Math 105 Teachers Guide

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1 What is Math 105?

Math 105 is the first in a sequence of mathematics courses for future elementary and middle school teachers. Its primary goal is to educate mathematically. We cannot expect schoolchildren to learn anything beyond the rote execution of algorithms if the first ten years of their education is at the hands of teachers who can't themselves go beyond this. By providing elementary educators with a good mathematical foundation, we hope to reverse the downward spiral into which many perceive mathematics education to have fallen. A secondary goal is to introduce students to the cooperative learning environment. Keep in mind, though, that the course is a content course, not a pedagogy course.

The content covered in Math 105 has a *how* and a *what* component. Students will be taught how to think and speak mathematically, many for the first time in their lives. They will solve problems they have not been shown how to do, they will learn to put their ideas into precise language and to prove their assertions when possible. They will learn to break down problems into smaller problems, use trial and error, generalize when appropriate, and to harness their (atrophied) common sense. It is the *how* content that determines the format of the course, necessitating the problem-solving theme and the socratic style of discussion.

The *what* content, i.e., the choice of topics, deserves some explanation as well. It is guided by the work of scholars such as Liping Ma as well as by the demands placed on elementary school mathematics teachers. The topics of place value, arithmetic algorithms, fractions and decimals need no justification. In addition, we believe that a solid mathematical base at roughly the high school level is esssential even for teachers who will be teaching children who are four to eight years old. These teachers will be given a great deal of independence in designing lessons and choosing aspects of the curriculum. Thus in addition to the topics listed above, we include three additional major topics. These are laced throughout the course: problem-solving, algebra and logic.

All three of these are in line with the 2000 NCTM Standards. The problem-solving curriculum is designed to get students thinking on their own and explaining coherently what they have done. The algebra content has a major and a minor focus, neither of these being algebraic manipulation. The major focus is word problems. In particular, these problems teach students how to assign variables and to write equations expressing relations between them. The minor focus is on algebra as a means of formulating and expressing generalizations. Finally, we expect students to learn to justify their solutions with some rigor. We assume this class is the first place they have done this, so we go easy on them at first, but we constantly harp on the level of rigor of their arguments (or ours) and urge them forwards.

Some specifics about the student body here at OSU are worth noting. College algebra (Math 148) is *not* a prerequisite for this course, which means that along with the problem solving will by necessity come some algebra review, and indeed a continual reinforcing of the use and meaning of algebraic notation. There may be some resistance on the part of some students to the way the class is run, though we have found the small group format by and large to be popular; resistance is more likely to come from the emphasis on problem-solving and lack of answers provided. The classroom format turns traditional norms upside down by making students take responsibility and authority for determining whether a problem has been solved correctly[1].

Teaching a course by means of cooperative learning, Socratic discourse, and the like involves decentralizing yourself (the instructor) as the focus for learning and authority. This is a skill which takes some time to develop, and it's not always easy. This is why it's best to start preparing to teach a course like this well in advance. Don't worry, though — you're not on your own. If you're expecting to teach in this sequence, you should already have looked into observing others' classes, and the rest of the informal instructor preparation program. Part of this is the extensive set of notes included in Section 3 of this guide. Reading and discussing them with folks who have had experience in the sequence will help defray a lot of the anxiety you may have.

This guide is designed to answer any questions you may have, large or small, about this class (although of course conferring with other people who have taught it is an irreplaceable resource), and to provide all the help you need with the details until you feel comfortable enough to stand on your own. Even once you've come into your own, it's a good idea to refer to the guide throughout your first time teaching the course, as we have collected the potential pitfalls of several semesters' worth of teaching it, and it's always best to be prepared. Section 2 gives a detailed blow-by-blow of each of the activities in the course pack (including solutions), along with suggested times and a couple of sample syllabi. Section 3 discusses how to teach this kind of class. Section 4 goes into detail on the write-ups: which activities have traditionally had write-ups

[1]We do not, in the end, withhold information and techniques they need to know or judgment on what is correct, but we do insist that they develop their own judgment and we train them to do so.

associated with them, how often to assign them, which ones are important to assign, how to make the assignments in class, and specifically what to look for in critiquing them. Section 5 discusses exams and evaluation, including a large bank of sample exams from a variety of past instructors. Finally, Section 6 details the organization of the directories and LaTeX files used to develop this guide and the course pack, so that you can take things you like and adapt them for your own handouts, exams and worksheets.

2 The activities

In this section we discuss each of the activities presented in the coursepack. We include the main point of the activity and the solution, as well as notes on how the worksheet has gone in the past and what directions one might want to take the discussion or what sorts of leading questions might be appropriate for groups that are stuck. If extra equipment is needed we include a reminder to that effect. An estimate of the number of (48-minute) hours is sometimes included as well for the longer worksheets. The notes for the early activities are more detailed since instructors may need more guidance when getting acclimated to the workshop format.

The July 1, 2003 version of the coursepack and guide has been edited to reflect the fact that no one in the past has ever made it through all the worksheets in the coursepack. This is because often an activity takes longer than anticipated, and even more often the instructor finds it necessary to insert a background reading or worksheet because the main worksheet was too difficult. Building on experience, we have added in as many of these background readings and worksheets as we felt necessary, and removed a few of the main worksheets to bring the total time back to 45 class sessions. Of course actual mileage may vary, so we've included the worksheet outtakes in the instructors guide. This provides instructors the flexibility to add them in as necessary, without having to used an untested activity. Still, instructors shouldn't hesitate to make up new worksheets or readings – all of the ones in the coursepack and guide were once new!

One final note on the use of Readings. The first version of this course was entirely worksheet based, with no auxilliary readings. We have kept the text-free approach as much as possible for various reasons. We have found that plunging into the workshop style is psychologically difficult for many students and that removing the text helps to put them in the right frame of mind. Also, removing the text emphasizes primacy of thinking, reasoning, understanding and communicating over remembering and absorbing. As mentioned above, some textual material became necessary. In Math 106 this takes the form of a long glossary. In Math 105, while definitions are in a way still the focus of the textual materials, students need to see these spelled out in lengthy examples. The present compromise is a set of Readings (Section 5 of the coursepack) that precede and accompany the worksheets. These are meant to be assigned as strictly out-of-class reading assignments. There are self-check questions in each Reading, which are meant to be answered at the time the reading homework is done. We have often found it necessary in the past to require that the answers to the self-check questions be out on the students' desks at the beginning of class, and that credit be given if the questions have been answered, correctly or incorrectly. This way, it is possible to spend a small amount of class time discussing their questions on the Reading and their answers to the self-checks without having to allot class time to a first pass at the material in the Reading. The worksheets themselves are marked with the names of any Readings that are meant to be done before the worksheet is started. Since the students may not know what day a particular worksheet will be begun it is important for the instructor to **look ahead to see when a Reading will be needed and assign it at least a few days in advance of that.**

2.1 Problem-solving and abstract thinking

Several readings should be assigned on the first day of class, to be read ASAP. These include Sections 6.1, 6.2 and 6.3. The most important of these is Section 6.1, "Paradigm for abstraction and generalization". Instead of spending time discussing this, you will want to show them, on every worksheet, how they are stepping through this process. Several reminders occur in the activity notes to do this.

2.1.1 Poison

[Materials: counters or chips for use in play; time 2 hours]

This is a great first activity in basically every way. (1) The final and most general solution is somewhat sophisticated and will challenge the students' explanatory powers. On the other hand, it is easy to start solving the problem simply by playing the game, and almost every student will find that s/he gains understanding by doing this. (2) The solution will typically emerge piece by piece. Students will generate several theories, some correct and some not, and the ensuing discussions will be good prototypes for how to argue mathematics. (3) The ultimate authority lies in being able to win the game. The teacher cannot win against a student who thoroughly understands the game. It is a tremendous confidence builder for students to see that something they have figured out is correct no matter what the teacher says. When the point is reached where several students understand completely how to win, teachers often play it up by issuing a challenge: "OK, so-and-so thinks she can win against me, no matter what I do — I am going to try my hardest to beat her; the rest of you vote on who you think will win." This can be done at several stages: when they think they can win the 10 counter game with you going first, when they think they can win the n counter game if they get to say who goes first, and when they (erroneously) think they can win the 10 or n counter game no matter who goes first. Poison should be the basis for the first write-up, and you should insist on continuing the discussion of the solution until everyone claims to understand why the pattern works.

Solution: The winning strategy for the general n counter game is: if n is one more than a multiple of three, choose to go second and always do the opposite of what the first player does. If n is two more than a multiple of three, go first and take one counter, then always do the opposite of what the other player does. If n is a multiple of three, go first and take two counters, then always do the opposite of what the other player does. Here's why it works. In any of these cases, you first force the other person to move when the number is one more than a multiple of three. By doing the opposite of what the other person does, you will together have removed a total of three counters, thus arriving at one more than another multiple of three (the next smallest multiple). Thus you can force the other player to move when the number of remaining counters is 4 and finally 1, meaning that the other person ends up with the last counter.

Remember not to give any of this away in class! Old habits die hard, especially when the students become frustrated, but the whole point of the activity is that if the students struggle long enough they *will* discover the answer! Here is a list of typical milestones they will come to and blocks they will encounter.

Many students will fixate on odd-even parity and try to argue that the strategy depends on parity, even in the face of evidence to the contrary. It is hard to discover the $3n + 1$ pattern, since the number 3 does not appear in the game description and since the pattern $3n+1$ is harder to recognize than the patterns $3n$ or $2n+1$. Initially they all play 10 counter games, and the first thing they discover is that leaving the opponent with 4 is a winning move. Some will now say they have found the strategy. This opens the door to a discussion of what a winning strategy is, and why "leave 4" is not a complete strategy. Some students will think that a strategy can include requiring the opponent to make certain moves; this can also come up in the discussion of what a strategy is.

In general, the students will have a very hard time at first saying exactly what they mean, and being precise (you will get $3x+1$ vs. $3(x+1)$ confusions, for example). Don't get too frustrated by this; just encourage them, and try to help them come up by themselves with a clear formulation of their thoughts. You may want to differentiate explicitly between understanding how to win and being able to explain how to win. A student understands if they can beat a perfectly smart opponent (you) every time. A student can explain how to win if they can give you a set of instructions which you will win if you follow, even if you play dumb and try hard to misinterpret them. Depending on what they come up with, it can be fun to demonstrate this.

Prompts: If groups get stuck right away, tell them just to try playing the game a few times. If they say they don't know how, tell them not to worry about winning or losing, just try it. If they don't seem to be recording the play-by-play, ask if they see any overall pattern — this should lead to a request to see (or hear) specific sets of moves from previous games. If a group appears to have solved the problem much faster than the others, ask to hear what their solution is. If there's a flaw in it, ask a question that will bring this flaw out. If it only works for the ten-counter game, tell them to move on to an eleven-counter game, or even down to a nine-counter game. If they've really hit on the solution without having played the game before, ask them for detailed justifications to "convince" you their solution is right (don't tell them they're right!). You might want to bring a few copies of "Rat Poison" (from the Outtakes) so that if a group finishes early, having obtained a complete solution they can justify, you can tell them to move on to "Rat Poison".

Also, at each step, as you move around, when you ask where a given group is, also ask if everyone in the group believes (or understands, or agrees with) the response you receive. It's important that the group members help each other keep up, and it probably won't yet be obvious to you which students are the quiet ones.

Large group discussion: When it comes time to discussing the solution in large group, remember that this is the first time for everyone, and students will not be used to explaining themselves (or convincing each other, or even speaking up at all). You will have to keep prompting them with, "Why?" Also try to get as many as possible to come to the board to write even a partial strategy, or a table of recorded moves, etc. — the first activity sets the tone for the semester, and you want them to be used to writing on the board, and talking to each other rather than to you. Encourage the notion that ideas can be road-tested: if two students claim winning strategies, one going first and the other going second, play them against each other. Same if two students suggest different moves from a given position. This is also a good time to start working on their verbal skills. When a student says something essentially correct but poorly stated, write it on the board. If the student immediately wishes to restate it, good. If not, the class as a whole can comment on it and refine it.

One question you should ask at some point (not necessarily right away) is "What does it mean to have a winning strategy?" The answer should include things like: you are in control of the game, you can always force your opponent to lose provided you have choice of going first or second, etc.

The inevitable question "Why are we doing it this way?" may come up as soon as this first large-group discussion, so be prepared to offer a rational, non-confrontational explanation of why "we" (and you) believe in it. Use the problem they've just solved as an example, and try to get some people to admit it gave them a little self-confidence to see that they could eventually come up with the answer by themselves.

Reminder: At the end of Poison, show them how they have used problem-solving heuristics and have stepped through the Paradigm for abstraction and generalization. The problem-solving was to try a few examples (playing the game a few times) and observe that whoever was left with 4 counters would lose. Getting to the next step usually required making this observation explicit and precise. This gave them a template for the next observation, which was that the player left with 7 counters also loses. From here, they could conjecture and verify the pattern: $4, 7, 10, \ldots$. Having solved 10-counter poison, they could attempt to conjecture and verify patterns for larger games.

The Paradigm steps were as follows. (i) Solve 10-counter Poison. (ii) Observe the nature of the solution: the pattern $4, 7, 10, \ldots$. (iii) State this observation in words. (iv) State this observation mathematically: you lose if you must play when the number of counters is $3n + 1$ for some integer n; the winning strategy is to take 2 counters if your opponent takes 1 and 1 if she takes 2. (v) There are a variety of means of proving this. (vi) One possible generalization is: if players are allowed to pick from 1 to k counters each time ($k = 2$ in standard Poison) then you lose if you are forced to move and the number of counters is $(k + 1)n + 1$ for an integer n; the correct strategy is to take $k + 1 - x$ counters if your opponent takes x counters.

2.1.2 Time to weigh the hippos

[Time: 2 hours]

This starts out looking like a hard but otherwise ordinary algebra problem (here's your first shot at identifying their abilities with algebra) but ends up confusing almost everybody, because the problem has two solutions. Getting the groups to discover this fact, and to understand what it means in the context of the problem, is tricky but very important.

Solution: The first part of solving the problem is identifying variables, and this takes more work than you would expect. For one thing, students tend to assign variables randomly to hippos without thinking about what they mean (in the end, we really want to assign variables to "the weight of the lightest hippo", "the weight of the second lightest hippo", etc.). And even when the subject of assigning such meaning to the variables came up, several students were not sure if this was allowable. Also, the students usually fail to distinguish between the hippo itself and the hippo's weight.

The second, even more problematic, aspect of the solution is writing down equations. It is easy enough to assign random pairs of letters to each of the five given numbers, and unfortunately one of the simplest ways of doing so results in one of the two solutions. In fact, you have to be careful assigning summed pairs of variables to pair-masses, because doing so makes implicit claims about specific hippos. If Hippo X's weight appears in the equation with 312 kg and also the one whose right-hand side is unknown, for instance, we have a problem, because this says that Hippo X is simultaneously one of the two lightest hippos and one of the two heaviest hippos, which is impossible. It may take a surprisingly long time for the students to realize the trouble inherent in this method.

If we choose A to represent the mass of the lightest hippo (in kg), B the second lightest hippo's mass, C the second heaviest hippo's mass and D the heaviest hippo's mass (try to let the students come up with variable names), then we know that $A + B = 312$, $A + C = 356$, $B + D = 466$, and $C + D$ is the unknown mass. Since we don't know which of $A + D$ and $B + C$ is heavier, we don't know which of the two is 378 kg, and which is 444 kg. Thus we should try each of the two possible assignments, and see the result of each assumption. As it happens, both lead to meaningful solutions: If $A + D = 378$, then (after some algebra) we find $A = 112$, $B = 200$, $C = 244$, $D = 266$, and $C + D = 510$. This is the solution groups are most likely to find. On the other hand, if we let $A + D = 444$, then we find $A = 145$, $B = 167$, $C = 211$, $D = 299$, and $C + D = 510$. In the context of the problem, this is bad, because we cannot know what the individual hippo masses are, although we do know that the last pair mass is 510 kg regardless.

Prompts: If they have trouble just getting started, ask them what they're interested in finding out, and tell them to identify it mathematically. (Even so, at this point you may have to push them to make them think of variables.) Tell them to try to quantify the information given to them in the problem. If a group gets as far as the equations, you should probably make sure that they understand what they're doing, although of course it's your call as to whether or not to make them go back and re-examine their assumptions in claiming that "these are the equations" (they may only get confused, shaky as their progress is already). If they get stuck doing the algebra, you can either try to remind them of a particular technique (like substitution) for solving systems of linear equations, or tell them to sit tight with that (if the large group discussion is coming up) and think instead about whether there's any other way to write the equations. Finally, you're likely to have a "good" group or two which struggles through, finds one solution, and contents itself with it. If there's sufficient time remaining before the large group discussion, ask them to go back and write down the assumptions they made early on.

Large-group discussion: At least one group should have a solution by the time you go into large group. With luck, each of the solutions will have been found by at least one group. If this is so, have the two solutions put up on the board, and then ask the class to critique them, or say why one is right and one is wrong. Once they see that the algebra is right in both cases, someone may decide that there are an infinite number of solutions. Even if this doesn't happen, eventually you have to cement the notion of how to assign variable pairs to the pair masses. Preferably, of course, the students do the cementing, and as seen above, the crux is the assignment of the intermediate pair masses to variable pairs (following an ordering assumption along the lines of $A < B < C < D$).

The question of whether random assignment of letter pairs to pair masses always generates a real solution will come up, and it will take a while to separate the difference between a real permutation of pair assignments and one which effectively does a global variable replacement:

Original assignment	Equivalent assignment	Fundamentally different
$A + B = 312$	$D + C = 312$	$A + B = 312$
$A + C = 356$	$D + B = 356$	$A + D = 356$
$A + D = 378$	$D + A = 378$	$A + C = 378$
$B + C = 444$	$C + B = 444$	$B + C = 444$
$B + D = 466$	$C + A = 466$	$B + D = 466$
$C + D =???$	$B + A =???$	$C + D =???$

Once everybody agrees that there are exactly two solutions (and why), be sure to spend time discussing what this means in the context of the problem. Is it good or bad to have two solutions? How could you avoid having more than one? In what other context might it be good? If you carefully marked down the sequence of pairs of hippos and their weights, can you eliminate one of the solutions?

