TOTAL CURVATURE AND SIMPLE PURSUIT ON DOMAINS OF CURVATURE BOUNDED ABOVE

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ABSTRACT. We show how circumradius and asymptotic behavior of curves in CAT(0) and CAT(K) spaces ($K > 0$) are controlled by growth rates of total curvature. We apply our results to pursuit and evasion games of capture type with simple pursuit motion, generalizing results that are known for convex Euclidean domains, and obtaining results that are new for convex Euclidean domains and hold on playing fields vastly more general than these.

1. INTRODUCTION

The goals of this paper are twofold:

(1) We study *total curvature* of a curve (the integral of its curvature) in spaces of curvature bounded above, and relate the total curvature, the curve's circumradius function, the asymptotic behavior of the curve, and the domain's curvature bound.
(2) We apply these results to a foundational problem in *pursuit-evasion games*, where an evader moves in a domain and is followed by a pursuer along a *pursuit curve*. We study the *capture problem:* whether the pursuer ever catches (comes sufficiently close to) the evader. Although our total curvature and circumradius results are new even for convex Euclidean playing fields, our playing fields are vastly more general than these.

We assume that the reader is familiar with the basic notions of CAT(K) and Alexandrov geometry (see, e.g., [9, 10]) as well as the theme that results which are true in Riemannian spaces of sectional curvature bounded above are often true — and often have more transparent proofs — in the broader class of CAT(K) spaces. Alternatively, the reader may consult the short appendix containing the definitions and basic tools that we use. We hope that readers based in comparison geometry will find both the theorems on the asymptotics of total curvature, and their applications to the capture problem, of interest; and readers interested in pursuit-evasion games will find the power of comparison geometry compelling.

1.1. **Motivation.** The application to pursuit-evasion games requires some motivation and background. There is a significant literature on pursuit and evasion games with natural motivations coming from robotics, control theory, and defense applications [14, 18, 32]. Such games involve one or more *evaders* in a fixed domain being hunted by one or more *pursuers* who win the game if the appropriate capture criteria are satisfied. Such criteria may be physical capture (the pursuers move to

1991 *Mathematics Subject Classification.* 91A24, 49N75.
Key words and phrases. Pursuit-evasion games, CAT(0) geometry, bounded curvature, total curvature, pursuit curve.

where the evaders are located) [16, 17, 27] or visual capture (there is a line-of-sight between a pursuer and an evader) [14, 30]. The types of pursuit games are many and varied: continuous or discrete time, bounded or unbounded speed, and constrained or unconstrained acceleration, energy expenditure, strategy, and sensing. For a quick introduction to the literature on pursuit games, see, e.g., [20, 14].

The applications in this paper focus on one particular variable in pursuit games: the geometry and topology of the domain on which the game is played. We keep all other variables fixed and as simple as possible. Thus, there will be a single pursuer-evader pair, a single simple pursuit strategy ('move toward the evader'), and no constraints on acceleration or related system features.

The vast majority of the known results on pursuit-evasion are dependent on having Euclidean domains which are two-dimensional or, if higher-dimensional, then convex. There has of late been a limited number of results for pursuit games on surfaces of revolution [15], cones [26], and round spheres [21]. Our results are complementary to these, in the sense that we work with domains of arbitrary dimension, with no constraints on being either smooth or locally Euclidean.

There are several reasons for wanting to extend the study of pursuit-evasion games to the most general class of playing fields possible. The most obvious such application is in the generalization from 2-d to 3-d, in which the pursuit game is a model for physical pursuit, as well as in the expansion to nonconvex domains. For example, a closed, simply connected domain $\mathcal{D}$ with smooth boundary in $\mathbb{E}^3$ is $\mathrm{CAT}(0)$ if the tangent plane at every boundary point p contains points arbitrarily close to p that are not in the interior of $\mathcal{D}$. (This is a special case of the characterization of upper curvature bounds of manifolds with boundary in [3].) More generally, a domain in $\mathbb{E}^3$ is $\mathrm{CAT}(1)$ if it is not too far from convex, that is, its boundary is not too outwardly curved; see Example 12 below.

However, higher dimensional playing fields can also correspond to physical problems, via *configuration spaces* of physical systems. Consider the following (fanciful) example. If one wants to mimic the action of a dancer with a complex robot, one could attempt a generalized pursuit game in which the playing field is the configuration space of the dancer's (or robot's) motions. The dancer's configuration plays the role of the evader, and the robot's configuration plays the role of the pursuer. If the robot's goal is to mimic the dancer in real time with knowledge only of the dancer's instantaneous body configuration, then this translates into a simple pursuit problem with one pursuer and one evader. The results of this paper show that no matter how high the dimension of the configuration space, the pursuit strategy will be successful if the configuration space is $\mathrm{CAT}(0)$. It has been demonstrated recently that there is a significant class of configuration spaces in robotics and related fields which do have an underlying $\mathrm{CAT}(0)$ geometry [1, 8, 12, 13], rendering the cartoon example above a little less unrealistic. In like vein, work on consensus, rendezvous, and flocking [31] is a form of coordinated pursuit in which the evader is the consensus or rendezvous state(s).

2. Total curvature

For a curve in a $\mathrm{CAT}(0)$ space, successively stronger constraints on the total curvature function control long-term behavior.

2.1. **Definitions.**

Definition 1. The *total rotation* τ_σ of a polygonal (i.e., piecewise-geodesic) curve σ is $\sum_i (\pi - \beta_i)$, where the $\beta_i \geq 0$ are the angles at the interior vertices. The *total curvature* of any curve γ is the lim sup of τ_σ as $\mu_\sigma \to 0$, over all polygonal σ inscribed in γ, where μ_σ is the maximum segment length of σ.

In $\mathrm{CAT}(0)$ spaces, monotonicity of total rotation follows from triangle comparisons: if σ is inscribed in a polygonal curve γ, then $\tau_\sigma \leq \tau_\gamma$ [4]. Thus the total curvature of any curve γ in a $\mathrm{CAT}(0)$ space is the supremum of τ_σ over all polygonal σ inscribed in γ. Monotonicity of total rotation fails in $\mathrm{CAT}(K)$ spaces for $K > 0$, but a more subtle argument proves that the total curvature of an arbitrary curve is the limit of τ_{σ_n} for any sequence σ_n of inscribed polygonal curves with $\mu_\sigma \to 0$ [24, see Theorem 18 below].

In particular, the total rotation τ_σ of a polygonal curve coincides with its total curvature. Accordingly: *from now on, we denote the total curvature of an arbitrary curve γ in a $\mathrm{CAT}(K)$ space by τ_γ.*

Example 2. If γ is a unitspeed curve in $\mathbb{E}^n$, then τ_γ equals the length in the unit sphere of the curve γ'^{+} of righthand unit tangent vectors, with jump discontinuities replaced by great circular arcs [6, Theorem 5.2.2]. In particular, if γ is smooth in $\mathbb{E}^2$, so that $\gamma'(t) = (\cos\theta(t), \sin\theta(t))$, then $\tau_\gamma = \int \kappa$, where $\kappa = |\gamma''| = |\theta'|$ [6, Section 5.3].

