Math 106 Teachers Guide

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1 What is Math 106?

Math 106 is the second in a two-course sequence of mathematics courses for future elementary and middle school teachers. Its primary goal is to educate mathematically. We cannot expect schoolchildren to learn anything beyond the rote execution of algorithms if the first ten years of their education is at the hands of teachers who can't themselves go beyond this. By providing elementary educators with a good mathematical foundation, we hope to reverse the downward spiral into which many perceive mathematics education to have fallen. A secondary goal is to introduce students to the cooperative learning environment. Keep in mind, though, that the course is a content course, not a pedagogy course.

The content covered in Math 106 has a *how* and a *what* component. Students will be taught how to think and speak mathematically, many for the first time in their lives. They will solve problems they have not been shown how to do, they will learn to put their ideas into precise language and to prove their assertions when possible. They will learn to break down problems into smaller problems, use trial and error, generalize when appropriate, and to harness their (atrophied) common sense. It is the *how* content that determines the format of the course, necessitating the problem-solving theme and the socratic style of discussion.

The *what* content, i.e., the choice of topics, deserves some explanation as well. Geometric reasoning holds a special place in mathematics. Historically, the study of geometry reached an advanced level long before arithmetic was invented: the Greeks did geometry at a level exceeding that of most high school students today, yet they did not have the concept of equations, variables, place value or the number zero! The lesson to be learned from this is that geometry harnesses some fundamental cognitive abilities. Different children learn in different ways, but we expect that at least some children will find it crucial to harness their geometric abilities if they are to understand much of the mathematics they encounter in school. Math 106 is designed to help students become familiar with basic geometric concepts, to understand them in a more sophisticated way than they did before, to allow them to cope with mathematics that will arise in the classroom setting, and, along with Math 105, to give them a solid mathematical base. We believe a solid mathematical base at roughly the high school level is essential even for teachers whose students are four to eight years old. These teachers will be given a good deal of independence in designing lessons and choosing curriculum, and will also be shaping their pupils' attitudes toward the subject. Thus in addition to such obviously applicable topics as measurement of length, area and angle, we also include some eudlidean geometry and a study of rigidity and the relation of shapes under scaling and rigid motions. Finally, we expect students to learn to justify their solutions with some rigor, for which geometry, being the birthplace of mathematical rigor, is an ideal venue.

Some specifics about the student body here at OSU are worth noting. College algebra (Math 148) is *not* a prerequisite for this course, which means that along with the problem solving will by necessity come some algebra review, and indeed a continual reinforcing of the use and meaning of algebraic notation. To some extent this was done in Math 105, although not all students in 106 have taken 105. There may be some resistance on the part of some students to the way the class is run, though we have found the small group format by and large to be popular; resistance is more likely to come from the emphasis on problem-solving and lack of answers provided. The classroom format turns traditional norms upside down by making students take responsibility and authority for determining whether a problem has been solved correctly[1].

Teaching a course by means of cooperative learning, Socratic discourse, and the like involves decentralizing yourself (the instructor) as the focus for learning and authority. This is a skill which takes some time to develop, and it's not always easy. This is why it's best to start preparing to teach a course like this well in advance. Don't worry, though — you're not on your own. If you're expecting to teach in this sequence, you should already have looked into observing others' classes, and the rest of the informal instructor preparation program. Part of this is the extensive set of notes included in Section 3 of this guide. Reading and discussing them with folks who have had experience in the sequence will help defray a lot of the anxiety you may have.

This guide is designed to answer any questions you may have, large or small, about this class (although of course conferring with other people who have taught it is an irreplaceable resource), and to provide all the help you need with the details until you feel comfortable enough to stand on your own. Even once you've come into your own, it's a good idea to refer to the guide throughout your first time teaching the course, as we have collected the potential pitfalls of several semesters' worth of teaching it, and it's always best to be prepared. Section 2 gives a detailed blow-by-blow of each of the activities in the course pack (including solutions), along with suggested times and a

[1]We do not, in the end, withhold information and techniques they need to know or judgment on what is correct, but we do insist that they develop their own judgment and we train them to do so.

couple of sample syllabi. Section 3 discusses how to teach this kind of class. Section 4 goes into detail on the write-ups: which activities have traditionally had write-ups associated with them, how often to assign them, which ones are important to assign, how to make the assignments in class, and specifically what to look for in critiquing them. Section 5 discusses exams and evaluation, including a large bank of sample exams from a variety of past instructors. Finally, Section 6 details the organization of the directories and LaTeX files used to develop this guide and the course pack, so that you can take things you like and adapt them for your own handouts, exams and worksheets.

2 The activities

Before giving detailed advice and forecasts on conducting the activities, a few words on vocabulary are in order. Pages 10–19 of the coursepack are a glossary containing over 100 terms they will be expected to know by the end of the term. We handle these a few at a time, based on when they are needed. Typically, few days in advance of a worksheet that uses a set of vocabulary words, we tell the students to look these words up and attempt to understand them. We then allot a few minutes at the beginning of class for them to ask questions.

At the beginning of each section of activities is listed all the vocabulary they will need to know for that section. It is recommended that this be assigned as reading well in advance, so the students can ask questions, be quizzed on the vocabulary, and so forth, before beginning the worksheets on which the vocabulary is needed. It is recommended that a few of the vocabulary items not be included among those assigned for advanced reading, since there are worksheets whose aim is to have the students develop these ideas on their own. Specifically, the notions of length, are, volume and dihedral angle have questions devoted to them on various worksheets. The definition of angle, however, should be read and understood in advance. Some terms may be defined in two different ways. An angle can be the two bounding rays or the space they bound. A polygon may be a circuit of line segments or the space they bound. Some notions such as plane figure are defined only informally (the term is used in this course to denote both planarity and boundedness). These and other vocabulary words must be discussed as they arise, since common usage is sometimes different and it imiportant to have an agreement as to the convention in Math 106 as well as an understanding that future uses they encounter may differ. See also the Math 106 definitions of prism and pyramid, which are defined as (generalized) cylinders and cones which have polygonal bases.

A word about the organization of material. The content is built around several "big ideas": scaling, rigidity, transformation. The introduction of the dimensions in the order 2, 1, 3, 0 has the following pedagogical purpose. The hardest concepts in measurement have to do with units and dimensionality ("scaling"). Dealing with these in one dimension is pedagogically difficult because one dimesion is too easy and familiar. We start in two dimensions, where there are challenging problems from the outset (e.g., as to how one even defines area and measures it). After this, we start from scratch with linear measure and three-dimensional measure, then go on to angle. The section thereafter, on Euclidean reasoning begins with the idea of rigidity – when is an object rigid, that is, when is it determined up to congruence? The final section on spatial on transformation (rigid motions and symmetry) is self-explanatory.

2.1 Problem solving introduction

2.1.1 Photo Layouts

Vocabulary: aspect ratio, quadrilateral, rectangle, square.

Photo Layouts (Introductory Activity) This problem is intended to acclimate students into the problem-solving environment of the course- both mathematical and cultural. Students will be fairly tentative and apprehensive of what they're getting into.

The term *aspect ratio* is in their vocabulary list, and will not be familiar to most of them. The wording of the problem is meant to highlight the ambiguity in the phrase "the same shape ... but different size". Once they have red and understood the term *aspect ratio*, they need to make the connection that this term defines and clarifies what it means for rectangles to be the same shape. This might be a good spot for an early, quick whole-class discussion or announcement. Probably you will also need to discuss whether 1:2 and 2:1 are the same aspect ratios or different. Either convention is OK, but the problem clearly says they must allow 1:2 and 2:1 on the same page, whether or not they consider these to have the same aspect ratio.

Last time I wished I had prepared a printout of several rectangles of varying sizes, aspect ratios and orientations, so I could test their understanding of aspect ratio by asking them whether any of the rectangles shown had the same aspect ratio (or earlier in the discussion, the same "shape").

After this, some may have uncertainty as to what layouts are allowed. Refer them back to the text for this, as they need practice working with the text of a problem to understand what is being asked. As they begin to find solutions, pay attention to whether they realize that the problem asks for all solutions. Perhaps it is worth pointing out to them that even when the problem does not explicitly ask for all solutions, it is understood that a complete answer involves finding all solutions.

Most students will approach the problem by trial-and-error and come up first with the trivial layouts and ratios: vertical or horizontal partitions into congruent rectangles of aspect ratio $1 : n$ for $n \leq 4$. There is also a two by two subdivision into rectangles of aspect ratio 1.

Students need prodding even to think of a possible diagram for a nontrivial layout, and then need more prodding to go try to figure out what the shapes must be if these layouts are to be realized. They should be encouraged first to try some numbers for the sides of one of the rectangles in the layout, then to see if the other numbers are determined from the requirement that all the rectangles be similar and by the fact that the whole shape is a square. It will be difficult for them to get exact answers by trial and error, and will probalby take all of day 2 to explore these numerically in any effective way.

The step to algebra will not occur to them on their own either, but by suggesting they try x instead of a numerical guess on one of the simpler non-trivial diagrams where they've already tried at least two numerical guesses will give them a task they can handle.

2.2 Area

Vocabulary: polygon, vertex, edge; triangle, parallelogram, trapezoid; equilateral, equiangular, regular (polygon), isosceles, scalene; altitude; distance.

2.2.1 Pizzas

The major issue in this problem is how pizzas are priced: Should cost be proportional to one- or two-dimensional measurements? Many students will not see that there is a controversy and solve the problem their way (usually considering cost to be proportional to the diameter) without considering that there might be other views. Many students will see other groups using the other kind of measurement and either consider it to be an "equally viable" approach to pricing or consider the other approach to be "wrong". Students need to see that one must argue why their approach is a correct one rather than ignoring the issue and simply doing the computations.

Last quarter, six of seven groups computed a price proportionally to the diameter and one group computed a cost proportional to the area. The six groups got similar answers, though not entirely the same due to roundoff and a computational error in one case. Without tipping off what was correct, the instructors asked the students to justify what they did. Someone from one of the linear groups went first, which was easy since they had the confidence of a near-consensus. Then someone from the group that had the two-dimensional solution gave their argument. The logic of their argument apparently convinced everyone in the room to switch to their pricing scheme. It doesn't always work out this nicely, but apparently this problem is one where the correct logic does appear self-evident, which makes it a good problem for the beginning of the quarter.

A few notes: it was important to encourage the groups making a one-dimensional computation to proceed with their computation and come up with a good explanation for it. Only then could the logic of the two computations be fairly compared. This also served the purpose of explaining the type of proportionality computation they were making, so when it came up in the more difficult two-dimensional context, students did not get bogged down in understanding the basics of this kind of computation.

Obviously there is a danger with this problem that every group will do the one-dimensional computation. If so, the instructors have to lead the whole groupo to a contradiction. For example, you could ask what a pizza should cost that was three times the diameter of the original one. Then pull out some identical disks (say quarters out of people's pockets) and ask how an enterpreneur might re-sell pieces of a triple-scale pizza (it is easy to see how to cut seven of the original size out of a triple-size). This might lead to the question of whether we aim to give a volume discount or not, but perhaps this has been settled (in the negative) in the small or large group discussion already. (It is also way too much of a volume discount even if you do want to give a volume discount.) If, on the other hand, everyone tries the better solution, then don't worry, it will be a phenomenal quarter!

2.2.2 Paper Shapes

The idea here is to get students thinking about concepts and methods of measurement beyond one dimension and perhaps also how one might approach area with their future students (formula or space or other approach?). This should be used with the Carmen Curtis "Area" videotape, which provides good insight into the cognitive steps in understanding the definition and measurement of area.

Some students may measure and use formulas while others may cut and "puzzle" the solution together (students may call it "conservation of space"). But in the end, to argue for their solutions, students will be forced to identify that there is a quantity called "area" which is conserved when you rearrange shapes, whose measurement may be taken in a number of ways, and which is a useful measure of the space taken up by a two-dimensional object. The activity should go quickly.

2.2.3 Geoboard Areas

Last quarter we did not have them find the perimeters, just the areas. Probably we will do the same this quarter, perhaps coming back to the perimeters in a week or two. The activity is flexible, since if groups work at different rates, it is OK for some to complete most of the problems in the time it takes the slower groups to complete only two of them (I don't recommend stopping before every group has gotten at least two).

One main purpose of this exercise is to give a middle ground between using a formula and using cutting and rearranging. The easiest way to do most of these is to dissect the shapes into rectangles and triangles, then use knowledge of the rectangle and triangle area formulae. It is best at this point if they are allowed to use the rectangle formula (which, presumably, they justified in their defining discussion of area in the "paper shapes" worksheet) but may only use the triangle area formula if they can justify it. This could lead to a good discussion of a general theorem about triangle areas, or more likely, to a few separate justifications about why individual triangles fill out half the area of a rectangle containing them. Students can get good at this kind of *ad hoc* argumentation without being able to prove a general theorem. Last quarter we even took this one step further. We identified a seemingly true but unproven fact, namely that a certain region of a rectangle outside of the triangle was congruent to a certain piece of the triangle. We vowed to come back to this later in the section on deductive reasoning and indeed we did.

You should expect that students will use various strategies even within one problem, particularly with area and especially with part (f) (usually involving splitting up the diagram and surrounding with rectangles or looking at the complement of a shape within a rectangle). In whole class discussion, you might want to ask which method is more general and/or make informal arguments why, say, a 2 by 1 triangle can be pieced together into a 1 by 1 square. You may want to emphasize at the start the importance of defining a unit of length or area. Last term we discussed several of these at length (since we got involved in a discussion of what could be proved rigorously about congruences) and ended up assigning only part (f) for a writeup.

2.2.4 Measuring a Sector

This activity takes place before any comprehensive discussion of formulae. Nevertheless, some groups may try to use a formula. Those groups should be allowed to do so provided they can understand and explain the terms, and are willing to try to explain why the formula might be true. Probably they should be allowed to assume the area formula for a whole circle, as long as they acknowledge it is out of the blue for now.

The groups that choose to count squares should compare answers and account for discrepancies. Are the answers sufficiently accurate? Are there better and worse ways of dealing with inaccuracies? How should groups that chose different sizes of sector compare answers? In fact, in a previous quarter, we had groups whose sectors had height 3, 4 and 5 large squares, and we had some groups counting in units of large squares and some in small squares.

Make sure students compare figures with one another so they know whether they agree they have done the construction correctly (encourage the use of their compass for drawing).

2.2.5 Area Formulae

Don't have students beat their brains over this. The purpose is to illustrate that area exists independent of formulae, that the collection of formulae is somewhat arbitrary, and that there may be more than one formula for a given shape. Unusual formulae to look for include Heron's, law of sines, formulae for ellipses, trapezoids, regular polygons, etc. Last time, we explicitly told them it was just for fun, and that there would be a prize for the most formulae (not counting as distinct any two that were essentially the same). Later, we told them that a small few of these were important, and recommended (but didn't insist) that they know them.

2.2.6 Picture Proofs

The idea behind these problems is that one can cut out shapes in an area- preserving way and rearrange them into simpler shapes (like rectangles) for which we know how to calculate area. Some students may mistakenly consider this to be one problem (not 3) or have trouble recognizing in each pair that the left hand picture has been transformed into the right-hand picture. For each pair, it is important for the students to understand what is wanted.

In the previous quarter, we asked them to answer "What is this proving, what is it relying on as previous knowledge (e.g., area of rectangle or previous work on page), and how does the proof work?". This time, we put in a second page asking them if they can answer the following three questions on each of the three problems.

1. What statement does the picture purport to prove?
2. What re-arrangement of the picture proves it, and why?
3. What might you need to justify about how the pieces are supposed to fit together?

Students should be able to describe the transformation, give an informal argument why area is preserved or easier to calculate, and state how the right-hand picture yields the resultant area formula. These may be returned to for more rigorous proof later on (e.g., on first pair, that the triangle that you chop off the end is congruent to the triangle you glue in at the other end). (Note: Students may not know the formula (property) for circumference of a circle, which is needed on the third part.).

It is a good idea to save the answers to the last part for re-examination when you get to the section on deductive reasoning. In fact, when the problem is revisited at the end of the section on Euclidean reasoning, the discussion is often quite fruitful (though it seems unnecessary to most students until they come to terms with False Proofs). In case it is useful, the Solutions directory contains a handout (parallelogram.pdf) with three different proofs of parallelogram re-arrangement, all with different stipulations as to what is drawn by definition and what must therefore be concluded.