2.1.3 Dots and patterns

[Recommend: out of class assignment]

This is taken from the old 105 lab book. It is easy enough, at least at the begining, to serve as their first out-of-class assignment. Students will answer question 1 (a) by various methods, so you may want to take the opportunity to discuss shortcuts other than the explicit formula. In fact the explicit formula of $n(n+1)/2$ is hard to find if you haven't seen it before. Question 1 (b) is a classic example of when step 4 of the paradigm can be hard: just because a student has been able to state an observation doesn't mean that student has the facility to state it algebraically. The same goes for the question after problem 2, relating it to problem 1. By the way, some students will make a mistake and overlook the fact that n on problem 1 corresonds to $n+1$ on problem 2. The best way to point this out is to try plugging in a small number for n in the formula from 1 (b), then counting subsets to check that this correctly answer the number of 2-element subsets of a set of size n, and when it doesn't, trying to track down the error and correct it.

2.1.4 Glicks, Glucks and Dr. Seuss

The original version of this worksheet had Glacks and Glocks as well. In all, there were four logically different kinds of statement, and students had some difficulties (most, but not all, of which are ironed out on their own) distinguishing between them. On the present worksheet, the forms of the statements are "if Glick and A, then B" and "if A or B, then Gluck". Some of the questions will engender no discussion, but at least some should spark conversation.

In the next version of the coursepack and guide, the Glacks are going to be put back in. They may be found in the "Outtakes" section of this guide. The reason for re-including them is that the statement about Glacks is a defining statement, so is different from the others which are simply statments known to be true. It is instructive for students to stumble on this and to debate the interpretation of the Glacks statement.

Solutions: 1. (a) true (b) can't tell (c) can't tell (d) can't tell (e) can't tell (f) true 2. (a) can't tell (b) true (c) can't tell (d) false (e) can't tell (f) false 3. (a) true (b) can't tell (c) can't tell (d) can't tell (e) can't tell.

2.1.5 Gotta hand it to you

This one might fit in under an hour. When groups get started, go around and check that they understand the statement. Ask them how many handshakes took place before the judge arrived if only 4 people were on the float.

Many will want to use the formula they learned in Dots and Patterns. They will find that ignoring the unique role of the judge and solving for a non-integral number of people on the float to give 1625 handshakes doesn't work. They will be stuck solving or trying to solve this quadratic for hours if you let them! Nudge them by getting them to interpret the computations they are doing. E.g., "Does the number $57.51\ldots$ I see on your paper represent the number of people on the float?"

When they find an answer, make sure they realize that they must find all answers. As it happens, the problem is neatly set up so that for any integer number of handshakes there is precisely one solution. It would be nice to bring the discussion to this point. This may or may not work, but certainly you can get somewhere by insisting that they have found a solution other than the one you had in mind. This is plausible to many of them, especially to those who found it unnatural to introduce integer constraints when they had a formula that looked as if it worked for all real numbers. They will be wondering whether finding a nearby integer and then letting the leftover handshakes go to the judge can always be done in exactly one way. They should find it possible to convince you that for these particular numbers there is a unique solution. See if you can push them at least to formulate the statement that for any positive integer number of handshakes in place of 1625, there is a solution and only one solution.

How far you get through the paradigm may vary on this one. If you make it to the generalization stage, it is worth pointing out explicitly what they have done: taken the uniqueness statement, with its justification, and generalized this to any number of handshakes.

2.1.6 Banned book survey

[Time: 2 hours]

This problem, is most easily solved via Venn diagrams and sets. Placed as it is, before the readings on sets, it serves to introduce the utility of Venn diagrams and iron out some misconceptions as to their conventional meaning[2]. In particular, the idea that a number drawn where it is inside just one circle should count how many objects have that property but no other (rather than counting the total number of objects with the given property) is usually forgotten by students who might intuitively know it if asked. Students who agree on a meaning for their diagrams, conventional or not, will usually get pretty far. So the first thing the instructor needs to do is get the students to be clear on what their diagrams mean, without tipping off that there may be a problem. Ways to do this include asking one group member what their groupmate meant by writing a particular number in a particular location.

Since sets and Venn diagrams have not formally been studied, some students will probably use algebra with no diagram. A similar principle applies: those who are clear what their variables represent will have won half the battle.

Solution: The most straightforward solution is to draw a Venn diagram with three intersecting circles, each one representing one of the books. If we label each of the eight regions so created, we can solve the resulting linear equations without (one hopes) too much trouble. If we let the number of *Huckleberry Finn* readers be $A+B+E+F$, the number of Bobbsey Twins readers be $D+E+F+G$, and the number of *Lady Chatterly's Lover* readers be $B+C+F+G$, then the information given in the problem can be written $A+B+E+F = .53$, $D+E+F+G = .42$, $B+C+F+G = .36$, $E+F = .25$, $F+G = .14$, $B+F = .10$, and finally $H = .12$, so that (using $A+B+C+D+E+F+G+H = 1$) $A+B+C+D+E+F+G = .88$. The solution (in percentages) is $A = .24$, $B = .04$, $C = .18$, $D = .09$, $E = .19$, $F = .06$, $G = .08$. Thus 6% (F) of the people surveyed had read all three books. In general, with linear equations we need 8 equations to solve 8 unknowns, and that is what we have here. Thus the last piece of information was necessary.

Further notes: This worksheet also serves to introduce inclusion-exclusion, though it is really a more complicated instance than would normally be used to introduce this topic. There is a reading which is assigned by the start of this worksheet, namely "Counting revisited". It introduces inclusion-exclusion (limited to 2-deep nesting). After discussing that Reading, students should know that you can count $A \cup B$ by adding $|A| + |B|$ and then subtracting $|A \cap B|$ for the overcount. (In fact they will have seen that $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B \cap C|$ when all the pairwise

[2] At this point, we find it philosphically better to allow nonconventional use as long as they are aware of the convention; later, in "Describign sets of number (Section 2.4), they will need to start adhering to the convention.

intersections are the same, and will interpret that as subtracting off two of the three counts for the triplecounted items.)

Thus another solution, which some might try, is to add $|A|+|B|+|C|$, then subtract off $|A \cap B|$, $|B \cap C|$ and $|A \cap C|$, which correctly counts everything except the items in $A \cap B \cap C$. When they see that these have been counted with weight exactly $3-3=0$, they can add this number back in. Warning: in the past some groups have come close to this solution by voodoo. That is, they add $A|+|B|+|C|$ and describe this incorrectly as "total number who have read one book". They then subtract $|A \cap B| + |B \cap C| + |A \cap C|$ for no particular reason (perhpas they see this will put htem in the right ballpark), and start playing with numbers to see how close they are. It's hard to do anything useful with that, which is why we have sometimes introduce the alternative inclusion-exclusion-based solution: it demonstrates that adding $|A| + |B| + |C|$ and then subtracting off the pairwise intersections really does count something, and that you just have to think it through to see what.

2.1.7 Chickens, rabbits, temperature, driving, ages and headaches!

Everyone know that the hardest of all things to teach (from grade school through differential equations) is word problems. In fact the focus of the algebra in this course is entirely away from algebra skills and onto word problem skills: algebra as a means of describing and analyzing a problem (and to some extent, algebra as a means of stating generalizations). The problems on this sheet progress from straightforward to very difficult. It is a good idea to try and elicit at least two solutions to each, one more *a priori* algebraic, and one based more on trial and error. The trial and error one must be presented first or no one will take it seriously, so you need to snoop on the groups and make sure the first one you call on in the whole-class discussion uses the most elementary method.

When you are going around to get groups unstuck, keep in mind the variety of paths to a solution. Rather than prejudging what will work for them, try to see if there is anywhere that their present dead-end can usefully lead. For students having trouble getting anywhere, help them try trial and error, and then help them "algebreify" it (see the notes on parts 2 and 3 of the "Word problem review" worksheet).

1. The equations $C + R = 50$ and $2C + 4R = 120$ are pretty easiy to find and to solve, yielding $C = 40, R = 10$. If a group has trouble formulating or solving the equations, encourage them to solve the problem by trial and error: try a number for

C, see what one equation says for R, see if that works in the other equation; if not adjust the number and see if the adjustment helped; pick new numbers, attempting to move in the right direction until you hit a solution, and to make your guesses better educated as you go. The point of this is to give ownership of the problem to students who need it most. This is not an algebra skills class, so you don't need to get them through the annointed method. Rather, they need to learn that they can get through it themselves if they keep their head clear. Later, in the whole-class discussion, they will see the algebra, and will understand it much better for having several times plugged in the numbers that the algebra represents.

2. You can give them the conversion formula if you wish, or you can tell them that $0°\text{C} = 32°\text{F}$ and $100°\text{C} = 212°\text{F}$ and let them work it out. The first difficulty for them will be in understanding that two variables are necessary, namely the temperature measurement in degrees Celsius and the measurement in degrees Fahrenheit. Encourage them to use something other than F and C for the variable names, since the tendency is already for students to confuse the object with its measurement (e.g., temperature, which is the same no matter how it is measured, with number of degrees).

A second difficulty is seeing the two rather different equations: the equation $X = 2Y$ comes from the problem, while the equation $X = 32 + (9/5)Y$ is the conversion formula. In addition, some students may get the formula $X = 2Y$ reversed. This brings to mind the problem of expressing the relation "There are 12 students for every professor" as an equation. The problem is famous, having been used in research studies. The wrong answer of $12S = P$ is very common. This is probably because a professor is (informally) equated with 12 students. So if you think of S as "students" rather than "the numer of students", you will get the equation backwards. If you see many students reversing equations on this worksheet, you might want to take a break and have a whole-class discussion where you ask them quickly to write down (each on their own) an anser to the student-professor problem, then compare notes. Another reason to bring up this problem is if you need more motivation for them to be pedantic with their interpretations of variables. Distinguishing between "students" and "number of students" may seem silly to them if they have not tripped up on a problem of this sort.

3. The main difficulties again lie in assignment of variables and in finding hidden equations. They can see that in addition to the speeds, the distance is unknown. The clearest way to solve this is with three variables: slow speed, fast speed and distance. They then need to relate time, distance and speed in the standard way to

get $d = 5f, d = 6s, f = s + 13$, which has the solution $s = 65, f = 78, d = 390$. Some students may use only two variables, s and f, in which, in addition to the equation $f = s+13$, they will need to observe that $65s = 78f$ because the distance is the same no matter how fast you drive. It can of course be done as well with only one variable. This type of problem is difficult enough that we added the "Word problem review" worksheet to reinforce what they have learned (originally called "Word problems out the wazoo").

4. This one is really hard! Most people see it as having four relevant variables: Donna's and Janlen's ages now (D and J), and Donna's and Jalen's ages then (d and j) whenever that was! It takes a very clear head to see that the information may be encoded as

$$\begin{aligned} D &= 2j \\ d &= J + 30 \\ D - d &= J - j \end{aligned}$$

To get the first two equations, one must let go of wanting to know how long ago was the past being referenced, and simply state the relations between one present age and the other past age. For the third, one must recognize that the past begin referred to is equally long ago for both people (this specifically gets reinforced in the first problem of "Word problem review"). And to top it off, the system is underdetermined! To be honest, I have never gotten all the way through this problem. Usually the farthest I get is to have the class be able to tell, if I tell them two possible ages now, whether these are consistent with the statement of the problem.

5. Sometimes we've skipped this for time reasons, and sometimes not. They don't appear to learn all that much from it, but on the other hand, we have been stepping up the emphasis on their ability to design problems (see the worksheets "Why? part II" and "The sense of it"). To be good teachers, they will need to be able to invent problems for specific purposes. This is an easy place to start, though not as instructive as inventing a problem to illustrate a specific point.

2.1.8 Comparing without counting

They have already read "Counting revisited", so they are quite well prepared and this worksheet should go quickly. That Reading purposely refrained from discussing correspondence as a means for determining the direction of an inequality, in order that problem 1 on the present worksheet not be trivial. So the main point of problem 1 here is to point out that you can test $|A| < |B|$ by matching all elements of A one-to-one with a proper subset of B. This is a case, where if the students do not quickly come up with the answer, there is probably not much use lingering. Tell them the observation yourself and go onto the next problem.

Problem 2 is a good way of reinforcing some of what they have learned by reading "Intorduction to sets". You will now get to see whether they know what a subset is. They have studied two-element subsets at length in "Dots and patterns", which has in the past led some to believe that all subsets have two elements. Even though they have seen set notation used, they may have trouble writing down a new set. They may also need explicitly to discuss the extension property: a set written in any order is the same set, and there is no need to write one twice, since an object is either in the set or it's not. The main point of the problem is to see that there are one-to-one correspondences between the subsets of a set S and each of two halves of the subsets of a set $S \cup \{x\}$. They then pass to a recursive understanding that adding an element doubles the number of subsets, and from there to there being 2^n subsets of an n element set. This last observation is not an answer to any of the stated questions, but the whose-class discussion there as a generalization and crystallization of the foregoing discussion.

This gives a nice example of the paradigm, which you point out to them explicitly once the whole-class discussion is complete. Steps they ought to pass through in this last part of the discussion are: finding the number of subsets of a n-element set for $n = 1, 2, 3, 4$; observing the pattern recursively as doubling; stating this in English; stating this in mathematical language, leading to the statement of an explicit formula 2^n for the n^{th} size; justification of the doubing observation, which was done when arguing problem 2 (d), and of the explicit formula – here they may not feel it necessary to justify that a repeatedly doubled sequence yields 2^n, but you want at least to point out that an initial value is needed for this to follow – also, try checking the degenerate case $n = 0$; lastly, is it possible to generalize? here they might want to u8se what they know about Cartesian products to write a formula for the number of sequences of length n of elements from a k-element set, which generalizes counting subsets when you view a subset as a sequence of elements from $\{\text{yes}, \text{no}\}$ indexed by elements of the original set.

2.1.9 Word problem review

[Recommend: out of class assignment]

A solution is given in the on-line materials, so the notes here are restricted to pedagogical points. The way we have used this worksheet in the past is as out-of-class homework. They are supposed to have learned how to do this kind of problem on their own, but typically need a lot of practice. The worksheet is intentionally composed of two sets of similar problems (1 and 4 are similar and 2 and 3 are similar). If you have some students who are particularly bad at this, you can meet with them individually, walk them through 1 and 2, and have them solve 3 and 4 in front of you, with the knowledge that 3 is like 2 and 4 is like 1.

Problems 1 and 4 are probably more straightforward and emphasize variable naming. In particular, they require that a quantity that changes not be named the same thing at two different values. On problem 1, if you choose the four-variable approach, the equations pretty much find themselves. If you choose the two-variable approach, then you must be clear that only present ages are named as variables, and that past ages are referenced by present ages minus 1 year. Problem 4 is similar, with a now-and-then theme and a known relation. It is harder because the words "2 to 1 majority" and "1 seat" majority are unfamiliar to some and refer to different types of discrepancy (additive versus multiplicative).

Problems 2 and 3 emphasize how to assign variables and find equations in more difficult problems when you are having trouble getting started. The idea is to make guess and check methodical enough that it can lead reliably to algebra. The student must, as she goes through this, learn on her own how to check a particular numerical guess and see what other numers it leads to and whether they are consistent. Having done this one or more times, the student then tries to replace the guess with a variable. Where before it was hard to say what the other quantities were in terms of this variable, now the student can refer to her own explicit computations to see how she got the next number. This leads eventually to more and more complicated expressions, one of which may finally be set equal to something, and an equation is derived! This method produces equations with reliatively few variables, since each time all expressions are computed in terms of the input data. While more variables are conceptually easier in a purely algebraic approach, this approach may be the only hope for students who can't seem to get to square 1.

Problem 2 and 3 relate best to the Paradigm for abstraction and generalization.

They begin with trial and error, observe how one input leads to determination of other quantities, they put into words how they made this inference, and they write it as an equation. The justification and further generalization steps are not present.

2.2 Whole numbers and their properties

This section has two distinct curricula. The first covers primes, factors, and the GCD and LCM. Our aim here is to give teachers familiarity with basic multiplicative properties of whole numbers. In this area, more than in any other, it will be important to have whole-class discussions that crystallize what the students have discovered on the worksheets. Major concepts such as unique prime factorization and the significance of the GCD are never addressed in the readings nor explicitly stated in the worksheets. Rather, the worksheets run the students through activities meant to prepare them for a deeper understanding of these ideas, and this preparation will be wasted if the discussions are not brought to fruition. The weekly summaries may be used to emphasize for which of the ideas the students will be held accountable; one might even include statements of important theorems in the weekly summary or an online handout (but not before the relevant activities are completed!).

The second curriculum in this section concerns properties of the whole number system and operations in the context of abstract properties of operations on sets. The dangers of too much abstration with too little motivation have been well recognized since the failure of the New Math in the 1960's. Nevertheless, to be effective teachers, the Math 105 students need to gain a little sophistication about the objects they are dealing with. It is important, for instance, to understand commutativity, since learning that addition and multiplication are commutative, and learning why, are recognizable cognitive steps in the education of young children. Learning the names of these formal properties helps prospective teachers to better recognize, categorize and understand these properties. More importantly, teachers need to go through the exercise of imagining "what if not" with these properties. It is difficult to teach a child to see that $7 + 4 = 4 + 7$ without ever having tried to see why one might think it not to be true.

2.2.1 All in the timing

This worksheet will take 1.5 to 2 days, depending on whether you cut any of it (many instructors don't work all the way through # 5) or assign any of it for homework. It will probably become evident that all or most of the students are familiar with the idea of making a factor tree. They are a little less clear on exactly what a factor tree accomplishes and to what degree factor trees are unique. It is best to have this discussion a little later, when they have all tried problems 1–3, since then the discussion can take full aim at the meaning of unique prime factorization.