Curves of finite total curvature in $\mathrm{CAT}(K)$ spaces are well-behaved, in the sense that they have unit-speed parametrizations, which have left and right unit velocity vectors at every point [24].

We are interested in how the asymptotic behavior of the *total curvature function*,

$$\tau(t) = \tau_{\gamma|[0,t]},$$

controls the function that measures the maximum distance from its initial point realized by the curve in a given time period:

Definition 3. The *circumradius function* of a curve γ is the real-valued function c, where $c(t)$ is the smallest number such that the path $\gamma|[0,t]$ lies in the ball of radius $c(t)$ about $\gamma(0)$.

2.2. **Growth rate of total curvature and circumradius.** The following theorem will be applied in Section 4 to pursuit-and-evasion games, to obtain a necessary condition for the evader to win, in terms of how far from home the evader wanders during given time periods. Theorem 4 generalizes a theorem of Dekster for Riemannian manifolds [11]. However, we use Reshetnyak majorization to obtain a simple argument that moreover holds for any $\mathrm{CAT}(0)$ domain.

Theorem 4. *For any curve γ, parametrized by arclength t, in a $\mathrm{CAT}(0)$ space, let τ and c be its total curvature and circumradius functions.*

(a) *If $\liminf_{t\to\infty} \tau(t)/t = 0$, then γ is unbounded.*
(b) *If $\tau \in O(t^a)$ for some $a \in (0,1)$, then $c \in \Omega(t^{1-a})$.*

Recall that $f \in \Omega(t^\lambda)$ means $f(t) \geq Bt^\lambda$ for some $B > 0$ and all t sufficiently large, while $f \in O(t^\lambda)$ means $f(t) \leq Bt^\lambda$ for some $B > 0$, and $f \in o(t^\lambda)$ means $\lim_{t\to\infty}(f(t)/t^\lambda) = 0$.

Example 5. Consider the spiral $\gamma(u) = (u\cos 2\pi u, u \sin 2\pi u)$ in $\mathbb{E}^2$. The total curvature function is linear in u, as is the circumradius, while the arclength t grows quadratically. This is case (b) for $a = \frac{1}{2}$, with $t/c(t)^2$ bounded.

Proof of Theorem 4. For part (a), we may suppose by approximation that any fixed initial segment $\gamma|[0,t]$ is polygonal. Subdivide $[0,t]$ into at most $\frac{\tau(t)}{\pi/2}+1$ subintervals so that the restriction γ_i of γ to each subinterval has total curvature at most $\pi/2$. (If any angles are less than $\pi/2$, we first refine the polygon by cutting across each such angle with a short segment to obtain two angles of at least $\pi/2$. Let ρ_i be the closed polygon consisting of γ_i and its chord σ_i. By Reshetnyak majorization, there is a closed convex curve $\widetilde{\rho}_i$ in $\mathbb{E}^2$ that majorizes ρ_i. Since a majorizing map preserves geodesics and does not increase angles (see Appendix), $\widetilde{\rho}_i$ is a closed polygon with the same sidelengths as ρ_i, consisting of a polygonal curve $\widetilde{\gamma}_i$ and its chord $\widetilde{\sigma}_i$, where the total curvature of $\widetilde{\gamma}_i$ is at most $\pi/2$.

Since $\widetilde{\gamma}_i$ is a convex curve in $\mathbb{E}^2$ having total curvature at most $\pi/2$, the ratio of its length to that of its chord is at most $\sqrt{2}$ (the ratio of two sides of an isosceles right triangle to its hypotenuse). Therefore

$$t \le \left(\frac{\tau(t)}{\pi/2}+1\right)\sqrt{2}\sup|\sigma_i|, \tag{2.1}$$

so

$$\frac{\tau(t)}{t} \ge \frac{\pi}{2}\left(\frac{1}{\sqrt{2}\sup|\sigma_i|}-\frac{1}{t}\right).$$

But if γ is bounded, so that $\sup|\sigma_i|<\infty$, it follows that $\tau(t)/t$ is bounded away from 0 for t sufficiently large. This proves part (a).

For part (b), if one substitutes $\tau(t)\le At^{\alpha}$ and $\sup|\sigma_i|\le 2c(t)$ in (2.1), it is immediate that $c(t)\ge Bt^{1-\alpha}$ for t sufficiently large. □

2.3. **Finite total curvature and asymptotic rays.** Total curvature also controls how close an infinite curve of finite total curvature must be to a geodesic ray. In the Riemannian setting, the conclusions of the following theorem were obtained by Langevin and Sifre [22] under stronger hypotheses. That is, they assume the pointwise curvature $\kappa(t)$ of a smooth curve γ satisfies $\kappa(t)\in O(t^{-1-\epsilon})$ in part (a), and the same with $t^{2+\epsilon}$ in part (b). Here again, we give a simple argument using CAT(0) techniques.

A curve γ is said to be *asymptotic* to a geodesic ray σ if the distance $d(\gamma(t),\sigma)$ from $\gamma(t)$ to the nearest point on σ is bounded. Now we show that a curve of finite total curvature τ_γ always has sublinear distance to some geodesic ray, to which it is asymptotic if the total curvature function approaches its limit τ_γ sufficiently rapidly.

Theorem 6. *Let γ be a curve, parametrized by arclength t, in a CAT(0) space. Suppose γ has finite total curvature $\tau_\gamma=\lim_{t\to\infty}\tau(t)$.*

(a) *Through any point p, there is geodesic ray σ such that $d(\gamma(t),\sigma)\in o(t)$.*
(b) *The circumradius function satisfies $c\in\Omega(t)$.*
(c) *If $\int_0^\infty(\tau_\gamma-\tau(t))dt<\infty$, then γ and σ are asymptotic.*

Proof. Again, we may assume γ is polygonal. Choose an increasing sequence $t_i\to\infty$, $i\ge 0$, $t_0=0$. Let σ_i be the geodesic joining $\gamma(0)$ and $\gamma(t_i)$. For $i\ge 1$, let ρ_i be the closed polygon made up of σ_{i-1}, $\gamma_i=\gamma|[t_{i-1},t_i]$, and σ_i. We denote by $\widetilde{\rho}_i$ a convex polygon in $\mathbb{E}^2$ that majorizes ρ_i, and by $\widetilde{\gamma}_i$ its subarc corresponding to γ_i. We suppose the $\widetilde{\rho}_i$ are arranged in a counterclockwise "fan", so that $\widetilde{\rho}_{i+1}$ and $\widetilde{\rho}_i$

intersect along the straight line segment $\tilde{\sigma}_i$ in each that corresponds to σ_i, and the points corresponding to $\gamma(0)$ coincide at the centerpoint $\tilde{O}$: see Figure 1.

