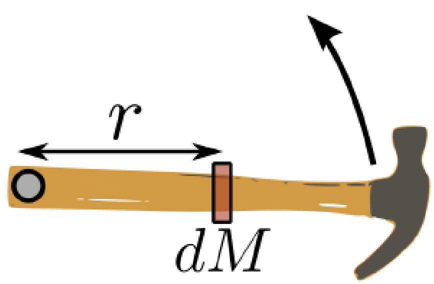
Consider first a particle of mass. The bigger the mass, the more resistant it will be to rotation about an axis. Similarly, the further the particle is from the axis, the more resistant it will be to rotation. For a point mass, the moment of inertia is given by

$$I = r^2 M,$$

where r is the distance of the particle from the axis of rotation, and M is the mass of the particle:

(Particle Animated GIF)

The next question is how to calculate the moment of inertia when all the mass is not at a single point. As in previous modules, the method will be to break the object into slices of mass, and consider the contribution of each slice to the moment of inertia:



Each slice can be thought of as an individual particle of mass which contributes to the moment of inertia. The contribution of the slice becomes the moment of inertia element dI:

$$dI = r^2\,dM.$$

Example

Consider a solid disc of radius R and constant density ρ rotated about its central vertical axis:

(Disk Rotating Around Diameter Animated GIF)

Compute its moment of inertia. (See Answer 1)

Example

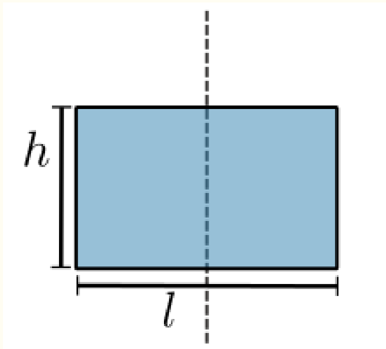
Consider a solid disc of radius R and constant density ρ rotated about its center:

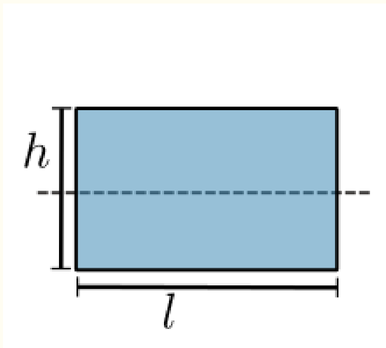
(Disk Rotating Around Center Animated GIF)

Compute its moment of inertia. (See Answer 2)

Example

Consider a rectangle of length l and height h. Compute the moment of inertia about the vertical axis through its center. Then compute the moment of inertia about the horizontal axis through its center. Hint: use symmetry to find the second answer from the first.





(See Answer 3)

41.2 Radius of gyration

Another property of an object, *radius of gyration*, denoted R_g, can be expressed in terms of the moment of inertia. Imagine replacing the object being rotated about an axis by a single point mass being rotated about that same axis. The radius of gyration is the radius at which the point mass has the same moment of inertia as the object. More specifically, $I = MR_g^2$, and solving for R_g gives

$$R_g = \sqrt{\frac{I}{M}}.$$

Note that because

$$I = \int r^2 \, dM$$

we can write

$$\begin{aligned} R_g &= \sqrt{\frac{\int r^2 \, dM}{\int dM}} \\ &= \sqrt{\overline{r^2}} \\ &= r_{RMS} \end{aligned}$$

So the radius of gyration is really the root mean square of the radius.

41.3 Higher mass moments

In the centroid module, we computed $\int x \, dM$ as part of computing the x-coordinate of the center of mass, $\overline{x}$.The moment of inertia I from this module is given by $\int x^2 \, dM$. These are respectively known as the first mass moment and the second mass moment (first and second referring to the powers of x).

There are higher mass moments: $\int x^n \, dM$, for $n \geq 3$, as well as the lower mass moment $\int x^0 dM$, which is just mass. These moments each give more information about how the mass of the object is distributed.

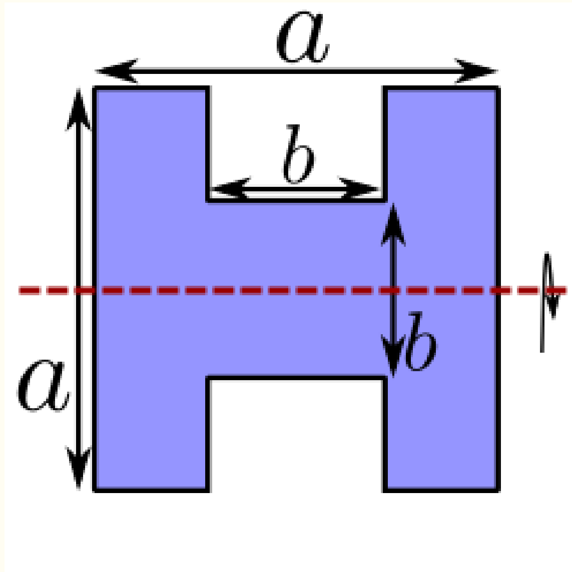
This is similar, in a sense, to how knowledge of the derivative of a function at a point leads to an approximation of the function using the Taylor series. The more derivatives one knows, the better the approximation. A logical question, then, is if one knows all the mass moments of an object, can one perfectly describe the distribution of mass?

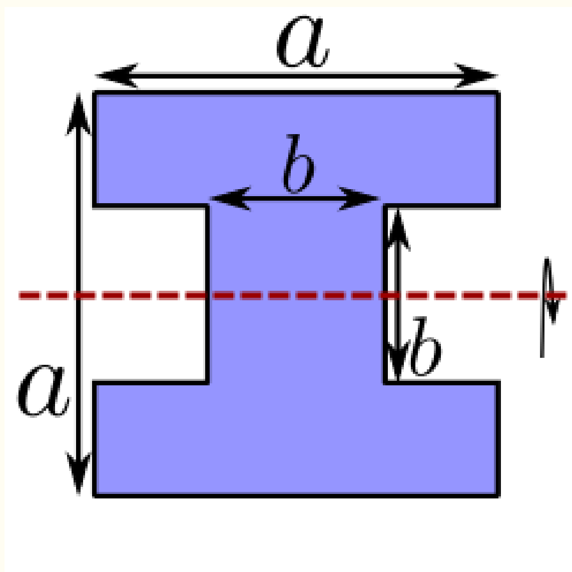
41.4 Additivity of moments

One nice feature of moments is that, being integrals, they are additive. This means that a complex region can be split into simpler regions for which we already know the moment of inertia, and these moments can be added to find the moment of inertia for the entire region.

Example

Compute the moment of inertia for each of the following figures about a horizontal axis through their centers.





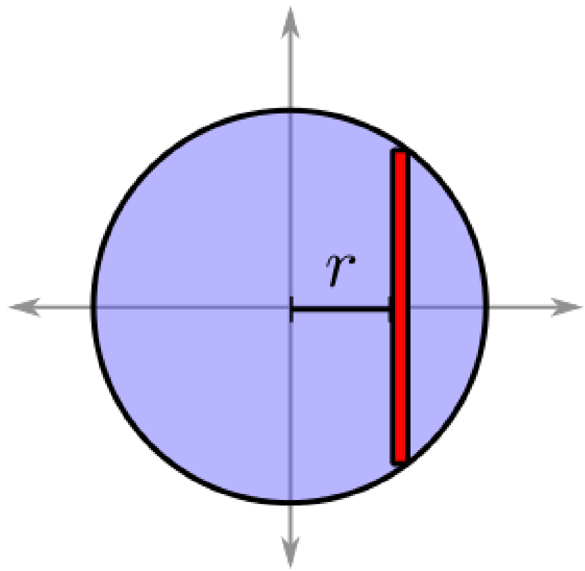
Which has the greater moment of inertia, the I-shaped figure or the H-shaped figure? (See Answer 4)

41.5 EXERCISES

- Consider a right triangle with vertices at (0,0), (5,0), (0,10). Consider rotating the triangle about the y-axis. The density is given by $\rho(x, y) = x$. Compute the moment of inertia. Compute the radius of gyration.