Set aside the first few minutes to discuss questions from the reading, because there will surely be some. In particular, remind them that they will be responsible henceforth for the vocabulary (prime, composite, relatively prime, GCD, LCM) as well as the divisor notation.

Problem 1 starts them off with something they can get their hands on. Many will use their calculators to see which are divisible by 7, but they will still need to work out factorizations. They are meant by the end to have decided for themselves that divisibility by 7 entails finding a 7 in the prime factorization but divisibility by 24 is not the same as finding a 24, or even a 4 and a 6. It may be touch and go whether at this point you can get to divisibility by 24 being the same as finding three 2's and a 3 in the prime factorization.

As pointed out in the discussion below of problem 2, it is important to have several passes at this discussion. First, the students must have ample time to explore. A brief whole-class discussion may be needed in order to get everyone on the same wavelength as to what a factor tree is. This or a subsequent discussion can get them thinking about the degree to which these are unique. They tend to understand that writing things in a different order is not an essential difference, so you can tease them by pretending to believe that 49×2 is a very different first step from 2×49 and will lead to a radically different factor tree; they will then claim that the trees won't be too different – you can just swap the two halves. Then you can pretend to be convinced that the factor trees are always the same. If they don't see $14 times 7$ as leading to a distinct factor tree, try one of the other numbers such as 144, where the factor trees can have very different shapes. Once they have seen that shape can vary, it is more meaningful for them to discover what is always constant, namely the multiset of prime factors at the leaves of the tree. We are aiming here at a deeper understanding of the Fundamental Theorem of Arithmetic (unique prime factorization) by giving them at least a fleeting though that it is not true.

Many students will attack problem 2 with a calculator. Most calculators will show 7^{29} and 13^{22} in scientific notation and will have them differing in the third digit after the decimal point. This leads them to believe the numbers are different. If this belief is too firm, you might need to shake it a little by having one student use a calculator to compute the decimal expansion of, say, $2^{1/1234}$, and then having another student key in (so as not to take advantage of digits stroed but not displayed) this number to the power 1234. Explain (if necessary) that this should be 2; the result should be off after about five decimal places on any caluclator with seven or eight digits of accuracy. The idea of course is that, whether or not a calculator can distinguish these, there is a quick and easy way to say "yes, these must be the same number" or "no, these numbers must be different". This discussion will be the first and simplest test of whether they have understood unique prime factorization, so the time sequence is important. They must try problems 1 and 2, then discuss them both in that order.

Problem 3 introduces the idea that the *number* of factors of a number depends only on the form of its prime factorization – all the powers of the primes, but not which primes they are. There are many possible followup questions, or intermediate questions one might prepare in case they are slow here. Examples would be (1) to give them two factor trees, one being of $p^2 \cdot q$ for some new primes p and q, and one being of $2^5 \times 3$, and asking them which will also have exactly 6 factors (intermediate question), or (2) to ask them to come up with another number that has exactly 6 factors (followup question – having them say how they came up with their answer should be interesting!).

Problem 4 is really an exercise in understanding the question – the verbal quantifier here is quite difficult for them. Unless you're ahead of schedule, either skip this, or get through the semantic portion of it, but there is no real need to spend enough time to actually arrive at an answer. Questions such as these are good to throw at faster students who have completed the remaining problems on the worksheet. Problem 5 is similar.

Problem 6 introduces a physical significance to the LCM. The second half of the problem is eminently skippable, serving only to highlight the difference between two possible questions one might ask. If left to their own devices, students are unlikely to mention LCM explicitly in their solution to problem 6, so you might want to lead the discussion there.

In problem 7, again it is the first question that is important and highlights the significance of the LCM. By the end of this problem, you need to have tackled the connection to the LCM explicitly, since they need fair warning that the LCM is important and that they are expected to understand its physical significance. The LCM appears as an object of more theoretical study in the next worksheet.

2.2.2 Counting factors

This worksheet explores in greater detail what was explored by problem 3 of the previous worksheet: how the number of factors is related to the form of the prime factorization and several special cases thereof.

Problem 1 shows that primes are characterized by having exactly two factors.

Problem 2 shows that having a number of factors which is a power of 2 is equivalent to having distinct prime factors. In the past students have caught onto this pretty well, but have struggled to put it into words; take several passes at verbalization if necessary.

Problem 3 sometimes stalls them for a while. First they can't think of any, then they incorrectly generalize from 4 and 9 to all squares. You must let them discover their own error if they do this, but then nudge them so they don't give up altogetherwhen they are so close.

Problem 4 takes the previous worksheet's problem 3 one step farther on the Paradigm – they must put their verbal understanding into more succinct notation.

Problem 5 is impossible unless they have begun to have a glimmer that if you have a number with a factors and one with b factors, then sometimes (i.e., if the proper factors are distinct) you can mutliply the numbers and get a number with $a \cdot b$ factors. You can lead them a little if necessary by asking them to list the factors of a product of two numbers, each of whose factor list is written on the board (and conveniently distinct); if they still don't see it, try it with the number 10 and 43 and ask them to list it in an order that will best convince you that you've gotten them all. If necessary, prompt them about what they know about Cartesian products (the "Counting revisited" reading and "Comparing without counting" worksheet).

Problem 6 extends problem 5 one or two more steps along the Paradigm. Point this out to them explicitly!

Problem 7 reinforces the limitation on the Cartesian product method.

2.2.3 Stamps

[Time: 2 hours]

This problem is, in the present version of the course, the only introduction the students will have to the significance of the GCD. For that reason, we felt it was worth including, even though past experience has shown progress on this problem can be frustratingly slow.

It revolves around relative primeness, although it will be some time before the students get any notion of this. The 4-cent and 9-cent problem eventually (beyond a certain point) allows one to make any amount of postage, while the 9-cent and 21-cent problem does not, for the simple reason that 9 and 21 are both multiples of 3, so only multiples of 3 can be made. It usually takes the students at least an hour to get even this far.

The crux of the problem in the beginning appears to be the way one keeps track of the amounts "makable" and "unmakable"; more than any other problem in this set of materials, the ease of solution depends directly on the method of representation chosen. Making two lists, one of each type of amount, gives little insight into the underlying structure. The optimal representation appears to be a grid-style list of all numbers (starting with 0 or 1), with each row of length one of the two stamp denominations (e.g., 9). One can then circle each of the "makable" amounts, and see that once a number in a given column is circled, all the numbers [physically] below that number in the same column are also circled, as they are multiples of the row length added to a "makable" number. An intermediate strategy is to make several lists, organized by one of the denominations, e.g., "multiples of 9", "4 plus multiples of 9", "8 plus multiples of 9", etc. (This latter can also manifest as a grid, which seems to be more helpful for the students.) There are, of course, many possible strategies, but students may choose one which gives them little insight for the problem and eventually get stuck. In this case it is your decision whether or not to give them a push. One instructor suggested a grid representation but let the students decide the width; another gave no suggestion, with the result that discussion took an extra hour before students came to an agreement.

Try to keep the students from getting discouraged, and if they tell you they cannot make any generalizations (as asked in the problem) after solving the first two questions, tell them to (a) look for the differences between the two pairs of numbers, or (b) make up some more pairs of small numbers and try answering the same question with them. (Ideally these ideas would occur to them naturally, but their problem solving skills are not likely to be honed this sharp yet.) In any case, realize that you will not get as far as a complete solution with explanation in class, and take what you can get (don't spend more than a week on this, interesting though it is). You want to get to the point where students recognize that the important thing is the relative primeness of the denominations involved, and why.

Solution: In complete generality, the answer is that given stamps of denomination a and b, one can make amounts which are multiples of GCF(a,b), and the last such multiple which cannot be made is LCM(a,b)$-a-b$. Thus if a and b are relatively prime, one can make every amount greater than $ab-a-b$. (Note: Every part of the preceding solution was originally discovered by a student!) Less generally, given denominations 4 and 9, one can make 4, 8, 9, 12, 13, 16, 17, 18, 20, 21, 22, and everything beyond (but not including) 23. Given denominations 9 and 21, one can make all multiples of 3 except 3, 6, 12, 15, 24 and 33. It should seem completely reasonable (though the students will probably not have thought of it in these terms) that since 9 and 21 are multiples of 3, you can only hope to make multiples of 3 using them.

As short a proof as I can find of why $ab-a-b$ is the last "unmakable" number given a and b relatively prime: Make a table of numbers in rows of b entries each:

$$\begin{array}{rrcrr}
1 & 2 & \ldots & b-1 & b \\
b+1 & b+2 & \ldots & 2b-1 & 2b \\
2b+1 & 2b+2 & \ldots & 3b-1 & 3b \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
(a-1)b+1 & (a-1)b+2 & \ldots & ab-1 & ab
\end{array}$$

Note that numbers in the same column differ by a multiple of b. Thus, the first (smallest, or highest-up in the table) multiple of a in each column will be the first "makable" amount in that column (by using, say, m a-stamps). Every number in the column below ma can be made by using m a-stamps and the same number of b-stamps as there are rows separating ma and the number in question (e.g., the number three rows below ma is $ma+3b$ and can be made with m a-stamps and 3 b-stamps).

Thus the question of being able to make all numbers is equivalent to finding a multiple of a in each of the b columns (except for the last one, if you like, since they're all multiples of b in that column). So we need at least the first b multiples of a to hit all the columns (i.e., one multiple of a for each of the b columns). Suppose the first b multiples of a *aren't* all in different columns, i.e., suppose there's one column with two multiples of a in it. Since the numbers are both multiples of a, their difference is a multiple of a. Since the numbers are in the same column, their difference is a multiple of b. Since a and b are relatively prime, their difference must be a multiple of ab. Thus the greater number (lower down in the column) is at least b multiples of a after the smaller number, and hence is not in the first b multiples of a. So the first b multiples of a do it, and eventually we can make anything.

Note also that the bth multiple of a is, as previously mentioned, not especially important because it hits the bth column, which were all multiples of b anyway. So the last column to be hit is the one hit by the $(b-1)$th multiple of a, $(b-1)a$. Thus the last number that cannot be made is the one directly above $(b-1)a$: $(b-1)a-b$, or $ab-a-b$.

There are several other similar "tabular" ways to look at this problem which you will undoubtedly encounter from your students. In this activity more than most, it's vital to keep your mind open to solutions which never would have occured to you (but still have some validity).

There are several levels of understanding to this problem: Level 1. With 4 and 9 you can eventually make everything; with 9 and 21 you can't. Level 2. The difference is due to 4 and 9 being relatively prime; with 9 and 21 you can only make multiples of 3. Level 3. In general you can eventually make every number precisely when the two numbers are relatively prime. Level 4. In general you can make every number beyond $ab-a-b$ if a and b are relatively prime; otherwise you can eventually make multiples of GCF(a,b). Level 5. Proving $ab-a-b$ is the last number one cannot make; or knowing the generalized formula (above). Level 6. Proving the generalized formula.

You should try to get the class discussion to Level 3. Some individuals will only get to Level 2; others will get as far as Level 4. One or two may get to Level 5. I myself have not bothered getting to Level 6.

2.2.4 Describing sets of numbers

Warning: some of the pdf versions of the coursepack have unusable versions of the figures for this one!

This worksheet uses their newly acquired number theoretic understanding as fodder for a basic discussion of sets and Venn diagrams. They have had some experience with diagrams in Banned Books, but only now begin a more formal study. In the

Banned Books worksheet you stressed that any diagram-writing convention is OK as long as you are clear and consistent about it. Now they need to learn what the conventional meaning of a Venn diagram is and to stop using it in nonconventional ways. As far as we know, each diagram is satisfied by only one letter from (a) to (j).

2.2.5 The locker problem

[Time: 1 to 1.5 hours]

Like the Stamps problem, this is a good problem for finding and recognizing patterns. It can come before the stamps problem if desired. The bottom line is that squares are the only numbers with an odd number of factors; if students have recently solved Related Problem 1 at the beginning of the unit, they may have made this observation and remember it (not likely, but just a warning). Here as with the stamps problem, there will be a multitude of ways of keeping track of which lockers are open and which closed, some of which will prove more useful than others. Here, however, students are usually able to find a representation that serves sufficiently well.

Solution: The number of times the door of a given locker is moved is equal to the number of factors of the locker number. All numbers have pairs of factors (and hence an even number of factors) and thus end up closed, except for the ten squares 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, which have a square root which only counts once (it's paired with itself). Students are likely to start off modeling just the first twenty (e.g.) lockers and going through and making the twenty first passes, and then trying to see the pattern. Eventually they will see that only squares end up open, but by the end of the large group discussion they should be able to say why square-numbered lockers end up open and the rest don't.

2.2.6 Clock arithmetic

The real content in this worksheet is probably in the Reading. The "bow-tie" operation will freak them out. Go over this to make sure they become comfortable with the abstraction template here. We recommend that you have them put their self-check answers on their desks at the start of class (if you don't always do this) and that you cruise around to see how they did before starting the whole-class discussion and question session on it. Once you get to the actual worksheet, it should be pretty quick.

2.2.7 Tarzan

This is an exercise in suspension of belief. In order to appreciate their implicit understanding of the properties of addition, they must consider in turn what might happen if each failed. By doing this, they reinforce their understanding of the abstract properties while having a few laughs. The properties of addition to which the parts of question 1 refer are: (a) closure; (b) that 0 is the identity element; (c) commutativity; (d) associativity.

Problem 2 demonstrates a method for adding one-digit numbers with carry. The explicit point of the problem is to practice justifying this in terms of known formal properties of addition. That's a good exercise, but the method analyzed is important as well. This addition technique is one commonly invented by young children on their own, and one which is often not discussed or even explicitly recognized by teachers in the U.S. (compare to curricula of other countries, where this is taught to teachers as a standard cognitive step in attaining fluency in mental addition). It is crucial in the discussion to mention this explicitly, as this is the only place in the curriculum to address this important skill. Such a discussion also paves the way for later worksheets where the students must play detective and figure out what a hypothetical child has done by looking at the scratchwork.

Problem 3 is a straightforward reinforcement of the definition of closure. Closure, being more obvious than commutativity for addition, and being almost a foregone conclusion in many contexts, is harder to come to grips with than commutativity, which is why we devoted a problem just to closure.

2.2.8 Negative numbers and the question "Why?"

There is more in this worksheet than you need to do. The important part is to get across to the students one or more models for negative numbers. This worksheet is almost an extended reading, but since the reading on models of operations was already quite extensive, we fashioned it as a worksheet. There is a mini-lecture on arithmetic and models that you might want to give now, or perhaps later when they first tackle place value and its relation to addition. The content is as follows.

- Primitive math begins with no formal arithmetic (no numeration system, no place value, no concept of addition, etc.) but only actual problems to be solved. I have 3 sheep, you have 8. How many will I have if I kill you and take all your sheep?

- Next comes the recognition that this problem comes out the same regardless of whether we are discussing sheep, apples or some other objects. We invent numeration to stand for the numbers involved and operations to stand for the combining we are doing.

- Next we come up with rules for doing this operation (for adding numbers in our numeration system). This rules are built on what we know from observations, but are ultimately simply self-consistent rules. The relation to future observations depends on our faith that the connections we observed in the past will extend analogously in the future. At this point we have moved to formal addition, along with a physical model that we *believe* will always correspond to it.

- Lastly, we start using the formal system to solve problems previously too hard to solve. For example, we might need to know in a hurry how many sheep we'd have if we combined tribal holdings of 378 sheep and 491 sheep.

On this worksheet, the point is that "why" questions about negative numbers and other such entities must be answered in terms of the physical systems they were meant to model. One can define formal systems of addition with negative numbers any way one wants, but to be useful they must be informed by outcomes of physical problems in which negative numbers arose. Here, we discuss three possible modeling origins of negative numbers, and see why each leads to the rule: **subtracting a negative number is the same as adding the corresponding positive number**. The second (charges) model is the most hands-on and most appropriate for a sliimmed down version of the worksheet. It comes from the old Math 105 lab book and requires only minimal equipment. But the comparison as well to the vector model and to a formal model (in which the main reason to invent negative numbers was so that all numbers could have additive inverses) is instructive if you can afford the time.

2.2.9 "Why?", part II

Except possibly on the "Chickens" worksheet, this is their first chance to invent a story problem for a specific purpose. We make it easy on them by making it multiple choice. Still, they find it tricky. Story 2 is the easiest to rule out, since it is clearly an illustration of how subtraction of a negative is like adding a positive. Nevertheless, when we have taught this in the past there have been some groups who chose it because it was simple and correct, even though it didn't model the right thing. Needless to say, they need the implicit reprimand this worksheet provides!

Ruling out story 1 is trickier. It appears to be a justification of why the product of two negatives is a positive. Further examination shows that it really justifies that the product of numbers with any signs is positive! More precisely, it shows that area is an unsigned unit, so is the product of unsigned lengths, so interpreting a product as area requires that you take absolute values before (or after) multiplying. This is a more subtle point, but still crucial to anyone who will ever have to anwer children's questions "Why?".

The last choice is the correct one. The only trick here is to see that the problem can be modeled as multplication of two negatives or of a negative and a positive, depending on whether you change over to upstream measurements before operating on the numbers. The point to be made here is that the latter is not incorrect: it is just that only one of the correct models for this problem illustratees the point you want to make.

2.2.10 Stupid number tricks

This worksheet could have been included a little later in the section on place value. It is placed where it is for two reasons: (1) so that the students are not tipped off as to the correct approach, and (2) in order that the idea of place value be motivated and exemplified before the in-depth study has begun.