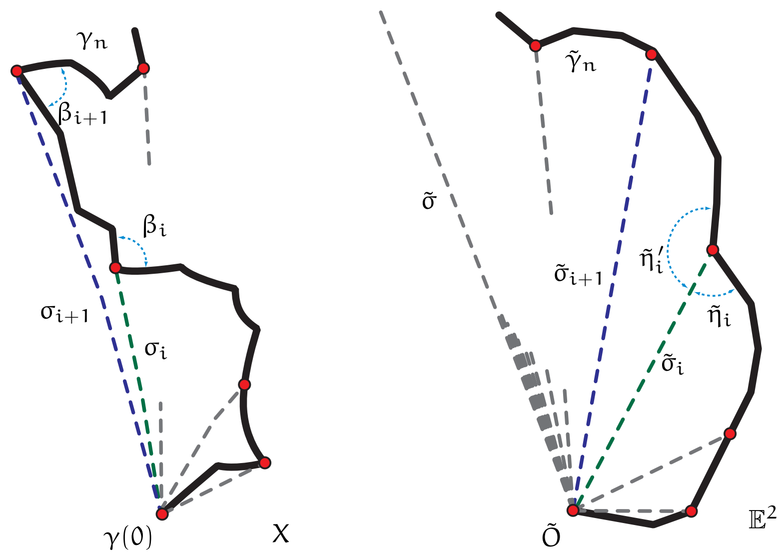

FIGURE 1. Polygons ρ_i and their convex comparisons, arranged to form a fan.

Consider the polygonal curve $\tilde{\gamma} : [0, \infty) \to \mathbb{E}^2$, parametrized by arclength, whose image is the union of the convex curves $\tilde{\gamma}_i$. The restriction of the majorizing map of $\tilde{\rho}_i$ to $\tilde{\gamma}_i$ maps onto γ_i and does not increase angles at vertices. Thus it does not decrease total curvature. Moreover, suppose $\tilde{\eta}_i$ and $\tilde{\eta}_i'$ are the interior angles of $\tilde{\rho}_i$ and $\tilde{\rho}_{i+1}$ at $\tilde{\gamma}(t_i)$. Similarly, suppose η_i is the angle at $\gamma(t_i)$ between the directions of σ_i and γ_i, and η_i' is the angle at $\gamma(t_i)$ between the directions of σ_i and γ_{i+1}. Then $\tilde{\eta}_i \geq \eta_i$ and $\tilde{\eta}_i' \geq \eta_i'$. By the triangle inequality for angles at $\gamma(t_i)$,

$$\tilde{\eta}_i + \tilde{\eta}_i' \geq \eta_i + \eta_i' \geq \beta_i, \tag{2.2}$$

where β_i is the angle at $\gamma(t_i)$ between the left and right directions of γ.

The angle between the initial and final tangents of $\tilde{\gamma}|[t_m, t_n]$ in $\mathbb{E}^2$ is the sum of the $\pi - (\tilde{\eta}_i + \tilde{\eta}_i')$, which could be negative, and the positive total curvatures of the convex polygonal curves $\tilde{\gamma}_i$ for $m + 1 \leq i \leq n$. Thus by (2.2), the angle between the initial and final tangents of $\tilde{\gamma}|[t_m, t_n]$ is no more than the total curvature of $\gamma|[t_m, t_n]$. Now take $t_m = 0$. Since $\tilde{\gamma}(0) = \tilde{O}$, the total angle at $\tilde{O}$ of the first n sectors of the fan is no more than the angle between the initial and final tangents of $\tilde{\gamma}|[0, t_n]$, and hence no more than τ_γ. Therefore the vertex angles of the $\tilde{\rho}_i$ are summable, and the angle between $\tilde{\gamma}$ and the ray from $\tilde{O}$ through $\tilde{\gamma}(t)$ converges to 0.

Let $\tilde{r}(t) = d(\tilde{O}, \tilde{\gamma}(t))$. By the First Variation Formula (A.1), the one-sided derivatives $d\tilde{r}/dt$ converge to 1. Thus we have $\tilde{r}(t)$ increasing, and if $A < 1$ then $\tilde{r}(t) > At$ for t sufficiently large. Furthermore, for each choice of sequence $t_i \to \infty$ our construction produces a function $\widetilde{r}$ satisfying $\widetilde{r}(t_i) = r(t_i)$, where $r(t) = d(O, \gamma(t))$. It follows that $r(t)$ also eventually increases and $r(t) > At$ for t sufficiently large.

Since the directions of the line segments $\widetilde{\sigma}_i$ at their basepoint $\widetilde{O}$ converge and $|\widetilde{\sigma}_i| \to \infty$, the $\widetilde{\sigma}_i$ converge to a Euclidean ray $\widetilde{\sigma}$ from $\widetilde{O}$. Let s be the arclength parameter on $\widetilde{\sigma}$. Since the angle at which a ray strikes $\widetilde{\gamma}$ converges to 0, it follows that for t sufficiently large, $\widetilde{\gamma}$ is the graph of a height function of order $o(s)$ over $\widetilde{\sigma}$. Hence $d(\widetilde{\gamma}(t), \widetilde{\sigma}) \in o(s(t)) = o(t)$.

Now since each $\widetilde{\rho}_i$ majorizes ρ_i, the intersections of the geodesics σ_i with any ball in M about $\gamma(0)$ converge to a geodesic. Therefore the σ_i converge to a geodesic ray σ in M. Furthermore, $d(\widetilde{\gamma}(t), \widetilde{\sigma})$ is realized by a line segment through the fan, infinitely partitioned by its intersections with a truncated sequence of the $\widetilde{\sigma}_i$. Since $\widetilde{\rho}_i$ majorizes ρ_i, $\gamma(t)$ is joined to σ by a path of no greater length. Therefore $d(\gamma(t), \sigma) \leq d(\widetilde{\gamma}(t), \widetilde{\sigma}) \in o(t)$, as claimed in part (a).

Part (b) follows from (a) trivially, given the linear circumradius of geodesic rays.

Since $d(\gamma(t), \sigma) \leq d(\widetilde{\gamma}(t), \widetilde{\sigma})$, it suffices for part (c) to show that the latter is bounded. The length of the projection of $\widetilde{\gamma}$ in $\mathbb{E}^2$ to a line normal to $\widetilde{\sigma}$ is obtained by integrating $\sin \beta(t)$, where β the angle between the righthand tangent of $\widetilde{\gamma}$ and the direction of $\widetilde{\sigma}$. Since

$$\sin \beta(t) \leq \beta(t) \leq \tau_{\widetilde{\gamma}|[t,\infty]} \leq \tau_{\gamma|[t,\infty]} \leq \tau_\gamma - \tau(t),$$

part (c) follows. □

Remark 7. By alternately gluing Euclidean and hyperbolic bands bounded by pairs of asymptotic geodesics, one can construct a CAT(0) space in which the curves of part (c) need not have *strict* asymptotes (the distance to which approaches 0 rather than merely being bounded). This construction is carried out in [22], although the CAT(0) nature of the resulting glued space is not mentioned.