2.2.7 The Apothem

It would be a good idea if the Area Formulae worksheet were due before the class was to start The Apothem. By chance, we did it this way last time, with the result that most groups had seen the formula. It was still a challenge to apply it correctly, after which we asked them to justify it. Some could, others had a lot of trouble. One group dissected into triangles but they were not the right ones. This was the hardest for us to deal with, since we didn't want to stomp on their good idea, but found it hard to continue from there. I think we eventually told them that it was a good idea, but this particular dissection didn't seem to be helping.

Groups that didn't start with the formula tended to try to gather some data numerically. If we got to them in time, we could try to encourage them to count all the squares in one triangular region first, hoping that this would prompt them then to multiply by the number of such reasons. History does not record whether this worked.

Some students may have difficulty making an algebraic generalization from the pentagon case. This is a good arena in which they can work on overcoming the problem of generalization and algebraic notation.

Almost eveyone had more difficulty making the generalization from a polygon to a circle, particularly the concept of replacing the perimeter with "circumference" and the number of sides growing large. Probably you should play this by ear and maybe drop it rther than push too hard.

2.3 Length, area and volume

Vocabulary: pyramid, prism, cone, cylinder; polyhedron, face, regular polyhedron, right cylinder; convex, circle, pentagon, hexagon, octagon, decagon, dodecagon, rhombus;

2.3.1 Need for Standard Units

It would be advisable to have the viewing of the linear measurement video duearound the same time as they do this worksheet.

Last quarter, we assigned Units of Measurement as homework, then had them do Need for Standard Units in class very briefly while also compare notes with the other members of their group on the homework. It turned out that "Need for Standard Uhnits" was not too interesting, but it seemed worth keeping in the coursepack since it ties in well with the linear measurement video. The key point to get out of this worksheet are that standard units are useful. The key points of the Units of Measurement worksheet are to be able to identify what dimension a unit is even without the tipoff of the word "square" or "cubic", to get some practice with unit conversion, and to make connections between units used in real life and the math we are doing in this section.

2.3.2 Units of Measurement

See discussion of previous worksheet.

2.3.3 Surface Area

The surface area as shown is 48 square units and the volume is 14 cubic units. There are multiple configurations which yield minimal surface area (38); all are as close to the sphere (or cube) as possible (a $2 \times 3 \times 3$ block with a pair of blocks stuck on somewhere). To maximize surface area (58), string the blocks out in the form of a thin rod. This is also not unique; one can have otherwise isolated blocks sharing one face with a side of the rod. It is important for students to report how they counted surface squares as well as know that they need to prove max results. The proof that

58 is maximum is realy not that hard, so this is good practice for them to attain some rigor in their argument. The proof that 38 is minimal should not be attempted – at least I don't know of any good proof.

At some point, you may want to force the more general question: what 3-dimensional shape encloses the most volume for a given surface area? Which shape uses the least surface area to enclose a given volume? Is there a shape which uses the most surface area? The physical example of a balloon, which doesn't "want" to stretch any more than it must, will bring home the principle if the students have difficulties. This discussion could be difficult but rewarding. Perhaps these ideas might be suggested to some of the faster small groups.

2.3.4 Length, Area and Volume

Since we started by working a lot with area, Problem 1 on defining a length for curved objects is the hardest and most essential here. In the spring, three definitions were given: (1) Place a string along the curve and straighten it out and measure the distance between endpoints, (2) Mark even units along the curve, making them small enough to approximate line segments, and (3) Measure with a trundle wheel, thus transferring all the units to a standard circle which can be marked evenly. The most obvious answer is you run a string along it, then straighten out the string and measure its length. But here it is meaningful to ask whether the notion of length of a curve makes sense in the absence of physical string. If you pursue this tack, it might be good if they have already read about or discussed the cognitive difference between length and distance. In absence of string, one can define length as a limit of sums of polygonal lengths, but these lengths are more likely to be conceptualized as distances.

Another purpose of this worksheet is to concentrate on the differences and similarities between the definitions of measure in various dimensions. We wanted to bring out the idea that objects in dimension n had a natural $(n-1)$-dimensional measure (surface area) but no natural measure in lower dimensions, but this point was not effectively made. They did at least see that the length of a three-dimensional object was more ambiguous that its area. It would be worth doing again if we could do a little better.

2.3.5 Volume: Eureka!

In retrospect, this was quite an important worksheet. Students had the most difficulty with # 2 and # 3. Some did numerical examples, yielding the answer to #2, but making the answer to #3 only approximate. It took a lot of nudging to get even the fastest groups to see that the inverse operation of cube root was relevant here. It might have been wise to prompt them more on where the 8 came from in #2 by asking them to consider perhaps what happens when one triples or quadruples the length.

The topic will be revisited in Scaling, but it needs to be seen multiple times, which is one important purpose of this worksheet. Another important purpose is to bring back physical intuition. Problem # 5, on Archimedes discovery of how to measure volume is not a great small group problem (since either you get it or you don't) but it is one of those pieces of knowledge we definitely want to cover in this course, so probably should not be skipped. The content of # 4, on prisms and cylinders, is also important conceptually.

2.3.6 Volume: Prisms and Cones

The first question here is more of an experiment than a problem. It could be done as homework. It is also probably the most interesting part of this sheet and the biggest logistical difficulty. If you can manage, though, it should prove well worth it: there's nothing like direct visual proof to make a student believe and remember.

I recommend inserting here an activity we did last quarter, where we showed how one instance of the pyramid formula could be proved, namely that a certain triangular pyrmaid is literally one sixth of a cube. We had each group assign one person to make (for homework) a tetrahedron whose four faces were two triangles $1 \times \sqrt{2} \times \sqrt{3}$ and two isosceles triangles with short side equal to 1. We provided the templates and they produced the solids. I then demonstrated how three fit to make a half-cube and that six fit to make a cube. It was fun but watch for one pitfall: You need three of each orientation. Next time, we'll have each group make two of them. They can use the posterboard from their coursepack.

Questions # 2 and # 3 both involve calculating the volume of a prism: Find the base area and multiply by height (or length). These were well worth doing, since many students who thought they understood the formulae had difficulty applying them.

2.3.7 The Length of a Square

Don't forget you'll need string!

This is intended to spark some controversy into how a square's space can be measured and in what units should be used. The best outcome would be for them to conclude that the linear measurement is actually the quotient of a volume (area of figure times diameter of string) by an area (cross-sectional area of the string). A somewhat more likely outcome would be for them to see the length as an area (4 square inches) divided by the width of the string.

Last quarter, most estimated the width of the string (1/16 inch) and figured it would take 64 inches of the string to fill the square, with some inaccuracies in exactly filling the square. We then had to prompt them to get to the outcome mentioned above. In any case, don't prompt them too much before they've had a good crack at question # 3.

2.3.8 Perimeter

This worsheet was inspired by a study that found that many students did not know that perimeter and area are not essentially the same thing. They believed that since both were size measures, the two would increase and decrease together. Needless to say, the idea of maximizing area for a given perimeter (the classical *isoperimetric problem*) did not make sense to them.

This worksheet was designed to get students thinking about different aspects of length and perimeter, particularly the idea that a fixed perimeter (area) does not guarantee knowledge of what the area (perimeter) is. Students may need help setting up and scaling the coordinate axes on # 2 and # 3. You might generalize the questions into, for example, what figure with a given area (perimeter) has minimal perimeter (area): a circle. You might also get into some algebraic representation of # 2 and # 3. This is a good place to brush up on graphing skills!

2.3.9 The Hungry Cow

This worksheet works well as an individual outside-of-class assignment. It is a good idea to do some assignments this way so that exams are not the students' first chance to do independent work.

The hardest part of this problem is drawing a diagram of where the cow can reach. You don't want to tell them outright to do this, at least at first, but should be attentive and question them on where they think the cow can and can't reach. As they come up with more systematic ways of answering these questions, we hope the idea of circular arcs will occur to them. You may need to remind them that the cow can reach grass on each side of the barn. For the final answer, it is fine to leave it in the form "X square feet of three-inch high grass".

2.4 Scaling

Vocabulary: congruent; similar.

Seeing as the vocabulary for Euclidean geometry is pretty extensive, you might want to go ahead and have them get started now on the following: point, ray, line, line segment, notation for writing these (see Basic Terms); angle, acute angle, obtuse angle, right angle, adjacent angles; bisect, trisect.

2.4.1 Scaling Worksheet

Be prepared for this worksheet to take two or three days. It is an important worksheet so allot the time now rather than rushing it later. After seeing some pretty disappointing results on last quarter's final exam, I'd say you need to stress the fact that they will be expected to know how lengths, areas and volumes change when you rescale a figure. Tell them this after, perhaps, a day of work on the worksheet.

The work on Eureka! helps, although students may have difficulty understanding how to (or the need to) algebraically prove using formulae what is asked in each question. Question # 4 will require students to split the region into pieces (linear and area) to argue the results are the same as in # 2. In the spring with the area problem, students had several distinct approaches to arguing this one, which we called (1a), (1b) and (2). Approach (1a) was to impose a grid which expands along with the drawing and then use the fact that the grid squares multiplied by 9 together with the fact that the ratio of an enclosing rectangle that gets covered same. Approach (1b) was the same except they used that the count of squares is the same. This seems more natural to us than (1a), but didn't to them. Approach (2) was to allow the grid to remain the same size, but to see then that the number of squares will multiply by 9.

Question # 6 uncovers misconceptions not previously uncovered in that there are 144 (not 12) square inches per square foot. We treated question # 8 as optional.

2.4.2 Changing Units

This continues the Scaling work, particularly extending # 6. It was suprising how many students could do this with a chain of numerical proportions, but then didn't realize that the answer they computed was suspiciously simple, nor could they explain once this was pointed out. The lesson, I'd say, is that they need to go through all these steps without interference: do the worksheet as is and don't give too much away at first.

Some students may be confused with weak ratio and proportion knowledge (e.g., setting up proportions backwards, etc). Identify this and help them.

Note that the reflection on units does not have an entry in this guide because it is a reflection rather than a worksheet, but it is quite important and should not be skipped. They tended to believe that Student A was wrong, Student B was right, and tht Student C was also right, but didn't see the need to resolve the apparent contradiction beteween B and C.

2.5 Angle

Vocabulary: similar, parallel, perpendicular, complementary angles, supplementary angles; collinear points, coplanar points;

2.5.1 Units of Angle

For discussion of dihedral angles, you really need a model (Polydrons or a home-made hinged dihedral angle).

Students with little previous work with radians may have difficulty dissecting the definitions involved in #2 . One might get into a discussion of "what does an angle measure?" similar to the analogous questions asked about length, area, and volume. In any case, *radian* is one of those vocabulary words that in the end, they must study and understand.

The last problem is to elicit the definition of a dihedral angle. If students come up with definitions other than the angle between lines in the planes perpendicular to the intersection line, then examine these carefully. Are they equivalent? To be a good answer, it must be well-defined: that is, not depend on some choice as to how to draw the configuration on the two faces. Don't forget to bring polydrons or some other manipulative for this discussion. Perhaps something larger than the polydrons would be best here.

2.5.2 Angle Measurement

Students should not have much trouble with the work here, although you might concentrate on hidden concepts involved in measuring angles, analogous to those that arose in linear measure (spring videotape) such as: no gaps between units, need for identical units, and starting point calibration. Also, in the spring, students identified thses concepts as important: placing the vertex in the pinhole, extending or copying an angle, to create something with identical measurement that is more measureable, knowledge of 180 and 360 degree angles as straight and all-the-way-around, knowledge that adjacent angles add, use of correct one of the two number markings. Questions # 2 and # 3 are meant specifically to emphasize various alignment issues, while the first angle is meant to force them to deal with the issue of extending or copying (since an extension goes off the page).

2.5.3 Angles of a Polygon

The student, of course, is neglecting that she is counting the central angles, totaling 360°, in her formula. The point is to emphasize the physical meaning of addition of adjacent angles so that they see physically that the angle sum computed is the interior angles plus 360°.

2.5.4 Trisection

First of all, they have to be told to do this one accurately, since the discrepancies are small. This is probably best done in small groups so as not to give away the answer. Secondly, many had trouble interpreting the instructions. They need to work through this on their own, though it can be frustrating. Thirdly, they don't always try a good range of figures, so they should be encouraged to do so. Many concluded last time that the middle angle would be the greatest and the other two would be equal.

A complete solution ought to explore not only whether all three angles are always the same, but whether they ever can be (No!), whether one can say which will be the greatest (it has to be the one containing or closest to containing the foot of the perpendicular) and which will be least (one of the ends, never the middle). We don't need to be sticklers for completeness here, but it should be encouraged.

2.5.5 Olentangy River

Last quarter we spent quite a bit of class time on this one. The students came up with a variety of solutions, which we put on the board for all to see to judge correctness and creativity/elegance. It took a while to get through each group's presentation. We also spent time in advance discussing criteria: potential accuracy of method, how well it can be justified, how elementary (for children), and how easy to carry out in practice.

I think it was worth the time, as it felt very real and vital, it provided some comic relief when one or two groups had to sweat on the carry-out-in-practice criterion, and it involved a variety of geometric ideas, quite relevant as lead-ins to the section on deductive reasoning. Perhaps it would have gone better if preceded by a discussion of similarity.

One could take this further by imposing new restrictions (e.g., your motion is restricted so you can only view the river from two places and can only see one landmark across the river). The idea here is that if they can determine that the river's width is the side of a triangle for which they can draw a similar triangle on their paper, and for which one side can be measured, they can complete the measurement by inferring from the similarity.

2.6 Deductive reasoning

Vocabulary: axiom, postulate, theorem; alternate interior angles, alternate exterior angles, vertical angles, exterior angles of a polygon, interior angles of a polygon (some of these definitions are in Supplemenatary reading at C5, not in the Vocabulary list); angle bisector, circumscribed circle, inscribed circle, median, centroid, perpendicular bisector, transversal.

2.6.1 Pythagorean Theorem

One of the big lessons here is that even the famous and somewhat impenetrable Pythagorean Theorem may be proved by students no more sophisticated than they are. A secondary idea is that if one is going to claim this theorem as part of one's knowledge base, one had better know a precise statement of it.

The idea of this particular worksheet is to get students to figure out what the pictures say, similar to the process in "Picture Proofs". This is more difficult that the OPicture Proofs, since it requires some algebra to see that the rearrangement identity proves the Pythagorean Theorem. Let the students discover this on their own as much as possible (they did pretty well last time).

One might come back to this activity during "False Proofs from True" to prove the theorem more rigorously, for example in showing that the interior square really is a square because of the two complementary angles which supplement each angle of the small square. Actually, there are two ways to do this, depending on whether one starts with the assumption that one has placed four congruent triangles inside a square and needs to show that the remaining figure is a square, or starts by placing four congruent triangles around a square and needs to show that the resulting figure is a square (and there are no doubt other interpretations).

2.6.2 Rigidity

These questions will be the students' first real exposure in the course to rigor and proof. First, have the students answer the questions in # 1 by picture. Then have them return to this in the worksheet "Applying Postulates and Theorems".

Question # 3 gets at the meaning of SSS as well, and some of the students found it very interesting that this application comes up all the time in real life (we found examples in the room: braces on the tables).

Last quarter only one group even got to question # 2, but if time permits more groups to get to it, it might be a good warm-up for the Proofs worksheet.

On # 1, students may consider (b) and (d) to be the same kind of problem. Students in the spring nicely used compasses (for circles) to make decisions.

2.6.3 Similarities

You're on your own for this one. But if you liked Rigidity, you'll probably like Similarities too. Be sure they know that a measurement can be a length or an angle (but each measurement only measures something on one of the triangles).

2.6.4 Applying Postulates and Theorems

This is a preliminary worksheet to doing euclidean proofs. Note that they can only apply a theorem to get a positive answer on one of the seven parts to question # 1: part (e) is an application of SSS.

Be sure to go over the theorems and postulates (pp. 150-152). The students should also read "Supplement C" (pp. 122-149) for a more in-depth look at what mathematical rigor is and what is expected of them.

2.6.5 Proofs

Doing these proofs is going to be hard as hell for most of the students, though in my experience, it is a difficulty that most of them find enjoyable: they feel they are gaining understanding albeit slowly and are justifiably proud when they or their group manage to come up with a correct proof. More importantly, most students can tell when they have a correct proof, a self-reliance which we try to foster.

On this worksheet, students can usually see the "picture", but often have difficulty seeing how to go about proving something is a parallelogram or square using the postulates and theorems. Difficulties include understanding what goals need to be set (e.g., on # 2, we want to show that the angles in EFGH are right angles and that the sides are congruent), that we need to look for postulates and theorems that help us toward the goals (e.g., on # 1, what postulates have parallel lines as a necessary condition), and, especially in these and subsequent problems, that one can take advantage of the triangle theorems to prove, for example, that two angles are congruent. It is important to ask students questions leading to these goals and actions after their initial attempts.