Students will struggle to understand this. In fact, effective understanding is almost impossible to attain without formal understanding. Groups that represent this algebraically can usually figure out why it works. For groups that don't, I have suggested doing the computation with several different numbers (each group member does one or two) and tabulating the results so that each computation, with all its intermediate results, occupies a column. This helps them to see that at a given stage, they have a number depending somehow algebraically on what they started with. If you ask them to use 10 and 100 as inputs, the algebraic dependence should be even clearer.

To explain their solution, they will need the fact that $10x$ looks like the integer x with a zero appended. Allow them to assume this without proof for now, but point it out as something we would like to see a proof of shortly, when we study place value.

2.2.11 Tarzan II

This extension of Tarzan I goes very quickly. The properties Tarzan is lacking are: (1) distributivity of multiplication over addition; (2) commutativity of multiplication (viewing taxation and discounting as multiplication by $1 + x/100$ and $1/2$ respectively); (3) reversal of all signs when subtracting a quantity in parentheses, so that if $\sum_{i=1}^{n} x_i = 100$ then $T - x_1 - \ldots - x_n = T - 100$.

2.3 Place value

We, the teachers of Math 105, all know that the use of other bases is often necessary to give students the perspective they need to understand their own base. This is the not the only reason to explore other bases, but it is the only one that many of the students will accept. With many populations of students, it is important to stress this at the beginning and throughout. We have also in the past stressed that once they have used other bases to understand base ten, the other based *per se* pretty much vanish from the curriculum (but the will still be tested on them!).

Some instructors find it useful to be very consistent about using spelled out words for numbers when using them in meta-numeration contexts, such as when specifiying a base for numeration. We have tried to make the worksheets consistent with this.

2.3.1 Throwing yourself off base

The name of this worksheet is a pun meant to reinforce the reason they are being made to study other bases.

Problem 1 is a straightforward exercise using a multiple-base monetary system that everyone in England had to use not too many generations ago.

Problem 2 is a counting problem. It really has more to do with Cartesian products than bases. Make sure everyone agrees on the definition of what constitutes a three-digit number.

Problem 3 is an introduction to divisibility tests. While the notion of divisibility has come up already, the means of testing has not. You will find that most students know how to test divisibility by 2, 5 and 10, and some by 4, 3 and 9. It is safe to forecast that no one will be able to justify any of these, and certainly not the 3 or 9 test. At this point, we are mainly looking for coherent and correct answers, not for justification. Ask for the justification, as always, but be content if they leave it as "believe, but unable to justify". Still, parts (c) and (d) ought to give enough data that they can come up with something interesting for part (e).

Problem 4 is a useful prelude to the rest of their formal study of bases. They should be told after they have done this that it is important and that they will be expected to know how to do this.

2.3.2 Arithmetic in other bases

This is really an extended self-check for the Introduction to Bases. In the past we have assigned it to be done in class onoy because at home they tend to get stuck early on and therefore don't do any of it. In class, you can ease them through a few problems untili they get they idea. By all means, don't feel compelled to spend more than 10 or 15 minutes on it. If they do a few of these, they've probably gotten all there is to get from the worksheet. One thing to watch out for is whether they are comfortable with what digits are allowed in each base, and particularly with what digits we choose (arbitrarily) to add to bases higher than ten.

2.3.3 Justify!

Our lofty goal for Math 105 was originally to have them understand complete justifications for all four basic arithmetic algorithms in base ten and to have them write these using correct mathematical language and grammar. We have pulled back substantially from this, not because this understanding is a bad goal but because it is hard to reach in any way other than having them essentially copy and not really learn the justifications.

This worksheet does a limited version for addition of 2-digit numbers of what we would have liked for all the operations. Still there are quite a few steps. To add ab and cd, they must

- Write these as $(10a + b) + (10c + d)$;
- Re-arrange (use associativity and commutativity) to get $10a + 10c + (b + d)$;
- Separate into cases $b + d \leq 9$ and $b + d \geq 10$;
- in the hard case, $b + d \geq 10$, re-arrange to get $(10a + 10c + 10) + (b + d - 10)$;
- In either case, factor out the 10 to get $10(a+c+\text{carry})+(b+d-10$ if applicable);
- Prove a lemma that $10x$ is written as $x0$ even if $x > 9$ (see Stupid Number Tricks) and use this to justify writing down $10(a + c + \text{carry})$ next to $b + d$ or $b + d - 10$ as applicable.

Whether or not they can do this in groups, you need to get through this as a whole class and possibly even make a handout of it afterwards.

2.3.4 A base four lesson

This one is optional. Advantages are that it emphasizes how to teach a regroup-and-carry approach to adding, which nicely complements the difficult algebraic derivation they have gone through. The manipulatives involved are ones they might actually have to use in classrooms. Drawbacks are that it will be a little vague what the goals are and whether they have been met, and that some students tend to spend oodles of time making this kind of assignment pretty (using colored pencils, graph paper, etc.) which will not enhance their absorption of mathematical content.

2.3.5 Big and little

Some of the content having to do with large numbers and exponentiation has been moved to Math 107, but a basic understanding of scientific notation and exposure to ideas of estimation remain in Math 105.

Problem 1 is there solely for exposure. They need to see that the reality of a billion is different from what they might guess as a shot in the dark. Those who think the most important aspect of mathematical preparation of teachers is along the lines of Paulos' "Innumeracy" will be disappointed that there aren't more worksheets like this one.

Problem 2 is meant to convey one specific idea: that it is easier to come up with valid estimates for quantities for which we have an observational basis for comparison, e.g., rate of hair growth per month rather than per second, and that often we may effectively estimate other quantities by breaking them down into multiplicative factors of the kind we can more easily estimate. Blades of grass per square inch, for example, is easier to picture mentally and count than blades per acre or square mile, and the size of the oval might be easier to estimate by estimating walking time and walking speed and multiplying. Problem 3 just puts this into words.

Problem 4 illustrates that when adding quantities of different magnitudes, one may ignore the smaller one entirely. Problem 5 points out that this is not true when multiplying! And problem 6 points out that, when a number is weritten in scientific notation, the exponent is a lot more important than the other number (what's it called?).

2.3.6 Negative place value

This could go before the worksheet on scientific notation (since the introduction of scinetific notation necessitates and understanding of negative powers of ten) or could go later, when the curriculum turns to decimals. In fact, thorough exploration of exponentiation is delayed to Math 107.

Here all that is needed is an understanding that $10^{-n} = 1/10^n$ to get the studetns started generalizing to other bases and to the general decimal representation. Students who don't know this fact explicitly may still be able to manufacture a procedural understanding sufficient to generalize decimal notation to other bases, going through the route of fractions, for instance, rather than decimals. It is best to take a pictorial, number line approach to problem 1 (e-f), since the visual depiction of subdivided intervals is a crucial complement to the students' verbal understanding of decimal representation.

Problem 2 makes sense only if the places for both positive and negative place value have already been explicitly identified. Thus you should make sure the discussion of Problem 1 at some point begins indexing the columns to the left of the decimals point as $0, 1, 2, \ldots$ and to the right as $-1, -2, -3, \ldots$. On problem 3, many students want to answer "Because the right way to do it is this, not that." Whether or not a complete and sophisticated answer is obtained here, be certain to point out that the response quoted above does not answer the question. To answer why X is wrong, it is seldom adequate to answer that Y is right: you need to show at what point the reasoning leading to X goes astray.

2.4 Division and fractions

This section of the curriculum is in a way the most straightforward, in that there is less "de-familiarization" to do. With fractions, as with whole number arithemtic, the students' manipulation skills are way ahead of their justification ability and modeling ability. On the other hand, whereas with whole number arithmetic is was difficult to find challenging problems on the basic operations (the most challenging problems were on the more novel number theoretic curriculum), it is easier with fractions to find challenging problems that bear directly on fractions, decimals, percentages and the associated algorithms. Thus this section is rather straightforward.

2.4.1 Ratio and Proportion problems

In problem 1, the main difficulty seems to be seeing that the natural answer to "how much less" is in terms of a proportion of the original amount rather than an additive (subtractive) displacement as the termm "less" seems to imply. Also, the total amount of pizza is not given, nor the number of slices per pizza, which throws many students for a loop. If this is troubling them, encourage them to make up numbers, solve the problem, then make up new numbers, solve again, compare a few such solutions and see what can be said in general.

Problem 2 is a straightforward proportion problem. This is a good point to make sure answers are convincingly delivered, units not forgotten, and so forth.

I am baffled by why problem 3 is hard for a lot of students. They have too many choices as to which substance will be the reference measure and which will be measured in proportion to proportion to the reference. Because of this, some reverse sweeter versus less sweet and do not know how to make the adjustment.

2.4.2 The sense of it

This is the only worksheet to emphasize physical units. On problem 1, try to contrast one story that adds inches to inches with another that adds feet to inches. The moral is, feet plus inches does make sense, but to compute a total, one must choose one unit and convert everything to that unit.

On problem 2, you will find that many students are not familiar with the most common unit in this vein, namely man-hours or person-hours. I think it is worth mentioning that this unit is in common use in the business as a unit of labor.

Problem 3 is meant to get at the unit of "unitless". They will learn more about this in Math 106, but for now what is important is that they see that it is invariant under changes of scale. If I weigh twice as much as you in pounds, then I weigh twice as much as you in kilograms.

Problem 4 is there mainly because this division of units tends to come up a lot in word problems and makes some students uneasy as to whether it is automatically justified. Here, we want to consider a simple such problem, solve it in a convincing way, notice that the units do cancel in the expected way (mi / MPH = hrs), and remember this in the future.

Problem 5 should illustrate that different units cannot be meaningfully added or subtracted. Don't let them spend too long on this one!

2.4.3 The cider press and the condominium

These are two problems where fractions must be understood to represent a proportion of a whole.

Students who are struggling with problem 1 need to be asked what happens when an apple is run through twice – how much of the possible juice will have been squeezed out of it? Once they understand that, it is simple to continue and compute the remaining juice after successive squeezes to be the powers of 2/3: $2/3, 4/9, 8/27, 16/81, \ldots$ and since $16/81 < 1/5$, this is sufficient to give the answer. Optional: to proceed along the paradigm, have them state in algebraic notation that the amount of juice left after n times through is $(2/3)^n$ of the original juice.

Problem 2 requires them either to find the hidden equation

$$(2/3)M = (3/5)W$$

or to see that the relevant way to compare 2/3 and 3/5 is to form a common numerator. If we rewrite the fractions as 6/9 and 6/10, then we can marry off six out of every nine men to six out of every ten women. Many students instead spend time writing the fractions as 10/15 and 9/15 and then don't know where to go from there, or spend time worrying about how many men or women there are in total. We usually tell students having the latter problem just to pick total numbers as they please, solve the problem, and then try to determine whether the answer depends on the numbers they put in. In this particular problem numbers of men and women can't be assigned independently but there is one degree of freedom in choosing a scale, so either the total number of men or the total number of women may be arbitrarily assigned. It is amazing how many students prefer to work with 100 men, even though this immediately forces 662/3 of them to be married.

2.4.4 Law and order

This worksheet deals with the number-line representation of fractions. I think it is a weak worksheet because the number-line representation is predicated on a linear unit, so it makes sense to add, subtract and compare these but not to multiply or divide them, at least if the answer is going to be placed back no the number line. It made the final cut primarily because, with this being the calculator era, there are a lot of students who have no idea of the relative sizes of anything not in decimal form. If you choose to do this worksheet, you should point out the abuse of units.

2.4.5 Lynna's arithmetic

Lynna is dividing 17 into 544 and correctly computing a quotient of 32. The justification is straightforward and there isn't much to write up. The hardest part is figuring out what she's doing. On this worksheet, the learining is in the process. Once a student has figured out what problem Lynna is doing, the student is much better positioned to understand the conventional long division algorithm. The conventional long division algorithm never appears in Math 105, though perhaps after this worksheet a brief discussion would be in order.

2.4.6 Visual operations

This worksheet is a good antidote to the abuse of units in Law and Order. It deals both with number-line representation and length representation. The point is that $x+y$ and $x-y$ can be drawn even when reference is permitted only to lengths already drawn. The ratio $x \div y$ can be stated as a number, though cannot be drawn since the is no reference unit. The product $x \cdot y$ cannot be drawn as a length but can be drawn as an area. On the number line, even less is possible without a zero reference point.

2.4.7 Fractions!

Problem 1: Students don't have too much trouble coming up with pictures of eighths subdivided into 24^{th}s. I find the arrangement of squares in a 3×8 rectangle a particularly useful diagram, since it is obvious when drawn this way that each eighth is a strip of three twenty-fourths and that $x/8 = (3x)/24$. But some will draw a line of eight intervals, each subdivided in three equal parts, or a pie-shaped chart, all of which are effective diagrams.

Problem 2: They all know how to comp are by cross-multiplying, so for some (not all!) the statement is easy. You need to demand a good proof here. The easiest route to this is to find a common denominator algebraically. Don't let them waste too much time trying algebraically to find the least common denominator.

Problems 3 and 5: These make sure they understand the why as well as the how when it comes to multiplication of mixed fractions via the distributive law. In both cases, the problem forces them not to convert to improper fractions. Those who can do #5 will enjoy explaining this seemingly daunting feat. Many get the idea of using the distributive law on a suitable truncation of #3, but then don't realize they can multiply $1\,1/2$ by 8604 in their heads.

Problem 4: This one is meant simply to require an explanation for the ubiquitout "common denominator" method of adding fractions. They must also address conversion between mixed and improper fractions, since adding the integer and fractional parts separately might end up with a mixed fraction with fractional part greater than 1.

Problems 6 and 7: Logically problem 7 comes before problem 6, but it works better to have them attack #6 and put it on hold for #7 if they get stuck. The idea is for them to understand the "flip and multiply" algorithm for division of fractions. It is easiest when dividing an integer by a unit fraction, as in problem 7, since one can argue that $1/n$ fits n times into 1, thus kn times into k. They then need to fill in a few steps in order to give a convincing explanation for #6.

Problem 8: This takes them through one more step of the Paradigm, expressing in algebraic notation a procedure they should understand pretty thoroughly by now.

2.4.8 Decimals

The most basic problem on this worksheet is problem 2. We assume they will not have too much trouble with it, but have placed it there so they will have a more complete record of what they know how to do.

Problem 1 makes them gather some data for an assault on the question of which decimals repeat. The only one with a long period is 13/23, which is the one they are meant to give up on. Every one other than this will terminate or repeat soon enough for them to see it repeats. Some students will probably do the manual computation on 13/23 long enough to see that it repeats. This should make the discussion interesting, as you can challenge whether they did this long computation correctly (look over their shoulders from time to time to make sure they do!) and try to elicit from them some kind of argument that it must eventually repeat. Perhaps you will save this part of the discussion until problem 4.

Problem 3 can be answered succinctly: fractions whose denominators have prime factorizations with only 2's and 5's. They should be able to formulate this eventually. The proof may be within their grasp as well: from problem 2 they know that the decimal for x terminates if and only if $x = y/10^n$ for some integers y and n. If $x = p/q$ in lowest terms, that means that $q|10^n$ for some n; but 10^n factors as 2^n5^n, so this is equivalent to q containing only 2 and 5 as prime factors.

Problem 4 challenges them to prove that all fractions must terminate or repeat as decimals. The idea is for them to see that the long division algorithm must get caught in a loop. Based on their data gathering in problem 1 and their further efforts on 5/17, they should have a glimmer of this already. If they don't yet get it, try working collectively on 351/487 and express doubt that it ever repeats. Perhaps even say that you go on forever getting different remainders. Surely someone will point out that you get a rerun eventually. Then you can say, "Yes I see you have to get a remainder eventually that has come up before, but most likely you get a completely different sequence after that."

Problem 5 is really a wimpy comromise between doing a worksheet on how to convert repeating decimals to fractions and skipping the topic altogether. As it stands it is not sufficient to teach students how to do this conversion and should probably be skipped. Perhaps the result could be discussed as something they will be accountable for knowing but will not be justified in class. Even better, there's always someone who wants to do a brief presentation for extra credit...

2.4.9 Percentages

The most natural way to answer problem 1 is to pick a variable P to be the price and to write the discounted price as $P - P \cdot \frac{35}{100}$. You can of course reduce this to $P \cdot (1 - .35)$ or even $(.65)P$, which is the least number of operations. The point here is for them to see that % means times 1/100 and that discounting by some percentage means subtracting a portion of the original, or multiplying by (1 - something). One subtlety it is good to convey is that percentages such as 35% are unitless. In some sense, they are in units of "percents", which are just 1/100's, but these are still just numbers, in other words, percents themselves are unitless.

Problem 2 is a straightforward problem in algebraic expression. We simply want them to find a description equivalent to "multiply by $(1 + x/100)$.

2.5 Outtakes

We include here ten worksheets that have been used at one time or another in this course and its precursors. Take a look at them now so you will know what is available should you want to add an extra worksheet in a topic area or substitute one. Three of them (Rat Poison, Glacks and Glocks, and Sets: extra problems) are sequels to existing worksheets, so if any of these activities (Poison, Glicks and Glucks, or Sets) seems to be going well and you want more material, it's there. Quarto Publishing, Frame Up and Four Cuts are general problem-solving activities, Whiz Kids is on place value, Septobasiland is on base seven, What *are* they doing is on deciphering children's arithmetic and 1/3 and 2/5 is on different meanings for fractions.

2.5.1 Rat poison

To win you have to be as sneaky as a rat!

RAT POISON has the same rules as POISON except that in RAT POISON you may take 1, 3, or 4 counters; you *cannot* take 2 counters. The team that takes the last counter is still the loser.

<u>Questions:</u>

1. 1. If RAT POISON starts with 10 counters, is there a winning strategy? What is it?

2. 2. Can you answer question 1 when the game starts with 20 counters? With any larger number of counters?

2.5.2 Quarto publishing

In Quarto publishing, four book pages are printed on each side of the large sheets of paper actually used. One side of a quarto sheet is shown in the illustration. Once the sheets are printed on both sides, they are stacked up, and the stack is folded twice: first from top to bottom on the dashed horizontal line, and then from side to side on the dashed vertical line. The tops of the pages are cut apart. Thus every quarto page produces eight book pages; see the illustration.