3. Simple pursuit on CAT(0) domains

The following rules define a basic discrete-time equal-speed pursuit game. (Continuous-time pursuit will be discussed in Section 6.) Let (X, d) denote a geodesic metric space (representing the domain on which the game is played). There is a single pursuer P and a single evader E starting at locations P_0 and E_0 respectively. At the i-th step, the evader moves from E_{i-1} to E_i, a point within distance D chosen by the evader. The pursuer moves to P_i, the point along a geodesic from P_{i-1} to E_{i-1} at distance D from P_{i-1}. The moves are illustrated in Figure 2. Four points, P_i, E_i, E_{i+1} and P_{i+1}, form a degenerate geodesic quadrangle with side lengths L_i, $d(E_i, E_{i+1}) \leq D$, L_{i+1}, and D, where $L_i = d(P_i, E_i)$ for each nonnegative integer i.

This type of motion, in which the pursuer moves in an unconstrained fashion in the direction of the evader, is called *simple pursuit*. Given P_0 and the sequence $\{E_i\}$, we say that P wins if $d(P_i, E_i) \leq D$ for some i; otherwise, E wins. We note that this instantaneous, memoryless strategy for pursuit is not necessarily the pursuer's optimal strategy [29, 20] — merely the simplest.

By the triangle inequality,

$$L_{i+1} \leq d(P_{i+1}, E_i) + D = L_i. \tag{3.1}$$

Thus $\lim_{i\to\infty} L_i = L_\infty$ exists, and the evader wins if and only if this limit is greater than D. Moreover,

$$\pi - \beta_i \leq \alpha_i \leq \widetilde{\alpha}_i, \tag{3.2}$$

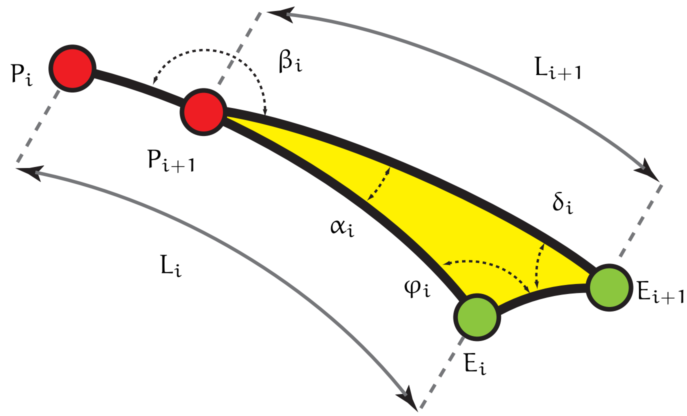

FIGURE 2. A degenerate quadrangle arising from a discrete-time capture problem.

where α_i is the angle between the geodesics joining P_{i+1} to E_i and E_{i+1}, and $\widetilde{\alpha}_i$ is the angle corresponding to α_i in the Euclidean triangle with the same sidelengths as $\triangle E_i P_{i+1} E_{i+1}$ (compare Definition 1). The first inequality in (3.2) is by the triangle inequality for the *angle distance* between the directions of geodesic segments with a common origin. These observations highlight the naturality of the CAT(0) definition in the context of pursuit problems.

A discrete-time *pursuit curve* $P(t)$ is obtained by joining the P_i by geodesic segments, where t has speed 1. Thus $P_i = P(iD)$, where D is the step size. The discrete-time *evader curve* $E(t)$ is defined similarly; however, since the evader's step sizes are assumed $\leq D$, on each geodesic segment the parameter t has $\triangle t = D$ and constant speed ≤ 1.

The following simple result is well known for convex Euclidean domains. Theorem 4 provides us with the immediate extension to CAT(0) domains:

Theorem 8. *For discrete-time simple pursuit on a complete* CAT(0) *domain, the domain is compact if and only if the pursuer always wins.*

Proof. If the evader wins, then by (3.1) we have $L_{i+1} \to L_\infty$ and $d(P_{i+1}, E_i) + D \to L_\infty$. Therefore the angle α_i vanishes in the limit, because the same is true for $\widetilde{\alpha}_i$. By (3.2), the total curvature of the P curve is sublinear. This curve is unbounded via Theorem 4, so the domain is noncompact.

Conversely, a noncompact CAT(0) domain contains an infinite geodesic ray, along which the evader and pursuer may move with constant separation and hence without capture. □

We now relate the total curvature functions $\tau^E(t)$ of the evader curve $E(t)$ and $\tau^P(t)$ of the pursuer curve $P(t)$. On the one hand, the evader may accumulate large total curvature by zigzagging, without much affecting the pursuer's total curvature. On the other hand, the pursuer's total curvature may exceed the evader's: if E runs along a geodesic ray and P does not start on the ray, then the evader's total curvature is 0 and the pursuer's is positive. The following result makes these observations precise. The proof uses only angle comparisons, and is added evidence that simple pursuit has an affinity for the CAT(0) setting.

Theorem 9. *For discrete-time simple pursuit on a* $\mathrm{CAT}(0)$ *domain, the total curvature functions of evader and pursuer satisfy*

$$\tau^P(t) \le \tau^E(t) + \pi.$$

Proof. Set $(\tau^P)^{n+1} = \tau^P\big((n+1)D\big)$, the total curvature of the pursuit curve to P_{n+1}, and similarly for $(\tau^E)^{n+1}$. Label the internal angles of $\triangle E_i P_{i+1} E_{i+1}$ by $\alpha_i, \varphi_i, \delta_i$ as indicated in Figure 2. Then

$$(\tau^P)^{n+1} = \sum_0^{n-1} (\pi - \beta_i) \le \sum_0^{n-1} \alpha_i \le \sum_0^{n-1} (\pi - \varphi_i - \delta_i),$$

since $\alpha_i + \varphi_i + \delta_i \le \pi$ by the $\mathrm{CAT}(0)$ condition.