41.6 Answers to Selected Examples

1. Because the distance to the axis is part of the inertia element, a good area element to use is a vertical rectangle, where every point has the same distance to the center axis. Let x be the distance from the central axis to the rectangle (thus, $r = x$):



The area of this rectangle, as has been computed several times previously, is $dA = 2\sqrt{R^2 - x^2}\,dx$. Then the mass element $dM = \rho\,dA$, and it follows that

$$\begin{aligned} dI &= r^2\,dM \\ &= 2x^2\rho\sqrt{R^2 - x^2}\,dx. \end{aligned}$$

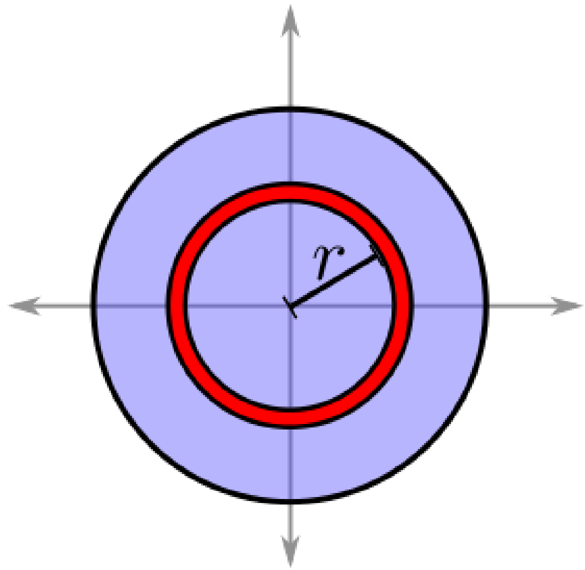
so integrating the inertia element gives

$$\begin{aligned} I &= \int dI \\ &= \int_{x=-R}^{R} 2\rho x^2\sqrt{R^2 - x^2}\,dx \\ &= 4\rho\int_{x=0}^{R} x^2\sqrt{R^2 - x^2}\,dx \end{aligned}$$

(using the fact that the integrand is an even function allows the final step). Now the substitution $x = R\sin\theta$, and some of the trig integral methods gives the answer $\frac{\rho\pi}{4}R^4$, which can also be written $\frac{1}{4}MR^2$, where $M = \pi R^2\rho$ is the mass of the disc.

(Return)

2. In this example, a good area element to use is a ring (also called an annulus), because every point in a ring has the same distance to the origin, which is the axis of rotation (one can imagine the axis sticking out of the page perpendicular to the center of the disc):



As before, $dM = \rho\, dA$. In this case, dA is the area of the ring, which is $2\pi r\, dr$ (the circumference of the ring times the width of the ring). It follows that

$$\begin{aligned} I &= \int dI \\ &= \int_{r=0}^{R} r^2 \rho 2\pi r\, dr \\ &= 2\pi\rho \int_{r=0}^{R} r^3\, dr \\ &= 2\pi\rho \frac{r^4}{4}\bigg|_{r=0}^{R} \\ &= \frac{\pi\rho R^4}{2}. \end{aligned}$$

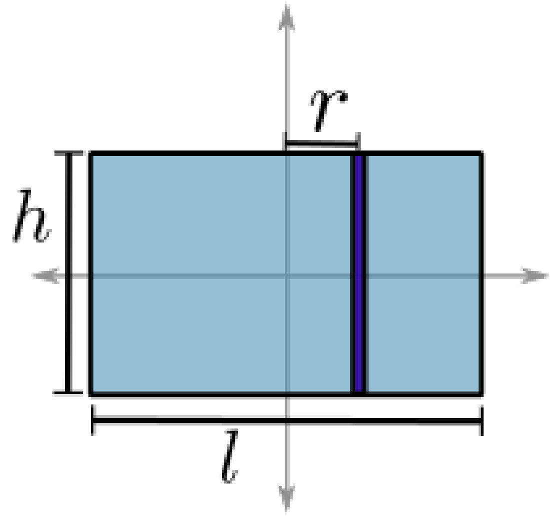
This can be expressed as $\frac{1}{2}MR^2$, where M is again the mass of the disc. Note that the answer in this example is twice that of the previous example. This can be explained (using the answer from the previous example) by noting that $r^2 = x^2 + y^2$ in this example. Therefore,

$$\begin{aligned} I &= \int r^2\, dM \\ &= \int (x^2 + y^2)\, dM \\ &= \int x^2\, dM + \int y^2\, dM, \end{aligned}$$

and these two integrals are, respectively, the moment of inertia about a vertical axis (from the previous example) and the moment of inertia about a horizontal axis. By symmetry, these are equal, which explains why this answer is twice the answer of the previous example.

(Return)

3. Center the rectangle at the origin. About the vertical axis, it is again best to use vertical rectangles. Let r denote the distance of this rectangle from the y-axis:



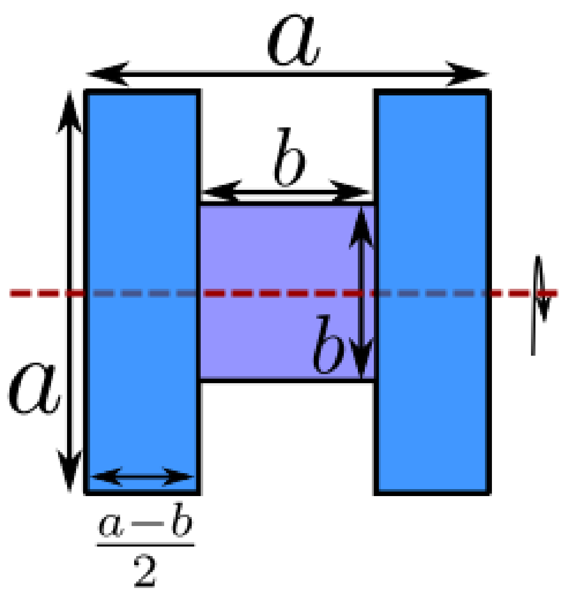
Then $r = x$, and $dM = \rho h\,dx$. Thus

$$\begin{aligned} I &= \int dI \\ &= \rho h \int_{x=-l/2}^{l/2} x^2\,dx \\ &= \rho h \frac{x^3}{3}\bigg|_{x=-l/2}^{l/2} \\ &= \frac{1}{12}\rho h l^3 \\ &= \frac{1}{12} M l^2, \end{aligned}$$

where $M = \rho l h$ is the mass of the rectangle. By symmetry, the moment of inertia about a horizontal axis through the center is $\frac{1}{12}Mh^2$.

(Return)

4. For the first figure, we can divide it into two rectangles (in light blue) and a square which are all being rotated about their horizontal center axis:



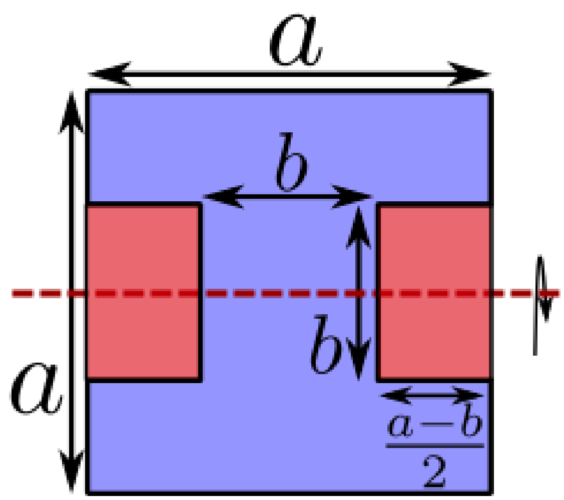
From the above example, we know that the moment of inertia for a rectangle about its horizontal axis is

$$\frac{1}{12}Mh^2 = \frac{1}{12}lh^3,$$

where l is the length and h is the height of the rectangle. So for each of the tall rectangles we have $I = \frac{1}{12}a^3\frac{a-b}{2}$ and for the square in the middle we have $I = \frac{1}{12}b^4$. Putting it together, we have the moment of inertia for the entire region is

$$\begin{aligned} I &= 2 \cdot \frac{1}{12}a^3\frac{a-b}{2} + \frac{1}{12}b^3 b \\ &= \frac{1}{12}(a^4 - a^3 b) + \frac{1}{12}b^4 \\ &= \frac{1}{12}\left(a^4 + b^4 - a^3 b\right). \end{aligned}$$

For the other region, we cannot divide it up into rectangles in the same exact way, because we do not know the moment of inertia for a rectangle rotated about an axis other than one through its center. Instead, we can take the entire square of side length a, and compute its moment of inertia. Then we can subtract off the inertia for the small rectangles we do not want to include, shown in red:



Again, using the fact from the previous example, the moment for the whole square is $\frac{1}{12}a^4$, and the moment for each of the smaller rectangles (which we will subtract) is $\frac{1}{12} \cdot \frac{a-b}{2} \cdot b^3$, so the moment of inertia for the whole region is

$$\begin{aligned} I &= \frac{1}{12}a^4 - 2 \cdot \frac{1}{12} \cdot \frac{a-b}{2} \cdot b^3 \\ &= \frac{1}{12}a^4 - \frac{1}{12}(ab^3 - b^4) \\ &= \frac{1}{12}(a^4 + b^4 - ab^3). \end{aligned}$$

So the I-shaped figure has the greater moment of inertia.