1. How many quarto sheets are needed for a book of 1240 pages?

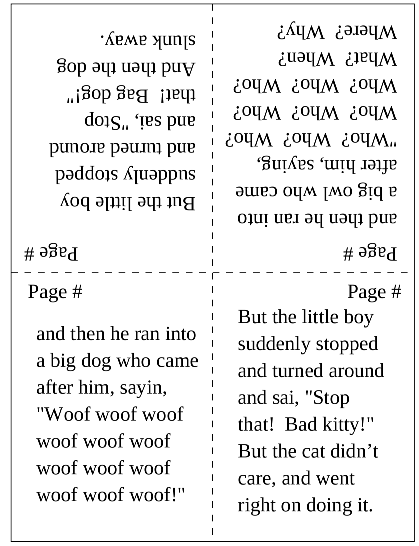
2. When the pages of this book are stacked and ready to be folded, what does the top sheet look like?

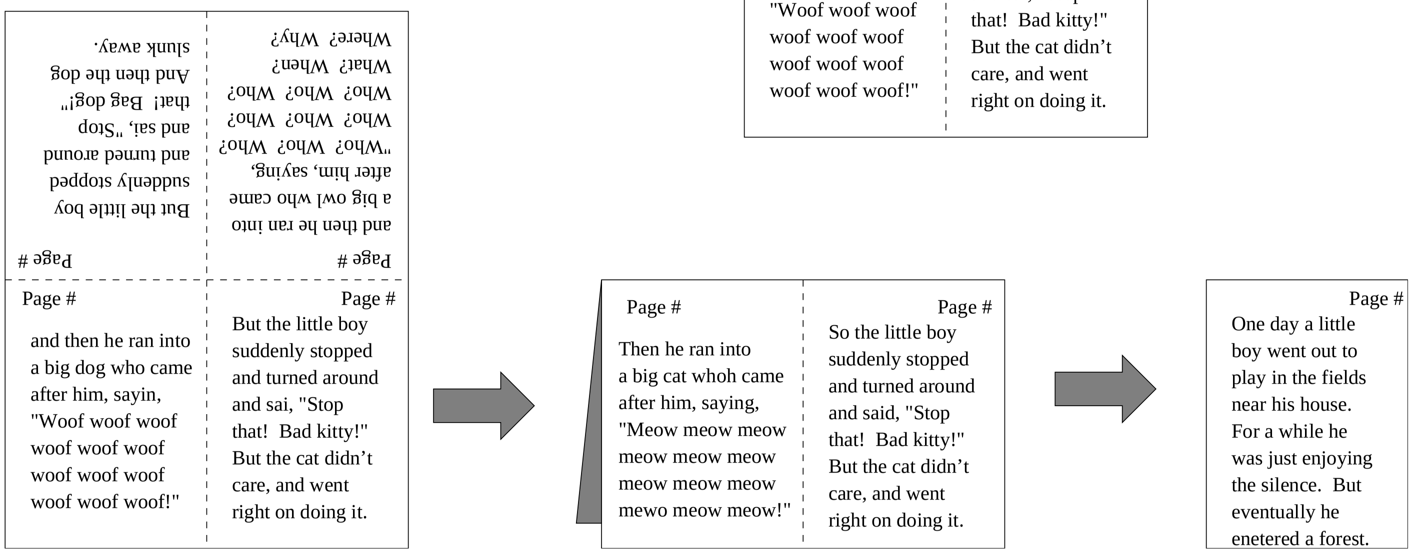
3. Draw a picture that shows where on the sheet the page numbers appear, and which are upside down.

4. What other page numbers on on the same quarto sheet as page 13?

At the right is a single quarto page.

The illustration below shows the three steps of folding.





2.5.3 A frame-up

I painted a picture that is 4 inches wider than it is high. If I put a 3-inch frame all the way around my picture, the area increases by 48 square inches. What are the dimensions of my picture?

2.5.4 Four cuts

If you were to take a block of cheese and make a cut all the way through, you would end up with two pieces. If you were then to make another cut without moving either of those pieces, you should be able to do it in such a way that you end up with four pieces. What is the maximum number of pieces you can have after three cuts (remember you can't move any of the pieces)? What about after four cuts? You should be ready to explain how to make the cuts so as to obtain the maximum possible number of pieces.

2.5.5 Glacks and Glocks

Read the following statement and indicate TRUE, FALSE, or CAN'T TELL for each of the conclusions listed.

1. A **Glack** is an odd number that either is less than 29 or is a divisor of 48.

Statement	True?	False?	Can't tell?
a. All Glack numbers less than 15 are odd.			
b. 14 is a Glack.			
c. All odd numbers less than 21 are Glacks.			
d. 16 is a Glack.			
e. No odd number greater than 27 is a Glack.			
f. Nine Glacks are prime.			

2. All **Glock** numbers are divisible by 2 and are multiples of 13.

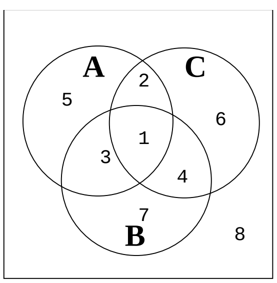
Statement	True?	False?	Can't tell?
a. 169 is a Glock.			
b. No prime number is a Glock.			
c. 52 is a Glock.			
d. All Glocks are even.			
e. All Glocks are divisible by 26.			
f. All numbers divisible by 26 are Glocks.			

2.5.6 Sets: extra problems

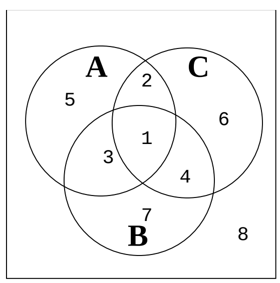
1. In a special ed classroom where there are to be 20 children, 9 of the children will need new desks high enough to accommodate wheelchairs, and seven of the students in the classroom will need desks fitted with electrical power strips for special equipment. If eight of the children can use already existing (lower) desks without power strips, how many of the new, higher desks need to be fitted with power strips?

2. In a given class of 25 students, there are a total of 11 boys, and a total of 7 left-handed children. If 9 of the boys are right-handed, how many of the girls are right-handed?

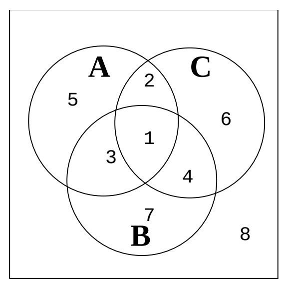
3. For each Venn diagram below, shade the numbered regions indicated, and find a way to express the region in terms of A, B and C using unions, intersections, and complements.



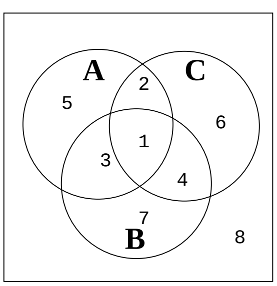
(a) 1, 2, 3



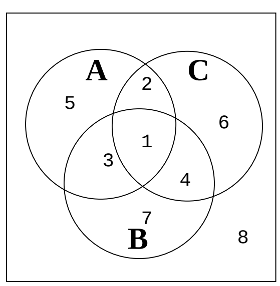
(b) 1, 2



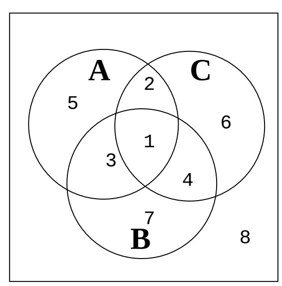
(c) 2, 3, 4



(d) 1, 2, 3, 4



(e) 5, 6, 7



(f) 1, 8

2.5.7 Whiz kids

1. Fletcher, age 10, can immediately tell you the square of any two-digit number ending in 5. He does NOT have them memorized. How does he do it?

2. In order to stay a step ahead of Fletcher, his mother Diana has found a new calculating trick. If you ask her to multiply 74 by 76, she will pause just a second or two and then tell you the answer. How does she do it? She can do the same for 24×26, or 44×46.

2.5.8 Driving in Septobasiland

The king of Septobasiland has proclaimed that the digits 7, 8 and 9 are never to be used in Septobasiland upon pain of death. As a result, the little wheels on the odometers of all the cars in Septobasiland have just the digits 0, 1, 2, 3, 4, 5, 6 on them. Thus, if the odometer registers 0006 and you drive one more mile, the odometer will register 0010, and if the odometer registers 0016 and you drive one more mile, it will register 0020.

1. If the odometer registers 0066 and you drive one more mile, what will the odometer register?

2. If the odometer registers 0325, how many miles has the car gone?

3. How many miles will the car have gone when the odometer turns over to all zeroes again?

4. After a car has gone nine hundred miles, what will its odometer register?

5. You and your companions go on a trip in Septobasiland, traveling together in two cars. The first car, which is new, starts with its odometer reading 0000, and the second starts with the odometer reading 1435. At the end of the trip, the odometer of the first car registers 0324. What will the odometer of the second car register?

6. Suppose again that the odometers of the two cars register 0000 and 1435 at the beginning of the trip. But this time at the end of the trip the odometer of the second car registers 2153. What will the odometer of the first car register?

2.5.9 What *are* they doing?

For each of the following students, try to figure out how they are computing. Then, try their algorithm on the given problems and describe why their algorithm works.

Adrian

$$\begin{array}{rrrr} & 5 & 2 & 8 \\ - & 2 & 8 & 3 \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{rrrr} & 5 & 12 & 8 \\ - & 3 & 8 & 3 \\ \hline & 2 & 4 & 5 \end{array}$$

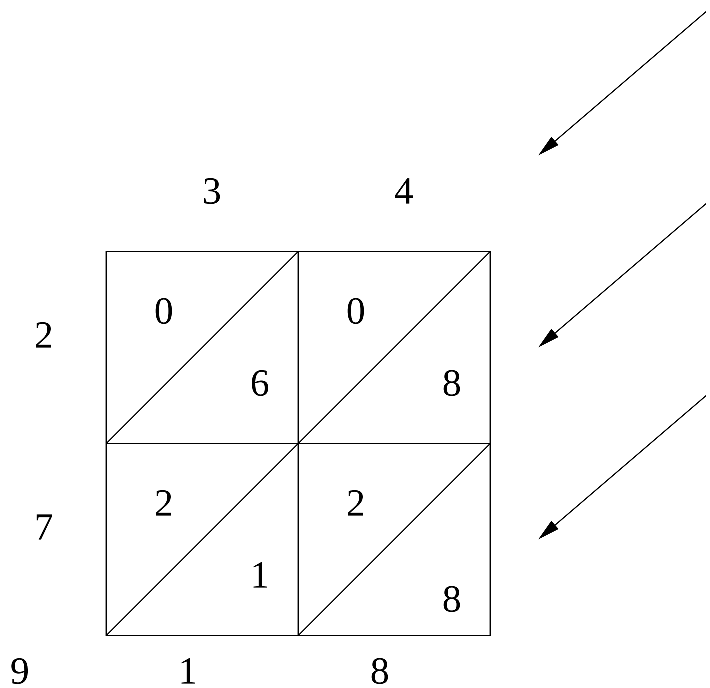
$$\begin{array}{rrrrr} & 2 & 0 & 0 & 3 \\ - & & 8 & 9 & 6 \\ \hline \end{array} \quad \text{becomes} \quad \begin{array}{rrrrr} & 2 & 10 & 10 & 3 \\ - & 1 & 9 & 10 & 6 \\ \hline & 1 & 1 & 0 & 7 \end{array}$$

Now use Adrian's algorithm to compute:

(a) $\begin{array}{rrrr} & 8 & 1 & 2 \\ - & 4 & 5 & 9 \\ \hline \end{array}$ (b) $\begin{array}{rrrrr} & 6 & 2 & 2 & 1 \\ - & 4 & 2 & 2 & 7 \\ \hline \end{array}$ (c) $\begin{array}{rrr} & 4 & 2_{five} \\ - & 1 & 3_{five} \\ \hline \end{array}$

Bill

27×34 becomes thus, $27 \times 34 = 918$.



Now compute the following products using this algorithm:

(a) 38×74 (b) 125×35

2.5.10 1/3 and 2/5

The following problems all involve the fractions $\frac{1}{3}$ and $\frac{2}{5}$; however, each has different answer. Explain what different roles the fractions are playing (use diagrams when appropriate) and what fraction operations are represented in the stories.

1. In your fourth grade class, $\frac{1}{3}$ of the students voted in the class elections. $\frac{2}{5}$ of the votes in your class were for Tammy. What proportion of the vote cast were for Tammy?
2. In the same class, $\frac{1}{3}$ of the boys get free lunch and on $\frac{2}{5}$ of the days, the only vegetable in the free lunch is catsup. What proportion of the lunches (at least) have catsup as the vegetable?
3. In the same class, $\frac{1}{3}$ of the students have a TI 82 and $\frac{2}{5}$ of them have a TI 86. Nobody has both, and nobody has any other graphing calculator. What proportion of the class has a graphing calculator?
4. In your class, $\frac{1}{3}$ of the students remembered to turn in their field trip permission slips. In the other fourth grade class, $\frac{2}{5}$ returned theirs. If both classes have the same number of students, what proportion of the fourth graders remembered to turn in their permission slips?
5. In chess club, 2 out of the 5 boys wear pocket protectors, and 1 out the 3 girls does. What is the proportion of chess club members that wear pocket protectors?

3 How to teach in a cooperative classroom

This section contains a great deal of information and advice regarding how to manage a cooperative classroom, on all levels. In fact, the amount of reading is rather overwhelming, so we suggest that at first you read just the first two sections, on the basics and on the beginning of the semester. The remaining sections, on small group dynamics, large group dynamics, and organizing yourself, can then be used as reference either as day one nears, or once the class has met a couple of times and you have some more specific questions. As with the rest of this guide, the recommended procedure is to browse first, then come back and zero in on the topics of greatest interest to you.

This guide was written for the Math 130-131-132 sequence at the University of Wisconsin. There are a few references that are specific to that situation, though the bulk of what is in here is general advice for anyone teaching in a group-learning, socratic classroom.

DISCLAIMER

In training one's self or others to teach a Socratic/Cooperative style class (henceforth SC), it seems no amount of preparation or advice can substitute for a certain kind of on-the-job training. The essential ingredients of on-the-job training are criticism and imitation. Experienced instructors visit the classrooms of new instructors, taking extensive notes on what they see: what went right, what went wrong, what might have worked better, and so on. New instructors also visit the classrooms of experienced instructors, taking equally careful notes on what went well or poorly, what might have worked better, and on what techniques they see used that they would like to use themselves. Some of this type of work can be done beforehand, via visits in the previous semester or videotaped classes, but it tends to be more valuable when it comes after the new instructor has had a chance to try teaching a class or two first. Currently in 130-131-132 we are trying for one visit each direction in the first two weeks, another in the second two weeks, plus at least one more during the term.

If you are not going to adhere to a plan as outlined above, then **don't expect the notes that follow to do much for you**. You can't learn to play the piano by reading books about it or by talking about technique with Rubenstein, so don't try to learn a significant new teaching skill without practice and guidance either. On the plus side, all the interested instructors that have taught SC classes here (admittedly a self-selected sample) have become pretty good at it, so you can be pretty confident that you'll be able to step right in and teach effectively this way even if you're not a virtuoso.

3.1 Basics

3.1.1 The philosophy

Unless you've been on Mars, you're probably aware of the age-old battle between "skills" advocates and "process" advocates. I'm from the "process" camp but I hope to avoid a lot of the partisanship that is prevalent in discussions of pedagogy and stick mainly to points both sides agree on. We want students to come away from (lower level) math classes with certain skills and attitudes. In particular we want them to reason and prove, to translate between words and symbols, to perform algebraic manipulations correctly and with understanding of the justification, and to solve problems other than clones of problems they have been shown how to do. Whether or not you believe specific skills and knowledge to be paramount (arithmetic of negative numbers, solution of quadratic equations, propositional logic, summation of common series), you undoubtedly want them to know these things in the ways mentioned above: verbally and symbolically, with justification and proof, well enough to apply to new situations.

The tenet underlying SC classes is that students need to learn that they can think for themselves, and that they will be able to learn properly if and only if they are forced or enticed to continue thinking things through on their own terms. Realize please that this does not apply to students who already have this skill. I don't think we need SC classes at the advanced undergraduate level, and they become increasingly inefficient at higher levels. If college admissions standards (or high school graduation standards) were what we'd like them to be, we wouldn't need SC classes in college at all. The students who benefit from SC classes are ones I would term remedial: those taking pre-calculus, and those in the Ed program who are required to take what is essentially junior high school mathematics.

Our philosophy with these students is to do anything we can to get them to think and speak mathematics, and then once we have them going, to exact from them a quality of mathematical reasoning that is higher than anything ever asked of them in a traditional course, thus ensuring that they learn the course content in a useful and permanent way. The meta-skill we emphasize is for the students to **know when they know something**, versus when they are just guessing or are confused. In the time-span of the course or sequence of courses we move from a "process is everything, choice of content matters little" approach to a stage where we cover the traditional content and expect students to focus on these topics and skills while applying the critical thinking they have learned in the first phase. The first phase is the harder phase for most instructors, since we have to be psychologists — and sometimes mind-readers — in addition to being mathematicians. These notes concentrate on this phase, though they apply to the other as well.

Perhaps the most controversial part of this approach is our unwillingness to tell students the answer. Some skills can only be acquired this way, and one can be overly dogmatic on this point. The basis for this is that much of this material is accessible to them, with a little help from us, and that our habitually providing answers will cause their problem-solving ability to atrophy, though it may increase their rate of skill acquisition (though we argue probably it won't). Thus we make every attempt when discussing an attempted solution in class not to tip off whether it is right or wrong until the whole class has had a chance to criticize it or register comprehension and agreement. Depending on the context, we do or do not in the end provide model solutions.