On the other hand, letting θ_i be the interior angle of the evader curve at E_i, we have

$$(\tau^E)^n = \sum_1^{n-1} (\pi - \theta_i) \ge \sum_1^{n-1} (\pi - \delta_{i-1} - \varphi_i),$$

since $\theta_i \le \delta_{i-1} + \varphi_i$ by the triangle inequality for angle distance between directions at E_i. Therefore

$$(\tau^P)^{n+1} - (\tau^E)^n \le \pi - \varphi_0 - \delta_{n-1}.$$

□

4. Escape

On a noncompact domain, the relevant question is whether the evader can escape when the pursuer adopts the simple pursuit-curve strategy, and, if so, what conditions lead to escape. Here we show that the pursuer still always wins if the circumradius of the evader does not grow fast enough, or, equivalently via Theorem 4, if the pursuit path is forced to curve too much. The proof of this necessary condition for escape uses an estimate on total curvature of pursuit curves that will be proved in Theorem 13 of the next section.

Theorem 10. *Suppose the evader wins a discrete-time simple pursuit on a* $\mathrm{CAT}(0)$ *domain* $\mathcal{D}$. *Then the total curvature* $\tau(t)$ *of the pursuit curve from* P_0 *to* $P(t)$ *satisfies* $\tau \in O(t^{\frac{1}{2}})$.

Proof, assuming Theorem 13. We invoke the facts that a $\mathrm{CAT}(0)$ domain is also a $\mathrm{CAT}(1)$ domain, and a rescaling of a $\mathrm{CAT}(0)$ domain is again a $\mathrm{CAT}(0)$ domain.

Theorem 13 states that for simple pursuit on a $\mathrm{CAT}(1)$ domain, if the evader wins and the initial distance $L_0 = d(P_0, E_0)$ is less than π, then the total curvature $\tau(t)$ of the pursuit curve from P_0 to $P(t)$ is $O(t^{\frac{1}{2}})$. Therefore we rescale the metric of $\mathcal{D}$ by π/L_0. Since angles are invariant under rescaling, then by Definition 1, the total curvature of a given segment of the pursuit curve is also invariant under rescaling. □

Corollary 11. *Let* c *denote the circumradius function of an evader's path on a* $\mathrm{CAT}(0)$ *domain. Then* $c \in \Omega(t^{\frac{1}{2}})$ *is a necessary condition for the evader to win in a discrete-time simple pursuit game.*

Proof. Combine Theorem 10 with part (b) of Theorem 4, using $a = \frac{1}{2}$. Thus the circumradius function of the pursuer is in $\Omega(t^{\frac{1}{2}})$. An asymptotic growth estimate of the circumradius of a curve P obviously also holds for any other curve E for which there is a uniform bound on $d(P(t), E(t))$. □

5. DOMAINS WITH POSITIVE CURVATURE BOUNDS

In applications, spaces with positive curvature are not merely possible but prevalent. In this section, we demonstrate that controlled amounts of positive curvature are admissible, as long as we control initial distances between the pursuer and evader.

First we provide a large class of nonconvex examples of CAT(K) domains in $\mathbb{R}^n$:

Example 12. A closed domain $\mathcal{D}$ in $\mathbb{R}^n$ with smooth boundary $\partial\mathcal{D}$, where $\mathcal{D}$ carries its intrinsic metric, is a CAT(K) space for $K > 0$ if it is supported at every $p \in \partial\mathcal{D}$ by a sphere of radius $1/\sqrt{K}$, that is, every point at distance $\leq 1/\sqrt{K}$ from $\mathcal{D}$ is the center of a closed ball that meets $\mathcal{D}$ in a single point.

Proof. By [2, Theorem 3], the hypothesis of supporting balls implies that geodesics of $\mathcal{D}$ of length $< \pi/\sqrt{K}$ are uniquely (and hence continuously) determined by their endpoints. By the Alexandrov patchwork construction (see [9, p.199]), it follows that $\mathcal{D}$ is a CAT(K) space. □

We now study the asymptotic behavior of total curvature of pursuit curves in CAT(K) spaces for $K > 0$. Rescaling the metric by the factor $1/\sqrt{K}$, we may assume $K = 1$.

Just as in the CAT(0) case, the triangle inequality for $\triangle P_{i+1}E_{i+1}P_i$ in a CAT(1) space easily implies that the distances $L_i = d(P_i, E_i)$ are monotonically non-increasing: see Figure 2. The condition for equality from one step to the next is that P_{i+1} and E_i are on a geodesic segment P_iE_{i+1}, and hence the angles $\angle P_iP_{i+1}P_{i+2}$ and $\angle P_{i+1}E_iE_{i+1}$ are both π.

Theorem 13. *On a* CAT(1) *domain, suppose the evader wins a discrete-time simple pursuit with initial distance* $L_0 = d(P_0, E_0) < \pi$*. Then the total curvature* $\tau(t)$ *of the pursuit curve from* $P(0)$ *to* $P(t)$ *is* $O(t^{\frac{1}{2}})$*.*

Proof. Since the distances $L_i = d(P_i, E_i)$ are monotonically nonincreasing, then all L_i are $< \pi$ and the triangles $\triangle P_{i+1}E_iE_{i+1}$ have perimeters $< 2\pi$. Thus, they have model triangles in the unit sphere. We number the angles of the triangles as in Figure 2, so that α_0 is the turning angle of P at the vertex P_1. As before we let the time step be D, and hence the first n segments of P have $n-1$ turning angles numbered $0, \ldots, n-2$ and total length $t = nD$. Let the angle corresponding to α_i be $\tilde{\alpha}_i$, so that $\alpha_i \leq \tilde{\alpha}_i$ by the CAT(1) condition. See Figure 3. Hence for $t \leq nD$,

$$\tau(t) = \sum_{i=0}^{n-2} \alpha_i \leq \sum_{i=0}^{n-2} \tilde{\alpha}_i.$$

Since the evader wins, we have $L_\infty > D$. We set $\Delta_i = L_i - L_{i+1}$.

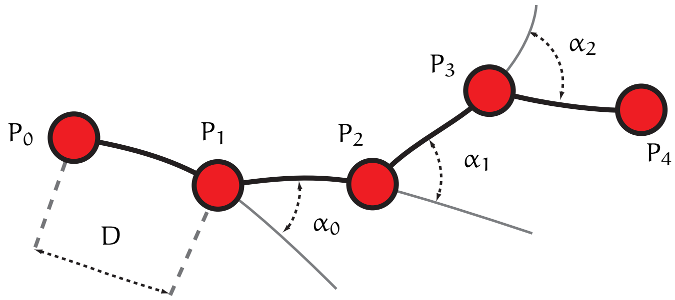

FIGURE 3. Notation for the pursuer's path.