This is important when considering whether to use an H-beam or and I-beam in construction. According to a fact mentioned in higher derivatives, the deflection $u(x)$ (the amount the beam sags at location x) satisfies the equation

$$EI\frac{d^4u}{dx^4} = q(x),$$

where E is the elasticity of the material (a constant), and $q(x)$ is a static load at location x along the beam. Because the I-beam has the greater moment of inertia, it follows that their deflection will be less, and so I-beams are more common in building construction.

(Return)

42 Fair Probability

Probability is the study of the likelihood of certain events occurring in a random experiment. A simple example is a coin flip. There are two outcomes: heads (H) or tails (T). If the coin is fair, then the probability of each outcome is $\frac{1}{2}$, written $P(H) = P(T) = \frac{1}{2}$. Another example is a roll of a standard die. There are six outcomes: 1 through 6. If the die is fair then the probability of each outcome is $\frac{1}{6}$.

In these types of problems, one can find the probability of an event occurring by counting the number of desired outcomes and dividing by the total number of outcomes.

Example

What is the probability that a pair of dice sums to seven or eleven? (See Answer 1)

Example

Alice and Bob play a game where they take turns flipping a fair coin, with Alice going first. The first player to get heads wins. What is the probability that Alice wins?
Hint: find the probability that Alice wins on her first flip, and the probability that she wins on her second flip, and the probability that she wins on her third flip, etc. Add up all these (infinitely many) probabilities to find the probability that she wins.
Second hint: For Alice to win on her second flip, it means that both Alice and Bob got tails on their respective first flips (otherwise the game would have ended in the first round). So the probability of Alice winning on her second flip is

$$P(\text{A got tails}) \cdot P(\text{B got tails}) \cdot P(\text{A got heads}) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \left(\frac{1}{2}\right)^3.$$

(See Answer 2)

42.1 Uniform distribution

The above examples refer to a fair coin and a fair die. A discrete experiment (i.e. the possible outcomes can be listed) is said to have the *uniform distribution* if the experiment is fair in the sense that every outcome is equally likely.

What about experiments which are not discrete? For instance, a spinner gives a point along the circumference of a circle, and the individual points of the circle cannot be enumerated. Throwing a dart at a circular dartboard likewise has as many outcomes as there are points in the interior of the disk. What does it mean for such an experiment to be *fair*, i.e., what does the uniform distribution mean in an experiment with continuous outcomes?

To answer this question, consider the probability of a range of outcomes of the experiment. So, for instance, what is the probability of the spinner landing in the first quarter of the circle? If the experiment is fair, then this probability should be the same as landing in any other quarter of the circle: $\frac{1}{4}$:

(Spinner Animated GIF)

Thus an experiment is *fair* (i.e. has the *uniform distribution*) if for any set of outcomes D,

$$P(D) = \frac{\text{volume of } D}{\text{total volume of all outcomes}}.$$

Here "volume" depends on the dimension of the experiment. For instance, the spinner has dimension 1 (where volume is really just the length) since any point on the circumference can be specified by a single value (say, the angle of the arrow relative to the positive x-axis). So a spinner is considered fair if the probability of the arrow landing in a certain range along the circumference equals the length of that range divided by the total circumference of the circle.

Length

Example

Find the probability that a randomly chosen angle θ has $\sin\theta > \frac{1}{2}$? (See Answer 3)

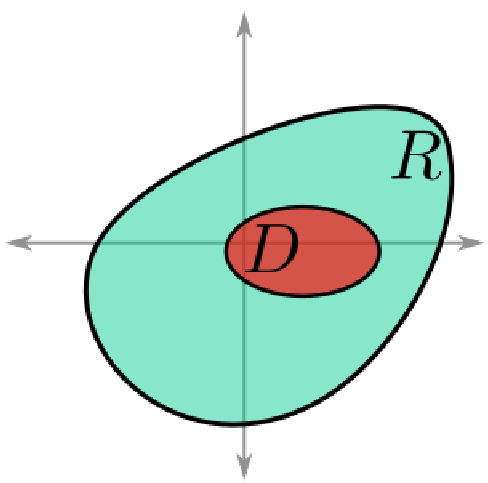
Example

Find the probability that a randomly chosen angle θ has $\tan\theta > 0$. (See Answer 4)

Area

In two dimensions, volume is really area, and so when computing the probability that a randomly chosen point in a region R in the plane lies within the region D, we have

$$P = \frac{\text{Area of } D}{\text{Area of } R}.$$



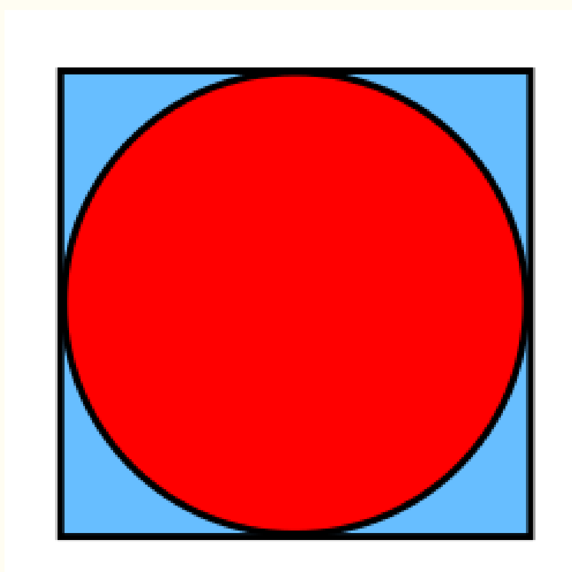
Example

A dartboard is circular with radius 9 inches:

The bullseye is a small circle at the center of the board. Find the radius of the bullseye so that the probability of hitting it is $\frac{1}{100}$ (assuming a throw hits the board uniformly at random). (See Answer 5)

Example

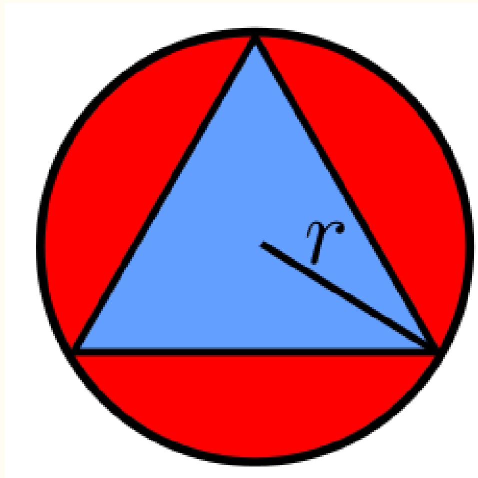
Find the probability that a randomly chosen point in a square lies within the circle inscribed in the square:



(See Answer 6)

Example

Find the probability that a randomly chosen point in a circle lies in the equilateral triangle inscribed in the circle:



Hint: the area of an equilateral triangle of side length s is

$$A = \frac{s^2\sqrt{3}}{4}.$$

(See Answer 7)

There are some probability problems that do not seem geometric in nature but can be solved by graphing the possible outcomes and taking the ratio of the areas.

