PERSISTENT HOMOLOGY AND EULER INTEGRAL TRANSFORMS

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ABSTRACT. The Euler calculus – an integral calculus based on Euler characteristic as a valuation on constructible functions – is shown to be an incisive tool for answering questions about injectivity and invertibility of recent transforms based on persistent homology for shape characterization.

1. INJECTIVE TRANSFORMS BASED ON PERSISTENT HOMOLOGY.

The past fifteen years have witnessed the rise of Topological Data Analysis as a novel means of extracting structure from data. In its most common form, data means a point cloud sampled from a subset of Euclidean space, and structure comes from converting this to a filtered simplicial complex and applying persistent homology (see [2, 5] for definitions and examples). This has proved effective in a number of application domains, including genetics, neuroscience, materials science, and more.

Recent work considers an inverse problem for shape reconstruction based on topological data. In particular, [7] defines a type of transform which is based on persistent homology as follows. Given a (reasonably tame) subspace $X \subset \mathbb{R}^n$, one considers a function from $\mathbb{S}^{n-1} \times \mathbb{N}$ to the space of persistence modules over a field $\mathbb{F}$. For those familiar with the literature, this *persistent homology transform* records sublevelset homology barcodes in all directions ($\mathbb{S}^{n-1}$) and all gradings ($\mathbb{N}$). The paper [7] contains the following contributions.

(1) For compact nondegenerate shapes in $\mathbb{R}^2$ and compact triangulated surfaces in $\mathbb{R}^3$, the persistent homology transform is injective; thus one can in principle reconstruct the shapes based on the image in the space of persistence modules. The proof is an algorithm.
(2) It is claimed that the proof survives reduction to the Euler characteristic, so that knowing all Euler characteristics of the intersection of the shape with all half-spaces in $\mathbb{R}^2$ or $\mathbb{R}^3$ (resp.) yields a likewise injective transform.
(3) Certain results on *sufficient statistics* follow from this injectivity, which are then applied to shape characterization (see also [3]). This is effected by discretizing the Euler characteristic transform both in direction and along the filtration.

This note reformulates the persistent homology transform of [7] in terms of Euler calculus on constructible functions. Though a more abstract framework, the theory effortlessly permits the following results.

(1) The Euler characteristic reduction of the persistent homology transform extends to an integral transform on constructible functions.
(2) This integral transform has an explicit inverse, with no restrictions on dimension, manifold structure, or nondegeneracy (beyond constructibility).
(3) This integral transform is shown to be but one of several invertible transforms that characterizes shapes with topological data.

2. EULER CALCULUS.

Euler characteristic is an integer-valued "compression" of a finitely-nonzero sequence $V_\bullet$ of finite-dimensional vector spaces over a field $\mathbb{F}$ given by the alternating sum of dimensions. Among complexes, Euler characteristic is an invariant of quasi-isormorphism, meaning that for $C_\bullet$ a complex of vector spaces and $H_\bullet$ its homology, $\chi(H_\bullet) = \chi(C_\bullet)$. On compact cell complexes, χ is well-defined and a homotopy invariant. Euler characteristic is additive on compact cell complexes, meaning that for A and B such, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$.

It is profitable to pass from the realm of compact cell complexes to more general *definable* or *constructible* subsets of $\mathbb{R}^n$ by using compactly-supported cohomology. This, combined with results from *o-minimal structures* [8] makes it trivial to work with an additive and homeomorphism-invariant Euler characteristic on definable sets. For the reader unfamiliar with the o-minimal theory, it suffices to substitute *semialgebraic* for *definable* or *constructible* in what follows.

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For X a definable subset of Euclidean space, the *constructible functions* on X are functions $h\colon X \to \mathbb{Z}$ that have definable (and locally finite) level sets. The set of constructible functions, $\mathsf{CF}(X)$, has the structure of a sheaf with the obvious restriction maps.[1] The Euler integral on X is simply the functional

$$\int_X \cdot \, d\chi \colon \mathsf{CF}(X) \to \mathbb{Z} \quad \text{taking} \quad \mathbf{1}_\sigma \mapsto (-1)^{\dim \sigma} \tag{1}$$

for each (open) definable simplex σ. As all definable sets are finitely definably triangulated, the Euler integral is well-defined and additive. Euler calculus possesses a Fubini Theorem, a convolution operation, and much more. For a thorough introduction, see [4].

3. Euler-Radon transform & inversion.

The first application of Euler calculus to integral transforms was given by Schapira in a seminal paper [6] that defined a topological Radon transform and gave conditions for an inverse to exist. Our summary follows the reformulation in [1] to weighted kernels. Consider a pair (X, Y) of definable spaces and $K \in \mathsf{CF}(X \times Y)$ a kernel — a constructible function on the product. The Radon transform $\mathcal{R}_K \colon \mathsf{CF}(X) \to \mathsf{CF}(Y)$ is defined explicitly via the formula

$$(\mathcal{R}_K h)(y) = \int_X h(x)\, K(x, y)\, d\chi(x). \tag{2}$$

The principal result of [6] is the following. Consider a second kernel $K' \in \mathsf{CF}(Y \times X)$ with Radon transform $\mathcal{R}_{K'} \colon \mathsf{CF}(Y) \to \mathsf{CF}(X)$. If there are constants λ, μ such that

$$\int_Y K(x, y) K'(y, x')\, d\chi(y) = (\mu - \lambda)\delta_\Delta + \lambda, \tag{3}$$

for $\Delta \subset X \times X$ the diagonal, then

$$(\mathcal{R}_{K'} \circ \mathcal{R}_K)h = (\mu - \lambda)h + \lambda \left(\int_X h\, d\chi \right) \mathbf{1}_X. \tag{4}$$

Thus, when $\lambda \neq \mu$, one can recover h exactly from the inverse transform (followed by the appropriate rescaling).