Apply the spherical law of cosines (5.1) to the model triangles:

$$\begin{aligned} \cos D &\le \cos d(E_i, E_{i+1}) \\ &= \cos L_{i+1} \cos(L_i - D) + \sin L_{i+1} \sin(L_i - D) \cos \tilde{\alpha}_i \qquad (5.1) \\ &= \cos L_{i+1} \cos(L_i - D) + \sin L_{i+1} \sin(L_i - D) \\ &\qquad - \sin L_{i+1} \sin(L_i - D)(1 - \cos \tilde{\alpha}_i) \\ &= \cos(D - \Delta_i) - \sin L_{i+1} \sin(L_i - D)(1 - \cos \tilde{\alpha}_i) \\ &\le \cos D + \Delta_i \sin D - B_\infty \tilde{\alpha}_i^2/5, \qquad (5.2) \end{aligned}$$

where $B_\infty = \inf_i\{\sin L_{i+1} \sin(L_i - D)\}$. The last inequality depends on two elementary inequalities. To verify that $\cos(D - \Delta_i) \le \cos D + \Delta_i \sin D$, apply the Mean Value Theorem for $\cos x$ on the interval $D - \Delta_i \le x \le D$. To verify that $1 - \cos \tilde{\alpha}_i \ge \tilde{\alpha}_i^2/5$, use the identity $1 - \cos x = 2\sin^2(x/2)$. Then since $\sin(x/2)$ is concave on the interval $0 \le x \le \pi$, its graph is above the chord: $\sin(x/2) \ge x/\pi$. Since $\pi^2 < 10$, we have $2/\pi^2 > 1/5$.

Inequality (5.2) yields an inequality for $\tilde{\alpha}_i^2$:

$$\tilde{\alpha}_i^2 \le 5\Delta_i \sin D/B_\infty.$$

The following calculation completes the estimate for total curvature. It uses the Cauchy inequality

$$\tau(t) = \tau(nD) \le \sum_{i=0}^{n-2} \tilde{\alpha}_i \le \sqrt{(n-1)\sum_{i=0}^{n-2} \tilde{\alpha}_i^2},$$

and the telescoping sum

$$\sum_{i=0}^{n-2} \Delta_i = L_0 - L_{n-1} \le L_0 - L_\infty.$$

Hence

$$\tau(t) \le \sqrt{(n-1)5\sin D(L_0 - L_\infty)/B_\infty} \le C\sqrt{nD} = C\sqrt{t},$$

where

$$C = \sqrt{5(L_0 - L_\infty)/B_\infty}.$$

□

Example 14. The hypothesis $L_0 = d(P_0, E_0) < \pi$ in Theorem 13 is necessary. For example, let $\mathcal{D}$ be the complement in $\mathbb{R}^n$ of one or more disjoint open balls of radius > 1. Then $\mathcal{D}$ is a CAT(1) domain by Example 12. Start with P and E antipodal on the boundary of one of the balls. P moves toward E a distance $D < \pi$ around the boundary, while E moves to the antipodal point; from then on, P and E can oscillate between the two antipodal pairs. The total curvature of P increases by π at each step, so grows linearly rather than $O(t^{\frac{1}{2}})$.

Example 15. In CAT(K) domains for $K > 0$, there is no circumradius estimate analogous to that of Corollary 11. As a simple example of escape with bounded circumradius, consider the domain $\mathcal{D}$ of Remark 14. Let P and E travel at constant distance $< \pi$ apart around a local geodesic of $\mathcal{D}$ which is a great circle in one of the ball boundaries.

Bounded escape in the CAT(1) setting always exhibits some aspects of Example 15:

Proposition 16. *Let $\mathcal{D}$ be a compact* CAT(1) *space. Suppose the evader wins a discrete-time simple pursuit with initial distance $L_0 = d(P_0, E_0) < \pi$. Then there is a bilaterally infinite local geodesic in $\mathcal{D}$, any finite segment of which is the limit of segments of the pursuit curve.*

The proof is immediate from Theorem 13 and the following lemma:

Lemma 17. *Let $\mathcal{D}$ be a compact* CAT(1) *space. Suppose the total curvature function $\tau(t)$ of a curve $\gamma : [0, \infty) \to \mathcal{D}$ has sublinear growth. Then there is a bilaterally infinite local geodesic in $\mathcal{D}$, any finite segment of which is the limit of segments of γ.*

We draw on the following work of Maneesawarng et al:

Theorem 18. *In a* CAT(1) *space, total curvature has the following properties.*

(1) Semi-continuity [25]: *if a sequence of polygonal curves with total curvatures τ_m converges uniformly on the same parameter interval to a curve γ, then $\tau_\gamma \leq \liminf \tau_m$.*
(2) Continuity under inscription [23, 24]: *if a sequence of polygonal curves with total curvatures τ_m is inscribed in a curve γ so the maximum diameters d_m of the broken segments of γ approach 0, then $\tau_\gamma = \lim \tau_m$.*
(3) Length estimate [23, 24]: *Let γ be a curve from p to q, with $\tau_\gamma + d(p, q) < \pi$. Then $|\gamma|$ is at most the length of an isosceles once-broken geodesic in the unit sphere having the same total curvature and endpoint separation.*

Proof of Lemma 17. We claim that for any $\epsilon > 0$ and $T > 0$, there is a sequence $t_n \to \infty$ such that the total curvature $\tau(t)$ of γ satisfies $\tau(t_n + T) - \tau(t_n - T) < \epsilon$. Otherwise, any increasing sequence of such t_n would have a finite supremum, after which the growth of τ would be linear, contradicting the hypothesis. Choosing sequences $\epsilon_i \to 0$ and $T_i = 2i$, and selecting one t_i for each i, yields a sequence t_i satisfying

$$\tau(t_i + i) - \tau(t_i - i) \to 0, \; t_i \to \infty. \tag{5.3}$$

Let γ_{ik} be the the restriction of γ to $[t_i - k, t_i + k]$, $i \geq k$, reparametrized by arclength on $[-k, k]$. Writing $\tau_{ik} = \tau_{\gamma_{ik}}$, we have $\lim_{i \to \infty} \tau_{ik} = 0$ by (5.3).

By compactness of $\mathcal{D}$, the γ_{i1} have a subsequence $\tilde{\gamma}_{i1}$ that converges to a curve ρ_1. By Theorem 18 (1), $\tau(\rho_1) = 0$ and so ρ_1 is a local geodesic. Theorem 18 (3) implies $|\gamma_{i1}| \to |\rho_1|$, and so ρ_1 has length 2.

By construction, the $\tilde{\gamma}_{i1}$ for $i \geq 2$ extend to subsegments of the pursuit curve of length 4 that form a subsequence of the sequence γ_{i2}. From this subsequence we may extract a further subsequence $\tilde{\gamma}_{i2}$ that converges to a curve ρ_2 of length 4. As before, ρ_2 is a local geodesic. Since the restrictions to $[-1, 1]$ of the $\tilde{\gamma}_{i2}$ converge to ρ_1, then ρ_2 extends ρ_1.