Organization is also a problem. Often they have a correct proof except that it uses something they can't justify, and they don't yet have the concept of going after the proof of that as a separate lemma. Most groups, however, were able to learn this organizational step, with a little help from the instructor to make sure that the arguments did not end up being circular, and fit together correctly.

Students may present their proofs in a variety of ways (e.g., essay or line-by-line). It is important to emphasize to them that each statement made must have a reason given for its truth. In the spring, it appeared that those who presented short line-by-line proofs were more successful.

2.6.6 First Constructions

This worksheet tends to run itself pretty well. You'll need to allow at least two lessons for it, and be prepared to quite with some groups having gotten to perhaps only three out of five constructions: since these constructions are fundamental to what follows, better make sure this one is assigned for writeup, in its entirety if possible.

2.6.7 Circle Construction

This one is optional. We did not do it last quarter, but I have done it before and used it last quarter as an extra credit group exercise. The *ad hoc* group that did this for extra credit worked hard, understood it, and did an OK job explaining it.

The key is to see which circles can contain any two points, or rather, where must the center of a circle be if it is to contain two given points. Students can be guided to an answer for this by considering whether a point that is "way off center" can possibly be the center of a circle containing the two given points, then asking themselves why not. The step from two of the points to three is not as hard as this first step.

2.6.8 False Proofs from True

All along the way, students will struggle with proofs if the propositions to be proved are too obvious. This is especially true in the Proofs worksheet; unfortunately, it is necessary to put such a worksheet first because they don't yet know enough to do a proof of something that isn't obvious. Here, they need ot discover that, for example, there is no reason in the first picture to believe that the angles on the left formed by the upper and lower triangles in the second figure are supplementary. In general, they need to appreciate that pictures can be deceiving and that formal proof is needed to determine what and what is not deceiving.

Solving problem # 1 is more a case of detective work than of proof, but once they do figure out the mystery, they will suddenly have a much better understanding of the fact that something needs to be proved in the Pythagorean Theorem picture proofs (although from my experience, it is unlikely they will come up with this notion on their own). To get full value out of this worksheet, it is important to strike while the iron is hot, and time things so that after the contradiction in # 1 is explained, there is still a good bit of class time to make some headway on # 2.

This is also a good time to go back to anything that came up earlier requiring rigorous proof, such as the Picture Proofs and geoboard dissections.

2.7 Symmetry and rigid motions

Vocabulary: translation, reflection, rotation; flip, flip-glide; line of reflection, center of rotation, order of rotational symmetry; axis of rotation, plane of reflection; composition of rigid motions; tesselation, regular tesselation, semi-regular tesselation;

2.7.1 Symmetry of Planar Figures

This is the first formal exploration of symmetry in the course and there may be a number of questions regarding, for example, what is allowed and not allowed for a particular kind of symmetry in a particular figure. At some point, the students should become aware that the number of available final positions of any element of the object being rotated is the order of rotational symmetry (e.g., rotating a regular pentagon about its centroid, any given side can come to rest superimposed on either itself or one of the four other sides).

Notes: (1) the font on the letter "A" in question # 1 (a) is not a good one, since it make the figure not bilaterally symmetric when actually it is meant to be. (2) In question # 3, both are possible (e.g., a trapezoid and a parallelogram).

2.7.2 Rigid Motions

There is a parallelism here, as the students run through a similar gamut of questions on each of the three types of motion.

Describing the motion $T(x, y)$ may be tricky to some. You may suggest to investigate what happens to particular points, such as axis points.

Translation: The punch line here is to get students to figure out how to construct a translation using only compass and straightedge (Part (b)). Students may, at first, not realize that they must copy both distance and direction. Many will want simply to "eyeball" angles, copying what they think is the translation arrow(student's choice of which way), or sketching lines which look horizontal or vertical to them and then accepting them as so. Be sure to point out the problems inherent in this approach with respect to accuracy. Probably the simplest way is to draw a line that crosses through both the original figure and the translation arrow at some points, and then copy angles.

Rotation: Again, the construction is the tricky part. The key here is the addition of a segment from the center of rotation to one of the verticies (or point in question in part (b)) given, which will allow the rotation to be affected by copying the angle. Part (d) asks the opposite question. Asking them what kind of path each point must follow during the rotation may help. In fact, part (d) of the rotation question consumed the most class time of any single item, with many groups guessing something close but not right and spending a long time trying to prove it. It is worthwhile prompting them to check whether their guess is right! Here is where having done Circle Construction will pay off, since the idea is similar: the locus of centers of rotation that can send a point A to A' is the perpendicular bisector; if they realize this, then the only remaining step is to intersect two of the bisectors. (Will they realize that all four must intersect at a point?)

Reflection: Students will be able to perform compass-and- straightedge reflections without much difficulty. On (d), they may have difficulty: One suggestion may be to ask them what is needed to construct a line as well as what must be true about the line with respect to a point and its image under the reflection. This is a good warm-up to "Three Flips" and actually somewhat easier.

2.7.3 What Symmetries are Possible?

OK, here's the trick: (1) and (3) are not possible.

(1) If a plane figure has two lines of symmetry, then if the lines meet, that point is a center of rotational symmetry (doing the two reflections one after another induces this symmetry) while if they are parallel, then there is a translational symmetry and the figure can't be a plane figure, which is defined to be finite. This one was within their grasps, though it took a long time. It was necessary to prompt them to try drawing some figures with two lines of symmetry without worrying about whether the figure had rotational symmetry, then ask whether there was a rotational symmetry, after which it was more obvious that the intersection point turned out to be a center of rotational symmetry.

(2) The two rotational symmetries must have the same center. There are many examples.

(3) No plane figure has two different centers of rotational symmetry, regardless of whether it has a line of symmetry. For one thing, a center of rotational symmetry is of

necessity the center of gravity, so there can be only one in a finite figure. This proof is a little beyond them, so probably only problem # 1 should be assigned on this worksheet, even in small groups (though there is no harm in letting the fast groups get to # 2 and # 3).

2.7.4 Composing Rigid Motions

This is, in some sense, a follow-up to the previous two worksheets. (1) accomplishes a translation and (2) and (3) accomplish rotations. At this point, the immediate goal should be to reinforce the fact, previously discovered, that flips through two lines sequentially make a rotation through the point of intersection.

Eventually, the students should be asked when a given rigid motion can be accomplished with exactly two reflections, and when with exactly three. The answer, of course, is that it depends on orientation: translations, rotations, and an even number of reflections preserve orientation, while glide flips and an odd number of reflections reverse it.

2.7.5 Three Flips

This worksheet should be considered optional, especially since you now know how close you are to the end of the quarter. It is a classification, which is a proof at a higher level than most of the ones they have done. Nevertheless, the continuation breaks it down into bite-sized steps that they can follow even if they have trouble putting the pieces back together in their heads. If you do this one, be sure to discuss orientation first. Note: we can use polydrons as well as the cut out figures provided, though polydrons do not go past six sides.

Students may have difficulty keeping track of their constructions. You may want to suggest that they use a different color for each successive reflection. Note that each vertex, once used to determine a reflection, should not move again: Thus line l_2 should contain A''' and line l_3 should contain both A''' and B'''.

2.7.6 Tesselations of the plane

You will need to spend time at the beginning making sure students understand the new terms. Give a simple example or two (e.g., a grid of squares vs. a brick wall pattern in which each row is shifted over half a brick horizontally) before you let them go. Last quarter we got through only a fraction of which. This was too bad, since much of the material on tesselations was not covered, and tesselations do arise, with little or no mathematical motivation, in children's textbooks.

(1) There are only three regular tessellations, made with triangle, square, and hexagon, respectively. They can see this by examining the angle sum at any point.

(2) An exhaustive list of the eight semi-regular tessellations will take considerably longer. Let them discover as many as possible on their own, and get them to put them up on the board using colored chalk. Some will discover viable vertex arrangements that do not tessellate, such as (5, 5, 10); if they do not realize the flaw, ask them to continue the drawing so that you can see how the tessellation pattern works.

In discussing the solution, you should first get the students to accept or reject all that have been put up. Then ask how you can be sure you have all of them. At some point in agreeing that the valid ones presented do work, you should have verified (and/or have the students come up with the idea) that the sum of the vertex angles around a point add to 360 degrees (and in the process, students should have generated a table of vertex angle measurements on their own or through encouragement). Now we start from that same requirement, using and/or generating the formula given and writing down an equation such as that in the handout Analysis of Semiregular Tesselations. Ask how many polygons fit around a point; you should ascertain that it must be between three and six, with the only case of the last number being the regular triangle tessellation. Then, write the equation for three polygons and get the students to simplify it as shown. Then you can tell the students to turn the page and look at the table of solutions provided, noting that these have been found by exhaustive search, and that only some of the solutions actually tessellate. With luck, the students will have turned up all eight of the solutions.

Last quarter, students only did the first paragraph of the above recommendations. They did, however, come up with the criterion for the vertex angles and might have, with more time, gone into the algebraic representations described in the handout.

(3) All triangles will tessellate; students should be able to give some kind of

argument as to why. One way to see this is to take two copies of a triangle, flip one over, and join them to create a parallelogram.

(4) All quadrilaterals also tessellate, although some students will not believe this until you prove it to them. Wait until they have had a chance to argue the issue among themselves before you take a side. Have someone draw as ugly and "untessellatable" a quadrilateral as they can on the board, and then use colored chalk to make three more copies of it at one of the verticies. Label the interior vertex angles and show how to make the tessellation if none of the students can see how.

2.7.7 Symmetries of Solids

The most important point here is for them to understand what the symmetries mean. In fact last quarter we used this as essentially a study sheet for the final, promising one question very similar to the five on this sheet and leaving it to the students how many to do and to ask any questions they might have.

A hint for students: consider a particular face or edge of the solid in question; what must a plane of reflectional symmetry do to it? (Bisect it or not touch it.) How many places are there for it to "land" under a rotation?

During this activity, students should eventually find a methodical way to be sure they have found everything. For example, for the cube, there are axes which pass through pairs of opposite verticies. There are 8 verticies, hence 4 such axes.

2.8 Outtakes

Vocabulary: hypercube, hyperoctahedron, circumcenter (not in glossary); dilation; cross-section (not in glossary).

Included here are several worksheets that we have tried out but, mostly for reasons of time, were eliminated from the curriculum. These are not as well tested; brief notes are included after each one.

2.8.1 Geoboard Constructions

1. Make a geoboard figure with area 3, so that as few nails as possible are contacted.

2. Make a geoboard triangle with area 5.

3. Make a geoboard square with area 3.

Notes:

The first problem mikght be best done after the Pick's Formula worksheet, although it could also serve as a lead-in to it.

The second problem is somewhat routine and tests whether they can go backward with the triangle area formula.

The third problem is impossible, since the square root of three cannot be constructed on a geoboard. If they know the pythagorean theorem, this might be a fun discussion, but given our ordering of the material, we weren't prepared for that discussion yet.

2.8.2 Pick's Formula

In 1899, Georg Pick discovered a beautiful formula for calculating the area A of a polygon that can be formed on a geoboard from just two easily determined numbers:

I −− the number of geoboard points in the interior of the polygon, and

B −− the number of geoboard points on its boundary.

Your task is to re-discover the relationship among A, I and B.

Notes:

This activity requires students to collect and then organize data, which is definitely the most difficult part of the discovery process here. Some students may try to collect data in tables; others may draw graphs. The problem is in systematically rather than randomly look at cases, in particular holding one variable constant while letting the other two vary. Some students may eventually need a hint to consider half of B rather than B.

This is a pretty good worksheet, and can lead to a good thrill for those who do discover a formula without too much help. On the other hand, a proof is pretty far out of range even once the correct formula is found, and it doesn't add too much to their conceptual knowledge base.

2.8.3 Circle Construction

Draw any three non-collinear points on a piece of paper. Can you construct a circle containing all three on its circumference?

Notes:

One possible use for this is as an extra credit group exercise. The *ad hoc* group that did this for extra credit worked hard, understood it, and did an OK job explaining it.

The key is to see which circles can contain any two points, or rather, where must the center of a circle be if it is to contain two given points. Students can be guided to an answer for this by considering whether a point that is “way off center” can possibly be the center of a circle containing the two given points, then asking themselves why not. The step from two of the points to three is not as hard as this first step.

2.8.4 Constructing an Altitude

1. Construct an equilateral triangle.

2. Then construct one of its altitudes.

3. If the sides each have length 1, how long is the altitude?

Notes:

This is very straightforward, and can be used for extra credit or reinforcement for students who are struggling. Depending on where discussions of the Pythagorean Theorem and of similar right triangles have led, this worksheet could be redundant.

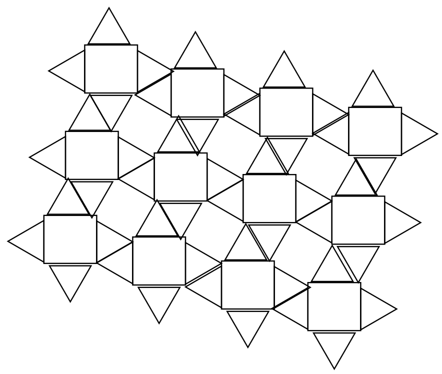
2.8.5 Wallpaper

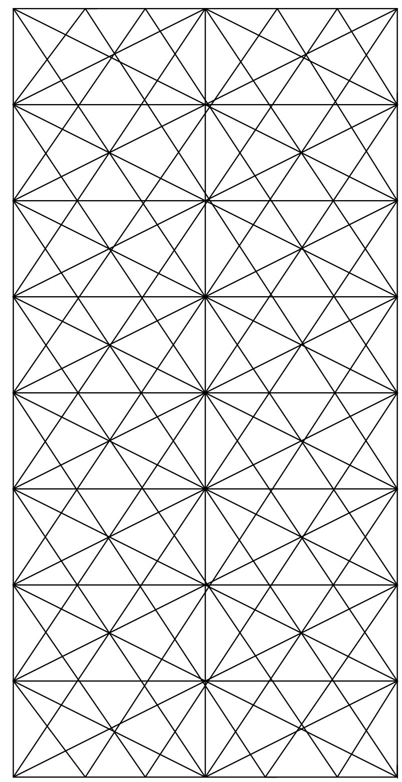
Infinite objects, such as tesselations, may have translational symmetries as well as symmetries of rotation and reflection. Infinite patterns with symmetries are called "wallpaper patterns". These patterns can be classified according to their symmetries. Here is something interesting: there are only 17 different types of symmetries of two-dimensional wallpaper patterns. The Wallpaper handout shows examples of all 17 types and list the symmetries of each type.

1. See if you can tell the type of each of these patterns: list the symmetries and then match to one of the 17 on the list.

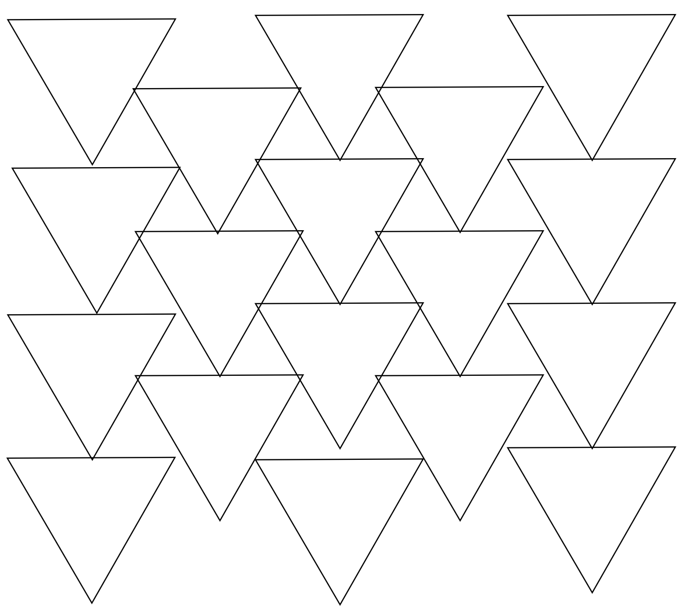
2. Find two wallpaper patterns outside of class, sketch them, and classify them.