3.1.2 Typical classroom mechanics

A usual 50 minute class consists of two kinds of time: some time when the students are working in groups of 3 or 4 on a problem or worksheet (set of problems) and some time when the entire class is discussing the problem set. Some instructors enjoy keeping to a familiar rhythm, spending the initial 25 to 30 minutes each day working in small groups, then spending the latter part of the lesson in a large group discussion detailing what the various small groups found, where they got stuck, and so forth. Often there are parts of the worksheet that are not covered in this phase; some of these are assigned for homework and some are discarded. Other instructors prefer to go back and forth a little more, starting in a small group, then convening the large group when most small groups are done with the first problem, discussing it a bit, then remanding the class into small groups, and so on. When a class meets for 75 minutes twice a week instead of 50 minutes three times a week, it is usually necessary to go back and forth this way, and it is also often convenient to continue a large group discussion from the end of one class at the beginning of the next.

During the small group working time, the instructor circulates among the groups, answering questions when necessary, doling out encouragement when necessary, challenging the students to justify what they claim to have figured out or to explain their half-baked ideas. Often the mere presence of the instructor encourages a renewed attack on a problem.

The large group discussion begins with the students explaining what they have done. Other students are required to listen carefully and to register agreement, disagreement or incomprehension. Once the explanation is comprehensible, those in disagreement are encouraged to justify their disagreement, with the aim of a resolution or synthesis. The instructor plays moderator as long as fruitful ideas are being produced, but slips into the role of leader when needed. For example, if no one challenges something wrong, or if there is a disagreement but it is too inarticulate to produce a good synthesis, then the instructor may rephrase what has been said so as to sharpen the contradiction or caricature a wrong approach, in a way that forces a light to dawn for at least some students.

Note also that in order for SC discussions to work, the class size must be limited, to no more than twenty or thirty maximum. The point is to have the class participate as a whole in the same conversation, but if you think back to social gatherings you've attended, even if everyone knows each other, it is difficult to keep everyone involved in the same conversation once you get more than a dozen or so people involved.

3.1.3 Highly recommended procedures

Before getting into specifics of classroom technique, here are a few simple procedures that make a large difference.

- *Nametags.* Have the students wear nametags each day until you know their names (in my case 2 or 3 weeks). When calling on students, call on them by name and in general attempt to use their names frequently. This serves two purposes. First, by attempting to learn their names, you create a separate mental category for each student, which helps you pay attention to how each student is doing and to their individual needs. Secondly, there seems to be a psychological advantage to students hearing their names. Coming around to a small group and asking "Amy, can you tell me what progress your group has made?" or asking in a large group discussion whether "Sam's objection to Cindy's idea" holds water elicits more of a response than the same questions without the names identified. Somehow, students are more prone to take their own beliefs seriously when names are attached.

- *Randomize groups.* I usually assign groups using playing cards randomly dealt: all the aces for a group, the twos form another, and so on. I re-form groups twice or thrice during the semester. Preventing students from choosing their buddies for a group helps them form connections, subject their ideas to the intellectual marketplace, and treat all the others in their groups fairly. The playing cards themselves lend an air of intrigue as students await the results of the lottery. If you decide not to use playing cards to change groups, you can assign the students the task of creating new groups such that no two "new" teammates have worked with each other before. (This can be done easily once, and with considerably more effort a second time.)

- *Get enough sleep.* Alertness is required on the part of the instructor. You'd be surprised what a difference this makes. You can fake it when you're lecturing, but try playing moderator when you can't concentrate or respond quickly, and you'll see what I mean.

- *Start the semester with a bang.* That is, don't spend the first day on administrative stuff and the second on some kind of review. Jump into an absorbing problem on day 1, preferably a tried-and-true chestnut, and fill in the administrative details later, when they've gotten the idea of what the class will be about. The tone of the first day's discussion sets an example that's hard to erase, so make sure it is as lively as possible.

- *Minimize the time you spend talking.* In particular, never talk for more than five minutes at a time — monologues shift students back into passive learning mode, and monologues of announcements tune students out. Of course, you may have to do a lot of talking in leading large-group discussions, but if you're talking by yourself for more than five minutes at a time without at least one student speaking up, you're not doing SC — you're lecturing.

- *Be yourself* at least up to a point. If you're the goofy type, be goofy; if you're serious and intense, let them feel the intensity; if you're understated and direct, be that way. You are on stage, and want to use your charisma, but don't pretend to be someone you're not.

3.2 Beginning the semester

3.2.1 The first day

In many ways, the first day can be one of the most important of the semester. It sets tone and precedent for the days that will follow, and lets the students know what to expect. Students will be anxious to know how their grades will be determined, and perhaps anxious in general to be in a math classroom. Your job that first day is to sell them on the class, to help them start to get comfortable. This job has two main parts: dealing quickly with administrative details and getting the students engaged in problem solving. Suggestions follow to help you with both of these parts.

You should already know all the administrative details about the class when you walk in the door the first day, and the quickest fair treatment for communicating these details is in the form of a handout. You may want to write your name and the class name, number, section, etc. on the blackboard before class starts, to make sure people are in the right place, but don't spend time in class writing all the details on the board. Rather, have them collected on a handout, so that you can skim the most important details (components of the grade, exam dates) and leave the students to peruse the rest at their leisure, *outside of class*.

The coursepack contains a psat syllabus which can serve as a sample handout – in fact you need to remove this and insert your own, or the students will be mightily confused!

Different instructors will have different ways of building the atmosphere they desire. Some instructors do best by jumping right into the first activity ("Poison") in Unit One. They then reserve a fraction of the second day's class to discuss the nature of the course, and why we do things the way we do, using the Poison lesson as an example. Other instructors are more comfortable devoting some time to "warm fuzzy" introductory activities. They will use a little class time to introduce everyone. To take care of any reticence on students' part to talk about themselves, try having everyone pair up with someone s/he doesn't know, and spend five minutes getting to know one another. (This also gets them used to talking in class!) Then go around the room having each student introduce his/her partner. Some instructors also have students fill out index cards with a few facts about the student (name, major, hometown, feelings about math, something unusual about the person, and perhaps suggested office hours); these can serve as backup in case someone forgets his/her partner's name. You can also add a couple of icebreakers such as "favorite cartoon character" and "least favorite vegetable".

If you spend time on introductions, then you will want to choose a short warm-up problem-solving activity that can be completed in the remaining time. Here's one (Pascal's Triangle) that has worked for us. First, make sure everyone has a nametag, and then put them into small groups. Put up on the board, or on a handout if you prefer, the first five or six rows:

```
        1
       1 1
      1 2 1
     1 3 3 1
    1 4 6 4 1
  1 5 10 10 5 1
      .  .  .
```

and ask the following questions: Can you find the rule used to generate Pascal's Triangle? What do the next two rows look like? What patterns do you recognize in it?

The groups will discover the rule quickly enough, and the last question is open-ended enough to allow discussion to continue until the bell rings.

3.2.2 Encouraging Good Habits from Day One

Encourage good habits early, to set precedents for the entire semester: Encourage people to talk, to offer ideas without worrying about whether they're "right" or not, to talk to each other (instead of to you) as much as possible, to write on the blackboard, to be precise about what they mean, and to get used to giving explanations and justifications to back up things they really believe. One way to encourage the latter three things is to "play dumb" and keep saying you're not sure what they mean — even if you do — as long as you think someone else in the class might not (which is usually true). Of course, you should give a disclaimer the first day of class explaining that you're not going to be giving them straight answers or telling them whether or not a solution is correct (though you should never let the book be definitively closed on a subject without making sure that there is closure, and the everyone understands that the solution on which the class has agreed is indeed correct). Part of this disclaimer will include the fact that you reserve the right to "play dumb" as long as you think something might not be clear to someone in the class.

Another habit to encourage from the beginning is to keep a running list of different problem solving strategies and techniques they have used. By the time you finish Unit One, they should already have a half dozen or more, and it might be worth taking a few minutes in class at that point to have them compare lists. Chapter 2 mentions another discussion to have around the end of Unit One, regarding group dynamics.

3.2.3 Resistance and "Why are we studying this?"

One of the questions which you as a teacher in the Math 130 sequence will hear over and over again is "Why are we studying this?" Why are we studying these topics which we will never be teaching? Why are we taking this course? These questions deserve an answer, and again this is something best discussed openly and early (although no doubt the question will recur). Among other things, elementary school teachers must lay the groundwork for their students' later experiences. In math, arithmetic is the foundation of algebra, and teachers need to know what's coming up in the years ahead, so that they can teach in a way that will later allow their students to look back and see the connections. Mathematics is all connected, and the sequence of K-12 math classes students take should be a natural progression, not an unrelated set of topics. Teachers also have to be able to anticipate precocious questions. Equally importantly, they must be comfortable and familiar enough with every subject they teach that their students will not all be turned off. Many, if not most, of the students in the Math 130 sequence have had bad experiences with math in the past, and very often this can be traced to a teacher or teachers who passed on their apathy or disinterest for the subject.

These past experiences will, as noted above and to varying extents, lead to resistance by the students to the way Math 130 forces them to shoulder responsibility for their mathematics. Here again, you as their instructor should bring this subject out in the open, early and as often as needed (although it can, at times, threaten to take up a lot of in-class time — try not to let this happen). Point out the first day that the reason for their doing this is not only to provide a model for them to use later is getting their own students to think for themselves, but that someday soon they will have to be the mathematical authority for a classroom full of inquisitive children, and that before they can be someone else's mathematical authority, they must be confident in being their own authority. There will invariably be one or two students in a section who do not "buy into" this philosophy, and in the end there may be nothing more you can do for them but to tell them that the class is designed this way because people who have had a lot of experience teaching math believe in it, and they'll have to play along for now and see what results come of it in the end.

Some students, especially the poorer (for want of a better term) ones, will be vocally and continually concerned for their grades. The most you can do is try, from the beginning, to make grade assignment rules as explicit as possible (but don't feel any obligation to explain down to the point level your grading of particular reports, any more than a composition teacher would), and try to reassure them. Don't let their angst transfer to you.

3.3 Class Composition and Small Group Dynamics

3.3.1 Doing the rounds

When you first assign a problem to work on in small groups, there may not be much you have to do. No one is stuck yet; no one needs your help. There is a lot you can accomplish in this time. First, you can quickly take attendance group by group — after you know the names this takes less than half a minute. After this, you will want to quickly "do the rounds". Visit each group once just to check that they have gotten down to work. On the first round, look for any trouble with the wording of the problem that may be holding people up. If it's part of their job to decipher it, encourage them to do so. If it's a mistake, or if you need to supply a definition, then make a quick announcement. On your second round you can linger longer. This is a good time to make a mental note of which groups are going faster than the others. It helps, during the subsequent large group discussion, to have a good idea of who has gotten how far. If a group has quickly and incorrectly or incompletely answered a problem and gone on to another, this is a good time to ask (innocently) for one of them to summarize for you what they found. The correct question on your part can cause them to re-examine what they've done without feeling that you've invalidated their answer (point of philosophy: you want them to be able to criticize their own work, realizing that mathematics will determine whether they are right, and that what they discover about this cannot be overruled by the teacher).

Example: A group has used an incorrect manipulation: $(a+b)^2 = a^2 + b^2$. You can "explain for those who are rusty on this", that this means "when you compute, e.g., $(7+3)^2$, in case you can't tell at a glance that $7 + 3 = 10$, squared $= 100$, you can use this nifty rule, getting $7^2 = 49$, plus $3^2 = 9$, which by the laws of algebra must also give you 100." If this doesn't elicit an objection, you can line up the sum as you speak:

$$\begin{array}{cr} & 49 \\ + & 9 \\ = & 100 \end{array}$$

When you do get an objection, ask them to pinpoint what went wrong, leading to, "Oh, you mean $(a+b)^2 = a^2 + b^2$ is not a universal rule of algebra?"

A reminder: Lurk early, lurk often. While students are in small groups, eavesdrop constantly. The students should get used to having you listen in to their conversations. This keeps you abreast of both (a) how far the class in general is getting and (b) which groups in particular are having difficulties, and where. If one group seems too conscious of your presence, then stand near another group nearby and appear to be watching them when in reality you're paying attention to the discussion in the other group. This part of the class is an important one for you as instructor and (presently) discussion moderator.

3.3.2 "Help, we're stuck"

The twin dangers here are that lazy students will say they're stuck so you do the work for them, while students who are truly stuck will lose morale if they have to sit idly during class. Asking if they have any ideas on what they might try will prove embarrassing if the answer is no. Sometimes I do this anyway. Sometimes I replace the problem with a smaller one: if you knew that $A = 15$ could you do the problem? Can you do the problem if you aren't required to make the number of cows and chickens the same? Sometimes I guess why they're stuck: So the problem is you don't really know the definition of average speed? An example of what might happen here is that they did know this but didn't think of going back to definitions as a way to proceed. Now when they say no that's not the problem, we know the definition of average speed, I say, "Oh, then you must be saying you don't have any way of determining the quantities defining speed, such as the time or the distance." They then say how they will proceed on this and I can smile and leave. In other cases, they are stuck because they don't really understand what's being asked. You can ask them to rephrase it, or ask them how they would check if someone else's answer (here you specify it) was right. It helps to have snooped enough so you have a good guess of where they're stuck. If not, you can ask them but won't always get reliable information.

3.3.3 Getting Groups to Work Together

The pre-service teachers tend to work well together. The pre-calculus population is less accustomed to this, and groups will degenerate into a lot of individuals ignoring each other, or occasionally explaining to each other. You may at times need to tell them explicitly "Janet has found what she thinks is an answer but Steve and Brenda apparently don't understand what she did, so Janet, you're going to have to explain it and see if you can convince Steve and Brenda." However, you should try other things first, before being this explicit. Ask Brenda what her group has found so far, and don't let anyone else answer for her. If she says she's stuck, ask if her whole group is stuck, and if not, tell her you'll come back in 3 minutes and ask her again for a summary of what her group has done. **Make sure groups are sitting in a tight circle**, not a line or a disarrayed cluster.

Sometimes you can give them specific tasks: Jason, finish the calculation you're doing; meanwhile Ann will add up Rob's numbers and will then check to see if they agree with yours; if not, it's up to all of you to judge which method if either was correct and why. If a group really has bad chemistry, change it. I've had students say to me: I just can't work with Judy — she won't listen and hogs the discussion. In that case, put Judy in with someone smarter or more aggressive than she. When re-forming the groups, I usually randomize again, but if there are trouble students, sometimes I stack the deck so that they get put with students who can handle them.

3.3.4 Free Riders

There are always lazy students who are content to let others do the work. If it's just laziness, I don't hesitate to reprimand them explicitly: "Adam — you're just staring into space and letting the other three figure out the problem; if the problems aren't challenging enough, then I can let you work faster in a group by yourself, but judging from your homework that's not the issue." On the other hand, if it's a student with a confidence problem who needs some nurturing, it's probably better to make a note of it and to continue to ask that student to explain what their group has done whenever you come around (you have to ask other students sometimes, or it gets too obvious). Involving the student as much as possible, with questions that are at their level but not patronizing, will often cure this. Also impress upon the others that it's their job to make sure the free rider is keeping up with the group since that person (you've just decided) will be in charge of the first group writeup.

3.3.5 Staying on Task

The less you have to reprimand here, the better. Make sure that when you tell them to get into groups they know exactly what they're supposed to start working on. Make the rounds quickly at the beginning so they don't start chatting, and keep an ear out for it later on. If a group continues to be bad about this, you can watch them, visit them more, chastise them humorously, make sure they don't get grouped together next time, etc., but if the whole class is bad, you should examine what you're doing that promotes doing something other than the math. You could be leaving them in small groups too long, while the slow groups finish. You could be joking with them too much during class time. I do also use reprimands, but sparingly — on day 3 this year I was in an impatient mood so I reprimanded the two groups that took more than a minute to form their groups and start working. It was something like "Sounds interesting whatever you're talking about, but you've got to get started on problem 1 — time's short this week!"

3.3.6 Students who are behind

First, it's a good idea to know what you do for them and what you can't. You can do a lot for these students in office hours, but it's not realistic to be able to spend more than (or even as much as) half an hour a week outside of class with any one student. So if they need help on a regular basis, suggest that they arrange for tutoring. You should find out before the semester what such resources are available. If they're doing passing work, but still underconfident, point out to them that if they continue to work at this level, they'll pass the class. Don't, however, make promises you can't keep about their grades; it's best not to prognosticate about their grades before they've taken an exam.

All that being said, you have to make sure they get the most out of their group work and don't drag down the group (they're as much afraid of this as you are). Here are some ways to build their confidence and make them more likely to participate to advantage. Spend some time around their group and be ready to pounce on those times when the lagging student, "Lenny", comes up with a good idea. Assign credit: if Jocelyn figures out how to do something with Lenny's idea, then it's "Lenny & Jocelyn's method". Try to give them constructive comments on their homework (I try to do this for everyone, but when time does not permit this, concentrate on students who need it). Making sure they are keeping up with their group is delicate — asking them to explain to you where their group is will help if they're not too far lost. Once they are far behind, consider placing them next time with a group of students that talk a lot, don't go all that fast, and are as kind as possible. At worst, you may have to settle for Lenny participating minimally during class and trying to catch up on his own at home.

Select Lenny to give a presentation, either solo or on behalf of his group. On a one-time basis you can invest a little extra time to help him with this to make sure it goes over well (have him practice it on you till he's confident enough to field questions).

Avoid asking Lenny really easy questions. This will make him feel like he must be dumb. These questions are scary in general since the reward for a correct answer is almost zero and the penalty for an incorrect answer is large. At your discretion, you might choose Lenny to answer questions that are not black and white, asked of the class at large: which of these problems was the hardest?, what kinds of thinking did you think were necessary on this problem?, and so on.