In this manner, we obtain local geodesics ρ_k of length $2k$ for any k, each an extension of the preceding one; and hence obtain a local geodesic ρ which by construction has the desired property. □

Remark 19. There may be no *periodic* local geodesic ρ with the property described in Proposition 16 or Lemma 17. To see this, consider the Thue-Morse infinite binary word, which we write as a sequence of the integers 1 and 2. Let $\mathcal{D}$ be the CAT(1) domain given by the complement in $\mathbb{R}^2$ of two disjoint open disks of radius 1. Let $\gamma : [0, \infty) \to \mathcal{D}$ be a local geodesic that winds around the two boundary circles $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$, according to the pattern dictated by Thue-Morse word. That is, the appearance of an integer $i \in \{1, 2\}$ indicates that γ makes positively oriented contact with $\partial\mathcal{D}_i$, and a subword consisting of $k > 1$ repeats of the integer i indicates that γ also consecutively performs $k - 1$ complete, positively oriented circuits around $\partial\mathcal{D}_i$. Our claim is immediate from the fact that no subword of the Thue-Morse word repeats three times in a row [7, Theorem 1.8.1].

6. Continuous-time pursuit

In the continuous version of simple pursuit, E moves along a rectifiable curve $E(t)$ parametrized with speed ≤ 1. It is assumed that $P(t)$ moves at constant unit speed, and for each t, the right-handed velocity vector $P'(t)$ exists and points along a geodesic from $P(t)$ to $E(t)$. Thus a continuous pursuit curve is a time-dependent gradient curve for the distance function from a moving point $E(t)$.

In the CAT(K) setting, we assume that the initial separation satisfies $L(0) < \pi/\sqrt{K}$. Then $L(t)$ is non-increasing, where $L(t) = d(P(t), E(t))$, as follows immediately from the corresponding fact for discrete pursuit and Theorem 20 below. Hence the geodesic from $P(t)$ to $E(t)$ is unique. The evader wins if and only if $\lim_{t\to\infty} L(t) > 0$.

Suppose we are given a rectifiable curve $E(t)$, with $t \geq 0$ and speed ≤ 1, and an initial pursuer position $P(0)$ and positive step size D. The *discrete-time pursuit game* $P_{D,i}$ *generated by the data* $\{E(t), P(0), D\}$ has evader sequence $E_{D,i} = E(iD)$ and initial pursuit point $P_{D,0} = P(0)$. As in section 3, there are corresponding broken geodesic pursuit and evader curves $P_D(t)$ and $E_D(t)$. These discrete-time curves do not form a continuous-time simple pursuit game unless the evader curve is a geodesic with $P(0)$ on a left-end geodesic extension. We denote the separation at time t by $L_D(t) = d(P_D(t), E_D(t))$.

Jun has provided a foundation for the theory of continuous simple pursuit and its approximation by discrete simple pursuit, including existence, uniqueness and curvature properties of continuous pursuit curves. In particular we use the following theorem:

Theorem 20 ([19])**.** *In a* CAT(K) *space, let* $E(t), t \geq 0$, *be a rectifiable curve with speed* ≤ 1, *and* $P(0)$ *be an initial pursuer position with initial separation* $L(0) = d(P(0), E(0)) < \pi/\sqrt{K}$. *Consider the corresponding discrete-time pursuit games* $P_{D(m),i}$ *with step sizes* $D(m) = 2^{-m}$. *Then the sequence of discrete-time pursuit curves* $P_{D(m)}(t)$ *for* $E_{D(m)}(t)$

converges to a continuous unit-speed pursuit curve $P(t)$ *for* $E(t)$, *uniformly on any initial arc* $t \leq T$. *Moreover,* $P(t)$ *is the unique pursuit curve with initial position* $P(0)$.

An immediate consequence of Theorem 20 is that if the continuous evader wins, then eventually the generated discrete evaders also win. Indeed, if m is sufficiently large then

$$L_{D(m)}(t) > L(t)/2 > D(m). \tag{6.1}$$

Therefore the continuous versions of Theorem 8 on compact domains, and Corollary 11 on escape circumradius functions c, follow directly from these discrete theorems and Theorem 20.

Moreover, Jun showed that our estimates from the proof of Theorem 13 can be used to prove the continuous version of that theorem. We do not have to assume that the evader wins in order to get a bound on total curvature of an initial arc $t \leq T = nD$ of the pursuer, only that $L_{n-2} > D$, i. e., the evader has not been caught yet. Then, evidently, we can replace C by the time-dependent multiplier

$$C_D(T) = \sqrt{\frac{5(L_0 - L_{n-1})}{B_D(T)}}, \tag{6.2}$$

where $B_D(T) = \min\{\sin L_{i+1} \sin(L_i - D) : i < n-1\}$.

Theorem 21 ([19]). *On a* CAT(1) *domain, suppose the evader wins a continuous simple pursuit with initial distance* $L_0 = d(P_0, E_0) < \pi$. *Then the total curvature* $\tau(t)$ *of the pursuit curve from* $P(0)$ *to* $P(t)$ *is* $O(t^{\frac{1}{2}})$.

Proof. By Theorem 20 and the inequalities (6.1) we know that for m sufficiently large, $E_{D(m)}(T)$ is not caught. Then the limit of the bound on $\tau_{D(m)}(T)$ given by (6.2) is

$$C(T)\sqrt{T} = \sqrt{\frac{5(L(0) - L(T))}{B(T)}}\sqrt{T},$$

where $B(T) = \min\{\sin^2 L(0), \sin^2 L(T)\}$. By Theorem 18 (1),

$$\tau(T) \leq \liminf \tau_{D(m)}(T) \leq \lim C_{D(m)}(T)\sqrt{T} = C(T)\sqrt{T}.$$

Since the evader wins, then $\tau(t) \leq C\sqrt{t}$ where $C = \lim_{T\to\infty} C(T) > 0$. □

The continuous case of Theorem 9 may also be reduced to the discrete case:

Theorem 22. *For continuous simple pursuit on a* CAT(0) *domain, the total curvature functions of evader and pursuer satisfy*

$$\tau^P(t) \leq \tau^E(t) + \pi.$$

Proof. The result is immediate from the following chain of inequalities.

$$\tau^P(t) \leq \liminf \tau^P_m(t) \leq \liminf \tau^E_m(t) + \pi = \limsup \tau^E_m(t) + \pi = \tau^E(t) + \pi.$$

The first inequality in the chain is by Theorem 18 (1); the second is by Theorem 9; the next (equality) is by Theorem 18 (2), which applies because the generated discrete evaders E_m are inscribed in E; finally the last is the definition of total curvature. □

Proposition 16 depended only on the asymptotic estimate for the total curvature of the pursuit curve, whose continuous version is Theorem 21, and on Lemma 17, which applies to any curve, so we immediately obtain the continuous case:

Proposition 23. *Let $\mathcal{D}$ be a compact* CAT(1) *space. Suppose the evader wins in continuous simple pursuit with initial distance $L_0 = d(P_0, E_0) < \pi$. Then there is a bilaterally infinite local geodesic in $\mathcal{D}$, any finite segment of which is the limit of segments of the pursuit curve.*

ACKNOWLEDGEMENTS

This work is funded in part by DARPA # HR0011-07-1-0002 and by NSF MSPA-MCS # 0528086. A preliminary and incomplete version of this work appeared in the proceedings of the conference *Robotics: Science and Systems 2006.*

We thank Alexandr Kostochka for directing us to the Thue-Morse sequence, and the referee for suggestions that improved the paper's readability.