(a)





(b)



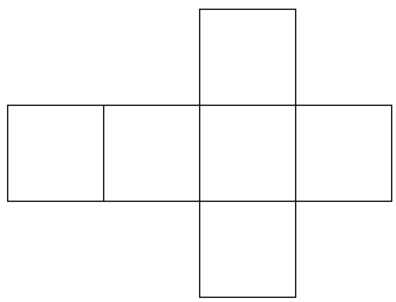
(c)

Notes:

The most important thing you need to know about this worksheet is that the list of 17 wallpaper patterns is not provided!

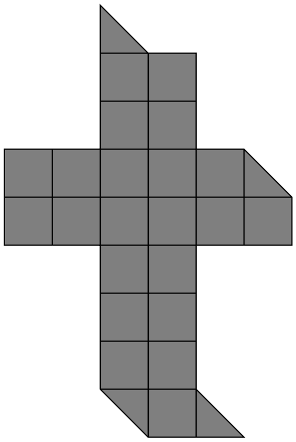
2.8.6 Folding up a Cube

1. The standard way of cutting out a 6-square region from a piece of graph paper so it will fold up into a cube is the cross shape:

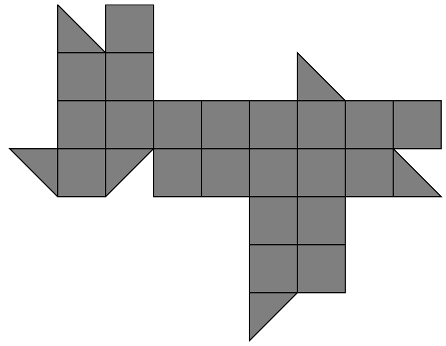
   

   There are several different shapes that will work, and that use 6 contiguous squares of graph paper. See how many you can find and draw them onto a clean sheet of graph paper. Don't count congruent patterns more than once.

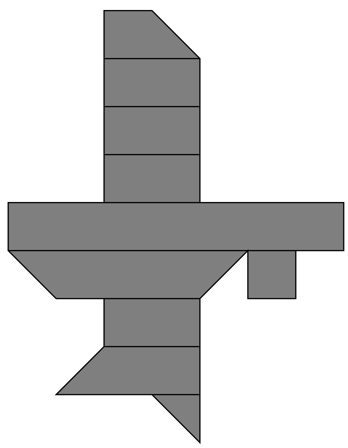
2. Which of the patterns on the next page can be folded up into a cube, with no gaps or overlap? The object is to try to visualize this, so please do not cut and test the figures until afterwards.



A



B



C

Notes:

This is the first of three worksheets on spatial visualization, which formed a mini-topic unit in a previous version of the course. I like this unit, since I think spatial visualization is a valuable skill. Also, I am very bad at it and some of the students are good at it, so it gives some students a chance to shine. The unit got cut because of time constraints, but if you like these three worksheets, you might be able to do them as quickies, interspersed throughout the quarter.

2.8.7 Hypercube!

Some people claim to be able to imagine a fourth spatial dimension. I don't know whether this is possible (I can't do it), but it is possible to infer things about what the fourth dimension would be like if there were one. If you extend the sequence: point, line segment, square, cube, by one more term, you get a four-dimensional cube, called a "hypercube" or "tesseract". How many vertices would it have? Edges? Faces? What more could you add to its description? If each side had length 3 inches, what would be its four-dimensional volume? Its three-dimensional "surface" area?

Notes:

ome students really enjoy speculating about the fourth dimension, even though it is sometimes disapointing to do so mathematically and realize it gives no spatial "sense" of the extra dimension. This worksheet would work well with one on the Euler characteristic, which we have not as yet included in this curriculum.

2.8.8 Dissection in 3-D

1. I have supplied you each with a cube of cheese. How can you cut this in one slice, so that each of the two new exposed faces is a perfect hexagon?

2. I want to cut a tetrahedron, with one slice, into two congruent polyhedra. In how many ways can this be done? Two ways count as different if the tetrahedron, marked by the places where it will be cut, is not congruent to the second marked tetrahedron.

3. Glue a face of a regular tetrahedron to a congruent face of a square-based pyramid. How many faces does the resulting solid have?

Notes:

All of the questions on this one are "old chestnuts" and can be found in books of problems or brain teasers. Again, it is rewarding for students whose strengths lie in this direction to get to try out their skills, though it does not tie in all that well with the conceptual themes of the course.

3 How to teach in a cooperative classroom

This section contains a great deal of information and advice regarding how to manage a cooperative classroom, on all levels. In fact, the amount of reading is rather overwhelming, so we suggest that at first you read just the first two sections, on the basics and on the beginning of the semester. The remaining sections, on small group dynamics, large group dynamics, and organizing yourself, can then be used as reference either as day one nears, or once the class has met a couple of times and you have some more specific questions. As with the rest of this guide, the recommended procedure is to browse first, then come back and zero in on the topics of greatest interest to you.

This guide was written for the Math 130-131-132 sequence at the University of Wisconsin. There are a few references that are specific to that situation, though the bulk of what is in here is general advice for anyone teaching in a group-learning, socratic classroom.

DISCLAIMER

In training one's self or others to teach a Socratic/Cooperative style class (henceforth SC), it seems no amount of preparation or advice can substitute for a certain kind of on-the-job training. The essential ingredients of on-the-job training are criticism and imitation. Experienced instructors visit the classrooms of new instructors, taking extensive notes on what they see: what went right, what went wrong, what might have worked better, and so on. New instructors also visit the classrooms of experienced instructors, taking equally careful notes on what went well or poorly, what might have worked better, and on what techniques they see used that they would like to use themselves. Some of this type of work can be done beforehand, via visits in the previous semester or videotaped classes, but it tends to be more valuable when it comes after the new instructor has had a chance to try teaching a class or two first. Currently in 130-131-132 we are trying for one visit each direction in the first two weeks, another in the second two weeks, plus at least one more during the term.

If you are not going to adhere to a plan as outlined above, then **don't expect the notes that follow to do much for you**. You can't learn to play the piano by reading books about it or by talking about technique with Rubenstein, so don't try to learn a significant new teaching skill without practice and guidance either. On the plus side, all the interested instructors that have taught SC classes here (admittedly a self-selected sample) have become pretty good at it, so you can be pretty confident that you'll be able to step right in and teach effectively this way even if you're not a virtuoso.

3.1 Basics

3.1.1 The philosophy

Unless you've been on Mars, you're probably aware of the age-old battle between "skills" advocates and "process" advocates. I'm from the "process" camp but I hope to avoid a lot of the partisanship that is prevalent in discussions of pedagogy and stick mainly to points both sides agree on. We want students to come away from (lower level) math classes with certain skills and attitudes. In particular we want them to reason and prove, to translate between words and symbols, to perform algebraic manipulations correctly and with understanding of the justification, and to solve problems other than clones of problems they have been shown how to do. Whether or not you believe specific skills and knowledge to be paramount (arithmetic of negative numbers, solution of quadratic equations, propositional logic, summation of common series), you undoubtedly want them to know these things in the ways mentioned above: verbally and symbolically, with justification and proof, well enough to apply to new situations.

The tenet underlying SC classes is that students need to learn that they can think for themselves, and that they will be able to learn properly if and only if they are forced or enticed to continue thinking things through on their own terms. Realize please that this does not apply to students who already have this skill. I don't think we need SC classes at the advanced undergraduate level, and they become increasingly inefficient at higher levels. If college admissions standards (or high school graduation standards) were what we'd like them to be, we wouldn't need SC classes in college at all. The students who benefit from SC classes are ones I would term remedial: those taking pre-calculus, and those in the Ed program who are required to take what is essentially junior high school mathematics.

Our philosophy with these students is to do anything we can to get them to think and speak mathematics, and then once we have them going, to exact from them a quality of mathematical reasoning that is higher than anything ever asked of them in a traditional course, thus ensuring that they learn the course content in a useful and permanent way. The meta-skill we emphasize is for the students to **know when they know something**, versus when they are just guessing or are confused. In the time-span of the course or sequence of courses we move from a "process is everything, choice of content matters little" approach to a stage where we cover the traditional content and expect students to focus on these topics and skills while applying the critical thinking they have learned in the first phase. The first phase is the harder phase for most instructors, since we have to be psychologists — and sometimes mind-readers — in addition to being mathematicians. These notes concentrate on this phase, though they apply to the other as well.

Perhaps the most controversial part of this approach is our unwillingness to tell students the answer. Some skills can only be acquired this way, and one can be overly dogmatic on this point. The basis for this is that much of this material is accessible to them, with a little help from us, and that our habitually providing answers will cause their problem-solving ability to atrophy, though it may increase their rate of skill acquisition (though we argue probably it won't). Thus we make every attempt when discussing an attempted solution in class not to tip off whether it is right or wrong until the whole class has had a chance to criticize it or register comprehension and agreement. Depending on the context, we do or do not in the end provide model solutions.

3.1.2 Typical classroom mechanics

A usual 50 minute class consists of two kinds of time: some time when the students are working in groups of 3 or 4 on a problem or worksheet (set of problems) and some time when the entire class is discussing the problem set. Some instructors enjoy keeping to a familiar rhythm, spending the initial 25 to 30 minutes each day working in small groups, then spending the latter part of the lesson in a large group discussion detailing what the various small groups found, where they got stuck, and so forth. Often there are parts of the worksheet that are not covered in this phase; some of these are assigned for homework and some are discarded. Other instructors prefer to go back and forth a little more, starting in a small group, then convening the large group when most small groups are done with the first problem, discussing it a bit, then remanding the class into small groups, and so on. When a class meets for 75 minutes twice a week instead of 50 minutes three times a week, it is usually necessary to go back and forth this way, and it is also often convenient to continue a large group discussion from the end of one class at the beginning of the next.

During the small group working time, the instructor circulates among the groups, answering questions when necessary, doling out encouragement when necessary, challenging the students to justify what they claim to have figured out or to explain their half-baked ideas. Often the mere presence of the instructor encourages a renewed attack on a problem.

The large group discussion begins with the students explaining what they have done. Other students are required to listen carefully and to register agreement, disagreement or incomprehension. Once the explanation is comprehensible, those in disagreement are encouraged to justify their disagreement, with the aim of a resolution or synthesis. The instructor plays moderator as long as fruitful ideas are being produced, but slips into the role of leader when needed. For example, if no one challenges something wrong, or if there is a disagreement but it is too inarticulate to produce a good synthesis, then the instructor may rephrase what has been said so as to sharpen the contradiction or caricature a wrong approach, in a way that forces a light to dawn for at least some students.

Note also that in order for SC discussions to work, the class size must be limited, to no more than twenty or thirty maximum. The point is to have the class participate as a whole in the same conversation, but if you think back to social gatherings you've attended, even if everyone knows each other, it is difficult to keep everyone involved in the same conversation once you get more than a dozen or so people involved.

3.1.3 Highly recommended procedures

Before getting into specifics of classroom technique, here are a few simple procedures that make a large difference.

- *Nametags.* Have the students wear nametags each day until you know their names (in my case 2 or 3 weeks). When calling on students, call on them by name and in general attempt to use their names frequently. This serves two purposes. First, by attempting to learn their names, you create a separate mental category for each student, which helps you pay attention to how each student is doing and to their individual needs. Secondly, there seems to be a psychological advantage to students hearing their names. Coming around to a small group and asking "Amy, can you tell me what progress your group has made?" or asking in a large group discussion whether "Sam's objection to Cindy's idea" holds water elicits more of a response than the same questions without the names identified. Somehow, students are more prone to take their own beliefs seriously when names are attached.

- *Randomize groups.* I usually assign groups using playing cards randomly dealt: all the aces for a group, the twos form another, and so on. I re-form groups twice or thrice during the semester. Preventing students from choosing their buddies for a group helps them form connections, subject their ideas to the intellectual marketplace, and treat all the others in their groups fairly. The playing cards themselves lend an air of intrigue as students await the results of the lottery. If you decide not to use playing cards to change groups, you can assign the students the task of creating new groups such that no two "new" teammates have worked with each other before. (This can be done easily once, and with considerably more effort a second time.)

- *Get enough sleep.* Alertness is required on the part of the instructor. You'd be surprised what a difference this makes. You can fake it when you're lecturing, but try playing moderator when you can't concentrate or respond quickly, and you'll see what I mean.

- *Start the semester with a bang.* That is, don't spend the first day on administrative stuff and the second on some kind of review. Jump into an absorbing problem on day 1, preferably a tried-and-true chestnut, and fill in the administrative details later, when they've gotten the idea of what the class will be about. The tone of the first day's discussion sets an example that's hard to erase, so make sure it is as lively as possible.

- *Minimize the time you spend talking.* In particular, never talk for more than five minutes at a time — monologues shift students back into passive learning mode, and monologues of announcements tune students out. Of course, you may have to do a lot of talking in leading large-group discussions, but if you're talking by yourself for more than five minutes at a time without at least one student speaking up, you're not doing SC — you're lecturing.

- *Be yourself* at least up to a point. If you're the goofy type, be goofy; if you're serious and intense, let them feel the intensity; if you're understated and direct, be that way. You are on stage, and want to use your charisma, but don't pretend to be someone you're not.

3.2 Beginning the semester

3.2.1 The first day

In many ways, the first day can be one of the most important of the semester. It sets tone and precedent for the days that will follow, and lets the students know what to expect. Students will be anxious to know how their grades will be determined, and perhaps anxious in general to be in a math classroom. Your job that first day is to sell them on the class, to help them start to get comfortable. This job has two main parts: dealing quickly with administrative details and getting the students engaged in problem solving. Suggestions follow to help you with both of these parts.

You should already know all the administrative details about the class when you walk in the door the first day, and the quickest fair treatment for communicating these details is in the form of a handout. You may want to write your name and the class name, number, section, etc. on the blackboard before class starts, to make sure people are in the right place, but don't spend time in class writing all the details on the board. Rather, have them collected on a handout, so that you can skim the most important details (components of the grade, exam dates) and leave the students to peruse the rest at their leisure, *outside of class*.

A sample handout is given on the next page. A few notes regarding it follow.

Math 130

Section 1, 1:20-2:10 PM MWF, B139 Van Vleck

Instructor: Christopher Kribs

Office: 822 Van Vleck

Phone: 262-0537

email: kribs@math.wisc.edu

WWW: http://math.wisc.edu/~kribs

Office Hours: after class and TBA

Text materials: available at Bob's Copy Shop in University Square ($8)

Midterm date: Tuesday, March 26, 5:15-7:00 PM

Final exam: Wednesday, May 15, 2:45-4:45 PM

This course is the first in a sequence of three designed to help you develop your intuitive reasoning and problem-solving skills. We will spend most of our time in class working in small groups on various problems, usually from the course packet. The problems are designed to be both interesting and nontrivial, so you should be prepared to spend some serious time on them (in class, for the most part). You will almost certainly get stuck on a problem at some point in the semester, but don't be discouraged — if you persevere, you'll get through! We will usually discuss the problems in a large group after most groups have finished them. Sometimes you will be asked to write up your ideas and solutions, but *always* you are expected to think about the problems, participate in solving them, and communicate your ideas with others. Communicating your ideas to others is as important as developing them in the first place.

Note that this is a math *content* course, and not a pedagogy course. We hope that taking this course will help you be a better teacher, but more by setting an example rather than teaching you math methods.

GRADES: Your grade for the course will be determined by two exams (20% each), by attendance and participation (20%), and in large part by written work you will turn in (40%).

The exams will be similar in nature to the problems we work in class, but short enough that you should be able to complete them in the time given. A sample exam will be distributed before the actual exam in order to give you a closer feel for it, though you should *not* expect it to serve as an exact blueprint for the real thing. There will be a midterm and a final exam (locations TBA); the dates are given above. Please mark these dates and times on your calendar now so as to avoid conflicts. In the event that a conflict arises, *please* see me as soon as possible so that we can resolve it.

Attendance and participation are a significant part of your grade because this course is more an experience than a set of material to be learned. Most of what I hope will happen for you in Math 130 will take place inside the classroom, working in groups and talking with others. You may miss up to 3 days for reasons of health, religion, etc. without penalty. Arriving late or leaving early counts as half an absence. If you have special needs, please see me in the first two weeks of the term.

It is also in your interest to participate in the group problem solving sessions since active learning is better than passive learning. Participation includes both small and large group work. If you don't feel comfortable answering questions, *ask* some of your own: that spurs discussion as much as an answer, and you won't be the only one with that question.

The written work will have two components: write-ups (also called problem reports) and reflections. A write-up is a detailed solution to a problem we discussed in class. These write-ups should be readable independently of any worksheet on which they are based, in good English and either legibly handwritten in ink or word-processed. They should always include the following: 1. a statement of the problem at hand, 2. any strategies you used to attack the problem, 3. the solution you obtained, with an explanation of how you got it (and how you know it is complete), and 4. a conclusion that says what we can take with us from the problem. Communication of what you understand (even if it's not a complete understanding) is at least as much the point as finding the solution.