3.3.7 Students who are ahead

Having such a student can be a real boon if they are gifted teachers as well. If they have a good feel for how to explain things and help others, they will make your class run more smoothly than you can, on their own. Even in this case, **avoid treating them in front of the class as a reliable source for right answers**. You don't want to create a situation where calling on them is tantamount to telling the class something yourself. It is OK though, to treat them as a reliable source for intelligent commentary.

If a student is obviously head and shoulders above the rest from day 1, you may consider exempting them from the course. That is, if they can do the worksheets on their own then they can probably pass the exemption exam, go on to the next course, and leave you with a more homogeneous class. Recommend that they see the course coordinator for an exemption exam; everyone will benefit from this. Later in the semester this is less of a good idea, though I've done it.

Assuming "Einstein" stays in the class and is not self-policing, you need to keep an eye on Einstein's group to make sure that Einstein is not explaining things to the others before they have a chance to figure it out themselves. Let Einstein explain things at the board in situations where you know there will be some wrong or unclear stuff in the explanation. Make sure though, that you give Einstein as much encouragement for what was right and clear as you would another student. If Einstein is a loner and tends to work fast but not share with the others, that will probably work out fine. You can explicitly designate Einstein as a group of 1 next time, or simply allow a *de facto* group of 1 to form. Sometimes, you can try asking Einstein explicitly to figure out a hint to give the rest of the group as to how to proceed but that won't completely solve the problem for them. It will make Einstein summon up teaching skills that are worthwhile in general, so it's worth a try, but be aware that Einstein may not be capable of this. In any case, don't let students disparage themselves in comparison to Einstein. You can say, "I see, because Einstein solved this problem in 5 minutes and you can't, you're going to give up?"

3.4 Managing Socratic Discussions

This is the hardest part, and the part of teaching SC classes that improves the most with practice. When observing someone else's Socratic discussions, try to imagine what would have happened had they made different choices (told or not told the students something, came up with a good counter-question, decided to pursue or not to pursue a student's line of reasoning).

3.4.1 Dead Ends

When an idea is proposed, the instructor will usually know right away whether it will lead anywhere. If it won't, there is a strong temptation to discourage the students from pursuing the idea. This may be a mistake. Probably the best thing that can happen in a Socratic discussion is a *flaming dead end*, meaning that an incorrect line of reasoning leads to a consequence so patently false that the students are compelled to re-examine the road that got them there. If you see your students headed for one of these, then all you really need to do is encourage them to get there without undue delay. Some things you might want to do are: get them to explicitly reaffirm the wrong assumption, so that they will remember it later and be able to pinpoint it; shut down any further sidetracks that branch off of this one (e.g., "OK, that's a good idea, but first we'll finish pursuing this one"); hasten their demise by keeping the pace brisk, perhaps doing some of the arithmetic for them or providing clarifying paraphrases.

You can influence how flaming a dead end will be by making the issue more concrete: ask them to illustrate their result with actual numbers (e.g., "so if the initial weight was 250 grams, then we see the final weight of $400 - 2w$ comes out to be what? Oh, I see, -100 grams... rather on the light side.").

Perhaps you will need to summarize their findings, juxtaposing two findings that are contradictory, or in the case that they have contradicted some of the given information, you may need to restate the givens by saying, e.g., "so you have now proved that the only whole number between 100 and 500 having no two digits the same and satisfying blah blah blah is 337." To further rub it in, it often works well to insist that you believed their method and there must be some other mistake: "Ah, you've proved that the long division we did was wrong — can anyone find where?" (of course it is actually correct), or "Ah, you've proved that when you use variables with subscripts, the usual method of solving linear equations doesn't work" (if you can trust them to fight back on this one).

Other kinds of dead end are less useful. Perhaps they are following a reasonable line of inquiry but it doesn't get them anywhere: looking for a nonexistent pattern, introducing too many variables, classifying according to an ineffective scheme. A reasonable goal in this case is to get them to figure out that they're stuck. If you tell them (or indicate in any of 1000 nonverbal ways) that their idea won't work, they will learn to look to you for validation of their ideas, whereas if they reach a dead end themselves and consciously decide to look for another approach, they have learned something valuable about problem-solving. That being said, there are ways to reduce the amount of time spent following a dead end. One trick is to decide after hearing a suggested approach whether to follow it immediately or whether to treat it as one of many to be written on the board before the class decides which to follow. If Jenny reports finding a pattern starting $4, 6, 8, 12$ and reports her reasoning as to what is likely to come next, you get to choose between (a) getting the whole class involved in speculating about the next number or (b) writing on the board "Idea:

look for a pattern". The key feature of this trick is that you're not giving anything away. Approach (a) is reasonable in some contexts, where the discussion of pattern promises to have some depth, and more importantly, approach (b) is something you sometimes use when the approach offered is correct. In fact you should make sure to use (b) on occasions where there were multiple interesting approaches but the first one offered happens to be the best: you catalogue every group's approach before asking the class to pick one and follow it.

Another way to expedite matters is to insist that the goals be well defined. Often when a bad approach is put into words, it comes out sounding discouraging: we thought we'd name as many variables as possible and then hope that inspiration struck; we decided that if n was equal to 5 the solution was obvious but we don't know how to do it for any other value. Sometimes mild discouragement doesn't work. I remember a worksheet designed to get them to invent the binary number system by asking them to come up with a scheme for representing all numbers with 1's and 0's. This cost a full day of discussion of the relative merits of various schemes, none of which had anything to do with binary. The instructor was very successful that semester, and in my opinion the investment of days such as that one paid off when students continued to work hard throughout the course because they didn't feel that the instructor was going to provide the answer for them. This takes guts, and doesn't work too well if the instructor conveys a growing uneasiness about the whole project. So if you're not up to following the wind, you're probably better off treating it the same as the situation in the next paragraph.

The least promising dead end is a total lack of ideas. Probably it's best not to convene a discussion at this point but to continue working in small groups where you can ask questions that elicit further work and break the impasse. But suppose a class discussion on a certain problem fizzles out midway. This might be a good time to drop it. If it's not essential that they end up knowing how to do the problem, and they don't have a realistic shot at finishing it for homework, perhaps make it into an extra credit assignment. If it is essential, consider dropping it for now and writing a worksheet for the next class that will lead them to it in more manageable steps. You probably need more time to solve this problem than you have on the spot.

3.4.2 How to Listen

You need to listen to students and they need to listen to each other. Tom Lester once told me of a study showing that the average amount of time between when a teacher asks a student a question and when the teacher prompts the student or gives up on them is 2 seconds. Two seconds is longer than it sounds, but nowhere near long enough to formulate a coherent thought unless you were already thinking it before the question was asked. There may not be anything you can do about the sound-bite trend in TV reporting, but there's a lot you can do about it in your classroom. The first thing to try is waiting. Don't nod yes or no, or say uh huh, or give the student any feedback at all until they have finished saying what they wanted to say. Then wait five or ten more seconds. The odds are that the student will, after pausing for breath, realize that they are not finished and continue. If not, at least the other students will have had a chance to think about what they just heard. If you're uncomfortable with this long a pause, try pacing or holding eye contact with the respondent as if you expect them to continue, or act as if you're trying to digest what they've just told you. In fact often you really will need time to think. If they said something that was wrong in a puzzling way, see if you can figure out what they really meant. Students will only listen to each other if you set an example, so make sure you don't respond without having really heard.

Students are also more likely to listen to each other if they feel that they are responsible for having understood it. In small groups they are more likely to feel this, but at a ratio of twenty or thirty to one, many feel that they can just take notes and sort it out later or not at all. They may also feel they have no right to interrupt since everyone else obviously understands. You can counter this by demonstrating an expectation that each student understands what each other student has said. After one student says something the slightest bit unclear, ask another to repeat it in her own words. This is a good time to pick on students rather than have them raise their hand to volunteer a paraphrase. If student B can't paraphrase what student A said, it's not necessarily student B's fault. Student B can ask student A to clarify if necessary, or ask for volunteers for someone else to clarify. Make sure to go back and find out whether student C's clarification of student A's remark did in fact help student B. After a little experience you'll know better when to go through this routine. If student B simply wasn't listening, they might feel reprimanded, but that's OK. It doesn't really work when the remark was clear in the first place, although it doesn't hurt to get a quick affirmation from the whole class that it is clear so far. The basic standard you are setting is that the discussion involves the whole class and is not a collection of one-on-one dialogues between the teacher and individual students.

3.4.3 Staging

Your expectations of the nature of a class discussion are conveyed nonverbally as much or more than verbally. Most instructors ask the students to rearrange their desks into a large U shape for any but the briefest class discussions. A subtler but important technique is to put as much of the class as possible between you and the respondent. If you call on a student on the left side of the room, walk over to the right side as you're doing so. As the words flow between you and the respondent, the almost physical presence of a stream flowing between the two of you will wash over the students in between. You and the respondent also keep eye contact with the rest of the class this way.

It is often a good idea to get students to come up to the blackboard. Students will give longer monologues at the blackboard, so be prepared to be a more active moderator if the student is losing the rest of the class. Be careful not to make having the right solution a pre-requisite for coming to the board, lest the students stop thinking critically and accept any blackboard demonstration as a surrogate for your telling them something. When a student is at the board, I try to take up a position in the back or on the perimeter of the room. Sometimes I sit in the student's seat. This has the effect of including the rest of the class, as above, and also gives me a new vantage — you'd be surprised at what you see this way.

Other body language to be aware of is whether you are passing judgment on what you hear. Do your eyes flit impatiently with wrong answers? Do you gesture in agreement with right answers? Do you angle your body to the board as if to write down something correct, then pause if it's not what you wanted? If you've chosen to teach SC style classes, it's because you want the students to develop their own judgment, so avoid this kind of tip-off.

A related topic is the use of intentional errors. These are a hit with kids, in a slapstick sort of way, but adult students tend to feel patronized. Instead, I substitute the mischievous lie. If a student tells me the found all five regular polyhedra I may say, "Ah, so you haven't found the other two then?" They can often sense that I'm putting them on, but will still take the bait and try to prove that there aren't any more. On a problem best solved by assigning a variable to a certain quantity, I once told a group of frustrated students that I'd tell them the value of one quantity for free if they could decide which quantity they wanted to know. I planned to lie and tell them it was 10 when it was really 6. In fact I had tried this previously with success: the students figured out that the assumption of 10 led to a contradiction and were able to figure out the unique value that didn't lead to a contradiction. This time it was even more successful. By the time I came back to give them their free question, they had figured out what quantity they wanted to be told, had put in x for this, and had gotten the solution (well ahead of the rest of the class).

3.4.4 Asking the Right Questions

When my teaching is evaluated by my students, they often say that I never answer questions, or answer them with another question. I take it as a compliment even though it isn't meant as one. The most common such interchanges are

> Student: Is this right?
> Instructor: I don't know. Does it sound right to you? Can you elaborate?
>
> Student: What should we do from here?
> Instructor: What do you think? Does anyone have any ideas?
>
> Student: Can we say blah blah blah?
> Instructor: I don't know, can you?

There is a certain amount of this you can get away with, depending on your personality and theirs. If you start sounding like "Eliza"[3], you won't get good results. Instead, try to ask them useful questions related to the specifics of what they've said. In the three above scenarios, try respectively

> Are you asking if your computation is correct, or if it will prove useful?
>
> Is there a problem-solving strategy that you know that might work here?
> Why don't these equations tell you what x is?
>
> If you're wondering whether you can assign the variable z to be the average

[3]The computer program imitating a nondirectional therapist, an early (and crude) approximation to something that could pass the Turing test.

of all the prices, the answer is yes, but you haven't yet said whether we know anything about z.

When observing other classes, this is where you should let your imagination run free. Imagine what questions they might have asked. Your hindsight now will be your foresight tomorrow.

The question "do you understand?" is the most often abused. (Notice that this is virtually the only question in the repertoire of the conventional lecturer and rarely elicits an honest response.) Some better variants are: can you say that in your own words? could you do what John just said with different numbers? do you agree or disagree? in what way is this similar to what so-and-so did? These are all comprehension questions, testing whether the respondent comprehends instead of asking point blank for a Yes/No as to whether the respondent comprehends.

Good questioning can help to reach flaming dead ends. Ask what happens when $x = 5$, or whether their purported method works for all starting data and not just what was given. If a student gives a vague definition, find a borderline case and ask how their definition applies in that case. Try also questions that goad by disingenuity. If their method is more general than they realize, ask how they got lucky enough to try their method on a square rather than a pentagon or hexagon for which it "probably wouldn't have worked".

3.4.5 Order versus Chaos

Ideally your students will be eager to answer your questions and discuss their ideas, but will listen patiently and attentively to each other and to you. If students are not willing to speak up and discuss their ideas, you need to loosen them up. It is a bad sign, for example, if the students are not happily chatting away when you enter the room five minutes before class, and are sitting in silence or whispering. In this case, you have probably done too well at eliminating chaos. Try assigning an activity in small groups where different groups are doing different things and they need to walk across the room to share information with each other. For example, there's a worksheet in Math 112 on infinite series where they approximate numerically some infinite sums and try to form conjectures based on each other's conclusions. Assigning a group project where they have to work outside of class together can tighten the bonds and make people feel more comfortable talking. When leading class discussions, be freer and more willing to follow the students' ideas wherever they lead. Dispensing with hand-raising and having students just call out can quicken the pace.

A classroom that's too chaotic is a problem also. If you have to call the students to attention more than once before they pipe down and listen, or if there is crosstalk during class discussions, you probably ought to do something about it. You can address this explicitly, asking the students to pay attention to you and to each other; you have to be consistent about this or they won't believe you mean it. Indirect methods of dealing with this are, however, usually more effective and should be tried first or at least in parallel. Insist on an orderly formation of desks into a U shape before a class discussion, rather than having them minimally perturb the small-group seating arrangements. It takes an extra minute, but it's worth it. In fact tell them you're pressed for time so they have to rearrange the desks in 30 seconds. A snappy set change will set the tone for what follows. When you observe crosstalk, try to get one of the crosstalkers up to the board to explain something, or to comment on what's just been said. By maximally involving that student in the lesson, you'll eliminate most of the off-task crosstalk, and the on-task crosstalk can probably be lived with. Another chaos reduction technique is to give them a more rigid idea of the structure of each class. Say you're going to spend 16 minutes in groups before convening a class discussion and then stick to it with absurd precision. The more aware they are of the structure of the lesson, the more they will stick to the tasks at hand.

The main point of this section is that you should make a conscious effort to optimize your position on the order-chaos axis, and that increasing order or chaos in the physical arrangements or chronological structure or types of assigned activities can help you change the balance in your class discussions.

3.5 Organization (yours)

SC style classes have more inherent disorganization than traditional classes, which is why you need to pay particular attention to organization.

3.5.1 Your Records

Students, at Wisconsin more than at other places I've been, react extremely positively to the appearance of organization. Probably the pre-service teachers are particularly impressed by this since they are consciously judging you on your pedagogical techniques. **Be a compulsive record-keeper.** I find it very helpful to reserve the 20 minutes after each class for writing a short summary of what happened in class that day. That way, if I've told a student I'd find them an extra-credit assignment, or if I've promised to bring something to class next time or promised that the next class would begin with a discussion of something, I can write this down along with other notes as to what I have in mind for the next class. In an SC style class there is a greater opportunity for unexpected things to happen, and therefore a greater need to write down what did happen.

Keeping date records of the homework you've assigned, both due dates and the date it was assigned, is essential. Whether or not you accept late homework is up to you, but it is certainly better to have students ask solicitously in advance for you to accept their late papers, which you will probably grant, than for students to assume it's OK and be upset if you don't grant them an extension after the fact. They are more likely to do the latter if they get the idea that you yourself don't remember when the homework was due. In fact, if I arrive at class early, I often take the opportunity to write up a reminder of what is due when.

Organization (and the appearance thereof) also helps students perceive you as self-confident, a quality which is crucial for an instructor in such a comparatively free-form classroom. Of course, one goal of the course is for them to assume that self-confidence themselves by the time they leave your classroom.

3.5.2 Grading

The message here is the same as in the previous section. Include in your course packet a clearly defined grading policy, specifying what portions of the grade are from homework, exams, quizzes, projects, group work, attendance, or whatever else you grade on. Give it enough thought so that you remember it easily, and can answer their questions immediately. If historical grade distribution data is available, you would be well advised to stick to it, since this will help to allay fears that the unusual pedagogical style will adversely affect their grade.

3.5.3 The Bell

When the bell rings at the end of class, everything becomes exponentially harder. My advice is to watch the clock like a hawk, so that you can make sure to wrap up the discussion at 2 minutes before the end. The discussion usually leaks over an extra minute, giving you one minute to say any summary comments or give instructions on homework, etc. This can be an important routine even when you have little to say: it makes you seem organized and on top of things.

If you can tell you are going to want to go overtime, because of a red-hot discussion you want to complete or something that's necessary so they can do their homework, announce to them 5 or 10 minutes ahead of time that you will probably be going overtime. It's best if you've let them go a minute or two early once before and have mentioned at the time that you're banking those minutes for such an occasion as this. If they're working in small groups at the end of class, it's less crucial but you still may want to halt them 1 or 2 minutes before the bell for closing remarks — it's even harder to get their attention after the bell when you don't already have the stage.

In short: **don't ever be surprised by the bell**.

4 Writeups

The write-ups (also called problem reports) are an important part of the course, because they force the student to communicate his/her knowledge about the problem. One consequence is that students cannot hide shortcomings in their understanding of the problem; another, more to the point, is that the students will develop the ability to give clear explanations on paper. Theoretically, a write-up should explain a problem clearly enough from beginning to end that a student could hand it to a colleague at that same school, and the colleague would be able to understand the whole problem without consulting anyone or anything else. It may well take a while for some students to develop good written communication skills, but you should be able to convince them that it is well worth the effort (and indeed most students have indicated at course's end that they believe it was) — after all, if a clear written explanation is harder to give than a clear verbal explanation, then they should come out well-prepared to explain to their own students.