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APPENDIX A. BACKGROUND

Here we set out the definitions and results that are assumed in the paper. Further discussion may be found in [9, 10].

A.1. **Upper bounds on curvature and** CAT(K) **spaces.** We consider *length spaces*, which are metric spaces for which the distance between any two points is the infimum of pathlengths joining them.

A curve σ in a length space is a *geodesic* if $d(\sigma(t), \sigma(t')) = |t - t'|$ for any two parameter values t, t', and a *local geodesic* if it is a geodesic when restricted to some neighborhood of each of its parameter values. A length space is a *geodesic space* if any two points are joined by a geodesic, and a *C-geodesic space* if any two points with distance $< C$ are joined by a geodesic.

Spaces with curvature bounded above are spaces whose geodesic triangles are no 'fatter' than triangles with the same sidelengths in a model space of constant curvature, according to the following definition.

Definition 24. A geodesic metric space is CAT(0) if the distance between any two points of any geodesic triangle $\triangle pqr$ is no greater than the distance between the corresponding points of the *model triangle* $\triangle \tilde{p}\tilde{q}\tilde{r}$ with the same sidelengths in the Euclidean plane $M_0 = \mathbb{E}^2$.

A $(\pi/\sqrt{K})$-geodesic metric space is CAT(K) for $K > 0$ if the distance between any two points of any geodesic triangle $\triangle pqr$ of perimeter $< 2\pi/\sqrt{K}$ is no greater than the distance between the corresponding points of the model triangle $\triangle \tilde{p}\tilde{q}\tilde{r}$ with the same sidelengths in the 2-dimensional Euclidean sphere M_K of radius $1/\sqrt{K}$.

The significance of the perimeter bound is that a model triangle then lies in an open hemisphere and is unique up to congruence. These definitions may be unified by setting $\pi/\sqrt{K} = \infty$ if $K = 0$, as we will do from now on. For $K < 0$, CAT(K) spaces are defined similarly by taking $\pi/\sqrt{K} = \infty$ and M_K to be the hyperbolic plane of curvature K. CAT(K) spaces for $K < 0$ are automatically CAT(0) and that stronger assumption is not used in this paper. Note that rescaling a CAT(1)

space by multiplying all distances by $1/\sqrt{K}$, $K > 0$, yields a CAT(K) space, since rescaling M_1 yields M_K.

Since triangles with given sidelengths in the model spaces M_K become fatter as K increases, it is clear that a CAT(K_1) space is also a CAT(K_2) space for $K_2 > K_1$. It is an easy consequence of the definition that a CAT(0) space $\mathcal{D}$ (respectively, a CAT(K) space $\mathcal{D}$) has unique geodesics between any two points (respectively, any two points with distance $< \pi/\sqrt{K}$), and these geodesics vary continuously with their endpoints. In particular, $\mathcal{D}$ (respectively, the open ball of radius $\pi/\sqrt{K}$ about any point in $\mathcal{D}$) is simply connected.

For a simple example, take $\mathcal{D}$ to be the Euclidean plane with one or more disjoint open circular disks of radius 1 removed, where $\mathcal{D}$ is equipped with the length metric. Then $\mathcal{D}$ is not a CAT(0) space but is CAT(1), since the CAT(1) perimeter condition excludes any triangle that encloses a removed disk, and the remaining triangles are no fatter than even their Euclidean models. $\mathcal{D}$ is not simply connected, while open balls of radius π are simply connected since they do not include any boundary circle. (Since $\mathcal{D}$ is locally CAT(0), its simply connected covering is a CAT(0) space, whose geodesics are sent to the local geodesics of $\mathcal{D}$ by the covering map.)

In a CAT(K) space, the *angle* $\alpha \in [0, \pi]$ between two geodesic segments starting from a common endpoint is well-defined: it is the greatest lower bound of the corresponding angles in model triangles for triangles formed by initial subsegments of the two geodesics together with the attached third side. The thinness condition implies that these model angles descend monotonically as the subsegments are shortened, with the greatest lower bound equal to the limit.

A.2. **Reshetnyak's Majorization Theorem.** An effective tool in CAT(K) geometry is *Reshetnyak majorization*, which extends the defining comparison property of CAT(K) spaces from triangles to closed curves:

Theorem 25 (Reshetnyak [28]). *Let γ be a closed, arclength-parametrized curve of length $< 2\pi/\sqrt{K}$ in a* CAT(K) *space X. Then there is a closed, arclength-parametrized curve $\widetilde{\gamma}$ which bounds a convex region D in M_K, and a distance-nonincreasing map $\varphi : D \to X$ such that $\gamma = \varphi \circ \widetilde{\gamma}$.*

Lemma 26. *In the setting of Theorem 25:*

(1) *If a subarc $\gamma| [a, b]$ is a geodesic then the corresponding subarc $\widetilde{\gamma}| [a, b]$ is a geodesic.*
(2) *If $\gamma| [a - \epsilon, a]$ and $\gamma| [a, a + \epsilon]$ are geodesics, then the angle between them at $\gamma(a)$ is no greater than the angle between $\widetilde{\gamma}| [a - \epsilon, a]$ and $\widetilde{\gamma}| [a, a + \epsilon]$ at $\widetilde{\gamma}(a)$.*

Proof. (1): If $s = |\gamma|$, $\widetilde{s} = |\widetilde{\gamma}|$, $r = |\gamma(a)\gamma(b)|$, and $\widetilde{r} = |\widetilde{\gamma}(a)\widetilde{\gamma}(b)|$, then

$$\widetilde{r} \geq r = s = \widetilde{s} \geq \widetilde{r}.$$

Therefore $\widetilde{s} = \widetilde{r}$.

(2): Immediate from the definition of angle, since $|\gamma(t)\gamma(t')| \leq |\widetilde{\gamma}(t)\widetilde{\gamma}(t')|$ for $t < a$, $t' > a$. □

A.3. **First Variation Formula.** The First Variation Formula for CAT(K) spaces governs the rate of change of distance $r(t)$ from a point O to a unit-speed geodesic

$\gamma(t)$, $t \geq 0$, assuming $0 < r(0) < \pi/\sqrt{K}$:

$$\left.\frac{dr}{dt}\right|_{0+} = -\cos\alpha, \tag{A.1}$$

where α is the angle at $\gamma(0)$ between γ and the geodesic between O and $\gamma(0)$.

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