I will also sometimes ask you to write a reflection on a rather less concrete issue, like "What does it mean to get stuck?" These essays, usually a page or two in length, will be graded more loosely, more on how much thought went into it than on organization and content.

I will let you know at the time I assign written work when it is due, but typically it will be due in class a week from the time it is assigned, and you will have roughly one assignment due per week.

Optional reference text: Billstein, Libeskind and Lott, *A problem solving approach to mathematics for elementary school teachers*. Second edition (1984). Menlo Park, CA: The Benjamin/Cummings Publishing Company.

These are *your* blackboards. Own the classroom.

Notes:

- Even though this course is designed to be run entirely from the activity set and class discussions, some students will feel lost without a textbook, for definitions, examples, and the like. The text listed in the sample handout, by Billstein, Libeskind and Lott, covers much of the same material as both Math 130 and Math 131 (the geometry course). Be careful, however, not to list the text alongside the coursepack, nor to give the impression that students who do not consult a text will miss anything. If the whole class seems lost on something, better to spend ten minutes in class explaining it or distribute a handout with examples rather than refer everyone to a textbook.
- As noted in the handout (and numerous times in this guide), Math 130 is a math content course, not a math methods course. Make sure the students understand that.
- See Chapter 6 of this guide for more discussion of how to determine course grades.

Different instructors will have different ways of building the atmosphere they desire. Some instructors do best by jumping right into the first activity ("Poison") in Unit One. They then reserve a fraction of the second day's class to discuss the nature of the course, and why we do things the way we do, using the Poison lesson as an example. Other instructors are more comfortable devoting some time to "warm fuzzy" introductory activities. They will use a little class time to introduce everyone. To take care of any reticence on students' part to talk about themselves, try having everyone pair up with someone s/he doesn't know, and spend five minutes getting to know one another. (This also gets them used to talking in class!) Then go around the room having each student introduce his/her partner. Some instructors also have students fill out index cards with a few facts about the student (name, major, hometown, feelings about math, something unusual about the person, and perhaps suggested office hours); these can serve as backup in case someone forgets his/her partner's name. You can also add a couple of icebreakers such as "favorite cartoon character" and "least favorite vegetable".

If you spend time on introductions, then you will want to choose a short warm-up problem-solving activity that can be completed in the remaining time. Here's one (Pascal's Triangle) that has worked for us. First, make sure everyone has a nametag, and then put them into small groups. Put up on the board, or on a handout if you prefer, the first five or six rows:

```
        1
       1 1
      1 2 1
     1 3 3 1
    1 4 6 4 1
  1 5 10 10 5 1
      .  .  .
```

and ask the following questions: Can you find the rule used to generate Pascal's Triangle? What do the next two rows look like? What patterns do you recognize in it?

The groups will discover the rule quickly enough, and the last question is open-ended enough to allow discussion to continue until the bell rings.

3.2.2 Encouraging Good Habits from Day One

Encourage good habits early, to set precedents for the entire semester: Encourage people to talk, to offer ideas without worrying about whether they're "right" or not, to talk to each other (instead of to you) as much as possible, to write on the blackboard, to be precise about what they mean, and to get used to giving explanations and justifications to back up things they really believe. One way to encourage the latter three things is to "play dumb" and keep saying you're not sure what they mean — even if you do — as long as you think someone else in the class might not (which is usually true). Of course, you should give a disclaimer the first day of class explaining that you're not going to be giving them straight answers or telling them whether or not a solution is correct (though you should never let the book be definitively closed on a subject without making sure that there is closure, and the everyone understands that the solution on which the class has agreed is indeed correct). Part of this disclaimer will include the fact that you reserve the right to "play dumb" as long as you think something might not be clear to someone in the class.

Another habit to encourage from the beginning is to keep a running list of different problem solving strategies and techniques they have used. By the time you finish Unit One, they should already have a half dozen or more, and it might be worth taking a few minutes in class at that point to have them compare lists. Chapter 2 mentions another discussion to have around the end of Unit One, regarding group dynamics.

3.2.3 Resistance and "Why are we studying this?"

One of the questions which you as a teacher in the Math 130 sequence will hear over and over again is "Why are we studying this?" Why are we studying these topics which we will never be teaching? Why are we taking this course? These questions deserve an answer, and again this is something best discussed openly and early (although no doubt the question will recur). Among other things, elementary school teachers must lay the groundwork for their students' later experiences. In math, arithmetic is the foundation of algebra, and teachers need to know what's coming up in the years ahead, so that they can teach in a way that will later allow their students to look back and see the connections. Mathematics is all connected, and the sequence of K-12 math classes students take should be a natural progression, not an unrelated set of topics. Teachers also have to be able to anticipate precocious questions. Equally importantly, they must be comfortable and familiar enough with every subject they teach that their students will not all be turned off. Many, if not most, of the students in the Math 130 sequence have had bad experiences with math in the past, and very often this can be traced to a teacher or teachers who passed on their apathy or disinterest for the subject.

These past experiences will, as noted above and to varying extents, lead to resistance by the students to the way Math 130 forces them to shoulder responsibility for their mathematics. Here again, you as their instructor should bring this subject out in the open, early and as often as needed (although it can, at times, threaten to take up a lot of in-class time — try not to let this happen). Point out the first day that the reason for their doing this is not only to provide a model for them to use later is getting their own students to think for themselves, but that someday soon they will have to be the mathematical authority for a classroom full of inquisitive children, and that before they can be someone else's mathematical authority, they must be confident in being their own authority. There will invariably be one or two students in a section who do not "buy into" this philosophy, and in the end there may be nothing more you can do for them but to tell them that the class is designed this way because people who have had a lot of experience teaching math believe in it, and they'll have to play along for now and see what results come of it in the end.

Some students, especially the poorer (for want of a better term) ones, will be vocally and continually concerned for their grades. The most you can do is try, from the beginning, to make grade assignment rules as explicit as possible (but don't feel any obligation to explain down to the point level your grading of particular reports, any more than a composition teacher would), and try to reassure them. Don't let their angst transfer to you.

3.3 Class Composition and Small Group Dynamics

3.3.1 Doing the rounds

When you first assign a problem to work on in small groups, there may not be much you have to do. No one is stuck yet; no one needs your help. There is a lot you can accomplish in this time. First, you can quickly take attendance group by group — after you know the names this takes less than half a minute. After this, you will want to quickly "do the rounds". Visit each group once just to check that they have gotten down to work. On the first round, look for any trouble with the wording of the problem that may be holding people up. If it's part of their job to decipher it, encourage them to do so. If it's a mistake, or if you need to supply a definition, then make a quick announcement. On your second round you can linger longer. This is a good time to make a mental note of which groups are going faster than the others. It helps, during the subsequent large group discussion, to have a good idea of who has gotten how far. If a group has quickly and incorrectly or incompletely answered a problem and gone on to another, this is a good time to ask (innocently) for one of them to summarize for you what they found. The correct question on your part can cause them to re-examine what they've done without feeling that you've invalidated their answer (point of philosophy: you want them to be able to criticize their own work, realizing that mathematics will determine whether they are right, and that what they discover about this cannot be overruled by the teacher).

Example: A group has used an incorrect manipulation: $(a+b)^2 = a^2 + b^2$. You can "explain for those who are rusty on this", that this means "when you compute, e.g., $(7+3)^2$, in case you can't tell at a glance that $7 + 3 = 10$, squared $= 100$, you can use this nifty rule, getting $7^2 = 49$, plus $3^2 = 9$, which by the laws of algebra must also give you 100." If this doesn't elicit an objection, you can line up the sum as you speak:

$$\begin{array}{cr} & 49 \\ + & 9 \\ = & 100 \end{array}$$

When you do get an objection, ask them to pinpoint what went wrong, leading to, "Oh, you mean $(a+b)^2 = a^2 + b^2$ is not a universal rule of algebra?"

A reminder: Lurk early, lurk often. While students are in small groups, eavesdrop constantly. The students should get used to having you listen in to their conversations. This keeps you abreast of both (a) how far the class in general is getting and (b) which groups in particular are having difficulties, and where. If one group seems too conscious of your presence, then stand near another group nearby and appear to be watching them when in reality you're paying attention to the discussion in the other group. This part of the class is an important one for you as instructor and (presently) discussion moderator.

3.3.2 "Help, we're stuck"

The twin dangers here are that lazy students will say they're stuck so you do the work for them, while students who are truly stuck will lose morale if they have to sit idly during class. Asking if they have any ideas on what they might try will prove embarrassing if the answer is no. Sometimes I do this anyway. Sometimes I replace the problem with a smaller one: if you knew that $A = 15$ could you do the problem? Can you do the problem if you aren't required to make the number of cows and chickens the same? Sometimes I guess why they're stuck: So the problem is you don't really know the definition of average speed? An example of what might happen here is that they did know this but didn't think of going back to definitions as a way to proceed. Now when they say no that's not the problem, we know the definition of average speed, I say, "Oh, then you must be saying you don't have any way of determining the quantities defining speed, such as the time or the distance." They then say how they will proceed on this and I can smile and leave. In other cases, they are stuck because they don't really understand what's being asked. You can ask them to rephrase it, or ask them how they would check if someone else's answer (here you specify it) was right. It helps to have snooped enough so you have a good guess of where they're stuck. If not, you can ask them but won't always get reliable information.

3.3.3 Getting Groups to Work Together

The pre-service teachers tend to work well together. The pre-calculus population is less accustomed to this, and groups will degenerate into a lot of individuals ignoring each other, or occasionally explaining to each other. You may at times need to tell them explicitly "Janet has found what she thinks is an answer but Steve and Brenda apparently don't understand what she did, so Janet, you're going to have to explain it and see if you can convince Steve and Brenda." However, you should try other things first, before being this explicit. Ask Brenda what her group has found so far, and don't let anyone else answer for her. If she says she's stuck, ask if her whole group is stuck, and if not, tell her you'll come back in 3 minutes and ask her again for a summary of what her group has done. **Make sure groups are sitting in a tight circle**, not a line or a disarrayed cluster.

Sometimes you can give them specific tasks: Jason, finish the calculation you're doing; meanwhile Ann will add up Rob's numbers and will then check to see if they agree with yours; if not, it's up to all of you to judge which method if either was correct and why. If a group really has bad chemistry, change it. I've had students say to me: I just can't work with Judy — she won't listen and hogs the discussion. In that case, put Judy in with someone smarter or more aggressive than she. When re-forming the groups, I usually randomize again, but if there are trouble students, sometimes I stack the deck so that they get put with students who can handle them.

3.3.4 Free Riders

There are always lazy students who are content to let others do the work. If it's just laziness, I don't hesitate to reprimand them explicitly: "Adam — you're just staring into space and letting the other three figure out the problem; if the problems aren't challenging enough, then I can let you work faster in a group by yourself, but judging from your homework that's not the issue." On the other hand, if it's a student with a confidence problem who needs some nurturing, it's probably better to make a note of it and to continue to ask that student to explain what their group has done whenever you come around (you have to ask other students sometimes, or it gets too obvious). Involving the student as much as possible, with questions that are at their level but not patronizing, will often cure this. Also impress upon the others that it's their job to make sure the free rider is keeping up with the group since that person (you've just decided) will be in charge of the first group writeup.

3.3.5 Staying on Task

The less you have to reprimand here, the better. Make sure that when you tell them to get into groups they know exactly what they're supposed to start working on. Make the rounds quickly at the beginning so they don't start chatting, and keep an ear out for it later on. If a group continues to be bad about this, you can watch them, visit them more, chastise them humorously, make sure they don't get grouped together next time, etc., but if the whole class is bad, you should examine what you're doing that promotes doing something other than the math. You could be leaving them in small groups too long, while the slow groups finish. You could be joking with them too much during class time. I do also use reprimands, but sparingly — on day 3 this year I was in an impatient mood so I reprimanded the two groups that took more than a minute to form their groups and start working. It was something like "Sounds interesting whatever you're talking about, but you've got to get started on problem 1 — time's short this week!"

3.3.6 Students who are behind

First, it's a good idea to know what you do for them and what you can't. You can do a lot for these students in office hours, but it's not realistic to be able to spend more than (or even as much as) half an hour a week outside of class with any one student. So if they need help on a regular basis, suggest that they arrange for tutoring. You should find out before the semester what such resources are available. If they're doing passing work, but still underconfident, point out to them that if they continue to work at this level, they'll pass the class. Don't, however, make promises you can't keep about their grades; it's best not to prognosticate about their grades before they've taken an exam.

All that being said, you have to make sure they get the most out of their group work and don't drag down the group (they're as much afraid of this as you are). Here are some ways to build their confidence and make them more likely to participate to advantage. Spend some time around their group and be ready to pounce on those times when the lagging student, "Lenny", comes up with a good idea. Assign credit: if Jocelyn figures out how to do something with Lenny's idea, then it's "Lenny & Jocelyn's method". Try to give them constructive comments on their homework (I try to do this for everyone, but when time does not permit this, concentrate on students who need it). Making sure they are keeping up with their group is delicate — asking them to explain to you where their group is will help if they're not too far lost. Once they are far behind, consider placing them next time with a group of students that talk a lot, don't go all that fast, and are as kind as possible. At worst, you may have to settle for Lenny participating minimally during class and trying to catch up on his own at home.

Select Lenny to give a presentation, either solo or on behalf of his group. On a one-time basis you can invest a little extra time to help him with this to make sure it goes over well (have him practice it on you till he's confident enough to field questions).

Avoid asking Lenny really easy questions. This will make him feel like he must be dumb. These questions are scary in general since the reward for a correct answer is almost zero and the penalty for an incorrect answer is large. At your discretion, you might choose Lenny to answer questions that are not black and white, asked of the class at large: which of these problems was the hardest?, what kinds of thinking did you think were necessary on this problem?, and so on.

3.3.7 Students who are ahead

Having such a student can be a real boon if they are gifted teachers as well. If they have a good feel for how to explain things and help others, they will make your class run more smoothly than you can, on their own. Even in this case, **avoid treating them in front of the class as a reliable source for right answers**. You don't want to create a situation where calling on them is tantamount to telling the class something yourself. It is OK though, to treat them as a reliable source for intelligent commentary.

If a student is obviously head and shoulders above the rest from day 1, you may consider exempting them from the course. That is, if they can do the worksheets on their own then they can probably pass the exemption exam, go on to the next course, and leave you with a more homogeneous class. Recommend that they see the course coordinator for an exemption exam; everyone will benefit from this. Later in the semester this is less of a good idea, though I've done it.

Assuming "Einstein" stays in the class and is not self-policing, you need to keep an eye on Einstein's group to make sure that Einstein is not explaining things to the others before they have a chance to figure it out themselves. Let Einstein explain things at the board in situations where you know there will be some wrong or unclear stuff in the explanation. Make sure though, that you give Einstein as much encouragement for what was right and clear as you would another student. If Einstein is a loner and tends to work fast but not share with the others, that will probably work out fine. You can explicitly designate Einstein as a group of 1 next time, or simply allow a *de facto* group of 1 to form. Sometimes, you can try asking Einstein explicitly to figure out a hint to give the rest of the group as to how to proceed but that won't completely solve the problem for them. It will make Einstein summon up teaching skills that are worthwhile in general, so it's worth a try, but be aware that Einstein may not be capable of this. In any case, don't let students disparage themselves in comparison to Einstein. You can say, "I see, because Einstein solved this problem in 5 minutes and you can't, you're going to give up?"

3.4 Managing Socratic Discussions

This is the hardest part, and the part of teaching SC classes that improves the most with practice. When observing someone else's Socratic discussions, try to imagine what would have happened had they made different choices (told or not told the students something, came up with a good counter-question, decided to pursue or not to pursue a student's line of reasoning).

3.4.1 Dead Ends

When an idea is proposed, the instructor will usually know right away whether it will lead anywhere. If it won't, there is a strong temptation to discourage the students from pursuing the idea. This may be a mistake. Probably the best thing that can happen in a Socratic discussion is a *flaming dead end*, meaning that an incorrect line of reasoning leads to a consequence so patently false that the students are compelled to re-examine the road that got them there. If you see your students headed for one of these, then all you really need to do is encourage them to get there without undue delay. Some things you might want to do are: get them to explicitly reaffirm the wrong assumption, so that they will remember it later and be able to pinpoint it; shut down any further sidetracks that branch off of this one (e.g., "OK, that's a good idea, but first we'll finish pursuing this one"); hasten their demise by keeping the pace brisk, perhaps doing some of the arithmetic for them or providing clarifying paraphrases.