Example

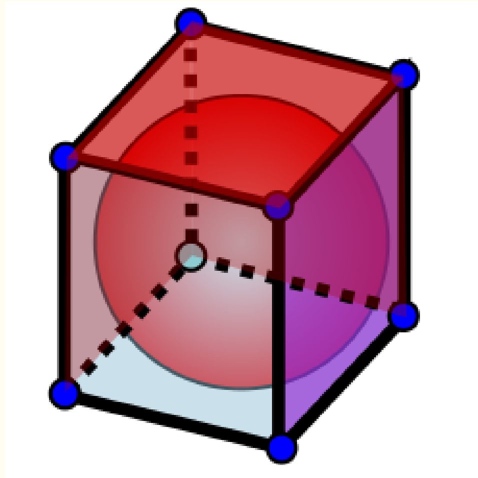
Xander and Yolanda want to meet up to study calculus. Each friend will arrive at the library at some random time between 5 pm and 6pm, wait 20 minutes for the other person, and then leave if the other person does not arrive in that time. Find the probability that the friends successfully meet up.
Hint: Let x be the number of minutes after 5 pm that Xander arrives and y be the number of minutes after 5 pm that Yolanda arrives. Now plot the possible arrival times as a region in the plane and determine the region which corresponds to them successfully meeting up. (See Answer 8)

Volume

Finally, in dimension 3, volume is volume as we traditionally know it. In this case, we imagine picking a point from within a 3D region and know the probability that the point lies within some subset of that region.

Example

Find the probability that a randomly chosen point from within a cube lies within the inscribed sphere:



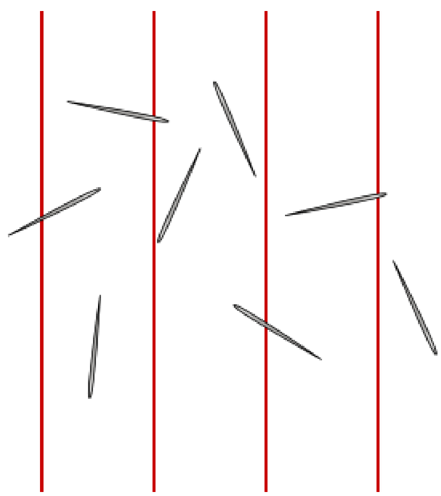
(See Answer 9)

Example

What is the probability that a randomly chosen point in a ball lies within 10% of the boundary (as measured by radius)? (See Answer 10)

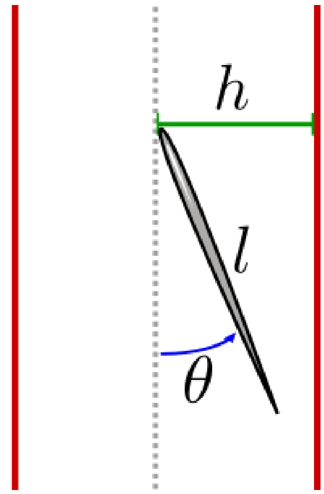
42.2 Buffon needle problem

The Buffon needle problem, named after the Count of Buffon, asks for the probability that a needle of length l, dropped uniformly at random onto a sheet with parallel lines spaced l units apart, will cross a line.

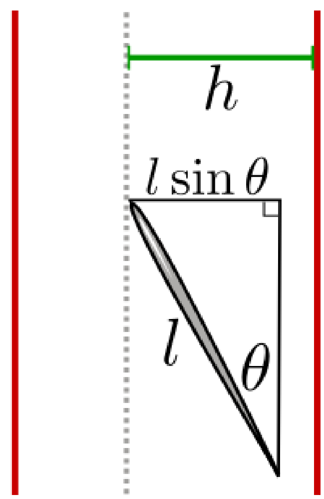


To simplify the problem, consider two parameters which determine whether the needle crosses:

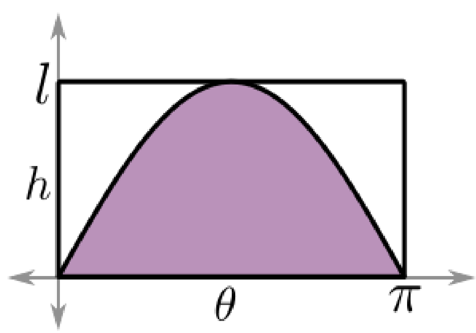
1. h, the distance from the left tip of the needle to the next line to its right
2. θ, the angle that the needle makes with a vertical line:



Note that $0 \leq h \leq l$ and $0 \leq \theta \leq \pi$. Now, for what values of h and θ is there a crossing? Note that by right triangle trigonometry, the horizontal distance from the left end of the needle to the right end of the needle is $l \sin\theta$:



Thus, there is a crossing if $h \leq l \sin\theta$, and there is no crossing if $h > l \sin\theta$. Graphing this inequality shows that the region below the curve (shown in purple) is where a crossing occurs. The region above the curve is where a crossing does not occur.



Dropping a needle at random is like randomly picking a point in this rectangle. Thus, the probability of a random needle creating a crossing equals the probability of randomly picking a point below the curve in the above rectangle. That probability is given by dividing the area under the curve by the area of the entire rectangle.

$$\begin{aligned} P(\text{crossing}) &= \frac{\int_0^\pi l \sin\theta \, d\theta}{l\pi} \\ &= \frac{1}{\pi}\left(-\cos\theta\Big|_0^\pi\right) \\ &= \frac{2}{\pi}. \end{aligned}$$

42.3 Answers to Selected Examples

1. By listing the desired outcomes, one finds that $(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)$ are the possible pairings which give 7, and $(6,5)$ and $(5,6)$ are the possible pairings which give 11. So there are 8 desired outcomes. The total number of outcomes is 6×6 (six outcomes for the first die paired with each of the six outcomes for the other die). So the probability is

   $$\frac{\#\text{ desired}}{\#\text{ total}} = \frac{8}{36} = \frac{2}{9}.$$

   (Return)

2. Proceeding as the hint suggests, we look for a pattern.

   $$P(\text{A wins on 1st flip}) = P(\text{A gets heads}) = \frac{1}{2}.$$

   And then

   $$\begin{aligned} &P(\text{A wins on 2nd flip}) \\ &= P(\text{A gets tails}) \cdot P(\text{B gets tails}) \cdot P(\text{A gets heads}) \\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \\ &= \left(\frac{1}{2}\right)^3. \end{aligned}$$

   Next,

   $$\begin{aligned} &P(\text{A wins on 3rd flip}) \\ &= P(\text{A gets tails}) \cdot P(\text{B gets tails}) \cdot P(\text{A gets tails}) \cdot P(\text{B gets tails}) \cdot P(\text{A gets heads}) \\ &= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \\ &= \left(\frac{1}{2}\right)^5. \end{aligned}$$

   In general, for Alice to win on the nth flip, she must get a head on that flip, *and* both Alice and Bob must have gotten tails on each of their previous $n-1$ flips. Thus, there are a total of $2(n-1)+1 = 2n-1$ coin flips that must come out in a precise way, and the probability of each of these is $\frac{1}{2}$, so we have

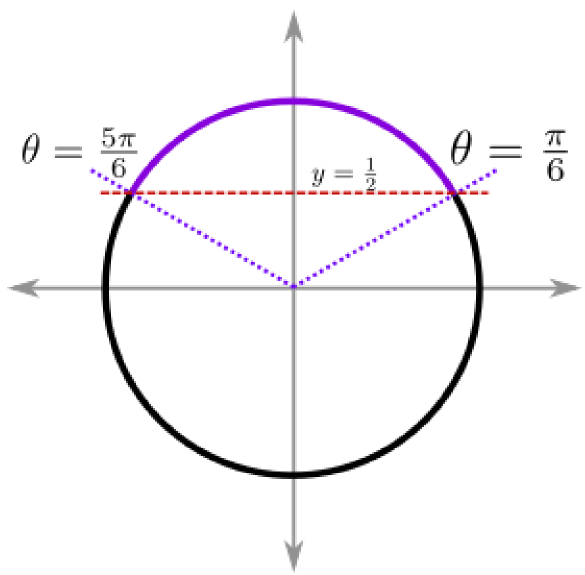
   $$P(\text{A wins on } n\text{th flip}) = \left(\frac{1}{2}\right)^{2n-1}.$$