This section of the guide discusses how to assign write-ups, which write-ups to assign, and how to grade them — although these are, in the end, only guidelines. At the end are two sample handouts you could give students to help them get used to writing math, and writing problem reports.

4.1 How, and how often?

The latter question is perhaps answered more quickly than the former. In general, you might want to aim for one write-up per week. More than this will cause you either to spend long hours grading (q.v.) or to fall behind in your grading; fewer than one every two weeks will not give the students the practice they need in developing their communication skills, and will also make you skip some important write-ups. Overall, you probably want a pretty even balance among individual write-ups, reflections, and other written work (including group write-ups and other homework). You may want to skip assigning a write-up the week of an exam.

As far as *how* to assign the write-ups:

1. Set some guidelines on the first day of the semester (see also Chapter 3 for this). Among other things, set a length of time between the date an assignment is made and the date it is due. These write-ups take a lot of time for the students, so they should have five to seven days in general (possibly less for the reflections, which are treated in Chapter 5). You probably don't want to set a single official length for write-ups, but between two and six pages is probably the norm. Do set rules for format, e.g., will you accept reports handwritten in pen? in pencil? Of course, also mention what you will be looking for in general, though more detailed specifics will come later, when you make assignments (see 3. below). [Part, if not all, of your guidelines may come in the form of a handout. See the example first day handout in Chapter 3, as well as the one given in the last part of this section.]

2. The write-up should almost always be on a problem that you have just finished discussing in class. It is a bad idea to assign a write-up on an activity on which you ran out of time, saying that the students should finish up on their own for the write-up. If the class discussion stops short of a full analysis of some aspect of the problem and you expect the students to address it in the writeup, it should be something you expected to leave for them to think about, not something you ran out of time for. Some suggested write-ups, such as Squares & Paths at the end of Unit One, are specifically designed as out-of-class problems, but you should make such assignments very sparingly.

3. You should discuss the write-up with the class for five minutes or so when you assign it, to make sure everyone knows what you want from them. This is especially important in the beginning, when students are unsure how to explain themselves, and in what generality they should present the solution. One way to do this is to ask the students, "What do you think are the important things about this problem, which we should include in the write-up?" Then write on the board the suggestions students make, adding your own if necessary once the students finish responding. (Of course, this means you have to decide what you want in the write-up before class!) Be willing to let the students argue for or against including certain items, and be careful not to make them include *too* much — you don't want to read a bunch of ten-page write-ups!

4. Most of the write-ups you will want to assign as individual write-ups: this is one of your big opportunities to evaluate an individual as opposed to his/her small group. However, assigning group write-ups can reduce work on both ends (the students' in writing and yours in grading), as well as being a nice change of pace sometimes. You might want to be especially observant the first time you do this, to see that each group member makes a more or less equal contribution. Again, use these in moderation.

4.2 Which ones?

Problems on which write-ups are assigned should be significant ones, where there is some complexity to the solution, and consequently a story to tell, both in the finding and in the explanation of what was found. You will often have spent more than one full day on the problem in class.

Here is a list of the problems on which write-ups were assigned in the Spring quarter of 2002, not including one problem which has been removed from the coursepack. Rubrics have been developed for all of these and are available in PostScript or pdf.

1. Photo Layouts (introductory practice writeup, graded but not counted)
2. Pizzas
3. Geoboards, part (f) only
4. Picture Proofs
5. Length of a Square
6. Surface Area
7. Scaling
8. Changing Units
9. Trisection
10. Rigidity
11. Proofs
12. First Constructions
13. Rigid Motions

Of course, you may feel free to add others which generated good in-class discussions, or omit one or two of those mentioned above.

4.3 Grading write-ups

You should make it clear to your students both before and after grading a given assignment what you were looking for, and how you determined the grade in general. However, unlike a calculus exam, you should *not* feel any obligation to give them an explicit point-by-point explanation of their grade, any more than a composition teacher has to explain the accumulation of good and bad points that resulted in giving a paper a B. Instructors in this course have historically used numerical scores rather than letter grades in grading write-ups, but you should do whatever feels comfortable.

Everyone has a different grading style, but one way to go about it is first to make a checklist of things you're looking for, or sections of the write-up, and then read/mark only that one section of all of the papers, to ensure that your comments will be consistent. Then, after you've examined each part of the write-up this way, go back and read each paper individually, skimming where you've made comments, and get an overall impression from which you can assign a total grade. When all papers have been graded, sort them by grade and flip through them to see if any appear to be "out of sequence", in which case you might want to take another look at those. Again, don't get too bogged down at the point level. The time required to grade write-ups properly increases more than linearly with the number of papers you have to grade, so you'll be looking for a middle ground in which each paper gets a fair reading but you don't spend fifteen or twenty hours grading.

Grading criteria: Here again, it's up to you. However, if we follow the ideas expressed in the sample handouts below, you'll be looking, in general, for:

1. A clear paraphrase of the problem description

2. An account of the method(s) used to solve the problem, including any major milestones or blocks, as well as "dead ends" which nevertheless proved enlightening in some way

3. A clear statement of the solution, as complete and general as appropriate, including an interpretation within the problem context

4. A clear explanation of why the solution works (or is the only, or complete, solution)

Since the nature of different problems can be quite different (compare Time to Weigh the Hippos with A Base Four Lesson), these elements may take on different forms.

Problem descriptions should be in the writer's own words, and give a complete enough description that the reader need not consult anything else (like the course materials) to understand it. Also, give credit for creativity, both in problem-solving methods and in writing up the solution.

One important element of the evaluation process is giving your students good feedback when you return their papers, both in written comments on their papers and verbally in class (in more generality). Encourage them; point out where, on the whole, the write-ups were strongest, and where they were weakest. It may help to give a handout back along with the very first problem report, so that they can see some of the elements of a good write-up. If you decide to make one, try to include good (but not bad) elements from specific papers (without naming names). *Don't* hand them a complete ideal problem report, as it tends to give them the impression there is/was a unique ideal which they were all supposed to guess. (It also gives students who take this course in later semesters an easy out for the first assignment.)

Here, for example, is an excerpt from a handout giving feedback on the problem report on "Poison" from Math 105. (This is not the entire handout!)

The winning strategy

> "The second player's strategy for playing with ten pennies is to watch what the first player does and do the opposite. For instance, if the first player chooses 2 pennies, then the second player chooses 1 and vice versa."

It doesn't take a lot of space to do this. Note that the writer makes it clear that this is the ten penny strategy, not the solution for 11 or 12 or 210.

The reason it works

> "If you pick opposites on every move, the total number [of] pennies taken by both of us on one turn is 3. There are 3 sets of 3 in ten. (3x3=9)

> However, you have to add 1, the poison penny, to make ten. That explains why when you get down to 4, if you pick first, you will have to lose. In 4 there is one set of 3 and the poison 1....
>
> By taking opposites, you take three pennies in each turn. Since 3x3 is 9, plus the poison #1 equals 10. For example, in my Group, if [my opponent] takes 1, I will take two. Now 7 are left. 7 is 3x2 plus the poison #1. [My opponent] takes 1, then I take two. Now there are 4 left. 4 is 3x1 plus the poison #1. She takes one, then I take two. She is left with the poison #1. Therefore, the person moving second, which was me, wins!"

Okay, this could have been better. Neither explanation is completely clear (and this writer did lose some points for clarity), although together they tell the story well. Again, writing out the example game helps.

The general solution

> "In order to win the 210 penny game, you would want to move first and take out two pennies. This will leave you with 208 pennies still on the desk. 208 divided by 3 gives you 69 with a remainder of 1, the poison penny. Just as 10 divided by 3 gives you 3 with a remainder of one. From here on, the team that moved first would take the opposite of the second team to win the game."

This was from a different paper than the previous, so the explanation of why the 10 penny strategy was a winner was somewhat different. But both of these students, by focusing on why the "opposites" strategy was a winner in the 10 penny game, figured out how to win the big game. I don't think it's coincidence that the two best explanations of why the "opposites" strategy works accompanied the two best general solutions.

5 Exams

5.1 What should they be like?

An obvious question to ask, given the cooperative nature of the sequence, is: Can the exams be given in groups? In fact, students will most likely have become comfortable working in groups by the time of the first exam, and you can be certain that a chorus will rise up to ask this question. My reason for not giving group exams is that I want to make sure that a reasonable portion of the student's grade is based on work they did without help. There is some research showing that students perform better in cooperative learning classrooms when exams are individual[4], but given the group nature of most of the rest of the work, I think individual accountability is a much stronger reason for sticking to individual exams. You may wish to point out that the students will not have their groupmates with them during their careers as teachers.

Exams, of course, invariably cause anxiety in many students, and you may want to make at least one of your exams open-notes (perhaps the final, if the midterm is closed-notes). This option also allows you to ask more detailed questions than you might be able to otherwise: questions about particular activities, or variations on problems assigned as write-ups. In either case, do be careful to write an exam that is at the level of the students: On one hand, it's easy to get carried away and overestimate what your students can handle. On the other, sometimes your students will surprise you (pleasantly).

One more note: You should probably not give more than one midterm plus the final, as time is precious, and using cooperative learning means a sacrifice on the content-versus-time scale. Exams can disrupt the flow of a class, and it's already difficult to cover as much as you'd like.

5.2 Final grades

The course grade will typically have at least one component — attendance and participation — which has historically not been present in traditional math classes. It

[4]Slavin, R.E. Research for the future: Research on Cooperative learning and achievement, what we know, what we don't know. *Contemporary Educational Psychology* **21** (1996): 43-49.

is important to include this, however, as the process and experiences that take place inside the classroom are the most significant part of the course, and a student cannot really learn what we want him/her to learn without being present. Furthermore, students in the 105-106 sequence respond to attendance grading with a more than 90helps keep the groups stable. Typically you should take attendance every day, which can be done unobtrusively, by counting heads five minutes or so into the period, when everyone is working in small groups. After a couple of weeks you'll be able to tell more or less at a glance who's missing. You should set clear guidelines for attendance at the beginning of the term and stick by them (see, for example, the sample first-day handout in Chapter 3).

Measuring class participation is more subjective. By halfway through the semester, you should be able to tell who is pulling his/her weight in the small groups, and should encourage those who are not to participate more. For large group participation, you may want to keep track of who says something, and mark it down after class (don't do it in class — students don't like the idea that you're taking notes on what they do or say). This method requires some good memory on your part, but after a few weeks you'll see who's speaking up and who's not, and can encourage those who don't to participate more. One way of doing so without putting pressure on to know the answer to a question is to call a student up to the blackboard to serve as scribe in writing down others' suggestions during a brainstorming type of discussion. However you decide to do it, you really will have to be diligent about it from the first day.

As a help in deciding how to break down the course grade into components, here is the breakdown used by the most recent instructors in 105 and 106.

- Attendance: 15%
- Written work, including quizzes: 45%
- Midterm exam: 15%
- Final exam: 25%

5.3 Exam and Study Guide Database

Midterm Exam, Math 105, Au02

NAME:

1. (2,2,1) Compute the LCM of each pair of numbers. (You may state the answer in the same form that the numbers are given.)

   (a) 68 and 72

   (b) $2^3 \times 3^4 \times 7^2 \times 11^2$ and $3 \times 5 \times 7^3$

   (c) x and x^2 where x is a positive integer

2. (5) A gear with 72 teeth turns a gear with 68 teeth. How many rotations must each gear make before both gears are simultaneously back in their starting position?

3. (1,2,3,2,2) Let S be the set of all subsets of the set $\{a, b\}$.

   (a) List the elements of S.

   (b) Make a table for the binary operation $\cap$ on the set S.

   (c) Is S closed under $\cap$? Is this operation commutative? associative?

   (d) Does S have an identity element for the operation $\cap$ and if so, which element?

   (e) Is there an inverse element for $\{a\}$ with the operation $\cap$, and if so, what element?

4. (6) In a room, there are three students for every professor. A professor walks in; now there are 2.5 students for every professor. Write equations to represent your knowledge and determine how many people are in the room now. [Only 2 points on this problem will be given for a solution by trial and error solution with no equations.]

5. (1,1,1,1) Mark each statement True, False, or Can't Tell, assuming you know that all Glyck numbers over 100 are even or prime:

   (a) 105 is a Glyck

   (b) 75 is a Glyck

   (c) There are no Glycks

   (d) There are no three-digit Glycks ending in a 5

Final Exam, Math 105, Au02

NAME:

1. (2,2,2,2)

   (a) Find the prime factorizations of these two numbers:
   - 777,000
   - 74,000,000

   (b) Compute the GCD of these two numbers.

   (c) How many factors does 74,000,000 have in all?

   (d) Write down a number with exactly twice this many factors.

2. (4) Order these numbers from greatest to least and give one-sentence explanations for why the greatest is the one you chose and why the least is the one you chose.

   (a) $.0000000000012345 + .000000000009876$

   (b) $.0000000000012345 \times .000000000009876$

   (c) $.0000000000012345 \div .000000000009876$

3. (6) All 25 cars recently sold at Vic's Used Car Lot have either anti-lock brakes or four-wheel drive. 16 of the cars have anti-lock brakes and 13 have four-wheel drive.

   (a) How many cars have both anti-lock brakes and four-wheel drive?

   (b) Draw a Venn diagram illustrating the situation

4. (4) Find two prime numbers that differ by 7, or prove that there aren't any two such numbers.

5. (3) Two room-mates are in a grocery line. The first buys milk and eggs and the second buys beer. The next two customers are also room-mates. The first buys milk and the second buys eggs and beer. Tarzan, the cashier, is surprised to see that both households spent the same total amount. Of which law is Tarzan apparently unaware?

   (a) The commutative law for multiplication

   (b) Order of operations

   (c) The associative law for addition

   (d) The commutative law for addition

   (e) That multiplication distributes over addition?

   No explanation is needed – just circle the correct letter.

6. (3) Find a fraction lying between $\frac{22}{7}$ and $\frac{355}{113}$. Give a one-sentence argument that your fraction does indeed lie between the other two.

7. (2) How many license plates are possible in a state whose plates have two letters followed by four numbers?

8. (4) A student comes across the fraction $\frac{120}{190}$ and simplifies it to $\frac{12}{19}$ by "cancelling the zeros". Later, the student comes across $\frac{113}{263}$ and simplifies it to $\frac{11}{26}$ by "cancelling the threes". Your job is to explain to the student why cancelling the zeros is OK but cancelling the threes is not. Your explanation must be general, not simply showing it does or does not work for one or several examples.

9. (4) It is often said that 10% of the world's population owns 90% of the wealth. Suppose that each of the wealthy 10% is equally wealthy and each of the poor 90% is equally poor. How many times wealthier is a wealthy person than a poor person?

10. (2*) *Extra Credit Problem*: Describe which fractions become terminating decimals in base six (decimals in base six are properly called "heximals").

6 Materials and how to use them

6.1 On-line handouts

Files are included for several types of on-line handouts we developed as the quarter went along. They may be found under the "more stuff" link.

First, every week we gave out a summary of what they were supposed to have gotten out of that week's activities. We did this partly because the material is difficult enough for some students that they run out of gas before getting to the end of a problem, let alone having time to think it over and assimilate the ideas. They are liable, even when looking back later, to miss the point. Also, there is little redundancy in the material so it helps for them to have a list of general ideas exemplified by each worksheet. They find this useful when studying for exams. It is also a way of conveying to them what they are responsible for learning.

Each instructor needs to put out their own such summary of course, since at the very least, the pace will vary from class to class. We provide the html file in the hope that it will be useful for cutting and pasting, or for prompting the rushed instructor to notice all the ideas embedded in the worksheets and point them out to the students. **At present, only the first six weeks of summaries are available.**

The second electronic hadout concerned guidelines. We made this mandatory reading on Day 1. This concerned proper etiquette for group discussions and for writeups. The third concerned the proper format for writeups.

In addition, consider making the Readings in Sections 6.1, 6.2 and 6.3 available online or bringing them as handouts the first day, as they are important and students may not get coursepacks immediately after the first class.

6.2 Solutions and such

During the course of the quarter, we handed out model solutions at various times. These are collected in the Solutions subdirectory, and of course, are accessible only by permission. In some cases, we gave out contrasting bad solutions – these are included for what they're worth. The good and bad solutions were contrastingly annotated in pen, but this annotation is not on the online versions.

Additionally, scoring rubrics were posted religiously after each assignment was due. We tried to make sure the rubric was posted before the re-do was due, since we felt that students doing the re-do were the ones who missed substantial points (except at first when they all want to do re-do's) and therefore they needed to be hit over the head with what our expectations were and why they missed points. Since improving their communication skills is the paramount mission in this course, and since the rubric highlights deficiencies here, we made sure that papers were graded in such a way that they could see exactly where they lost points on the rubrics: scores would be written as $(3 + 1 + 2) + (2 + 2) + 6 + 2$ if the rubric identified four scoring categories, with three subcategories in the first and two in the second. All the rubrics are included in the Rubrics file in the Solutions directory. Source code is in the LaTeX directory. **At present, rubrics are only available for nine of the worksheets that were assigned for writeup.**

6.3 LaTeX materials

If you want to use any of the worksheets, solutions, and so forth, in some form other than exactly what is provided, you will probably want the LaTeX source files. The LaTeX directory has source files for the coursepack and for all the auxilliary materials you might want to use – exams, solutions and so forth. You may download them as needed and adapt them at will. You will need to obtain permission to access the LaTeX directory.

At the very least, even if you don't know how to mess with LaTeX, you will need to remove the two-page sample syllabus that's in the courspack and substitute your own syllabus.

Figures were made using xfig and exported both in eps and pdf. All figures are in the figures subdirectory of the LaTeX directory.

All contributions of new materials will be gratefully accepted by e-mail to pemantle@math.upenn.edu. More grist for the exam problem mill would be particularly useful.