You can influence how flaming a dead end will be by making the issue more concrete: ask them to illustrate their result with actual numbers (e.g., "so if the initial weight was 250 grams, then we see the final weight of $400 - 2w$ comes out to be what? Oh, I see, -100 grams... rather on the light side.").

Perhaps you will need to summarize their findings, juxtaposing two findings that are contradictory, or in the case that they have contradicted some of the given information, you may need to restate the givens by saying, e.g., "so you have now proved that the only whole number between 100 and 500 having no two digits the same and satisfying blah blah blah is 337." To further rub it in, it often works well to insist that you believed their method and there must be some other mistake: "Ah, you've proved that the long division we did was wrong — can anyone find where?" (of course it is actually correct), or "Ah, you've proved that when you use variables with subscripts, the usual method of solving linear equations doesn't work" (if you can trust them to fight back on this one).

Other kinds of dead end are less useful. Perhaps they are following a reasonable line of inquiry but it doesn't get them anywhere: looking for a nonexistent pattern, introducing too many variables, classifying according to an ineffective scheme. A reasonable goal in this case is to get them to figure out that they're stuck. If you tell them (or indicate in any of 1000 nonverbal ways) that their idea won't work, they will learn to look to you for validation of their ideas, whereas if they reach a dead end themselves and consciously decide to look for another approach, they have learned something valuable about problem-solving. That being said, there are ways to reduce the amount of time spent following a dead end. One trick is to decide after hearing a suggested approach whether to follow it immediately or whether to treat it as one of many to be written on the board before the class decides which to follow. If Jenny reports finding a pattern starting $4, 6, 8, 12$ and reports her reasoning as to what is likely to come next, you get to choose between (a) getting the whole class involved in speculating about the next number or (b) writing on the board "Idea: look for a pattern". The key feature of this trick is that you're not giving anything away. Approach (a) is reasonable in some contexts, where the discussion of pattern promises to have some depth, and more importantly, approach (b) is something you sometimes use when the approach offered is correct. In fact you should make sure to use (b) on occasions where there were multiple interesting approaches but the first one offered happens to be the best: you catalogue every group's approach before asking the class to pick one and follow it.

Another way to expedite matters is to insist that the goals be well defined. Often when a bad approach is put into words, it comes out sounding discouraging: we thought we'd name as many variables as possible and then hope that inspiration struck; we decided that if n was equal to 5 the solution was obvious but we don't know how to do it for any other value. Sometimes mild discouragement doesn't work. I remember a worksheet designed to get them to invent the binary number system by asking them to come up with a scheme for representing all numbers with 1's and 0's. This cost a full day of discussion of the relative merits of various schemes, none of which had anything to do with binary. The instructor was very successful that semester, and in my opinion the investment of days such as that one paid off when students continued to work hard throughout the course because they didn't feel that the instructor was going to provide the answer for them. This takes guts, and doesn't work too well if the instructor conveys a growing uneasiness about the whole project.

So if you're not up to following the wind, you're probably better off treating it the same as the situation in the next paragraph.

The least promising dead end is a total lack of ideas. Probably it's best not to convene a discussion at this point but to continue working in small groups where you can ask questions that elicit further work and break the impasse. But suppose a class discussion on a certain problem fizzles out midway. This might be a good time to drop it. If it's not essential that they end up knowing how to do the problem, and they don't have a realistic shot at finishing it for homework, perhaps make it into an extra credit assignment. If it is essential, consider dropping it for now and writing a worksheet for the next class that will lead them to it in more manageable steps. You probably need more time to solve this problem than you have on the spot.

3.4.2 How to Listen

You need to listen to students and they need to listen to each other. Tom Lester once told me of a study showing that the average amount of time between when a teacher asks a student a question and when the teacher prompts the student or gives up on them is 2 seconds. Two seconds is longer than it sounds, but nowhere near long enough to formulate a coherent thought unless you were already thinking it before the question was asked. There may not be anything you can do about the sound-bite trend in TV reporting, but there's a lot you can do about it in your classroom. The first thing to try is waiting. Don't nod yes or no, or say uh huh, or give the student any feedback at all until they have finished saying what they wanted to say. Then wait five or ten more seconds. The odds are that the student will, after pausing for breath, realize that they are not finished and continue. If not, at least the other students will have had a chance to think about what they just heard. If you're uncomfortable with this long a pause, try pacing or holding eye contact with the respondent as if you expect them to continue, or act as if you're trying to digest what they've just told you. In fact often you really will need time to think. If they said something that was wrong in a puzzling way, see if you can figure out what they really meant. Students will only listen to each other if you set an example, so make sure you don't respond without having really heard.

Students are also more likely to listen to each other if they feel that they are responsible for having understood it. In small groups they are more likely to feel this, but at a ratio of twenty or thirty to one, many feel that they can just take notes and sort it out later or not at all. They may also feel they have no right to interrupt since everyone else obviously understands. You can counter this by demonstrating an expectation that each student understands what each other student has said. After one student says something the slightest bit unclear, ask another to repeat it in her own words. This is a good time to pick on students rather than have them raise their hand to volunteer a paraphrase. If student B can't paraphrase what student A said, it's not necessarily student B's fault. Student B can ask student A to clarify if necessary, or ask for volunteers for someone else to clarify. Make sure to go back and find out whether student C's clarification of student A's remark did in fact help student B. After a little experience you'll know better when to go through this routine. If student B simply wasn't listening, they might feel reprimanded, but that's OK. It doesn't really work when the remark was clear in the first place, although it doesn't hurt to get a quick affirmation from the whole class that it is clear so far. The basic standard you are setting is that the discussion involves the whole class and is not a collection of one-on-one dialogues between the teacher and individual students.

3.4.3 Staging

Your expectations of the nature of a class discussion are conveyed nonverbally as much or more than verbally. Most instructors ask the students to rearrange their desks into a large U shape for any but the briefest class discussions. A subtler but important technique is to put as much of the class as possible between you and the respondent. If you call on a student on the left side of the room, walk over to the right side as you're doing so. As the words flow between you and the respondent, the almost physical presence of a stream flowing between the two of you will wash over the students in between. You and the respondent also keep eye contact with the rest of the class this way.

It is often a good idea to get students to come up to the blackboard. Students will give longer monologues at the blackboard, so be prepared to be a more active moderator if the student is losing the rest of the class. Be careful not to make having the right solution a pre-requisite for coming to the board, lest the students stop thinking critically and accept any blackboard demonstration as a surrogate for your telling them something. When a student is at the board, I try to take up a position in the back or on the perimeter of the room. Sometimes I sit in the student's seat. This has the effect of including the rest of the class, as above, and also gives me a new vantage — you'd be surprised at what you see this way.

Other body language to be aware of is whether you are passing judgment on what you hear. Do your eyes flit impatiently with wrong answers? Do you gesture in agreement with right answers? Do you angle your body to the board as if to write down something correct, then pause if it's not what you wanted? If you've chosen to teach SC style classes, it's because you want the students to develop their own judgment, so avoid this kind of tip-off.

A related topic is the use of intentional errors. These are a hit with kids, in a slapstick sort of way, but adult students tend to feel patronized. Instead, I substitute the mischievous lie. If a student tells me the found all five regular polyhedra I may say, "Ah, so you haven't found the other two then?" They can often sense that I'm putting them on, but will still take the bait and try to prove that there aren't any more. On a problem best solved by assigning a variable to a certain quantity, I once told a group of frustrated students that I'd tell them the value of one quantity for free if they could decide which quantity they wanted to know. I planned to lie and tell them it was 10 when it was really 6. In fact I had tried this previously with success: the students figured out that the assumption of 10 led to a contradiction and were able to figure out the unique value that didn't lead to a contradiction. This time it was even more successful. By the time I came back to give them their free question, they had figured out what quantity they wanted to be told, had put in x for this, and had gotten the solution (well ahead of the rest of the class).

3.4.4 Asking the Right Questions

When my teaching is evaluated by my students, they often say that I never answer questions, or answer them with another question. I take it as a compliment even though it isn't meant as one. The most common such interchanges are

> Student: Is this right?
> Instructor: I don't know. Does it sound right to you? Can you elaborate?
>
> Student: What should we do from here?
> Instructor: What do you think? Does anyone have any ideas?
>
> Student: Can we say blah blah blah?
> Instructor: I don't know, can you?

There is a certain amount of this you can get away with, depending on your personality and theirs. If you start sounding like "Eliza"[2], you won't get good results. Instead, try to ask them useful questions related to the specifics of what they've said. In the three above scenarios, try respectively

> Are you asking if your computation is correct, or if it will prove useful?
>
> Is there a problem-solving strategy that you know that might work here? Why don't these equations tell you what x is?
>
> If you're wondering whether you can assign the variable z to be the average of all the prices, the answer is yes, but you haven't yet said whether we know anything about z.

When observing other classes, this is where you should let your imagination run free. Imagine what questions they might have asked. Your hindsight now will be your foresight tomorrow.

The question "do you understand?" is the most often abused. (Notice that this is virtually the only question in the repertoire of the conventional lecturer and rarely elicits an honest response.) Some better variants are: can you say that in your own words? could you do what John just said with different numbers? do you agree or disagree? in what way is this similar to what so-and-so did? These are all comprehension questions, testing whether the respondent comprehends instead of asking point blank for a Yes/No as to whether the respondent comprehends.

Good questioning can help to reach flaming dead ends. Ask what happens when $x = 5$, or whether their purported method works for all starting data and not just what was given. If a student gives a vague definition, find a borderline case and ask how their definition applies in that case. Try also questions that goad by disingenuity. If their method is more general than they realize, ask how they got lucky enough to try their method on a square rather than a pentagon or hexagon for which it "probably wouldn't have worked".

[2] The computer program imitating a nondirectional therapist, an early (and crude) approximation to something that could pass the Turing test.

3.4.5 Order versus Chaos

Ideally your students will be eager to answer your questions and discuss their ideas, but will listen patiently and attentively to each other and to you. If students are not willing to speak up and discuss their ideas, you need to loosen them up. It is a bad sign, for example, if the students are not happily chatting away when you enter the room five minutes before class, and are sitting in silence or whispering. In this case, you have probably done too well at eliminating chaos. Try assigning an activity in small groups where different groups are doing different things and they need to walk across the room to share information with each other. For example, there's a worksheet in Math 112 on infinite series where they approximate numerically some infinite sums and try to form conjectures based on each other's conclusions. Assigning a group project where they have to work outside of class together can tighten the bonds and make people feel more comfortable talking. When leading class discussions, be freer and more willing to follow the students' ideas wherever they lead. Dispensing with hand-raising and having students just call out can quicken the pace.

A classroom that's too chaotic is a problem also. If you have to call the students to attention more than once before they pipe down and listen, or if there is crosstalk during class discussions, you probably ought to do something about it. You can address this explicitly, asking the students to pay attention to you and to each other; you have to be consistent about this or they won't believe you mean it. Indirect methods of dealing with this are, however, usually more effective and should be tried first or at least in parallel. Insist on an orderly formation of desks into a U shape before a class discussion, rather than having them minimally perturb the small-group seating arrangements. It takes an extra minute, but it's worth it. In fact tell them you're pressed for time so they have to rearrange the desks in 30 seconds. A snappy set change will set the tone for what follows. When you observe crosstalk, try to get one of the crosstalkers up to the board to explain something, or to comment on what's just been said. By maximally involving that student in the lesson, you'll eliminate most of the off-task crosstalk, and the on-task crosstalk can probably be lived with. Another chaos reduction technique is to give them a more rigid idea of the structure of each class. Say you're going to spend 16 minutes in groups before convening a class discussion and then stick to it with absurd precision. The more aware they are of the structure of the lesson, the more they will stick to the tasks at hand.

The main point of this section is that you should make a conscious effort to optimize your position on the order-chaos axis, and that increasing order or chaos in the physical arrangements or chronological structure or types of assigned activities can help you change the balance in your class discussions.

3.5 Organization (yours)

SC style classes have more inherent disorganization than traditional classes, which is why you need to pay particular attention to organization.

3.5.1 Your Records

Students, at Wisconsin more than at other places I've been, react extremely positively to the appearance of organization. Probably the pre-service teachers are particularly impressed by this since they are consciously judging you on your pedagogical techniques. **Be a compulsive record-keeper.** I find it very helpful to reserve the 20 minutes after each class for writing a short summary of what happened in class that day. That way, if I've told a student I'd find them an extra-credit assignment, or if I've promised to bring something to class next time or promised that the next class would begin with a discussion of something, I can write this down along with other notes as to what I have in mind for the next class. In an SC style class there is a greater opportunity for unexpected things to happen, and therefore a greater need to write down what did happen.

Keeping date records of the homework you've assigned, both due dates and the date it was assigned, is essential. Whether or not you accept late homework is up to you, but it is certainly better to have students ask solicitously in advance for you to accept their late papers, which you will probably grant, than for students to assume it's OK and be upset if you don't grant them an extension after the fact. They are more likely to do the latter if they get the idea that you yourself don't remember when the homework was due. In fact, if I arrive at class early, I often take the opportunity to write up a reminder of what is due when.

Organization (and the appearance thereof) also helps students perceive you as self-confident, a quality which is crucial for an instructor in such a comparatively free-form classroom. Of course, one goal of the course is for them to assume that self-confidence themselves by the time they leave your classroom.

3.5.2 Grading

The message here is the same as in the previous section. Include in your course packet a clearly defined grading policy, specifying what portions of the grade are from homework, exams, quizzes, projects, group work, attendance, or whatever else you grade on. Give it enough thought so that you remember it easily, and can answer their questions immediately. If historical grade distribution data is available, you would be well advised to stick to it, since this will help to allay fears that the unusual pedagogical style will adversely affect their grade.

3.5.3 The Bell

When the bell rings at the end of class, everything becomes exponentially harder. My advice is to watch the clock like a hawk, so that you can make sure to wrap up the discussion at 2 minutes before the end. The discussion usually leaks over an extra minute, giving you one minute to say any summary comments or give instructions on homework, etc. This can be an important routine even when you have little to say: it makes you seem organized and on top of things.

If you can tell you are going to want to go overtime, because of a red-hot discussion you want to complete or something that's necessary so they can do their homework, announce to them 5 or 10 minutes ahead of time that you will probably be going overtime. It's best if you've let them go a minute or two early once before and have mentioned at the time that you're banking those minutes for such an occasion as this. If they're working in small groups at the end of class, it's less crucial but you still may want to halt them 1 or 2 minutes before the bell for closing remarks — it's even harder to get their attention after the bell when you don't already have the stage.

In short: **don't ever be surprised by the bell**.

4 Writeups

The write-ups (also called problem reports) are an important part of the course, because they force the student to communicate his/her knowledge about the problem. One consequence is that students cannot hide shortcomings in their understanding of the problem; another, more to the point, is that the students will develop the ability to give clear explanations on paper. Theoretically, a write-up should explain a problem clearly enough from beginning to end that a student could hand it to a colleague at that same school, and the colleague would be able to understand the whole problem without consulting anyone or anything else. It may well take a while for some students to develop good written communication skills, but you should be able to convince them that it is well worth the effort (and indeed most students have indicated at course's end that they believe it was) — after all, if a clear written explanation is harder to give than a clear verbal explanation, then they should come out well-prepared to explain to their own students.

This section of the guide discusses how to assign write-ups, which write-ups to assign, and how to grade them — although these are, in the end, only guidelines. At the end are two sample handouts you could give students to help them get used to writing math, and writing problem reports.

4.1 How, and how often?

The latter question is perhaps answered more quickly than the former. In general, you might want to aim for one write-up per week. More than this will cause you either to spend long hours grading (q.v.) or to fall behind in your grading; fewer than one every two weeks will not give the students the practice they need in developing their communication skills, and will also make you skip some important write-ups. Overall, you probably want a pretty even balance among individual write-ups, reflections, and other written work (including group write-ups and other homework). You may want to skip assigning a write-up the week of an exam.

As far as *how* to assign the write-ups:

1. Set some guidelines on the first day of the semester (see also Chapter 3 for this). Among other things, set a length of time between the date an assignment is made and the date it is due. These write-ups take a lot of time for the students, so they should have five to seven days in general (possibly less for the reflections, which are treated in Chapter 5). You probably don't want to set a single official length for write-ups, but between two and six pages is probably the norm. Do set rules for format, e.g., will you accept reports handwritten in pen? in pencil? Of course, also mention what you will be looking for in general, though more detailed specifics will come later, when you make assignments (see 3. below). [Part, if not all, of your guidelines may come in the form of a handout. See the example first day handout in Chapter 3, as well as the one given in the last part of this section.]