Adding these up for all n, and using the geometric series, gives

$$
\begin{aligned}
P(\text{A wins}) &= \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{2n-1} \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^5 + \cdots \\
&= \frac{1}{2} \cdot \left(1 + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^4 + \cdots\right) \\
&= \frac{1}{2} \cdot \left(1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right) \\
&= \frac{1}{2} \cdot \frac{1}{1 - 1/4} \\
&= \frac{2}{3}.
\end{aligned}
$$

(Return)

3. We can visualize the sine of the angle by considering a unit circle, and noting that sine is the y-coordinate of a point on the circle:



Then one finds that the angles for which $\sin\theta > \frac{1}{2}$ are

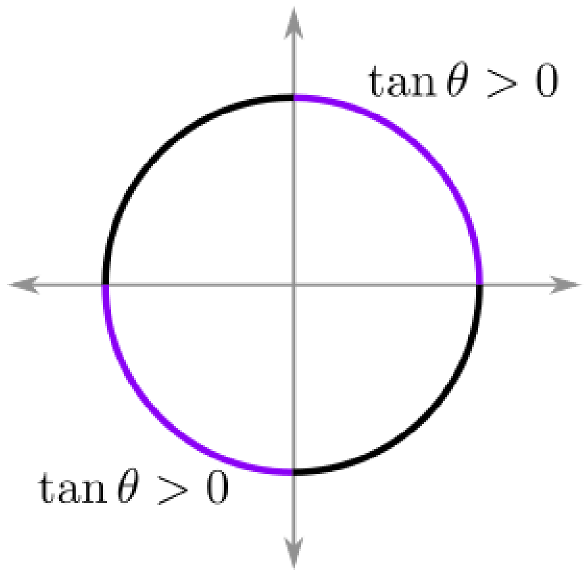
$$\frac{\pi}{6} < \theta < \frac{5\pi}{6}.$$

The length of this portion of the circumference of the circle is $\frac{4\pi}{6}$, and so the probability of a random angle θ satisfying $\sin\theta > \frac{1}{2}$ is

$$
\begin{aligned}
P &= \frac{4\pi/6}{2\pi} \\
&= \frac{1}{3}.
\end{aligned}
$$

(Return)

4. Note that tangent is positive when sine and cosine have the same sign, i.e. if sine and cosine are both positive or if sine and cosine are both negative. This corresponds to the first and third quadrants of the unit circle:



The length of each these arcs is $\frac{\pi}{2}$, and so the probability that $\tan\theta > 0$ is

$$\begin{aligned} P &= \frac{2 \cdot \pi/2}{2\pi} \\ &= \frac{1}{2} \end{aligned}$$

(Return)

5. Ignoring the unnecessary detail of the dartboard, let the radius of the bull's eye be r. Then

$$\begin{aligned} P\left((\text{ bullseye } \right) &= \frac{A(\text{ bullseye })}{A(\text{ board })} \\ &= \frac{\pi r^2}{\pi \cdot 9^2} \\ &= \frac{r^2}{81}. \end{aligned}$$

Setting equal to $\frac{1}{100}$ and solving gives $r = 0.9$ inches. (In reality, the bullseye is much smaller, but the numbers worked out nicer in this example).

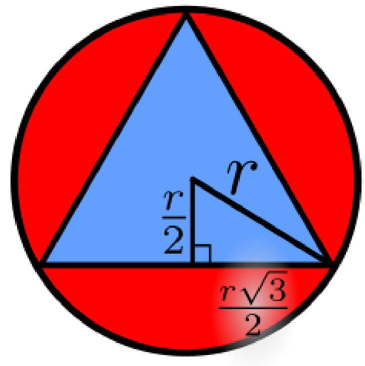
(Return)

6. If the radius of the circle is r, then the side length of the square is $2r$. Thus, the area of the circle is πr^2 and the area of the square is $(2r)^2 = 4r^2$. And so the probability that a point chosen at random within the square also lies within the circle is

$$\begin{aligned} P &= \frac{\pi r^2}{4r^2} \\ &= \frac{\pi}{4}. \end{aligned}$$

(Return)

7. By doing a little bit of right triangle trigonometry:



we find that the side length of the triangle is

$$s = r\sqrt{3}.$$

Therefore, the area of the triangle is

$$\begin{aligned} A &= \frac{s^2\sqrt{3}}{4} \\ &= \frac{3r^2\sqrt{3}}{4}. \end{aligned}$$

And so the probability of a point within the circle being within the triangle is the ratio of the areas:

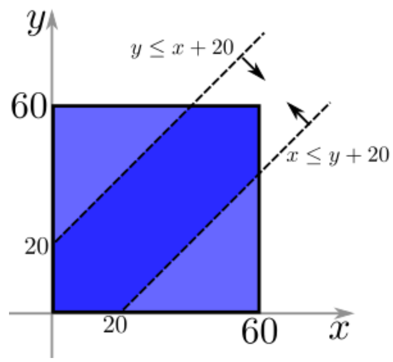
$$\begin{aligned} \frac{\text{Area of triangle}}{\text{Area of circle}} &= \frac{1}{\pi r^2}\frac{3r^2\sqrt{3}}{4} \\ &= \frac{3\sqrt{3}}{4\pi} \end{aligned}$$

(Return)

8. The possible outcomes form a square for $0 \leq x \leq 60$ and $0 \leq y \leq 60$. For the friends to meet, we must have that Yolanda arrives no later than 20 minutes after Xander *and* that Xander arrives no later than 20 minutes after Yolanda arrives. Mathematically,

$$\begin{aligned} y &\leq x + 20 \\ x &\leq y + 20 \end{aligned}$$

The two will successfully meet if and only if these two conditions are met. Graphing these inequalities, the points they have in common are shown below in dark blue:



So the probability of them meeting is the area of the dark blue region divided by the total area of the square. It is easier to determine the area of the region we do not want and subtract. There are two isosceles right triangles of side length 40, so the area of the region we do not want is

$$\text{bad area } = 2 \cdot \frac{1}{2} \cdot 40 \cdot 40 = 1600$$

Therefore, the probability of the friends meeting is

$$\begin{aligned} P &= \frac{\text{area of dark blue region}}{\text{total area}} \\ &= \frac{\text{total area} - \text{light blue area}}{\text{total area}} \\ &= \frac{3600 - 1600}{3600} \\ &= \frac{2000}{3600} \\ &= \frac{5}{9}. \end{aligned}$$

(Return)

9. If r is the radius of the inscribed sphere, then the side length of the cube is $2r$. Therefore, the volume of the sphere is $\frac{4}{3}\pi r^3$ and the volume of the cube is $(2r)^3 = 8r^3$. So the probability that a random point within in the cube lies within the sphere is

$$\begin{aligned} P &= \frac{(4/3)\pi r^3}{8r^3} \\ &= \frac{\pi}{6}. \end{aligned}$$

(Return)

10. Let r be the radius of the ball. Then the volume of the ball (the volume of all the possible outcomes) is $\frac{4}{3}\pi r^3$.

To find the volume of the desired outcomes, consider the volume of the undesired outcomes: those points which lie within 90% of the center. These points form a ball of radius $\frac{9}{10}r$, hence their volume is

$\frac{4}{3}\pi\left(\frac{9}{10}r\right)^3$. So the desirable outcomes have the complementary volume

$$\begin{aligned}
\text{volume of desired outcomes} &= \text{ total volume} - \text{ volume of undesired outcomes} \\
&= \frac{4}{3}\pi r^3 - \frac{4}{3}\pi(\frac{9}{10}r)^3 \\
&= \frac{4}{3}\pi r^3\left(1-(9/10)^3\right).
\end{aligned}$$

Thus, the probability of a point being within 10% of the boundary is

$$\begin{aligned}
\frac{\text{volume of desired outcomes}}{\text{total volume}} &= \frac{\frac{4}{3}\pi r^3(1-(9/10)^3)}{\frac{4}{3}\pi r^3} \\
&= 1-(9/10)^3 \\
&= 0.271
\end{aligned}$$

(Return)

43 Probability Densities

The last module dealt with the uniform distribution, where any one outcome is as likely as another. This module deals with experiments whose outcomes have different probabilities. For example, consider an unfair coin which has a $\frac{2}{3}$ probability of landing heads and a $\frac{1}{3}$ probability of landing tails. Another example is time spent on hold with customer service, where it is more likely that the call is answered in the first hour than in the second hour.

43.1 Random variable and probability density function (PDF)

A *random variable* X is a function whose output should be thought of as the outcome of an experiment. Associated with a random variable is a *probability density function* (PDF) $\rho(x)$, which is defined by $P(a \leq X \leq b) = \int_a^b \rho(x)dx$. That is, the probability that the random variable falls in a certain range of values is given by integrating the PDF over that range of values.

Phrased another way, we can think of probability P as the quantity we want to compute over a certain range of values, and the probability element is given by

$$dP = \rho(x)\,dx.$$

Example

Consider the spinner from the last module. The outcome of a spin is some angle (relative to the positive x-axis) between 0 and 2π. If X is the random variable which gives the output of a spin, then

$$P(a \leq X \leq b) = \frac{b-a}{2\pi},$$

since the spinner was assumed to be fair. This holds for all $0 \leq a \leq b \leq 2\pi$. Then the associated PDF is

$$\rho(x) = \begin{cases} \frac{1}{2\pi} & \text{if } 0 \leq x \leq 2\pi \\ 0. & \text{otherwise} \end{cases}$$

Note

Sometimes a PDF $\rho(x)$ is only defined on a certain domain D. D can be thought of as the set of all possible outcomes of the experiment X. In this case, it is assumed that $\rho(x) = 0$ for x not in that domain. So another way of defining the PDF for the spinner is $\rho(x) = \frac{1}{2\pi}$ for $0 \leq x \leq 2\pi$.

43.2 Properties of a probability density function

The following are defining properties of a PDF. In other words, a function $\rho(x)$ is a PDF on the domain D if and only if it satisfies these properties.

1. $\rho(x) \geq 0$ for all $x \in D$.
2. $\int_D \rho(x)\, dx = 1$.

The first property is necessary since probabilities must be non-negative. The second property reflects the fact that the random variable X associated with $\rho(x)$ must have *some* outcome in the domain D (since D is the set of all possible outcomes), and so integrating over all of these outcomes should give 1.

Note

If $\rho(x)$ is defined on some specific domain D, then the integral over that specific domain should equal 1. This is because $\rho(x) = 0$ outside of that domain, as mentioned in the above note.

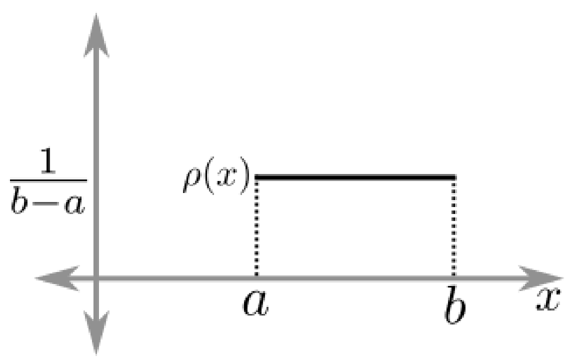
Example

Find the value of the constant c so that $\rho(x) = \frac{c}{1+x^2}$ for all x is a PDF. (See Answer 1)

43.3 Several specific density functions

Uniform density

Hinted at above and in the previous module, the *uniform density function* (or uniform distribution) on $[a, b]$ is given by $\rho(x) = \frac{1}{b-a}$ (and $\rho(x) = 0$ if x is not in $[a, b]$):



More generally, the uniform distribution on the domain D (whatever the dimension) is given by

$$\rho(x) = \frac{1}{\text{Volume of } D}.$$

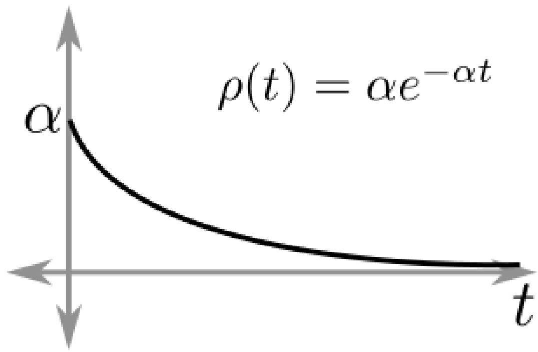
In dimension 0, where outcomes are discrete (as in the rolling of a die or the flipping of a coin), remember that volume is just counting. So in this case the probability of a particular outcome is

$$\rho(x) = \frac{1}{n},$$

where n is the number of outcomes in the domain D (e.g. $n = 6$ for the roll of a die; $n = 2$ for a coin flip).

Exponential density

Another density function used to model many common experiments is the *exponential density function*. This is actually a whole family of density functions given by $\rho(t) = \alpha e^{-\alpha t}$ for $t \geq 0$ and $\alpha > 0$ some constant. The reason a parameter t is used is that the exponential density is often used to model experiments with a time outcome.



Example

Show that the exponential density $\rho(t) = \alpha e^{-\alpha t}$ (for $t \geq 0$) satisfies the properties of a density function. (See Answer 2)

Example

Consider a call made to customer service at Acme company. The number of minutes spent on hold before the call is answered is often modeled with an exponential density function

$$\rho(t) = \alpha e^{-\alpha t}.$$

Find, in terms of α, the probability that the waiting time for a call is less than 30 minutes. (See Answer 3)

Example

Again consider customer service call waiting time at Acme company, and again assume an exponential density function

$$\rho(t) = \alpha e^{-\alpha t}.$$

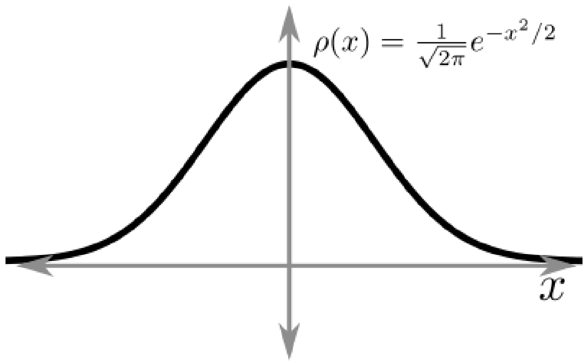
Suppose half of all customers are answered within 5 minutes. Find α and then find the probability that a call takes more than 10 minutes to be answered. (See Answer 4)

Gaussian density

The last probability density function is the '*Gaussian*, or *normal*, density function. This is an important density function and is expanded on in the next module. Like the exponential, the Gaussian density function usually has parameters (see the next module), but in its simplest form, the Gaussian is given by

$$\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

The Gaussian has all real x as its domain, but because it tails off so quickly in both directions, the probability of getting values far from the center (in this case $x = 0$) is very small.



43.4 EXERCISES

- Which of the following are probability density functions?

a. $f(x) = 1/2$ on $D = [0, 2]$

b. $f(x) = \dfrac{\sin(x)}{2}$ on $D = [0, 3\pi]$

c. $f(x) = 5e^{-2x}$ on $D = [0, \infty)$

d. $f(x) = \dfrac{1}{\pi(1+x^2)}$ on $D = (-\infty, \infty)$

e. $f(x) = \dfrac{x}{2}$ for $0 \le x \le 1$

$\dfrac{1}{2}$ for $1 \le x \le 2$

$\dfrac{3}{2} - \dfrac{x}{2}$ for $2 \le x \le 3$

43.5 Answers to Selected Examples

1. As long as $c \geq 0$, the first property for a PDF will be met, since $1 + x^2 > 0$ for all x. To satisfy the second property, compute

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{c}{1+x^2}\, dx &= c\left(\arctan(x)\Big|_{-\infty}^{\infty}\right) \\ &= c\left(\frac{\pi}{2} - (-\frac{\pi}{2})\right) \\ &= c\pi.\end{aligned}$$

Since this integral is supposed to be 1, we find that $c = \frac{1}{\pi}$.