2. The write-up should almost always be on a problem that you have just finished discussing in class. It is a bad idea to assign a write-up on an activity on which you ran out of time, saying that the students should finish up on their own for the write-up. If the class discussion stops short of a full analysis of some aspect of the problem and you expect the students to address it in the writeup, it should be something you expected to leave for them to think about, not something you ran out of time for. Some suggested write-ups, such as Squares & Paths at the end of Unit One, are specifically designed as out-of-class problems, but you should make such assignments very sparingly.

3. You should discuss the write-up with the class for five minutes or so when you assign it, to make sure everyone knows what you want from them. This is especially important in the beginning, when students are unsure how to explain themselves, and in what generality they should present the solution. One way to do this is to ask the students, "What do you think are the important things about this problem, which we should include in the write-up?" Then write on the board the suggestions students make, adding your own if necessary once the students finish responding. (Of course, this means you have to decide what you want in the write-up before class!) Be willing to let the students argue for or against including certain items, and be careful not to make them include *too* much — you don't want to read a bunch of ten-page write-ups!

4. Most of the write-ups you will want to assign as individual write-ups: this is one of your big opportunities to evaluate an individual as opposed to his/her small group. However, assigning group write-ups can reduce work on both ends (the students' in writing and yours in grading), as well as being a nice change of pace sometimes. You might want to be especially observant the first time you do this, to see that each group member makes a more or less equal contribution. Again, use these in moderation.

4.2 Which ones?

Problems on which write-ups are assigned should be significant ones, where there is some complexity to the solution, and consequently a story to tell, both in the finding and in the explanation of what was found. You will often have spent more than one full day on the problem in class.

Here is a list of the problems on which write-ups were assigned in the Spring quarter of 2002, not including one problem which has been removed from the coursepack. Rubrics have been developed for all of these and are available in PostScript or pdf.

1. Photo Layouts (introductory practice writeup, graded but not counted)
2. Pizzas
3. Geoboards, part (f) only
4. Picture Proofs
5. Length of a Square
6. Surface Area
7. Scaling
8. Changing Units
9. Trisection
10. Rigidity
11. Proofs
12. First Constructions
13. Rigid Motions

Of course, you may feel free to add others which generated good in-class discussions, or omit one or two of those mentioned above.

4.3 Grading write-ups

You should make it clear to your students both before and after grading a given assignment what you were looking for, and how you determined the grade in general. However, unlike a calculus exam, you should *not* feel any obligation to give them an explicit point-by-point explanation of their grade, any more than a composition teacher has to explain the accumulation of good and bad points that resulted in giving a paper a B. Instructors in this course have historically used numerical scores rather than letter grades in grading write-ups, but you should do whatever feels comfortable.

Everyone has a different grading style, but one way to go about it is first to make a checklist of things you're looking for, or sections of the write-up, and then read/mark only that one section of all of the papers, to ensure that your comments will be consistent. Then, after you've examined each part of the write-up this way, go back and read each paper individually, skimming where you've made comments, and get an overall impression from which you can assign a total grade. When all papers have been graded, sort them by grade and flip through them to see if any appear to be "out of sequence", in which case you might want to take another look at those. Again, don't get too bogged down at the point level. The time required to grade write-ups properly increases more than linearly with the number of papers you have to grade, so you'll be looking for a middle ground in which each paper gets a fair reading but you don't spend fifteen or twenty hours grading.

Grading criteria: Here again, it's up to you. However, if we follow the ideas expressed in the sample handouts below, you'll be looking, in general, for:

1. A clear paraphrase of the problem description

2. An account of the method(s) used to solve the problem, including any major milestones or blocks, as well as "dead ends" which nevertheless proved enlightening in some way

3. A clear statement of the solution, as complete and general as appropriate, including an interpretation within the problem context

4. A clear explanation of why the solution works (or is the only, or complete, solution)

Since the nature of different problems can be quite different (compare Time to Weigh the Hippos with A Base Four Lesson), these elements may take on different forms.

Problem descriptions should be in the writer's own words, and give a complete enough description that the reader need not consult anything else (like the course materials) to understand it. Also, give credit for creativity, both in problem-solving methods and in writing up the solution.

One important element of the evaluation process is giving your students good feedback when you return their papers, both in written comments on their papers and verbally in class (in more generality). Encourage them; point out where, on the whole, the write-ups were strongest, and where they were weakest. It may help to give a handout back along with the very first problem report, so that they can see some of the elements of a good write-up. If you decide to make one, try to include good (but not bad) elements from specific papers (without naming names). *Don't* hand them a complete ideal problem report, as it tends to give them the impression there is/was a unique ideal which they were all supposed to guess. (It also gives students who take this course in later semesters an easy out for the first assignment.)

Here, for example, is an excerpt from a handout giving feedback on the problem report on "Poison" from Math 105. (This is not the entire handout!)

The winning strategy

> "The second player's strategy for playing with ten pennies is to watch what the first player does and do the opposite. For instance, if the first player chooses 2 pennies, then the second player chooses 1 and vice versa."

It doesn't take a lot of space to do this. Note that the writer makes it clear that this is the ten penny strategy, not the solution for 11 or 12 or 210.

The reason it works

> "If you pick opposites on every move, the total number [of] pennies taken by both of us on one turn is 3. There are 3 sets of 3 in ten. (3x3=9)

> However, you have to add 1, the poison penny, to make ten. That explains why when you get down to 4, if you pick first, you will have to lose. In 4 there is one set of 3 and the poison 1....
>
> By taking opposites, you take three pennies in each turn. Since 3x3 is 9, plus the poison #1 equals 10. For example, in my Group, if [my opponent] takes 1, I will take two. Now 7 are left. 7 is 3x2 plus the poison #1. [My opponent] takes 1, then I take two. Now there are 4 left. 4 is 3x1 plus the poison #1. She takes one, then I take two. She is left with the poison #1. Therefore, the person moving second, which was me, wins!"

Okay, this could have been better. Neither explanation is completely clear (and this writer did lose some points for clarity), although together they tell the story well. Again, writing out the example game helps.

The general solution

> "In order to win the 210 penny game, you would want to move first and take out two pennies. This will leave you with 208 pennies still on the desk. 208 divided by 3 gives you 69 with a remainder of 1, the poison penny. Just as 10 divided by 3 gives you 3 with a remainder of one. From here on, the team that moved first would take the opposite of the second team to win the game."

This was from a different paper than the previous, so the explanation of why the 10 penny strategy was a winner was somewhat different. But both of these students, by focusing on why the "opposites" strategy was a winner in the 10 penny game, figured out how to win the big game. I don't think it's coincidence that the two best explanations of why the "opposites" strategy works accompanied the two best general solutions.

5 Exams

5.1 What should they be like?

An obvious question to ask, given the cooperative nature of the sequence, is: Can the exams be given in groups? In fact, students will most likely have become comfortable working in groups by the time of the first exam, and you can be certain that a chorus will rise up to ask this question. My reason for not giving group exams is that I want to make sure that a reasonable portion of the student's grade is based on work they did without help. There is some research showing that students perform better in cooperative learning classrooms when exams are individual[3], but given the group nature of most of the rest of the work, I think individual accountability is a much stronger reason for sticking to individual exams. You may wish to point out that the students will not have their groupmates with them during their careers as teachers.

Exams, of course, invariably cause anxiety in many students, and you may want to make at least one of your exams open-notes (perhaps the final, if the midterm is closed-notes). This option also allows you to ask more detailed questions than you might be able to otherwise: questions about particular activities, or variations on problems assigned as write-ups. In either case, do be careful to write an exam that is at the level of the students: On one hand, it's easy to get carried away and overestimate what your students can handle. On the other, sometimes your students will surprise you (pleasantly).

One more note: You should probably not give more than one midterm plus the final, as time is precious, and using cooperative learning means a sacrifice on the content-versus-time scale. Exams can disrupt the flow of a class, and it's already difficult to cover as much as you'd like.

5.2 Final grades

The course grade will typically have at least one component — attendance and participation — which has historically not been present in traditional math classes. It

[3]Slavin, R.E. Research for the future: Research on Cooperative learning and achievement, what we know, what we don't know. *Contemporary Educational Psychology* **21** (1996): 43-49.

is important to include this, however, as the process and experiences that take place inside the classroom are the most significant part of the course, and a student cannot really learn what we want him/her to learn without being present. Furthermore, students in the 105-106 sequence respond to attendance grading with a more than 90helps keep the groups stable. Typically you should take attendance every day, which can be done unobtrusively, by counting heads five minutes or so into the period, when everyone is working in small groups. After a couple of weeks you'll be able to tell more or less at a glance who's missing. You should set clear guidelines for attendance at the beginning of the term and stick by them (see, for example, the sample first-day handout in Chapter 3).

Measuring class participation is more subjective. By halfway through the semester, you should be able to tell who is pulling his/her weight in the small groups, and should encourage those who are not to participate more. For large group participation, you may want to keep track of who says something, and mark it down after class (don't do it in class — students don't like the idea that you're taking notes on what they do or say). This method requires some good memory on your part, but after a few weeks you'll see who's speaking up and who's not, and can encourage those who don't to participate more. One way of doing so without putting pressure on to know the answer to a question is to call a student up to the blackboard to serve as scribe in writing down others' suggestions during a brainstorming type of discussion. However you decide to do it, you really will have to be diligent about it from the first day.

As a help in deciding how to break down the course grade into components, here is the breakdown used by the most recent instructors in 105 and 106.

- Attendance: 15%
- Written work, including quizzes: 45%
- Midterm exam: 15%
- Final exam: 25%

5.3 Exam and Study Guide Database

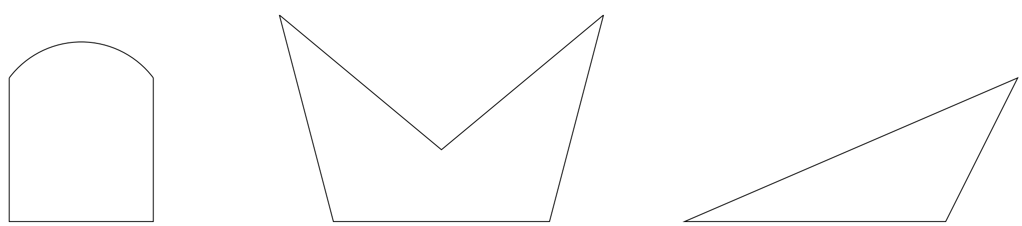
Additional study problems, Math 106, Sp02

Quiz 1, Math 106, Wi02

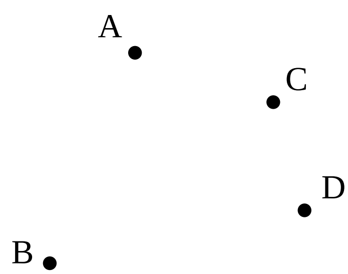
This is a short answer quiz. You do not need to justify your answers. Please consider lines that appear to be equal, perpendicular, etc., to be so.

1. For each of the shapes shown, write beneath it all the listed nouns which it could be and all the listed adjectives that describe it.

polygon	equilateral
quadrilateral	scalene
trapezoid	



2. True or false? The area of $\triangle ABC$ plus the area of $\triangle CBD$ equals the area of $\triangle ABD$.



3. With reference to the same figure, true or false? $\angle ABC$ is adjacent to an angle made by rays $\overrightarrow{BC}$ and $\overrightarrow{BD}$.

Quiz 2, Math 106, Wi02

This is a short answer quiz. You do not need to justify your answers.

1. Draw a pair of approximately vertical lines next to each other. Then draw a transversal to both of these. On this figure, label:
   (a) A pair of alternate interior angles
   (b) A pair of vertical angles
   (c) A pair of supplementary angles

2. Draw a small triangle, labeled $\triangle ABC$, with an obtuse angle at $\angle BAC$. On this figure, draw (you do not need to construct, just an approximate sketch is fine):
   (a) A median through B
   (b) An altitude through C
   (c) The perpendicular bisector of $\overline{AB}$
   (d) The circumscribed circle

Midterm Exam, Math 106, Winter 2003

You may use your notes. The last part of the last problem is a mini-writeup and requires justification. For the other problems, you may submit a one-sentence justification or none at all. Answers with justification may receive partial credit if wrong, but may also not receive full credit if correct.

1. [20 points] Plans for the new Worthington pool called for a shape of an isosceles right triangle (when viewed from the top), and a uniform depth of 12 feet. The floor and walls were to be tiled in 1 foot square ceramic tiles, and the rim of the pool in 1 foot long tempered rubber pieces. Due to budget problems, the pool was actually constructed as a 5/6 scale model of the original, but with tiles the same size as planned (ten inch tiles were not available). For each of the quantities below, select one of these answers:

   (i) stayed the same
   (ii) decreased by a factor of 5/6
   (iii) decreased by a factor of 25/36
   (iv) decreased by a factor of 125/216
   (v) none of the above

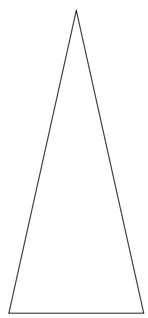
   (a) The angles of the triangle shape (viewed from above)...

   (b) The amount of water needed to fill the pool...

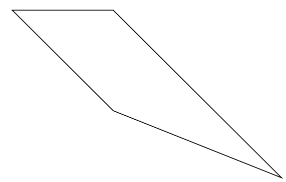
   (c) The number of rubber pieces needed to form the rim...

   (d) The number of tiles needed to tile the floor and walls...

2. [15 points] Give the most specific name you can for each of these shapes. As usual, assume things to be equal if nearly equal, et cetera.



(a)



(b)

The shape of the water in the pool in problem 1

(c)

3. [24 points] A student comes to you with the following proof for the formula of the volume of a pyramid.

    > The pyramid has the same base as a prism, but when you intersect it with a plane parallel to the base anywhere higher up, the intersection is smaller, decreasing to zero when you reach the apex. The average of this area of intersection is less tha $\frac{1}{2}$ because, for example, when you go halfway up the intersection is scaled by $\frac{1}{2}$ so has area $\frac{1}{4}$ the area of the base. We see therefore that we get about $\frac{1}{3}$ or the volume of the prism, or in other words, $V = (1/3)h \cdot A$.

    Circle the dot before each statement that would be a valid criticism of this student's proof.

    - The proof proves equally well that $V = (1/4)h \cdot A$.
    - The student never said what she proved.
    - The final formula will not be in units of volume.
    - The student used a formula for the volume of a prism but never said that this formula was assumed to be known.
    - The student assumed it was a right pyramid.
    - The quantities h and A were never defined or explained.

4. [26 points] A standard piece of paper in the U.S. is $8.5''$ by $11''$.

    (a) [4 points] What is its aspect ratio?

    (b) [4 points] If you cut it in half with a cut parallel to its short side, what is the aspect ratio of each new piece?

(c) [18 points] What aspect ratio can you start with, so that when you cut in half this way, you get two pieces with the same aspect ratio as the uncut paper?

1. Suzy neatly and evenly eats the top part of her ice cream cone, so what's left is a similar but smaller cone. If the new cone has half the volume of the original cone, what is its height relative to the original cone?

2. A proof is given of the statement that $\triangle ABC$ is isosceles. For each step you must determine whether an error occurs there, or whether the statement in that step is a correct consequence of the previous steps for the reasons stated. If a mistake is made at an early step, you should judge false statements at a later step correctly reasoned if they follow from the earlier incorrect statement for the reasons given. Mark a check or an X at the end of each line. (It will probably help you to draw as you read.)

   (a) Construct the perpendicular bisector from point A of the line segment BC (First Constructions, 3 and 4)

   (b) Let D be the place this bisector meets BC.

   (c) $BD = DC$ (definition of bisect)

   (d) $\angle ADB = \angle ADC$ (both are 90° be definition of perpendicular)

   (e) $AD = AD$

   (f) $\triangle ADB \cong \triangle ADC$ (SAS)

   (g) $AB = AC$ (CPCTC)

   Therefore $\triangle ABC$ is isosceles.

3. Circle the letter corresponding to the correct answer in the following problem. A 180° rotation around the point $(0,0)$ followed by a 180° rotation around the point $(3,5)$ is the same as:

   (a) A reflection in the line joining $(0,0)$ and $(3,5)$

   (b) A translation 3 to the right and 5 up.