(Return)

2. The exponential function is never negative, so one need only check the integral. One finds

$$\begin{aligned}\int_{t=0}^{\infty} \alpha e^{-\alpha t}\, dt &= \alpha \frac{1}{-\alpha} e^{-\alpha t}\Big|_{t=0}^{\infty} \\ &= -(0-1) \\ &= 1,\end{aligned}$$

as desired. So the exponential density is in fact a density.

(Return)

3. To find the probability that $0 \le X \le 30$, use the relationship between probability and the PDF, which is

$$\begin{aligned}P(0 \le X \le 30) &= \int_0^{30} \rho(x)\, dx \\ &= \int_0^{30} \lambda e^{-\lambda x}\, dx \\ &= -e^{-\lambda x}\Big|_0^{30} \\ &= -e^{-30\lambda} - (-1) \\ &= 1 - e^{-30\lambda}.\end{aligned}$$

(Return)

4. Since half of all customers are answered within 5 minutes, we have that

$$P(0 \le X \le 5) = \frac{1}{2}.$$

On the other hand, we know that this can be expressed as the integral of the density function, so we have

$$\begin{aligned}\frac{1}{2} &= \int_{t=0}^{5} \rho(t)\, dt \\ &= \int_{t=0}^{5} \alpha e^{-\alpha t}\, dt \\ &= -e^{-\alpha t}\Big|_{t=0}^{5} \\ &= -e^{-5\alpha} - (-1) \\ &= 1 - e^{-5\alpha}.\end{aligned}$$

So we have that

$$e^{-5\alpha} = \frac{1}{2}.$$

Taking the log of both sides, dividing by -5 and simplifying, we have

$$\begin{aligned}\alpha &= \frac{1}{-5} \ln\left(\frac{1}{2}\right) \\ &= \frac{1}{-5}(-\ln 2) \\ &= \frac{1}{5} \ln 2.\end{aligned}$$

For the second part, we want to know the probability of waiting more than 10 minutes. This is (leaving α as a constant for now)

$$\begin{aligned}
P(X \geq 10) &= \int_{t=10}^{\infty} \rho(t)\, dt \\
&= \int_{t=10}^{\infty} \alpha e^{-\alpha t}\, dt \\
&= -e^{-\alpha t}\Big|_{t=10}^{\infty} \\
&= 0 - \left(-e^{-\alpha \cdot 10}\right).
\end{aligned}$$

Now plugging in the value of α, we have

$$\begin{aligned}
P(X \geq 10) &= e^{-(\ln 2/5)\cdot 10} \\
&= e^{-2\ln 2} \\
&= 2^{-2} \\
&= \frac{1}{4}.
\end{aligned}$$

(Return)

44 Expectation And Variance

When performing an experiment, it is useful to know what the expected outcome will be as well as how much variation one can expect among the outcomes. The notions of expected outcome and variation are made formal in this module by the terms *expectation*,*variance*, and *standard deviation*.

This module will also show some of the connections of these statistical metrics with the applications of the previous modules.

44.1 Expectation

Consider a random variable X with probability density function (PDF) $\rho(x)$ defined on some domain D. The *expectation of* X, denoted by $\mathbb{E}$, is defined by

$$\begin{aligned}\mathbb{E} &= \int_D x\rho(x)\,dx \\ &= \int_D x\,dP,\end{aligned}$$

where dP is the probability element. The expectation of X is sometimes called the *mean of* X, the *expected value*, or the *first moment*. In some books it is denoted μ_X. It is best to think of the expectation as the number one gets by repeating the experiment many times and taking the average of the outputs.

The notion of expectation is more general than the mean because one can also take the expectation of a function of X. The *expectation of* $f(X)$ is defined by

$$\mathbb{E}[f(X)] = \int_D f(x)\rho(x)\,dx.$$

Example

Find the expectation of X, where X is uniformly distributed on the interval $[a, b]$. (See Answer 1)

Example

Recall that the random variable X is said to have the *exponential distribution* if the PDF associated with X is $\rho(t) = \alpha e^{-\alpha t}$ for $t \geq 0$, where $\alpha > 0$ is some constant. Find the expectation of the exponential distribution (in terms of α). (See Answer 2)

44.2 Variance

Consider a random variable X with PDF $\rho(x)$. The *variance of* X, denoted $\mathbb{V}$, is defined by

$$\begin{aligned}\mathbb{V} &= \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] \\ &= \mathbb{E}[X^2] - \mathbb{E}[X]^2.\end{aligned}$$

In the notation of the lecture,

$$\begin{aligned}\mathbb{V} &= \int_D (x - \mathbb{E})^2 \, dP \\ &= \int_D x^2 \, dP - \mathbb{E}^2.\end{aligned}$$

Note: it requires some calculation to show the second equality above holds. Either of the above expressions may be taken as the definition of variance, and the second one might be slightly simpler for the sake of computation. (See Justification 3)

Example

Compute the variance of the exponential density function $\rho(x) = \alpha e^{-\alpha x}$. (See Answer 4)

44.3 Standard deviation

Consider a random variable X with PDF $\rho(x)$. Then the *standard deviation of* X, denoted σ_X, is defined by

$$\begin{aligned}\sigma_X &= \sqrt{V[X]} \\ &= \sqrt{E[X^2] - E[X]^2} \\ &= \sqrt{\int_D x^2 \rho(x) \, dx - \left(\int_D x\rho(x) \, dx\right)^2}.\end{aligned}$$

Example

Find the standard deviation of X, where X is uniformly distributed over $[a, b]$. (See Answer 5)

44.4 Interpretations

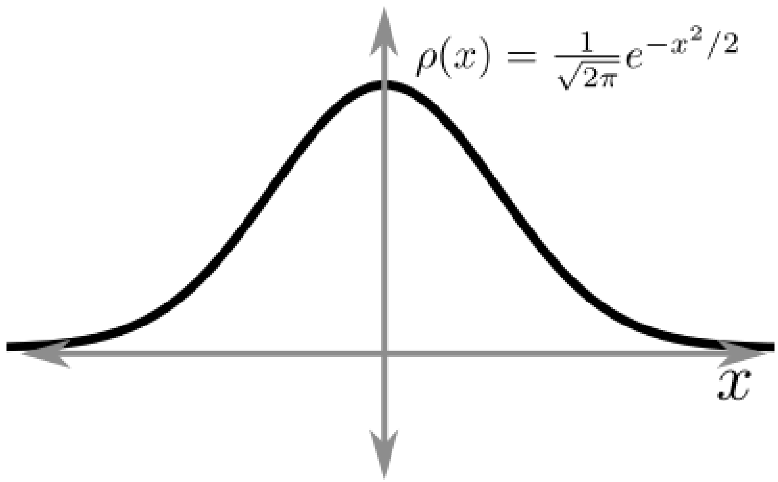
If one interprets the PDF $\rho(x)$ as the density of a rod at location x, then:

1. The mean, $\mu = \int x\rho(x) \, dx$, gives the center of mass of the rod.
2. The variance, $V = \int (x - \mu)^2 \rho(x) \, dx$, gives the moment of inertia about the line $x = \mu$.
3. The standard deviation, $\sigma = \sqrt{V}$, gives the radius of gyration about the line $x = \mu$.

44.5 The normal distribution

A random variable X is said to have the *normal distribution*, or to be *normally distributed*, with mean μ and standard deviation σ if its PDF is of the form

$$\rho(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}.$$



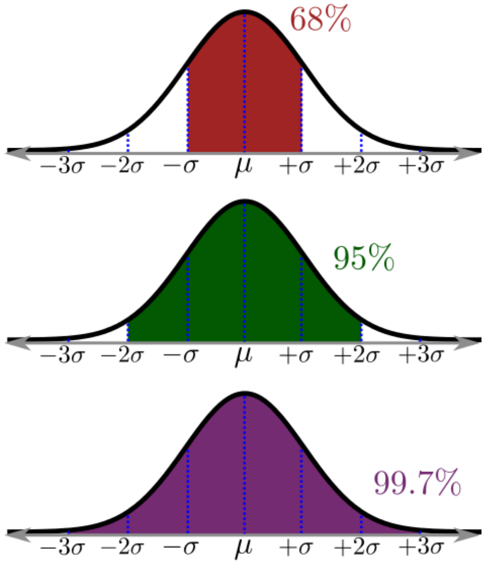
Due to its ubiquity throughout the sciences, the normal distribution is one of the most well-known probability distributions. However, because its PDF does not have an elementary anti-derivative, it is not easy to calculate exact probabilities associated with the normal distribution. Instead, there are is a rule of thumb which can be used.