   (c) The rotation $T(x,y) = (3 - x, 5 - y)$

   (d) The rotation $T(x,y) = (6 - x, 10 - y)$

   (e) A 360° rotation around the point $(3/2, 5/2)$

   (f) The translation $T(x,y) = (x + 6, y + 10)$

   (g) None of the above

4. Give a straightedge and compass construction of a triangle with angles $45°, 45°$ and $90°$. Then give a formal proof that your construction is correct.

5. How many axes of rotational symmetry does a cube have, and of what orders?

6. The vertices A, B and C of $\triangle ABC$ all lie on a circle, whose center, O, is on the line segment AC (see the figure). Which of the following must be true? A "Yes" or "No" is sufficient for each.

   (a) BO is the angle bisector of $\angle ABC$

   (b) O bisects the line segment AC

   (c) $\triangle AOB$ is isosceles

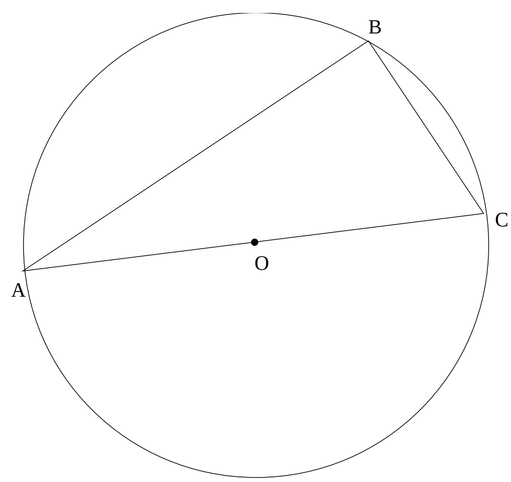
   (d) $\triangle BOC$ is isosceles

   (e) $\triangle ABC$ is isosceles

   (f) $\angle OAB = \angle OBA$

   (g) $\angle OCB = \angle OBC$

   (h) $\angle ABC = 90°$



7. A right regular hexagonal prism has regular hexagons for bases and rectangles for its other faces. (See model) How many planes of reflectional symmetry does it have? How many axes of rotational symmetry does it have, and of what order(s)?

8. A construction is given that purports to find a point on line segment AB that is one third of the way from A to B. It is then proven to work. For each step you must determine whether an error occurs there, or whether the statement in that step is a correct consequence of the previous steps for the reasons stated. If a mistake is made at an early step, you should judge false statements at a later step correctly reasoned if they follow from the earlier incorrect statement for the reasons given. Mark a check or an X at the end of each line.

   Construction:

   (a) Draw any line through A not containing the segment AB; call the line l.
   (b) Construct a parallel line k to l through B (First Constructions, 5)
   (c) Choose a point C other than A on l.
   (d) Draw a circle centered at B with radius AC and, among the two points of intersection of the circle with line k, let D be the one on the opposite side of k from the point C. (We have not worked with formal proofs for choosing things on opposite sides of lines, so you do not need to check this step.)
   (e) Mark off another identical length from D and call it E.
   (f) Let X be the intersection of CE with AB; this is the point you want.

   Proof:

   (a) $\angle AXC = \angle BXE$ (vertical angle theorem)
   (b) $\angle XAC = \angle XBE$ (alternate interior angles)
   (c) $\triangle AXC \sim \triangle BXE$ (AA theorem)
   (d) $AX/BX = AC/BE$ (CPSTP)
   (e) $AC/BE = 1/2$ (by construction)

   Therefore, $AX/BX = 1/2$, so AX is one third of the segment AB as claimed.

Final Exam, Math 106, Sp02

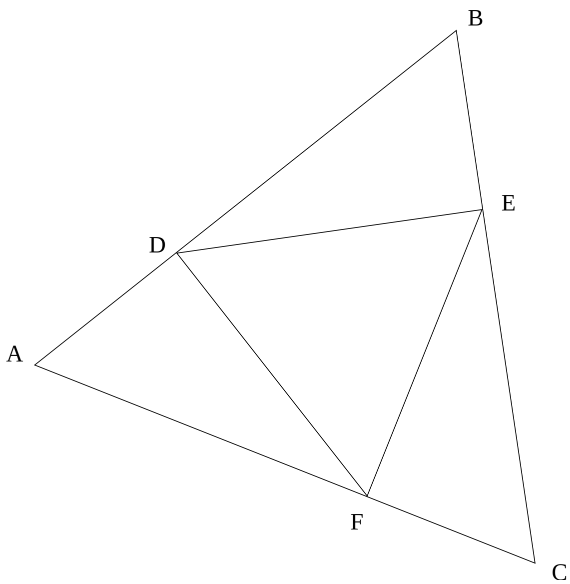
NAME:

1. (a) Construct a rhombus using your straightedge and compass (leaving enough pencil marks to indicate what you did).

   (b) How many lines of reflectional symmetery does it have?

   (c) Does it have a rotational symmetry and, if so, of what order?

2. In the following figure, $\triangle ABC$ is equilateral and the lengths AD, BE and CF are all equal. Prove that $\triangle DEF$ is equilateral.



3. How many planes of reflectional symmetry does a right regular pentagonal pyramid have? How many axes of rotational symmetry does it have, and of what order(s)?

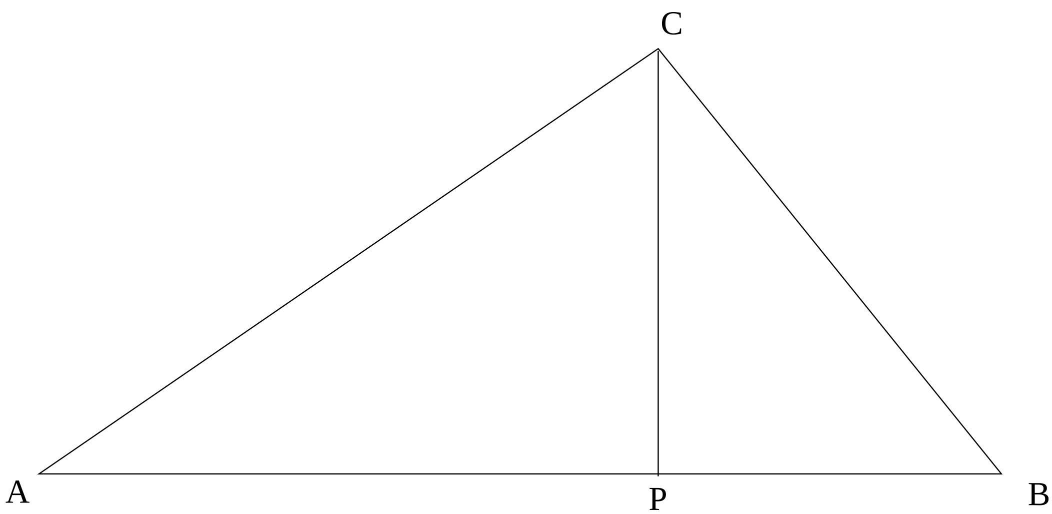
4. Given that $\triangle ABC$ is a right triangle with hypotenuse AB and altitude CP (as shown), which of the following must be true? A "Yes" or "No" is sufficient for each.

   (a) The area is divided into two equal parts

   (b) $\triangle APC \sim \triangle CPB$

   (c) CP is the angle bisector of $\angle ACB$

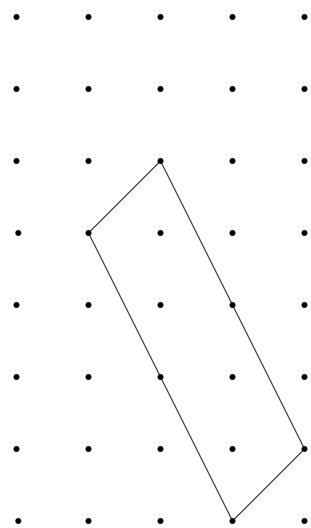
   (d) $AP/AC = BP/BC$



5. Is the transformation $T(x, y) = (y, x + 2)$ a rigid motion? If it is, carefully describe what rigid motion it represents in terms of reflections, rotations and translations. If not, then explain why it is not.

6. Compute the area of the following geoboard parallelogram. Write the answer in the blank.

_______________.



7. I have an irregularly shaped patio in my backyard, which I am covering with small tiles from a mail-order tilemaker in Italy. I have prepaid the postage on the order, according to the expected weight of the shipment. When it arrives, I see the tilemaker has mistakenly filled the order with mini-tiles, one third as big in every dimension. He has still given me the right amount to cover my patio.

   (a) By what factor did the weight of the shipment change, or did it stay the same?

   (b) By what factor did the number of individual tiles change?

   (c) The tiles on the border of the region are all treated with a special coating to prevent erosion. By what factor did the number of coated tiles change?

Senior Final Exam, Math 106, Wi03

NAME:

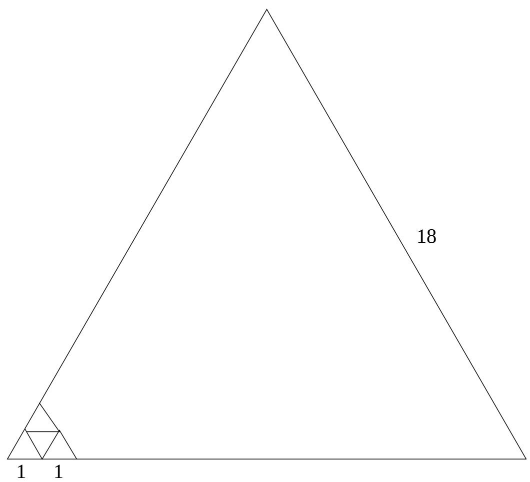
1. Cylinder B has twice the radius and half the height of cylinder A. Which has the greater volume? Explain why.

2. A proof is given of the statement that $\triangle ABC$ is isosceles. For each step you must determine whether an error occurs there, or whether the statement in that step is a correct consequence of the previous steps for the reasons stated. If a mistake is made at an early step, you should judge false statements at a later step correctly reasoned if they follow from the earlier incorrect statement for the reasons given. Mark a check or an X at the end of each line.

   (a) Construct the bisector of $\angle ABC$. (First Constructions, 2)
   (b) Let D be the point where the bisector intersects AC.
   (c) Extend the line BD and choose a point E on it.
   (d) $\angle ADE = \angle CDB$ (vertical angle theorem)
   (e) $AD = DC$ (CPCTC)
   (f) $BD = BD$
   (g) $\angle ABD = \angle CBD$ (definition of bisect)
   (h) $\triangle ABD = \triangle CBD$ (SAS theorem)
   (i) $AB = BC$ (CPCTC)

   Therefore $\triangle ABC$ is isosceles.

3. How many of the small equilateral triangles (side 1) will fit in the large equilateral triangle (side 18)? (See picture)



4. (a) Circle the letter corresponding to the correct answer in the following problem. A $180°$ rotation around the point $(a, 0)$ sends the point (x, y) to the point:

   i. $(-x, -y)$
   ii. $(-x + a, -y)$
   iii. $(x, -y)$
   iv. $(2a - x, -y)$
   v. $(y, -x)$
   vi. $(y + a, -x)$

(b) Circle the letter corresponding to the correct answer in the following problem. A reflection across the diagonal line $x = y$ followed by a reflection across the y-axis results in the single rigid motion described as follows:

   i. A reflection in the line bisecting the angle made by the other two
   ii. A reflection across the line $y = -x$
   iii. A rotation of $45°$ counterclockwise about the origin
   iv. A rotation of $90°$ counterclockwise about the origin
   v. A translation in the diagonal direction
   vi. None of the above

5. Give a straightedge and compass construction of a triangle with angles $30°, 60°$ and $90°$. Then give a formal proof that your construction is correct.

6. Which of the following three-dimensional objects have a symmetry that is a $120°$ rotation? A “Yes” or “No” is sufficient for each.

   (a) A square-based pyramid

   (b) A right regular hexagonal prism

   (c) A cube

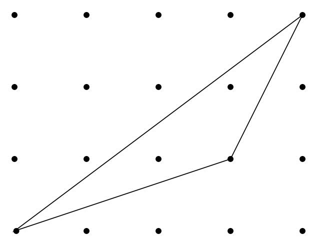
   (d) A regular octahedron

   (e) A right cylinder

   (f) A regular dodecahedron

7. Compute the area of the following geoboard triangle. Write the answer in the blank.

________________.



6 Materials and how to use them

6.1 To be packaged with the coursepack

A set of school supplies was bundled with the coursepack and students were required to bring them to class each day. To help them do this, we included a binder for the coursepack and a pouch that clipped into the binder to carry the supplies. The complete list of supplies included by the OSU bookstore was:

- coursepack (duplicated at the OSU copy center and sold at cost)
- $1\frac{1}{2}$ inch three-ring binder
- soft plastic clip-in zippered pouch
- compass
- protractor (the kind with the hole at the origin works best)
- scissors (should fit in the pouch)
- colored pencils
- small pencil sharpener
- 20 sheets of three-hole punched graph paper, five squares to the inch

6.2 Other stuff you'll need

Additionally, the classroom was stocked with the following supplies, which were needed at various points.

- geoboards
- chalkboard compass
- string
- measuring wheel or 25 foot measuring tape

There are two videotapes that I own only one copy of. I put them on reserve for the students to watch. They are classroom videos of a second grade teacher teaching discovery-method classes. One is on area and one is on length. The teacher's name is Carmen Curtis, the person I got the tapes from is Richard Lehrer, and he may be reached at

> Rich Lehrer, Department of Teaching and Learning, Peabody College, Vanderbilt University, Nashville, TN 37204

I highly recommend you obtain these tapes and require the students to watch them.

6.3 On-line handouts

Files are included for several types of on-line handouts we developed as the quarter went along.

First, every week we gave out a summary of what they were supposed to have gotten out of that week's activities. We did this partly because the material is difficult enough for some students that they run out of gas before getting to the end of a problem, let alone having time to think it over and assimilate the ideas. They are liable, even when looking back later, to miss the point. Also, there is little redundancy in the material so it helps for them to have a list of general ideas exemplified by each worksheet. They find this useful when studying for exams. It is also a way of conveying to them what they are responsible for learning.

Each instructor needs to put out their own such summary of course, since at the very least, the pace will vary from class to class. We provide the html file in the hope that it will be useful for cutting and pasting, or for prompting the rushed instructor to notice all the ideas embedded in the worksheets and point them out to the students. The suumaries file from Spring 02 as well as the first four weeks of summaries from Winter 03 are found under the "more stuff" link.

Several handouts concerned guidelines. The first, which was mandatory reading on Day 1, was on etiquette in small-group and whole-class discussions. The second addressed the proper format for a writeup and the third addressed this in more detail for those writeups in which formal Euclidean proofs were required.

Finally, we include a handout which was the instructors' response to the collective thoughts on teacher preparation. After reading their responses to that reflection, we thought it would be productive to address a response to the class, and this, along with a brief oral discussion, seemed the easiest way. It is included only as an example, of course, since actual dialogues on this will vary.

6.4 Solutions and such

During the course of the quarter, we handed out model solutions at various times. These are collected in the Solutions subdirectory, and of course, are accessible only by permission. In some cases, we gave out contrasting bad solutions – these are included for what they're worth. The good and bad solutions were contrastingly annotated in pen, but this annotation is not on the online versions.

Additionally, scoring rubrics were posted religiously after each assignment was due. We tried to make sure the rubric was posted before the re-do was due, since we felt that students doing the re-do were the ones who missed substantial points (except at first when they all want to do re-do's) and therefore they needed to be hit over the head with what our expectations were and why they missed points. Since improving their communication skills is the paramount mission in this course, and since the rubric highlights deficiencies here, we made sure that papers were graded in such a way that they could see exactly where they lost points on the rubrics: scores would be written as $(3+1+2)+(2+2)+6+2$ if the rubric identified four scoring categories, with three subcategories in the first and two in the second. All the rubrics are included in the Rubrics file in the Solutions directory. Source code is in the LaTeX directory.

6.5 LaTeX materials

If you want to use any of the worksheets, solutions, and so forth, in some form other than exactly what is provided, you will probably want the LaTeX source files. The LaTeX directory has source files for the coursepack and for all the auxilliary materials you might want to use – exams, solutions and so forth. You may download them as needed and adapt them at will. You will need to obtain permission to access the LaTeX directory.

At the very least, even if you don't know how to mess with LaTeX, you will need to remove the two-page sample syllabus that's in the courspack and substitute your own syllabus.

Figures were made using xfig and exported both in eps and pdf. All figures are in the figures subdirectory of the LaTeX directory.

All contributions of new materials will be gratefully accepted by e-mail to pemantle@math.upenn.edu. More grist for the exam problem mill would be particularly useful.