The 68-95-99.7 rule

Given a random variable X which is normally distributed with mean μ and standard deviation σ, the following hold:

1. $P(\mu - \sigma \leq X \leq \mu + \sigma) \approx .68$.
2. $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) \approx .95$.
3. $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) \approx .997$.

In other words, 68% of samples will fall within 1 standard deviation of the mean. 95% of samples will fall within 2 standard deviations of the mean. And 99.7% of samples will fall within 3 standard deviations. These rules, along with the symmetry of the normal PDF, can be used to approximate many probabilities relating to the normal distribution:



Example

The height of men in a certain population is normally distributed with mean $\mu = 70$ inches and standard deviation $\sigma = 2$ inches. If a man is chosen at random from the population, what is the probability that he is taller than 72 inches? (See Answer 6)

44.6 EXERCISES

- Compute the expected value of normally distributed random variable with probability density function $\rho(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$ on $-\infty < x < \infty$.

44.7 Answers to Selected Examples

1. Recall that the PDF associated with X is given by $\rho(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. Thus, the mean is given by

$$\begin{aligned}\mathbb{E} &= \int_a^b x \cdot \frac{1}{b-a}\, dx \\ &= \frac{1}{b-a} \frac{x^2}{2}\bigg|_a^b \\ &= \frac{1}{b-a} \cdot \frac{1}{2}(b^2 - a^2) \\ &= \frac{1}{b-a} \cdot \frac{1}{2}(b+a)(b-a) \\ &= \frac{1}{2}(a+b).\end{aligned}$$

(Return)

2. From the definition of expectation, one finds

$$\begin{aligned}\mathbb{E} &= \int_0^\infty t\alpha e^{-\alpha t}\, dt \\ &= \alpha \int_0^\infty t e^{-\alpha t}\, dt.\end{aligned}$$

Using integration by parts, with

$$\begin{aligned} u &= t & du &= dt \\ dv &= e^{-\alpha t} & v &= \frac{1}{-\alpha} e^{-\alpha t}, \end{aligned}$$

we find that

$$\begin{aligned}\alpha \int_0^\infty t e^{-\alpha t}\, dt &= \alpha \left(\frac{t}{-\alpha} e^{-\alpha t} - \int_0^\infty \frac{1}{-\alpha} e^{-\alpha t}\, dt \right) \\ &= \left(-t e^{-\alpha t} - \frac{1}{\alpha} e^{-\alpha t} \right) \Big|_0^\infty \\ &= (0-0) - (0 - \frac{1}{\alpha}) \\ &= \frac{1}{\alpha}.\end{aligned}$$

(Return)

3. Expanding out the expression and using the linearity of the integral, we find

$$\begin{aligned}\int_D (x-\mathbb{E})^2\, dP &= \int_D (x-\mathbb{E})^2 \rho(x)\, dx \\ &= \int_D x^2\, dP - \int_D 2x\mathbb{E}\, dP + \int \mathbb{E}^2\, dP \\ &= \int_D x^2\, dP - 2\mathbb{E} \int_D x\, dP + \mathbb{E}^2 \int\ dP \\ &= \int_D x^2\, dP - 2\mathbb{E}\cdot\mathbb{E} + \mathbb{E}^2 \\ &= \int_D x^2\, dP - \mathbb{E}^2.\end{aligned}$$

because $\int x\, dP = \mathbb{E}$ and $\int\ dP = 1$, by the definition of expectation and the definition of the probability density function, respectively.

(Return)

4. The variance requires us to compute

$$\begin{aligned}\mathbb{E}(X^2) &= \int_D x^2\, dP \\ &= \int_{x=0}^\infty x^2 \alpha e^{-\alpha x}\, dx.\end{aligned}$$

Using integration by parts, with

$$u = x^2 \qquad du = 2x\,dx$$
$$dv = \alpha e^{-\alpha x}\,dx \qquad v = -e^{-\alpha x},$$

we find

$$\int_{x=0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx = -x^2 e^{-\alpha x}\Big|_{x=0}^{\infty} + \int_{x=0}^{\infty} 2x e^{-\alpha x}\,dx.$$

This second integral can be done with integration by parts again, or we can use the fact that this is almost the integral for the expectation. Namely, we know

$$\int_{x=0}^{\infty} x\alpha e^{-\alpha x}\,dx = \frac{1}{\alpha},$$

and so by dividing through by α, we have

$$\int_{x=0}^{\infty} x e^{-\alpha x}\,dx = \frac{1}{\alpha^2}.$$

Putting this together, we have

$$\begin{aligned}\int_{x=0}^{\infty} x^2 \alpha e^{-\alpha x}\,dx &= -x^2 e^{-\alpha x}\Big|_{x=0}^{\infty} + \frac{2}{\alpha^2}\\ &= (0-0) + \frac{2}{\alpha^2}\\ &= \frac{2}{\alpha^2}.\end{aligned}$$

Finally, then, the variance is

$$\begin{aligned}\mathbb{V} &= \int_D x^2\,dP - \mathbb{E}^2\\ &= \frac{2}{\alpha^2} - \left(\frac{1}{\alpha}\right)^2\\ &= \frac{1}{\alpha^2}.\end{aligned}$$

(Return)

5. Again, recall that the PDF for the uniform distribution is $\rho(x) = \frac{1}{b-a}$ for $a \leq x \leq b$. Thus,

$$\begin{aligned}E[X^2] &= \int_a^b x^2 \frac{1}{b-a}\,dx\\ &= \frac{1}{b-a}\frac{x^3}{3}\Big|_a^b\\ &= \frac{1}{b-a}\cdot\frac{1}{3}(b^3-a^3)\\ &= \frac{1}{b-a}\cdot\frac{1}{3}(b-a)(b^2+ba+a^2)\\ &= \frac{b^2+ab+a^2}{3}\end{aligned}$$

From the previous example, $E[X] = \mu_X = \frac{a+b}{2}$. Thus,

$$\begin{aligned}
\sigma_X &= \sqrt{E[X^2] - E[X]^2} \\
&= \sqrt{\frac{b^2 + ba + a^2}{3} - \frac{b^2 + 2ba + a^2}{4}} \\
&= \sqrt{\frac{b^2 - 2ab + a^2}{12}} \\
&= \frac{b-a}{\sqrt{12}}.
\end{aligned}$$

(Return)

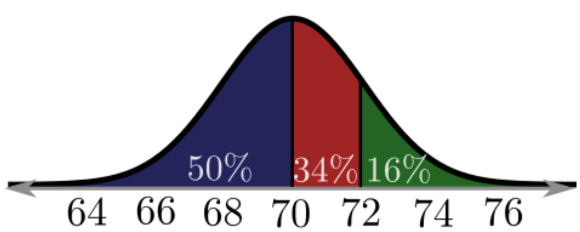
6. Let X be the height of a randomly chosen man. Then $P(68 \leq X \leq 72) = .68$ by the above rule. By symmetry $P(68 \leq X \leq 70) = P(70 \leq X \leq 72) = .34$. Also, by symmetry, $P(X \leq 70) = .5$. Thus,

$$\begin{aligned}
P(X \leq 72) &= P(X \leq 70) + P(70 \leq X \leq 72) \\
&= .5 + .34 \\
&= .84.
\end{aligned}$$

It follows that

$$\begin{aligned}
P(X > 72) &= 1 - P(X \leq 72) \\
&= 1 - .84 \\
&= .16.
\end{aligned}$$

This is best visualized by labeling the various regions under the normal curve with their areas:



So the probability that a randomly chosen man from the population is taller than 72 inches is .16.

(Return)