The point of this note is to show that working with Euler integral transforms is preferable to mapping a set into a space of persistence modules, as the Euler transform provides a more efficient representation that yields full invertibility, not merely injectivity.

4. Inversion for the sublevelset Euler integral transform.

The persistent homology transform of [7] is easily converted into a Radon integral transform. Let $X = \mathbb{R}^n$ and $Y = \mathbb{S}^{n-1} \times \mathbb{R}$ with kernel K the indicator function on the set $\{(x, (\xi, t)) \colon x \cdot \xi \le t\}$. Given the resemblance to sublevelset filtrations in persistent homology, we denote this the *sublevelset Euler integral transform*.

Theorem 5. *The sublevelset Euler integral transform $\mathcal{R}_K \colon \mathsf{CF}(X) \to \mathsf{CF}(Y)$ is invertible for all dimensions n.*

Proof. Consider as the dual kernel K' the indicator function of the set

$$\{(x, (\xi, t)) \colon x \cdot \xi \ge t\}.$$

One observes the following.

Denote by K_x the set of all (ξ, t) such that x lies in the halfspace $x \cdot \xi \le t$. Likewise with the dual fiber K'_x reversing the inequality. The intersection $K_x \cap K'_x$ is the set of (ξ, t) with the property that for each $\xi \in \mathbb{S}^{n-1}$, there is a unique t at which $x \cdot \xi = t$. Thus, $\mu = \chi(K_x \cap K'_x) = \chi(\mathbb{S}^{n-1}) = 1 - (-1)^n$.

For $x \neq x'$, the intersection $K_x \cap K'_{x'}$ is the set of all (ξ, t) such that $x \cdot \xi \le t$ and $x' \cdot \xi \ge t$. For fixed $\xi \in \mathbb{S}^{n-1}$, the set of compatible t is empty if $(x - x') \cdot \xi < 0$ and is a compact interval when $(x - x') \cdot \xi \ge 0$. Thus, $K_x \cap K'_{x'}$ is a compact contractible set, and $\lambda = \chi(K_x \cap K'_{x'}) = 1$.

As $\lambda \neq \mu$, the transform is invertible for all n. □

Corollary 6. *The persistent homology transform of* [7] *and the smoothed Euler characteristic transform of* [3] *are invertible on constructible subsets of $\mathbb{R}^n$ for all n.*

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[1] This structure, though very helpful for generating clean definitions, can be ignored by the reader for whom sheaves are unfamiliar.

5. Additional invertible transforms.

The sublevelset Euler integral transform is but one of several invertible transforms on $X = \mathbb{R}^n$. As the Euler calculus appears underutilized, and as these transforms are so simple to define and invert, it seems appropriate to recall some known invertible topological integral transforms.

(1) The original example of Schapira's inversion formula has Y equal to the affine Grassmannian of hyperplanes in $X = \mathbb{R}^n$. Thus, recording all Euler characteristics of all flat codimension-1 slices is an invertible transform (with self-dual kernel).

(2) The article [1] gives several other examples of invertible transforms, including the following. Let C be a compact convex definable subset of $X = \mathbb{R}^n = Y$ with kernel K the indicator function on the set $\{x - y \in C\}$. Thus, $\mathcal{R}_K$ is a constructible "blur" with filter C. This is an invertible transform for all n.

These examples are far from exhaustive. To close, we present a few novel invertible topological integral transforms.

(1) Schapira's original example with the affine Grassmannian has a stereographic variant. Let $X = \mathbb{D}^n$ be a closed ball and $Y = \partial\mathbb{D} \times \mathbb{R}^{\geq 0}$. The (self-dual) kernel is given as the indicator function on the set $\{||x-y|| = t\}$: one measures distance to a point on the boundary of X. The resulting transform is invertible for all n with $\mu = \chi(\mathbb{S}^{n-1})$ and $\lambda = \chi(\mathbb{S}^{n-2})$.

(2) The previous example can be modified to a sublevel/superlevel setting, analogous to the persistent Euler integral transform of this note. Keeping X and Y as before, one can set K to be the indicator function on the set $\{||x - y|| \leq t\}$ with the dual kernel K' reversing the inequality. This transform is invertible for all n with μ and λ unchanged. These two examples suggest generalizations to other geometric domains with boundary.

(3) Let $X = \mathbb{R}^n = Y$ with γ a codimension-0 cone in $\mathbb{R}^n$ with vertex at the origin that does not contain a half-space. Let $K = K'$ be the indicator function over the set $\{(x, y)\colon x - y \in (\gamma \cup -\gamma)$. Then, for all $n > 1$, this transform is invertible with $\mu = -1$ and $\lambda = 0$.

In the same manner that the persistent Euler integral transform is discretized (and smoothed) to vectorize shape data [7, 3], one can discretize any of the invertible Euler integral transforms defined above to use as a statistic for shapes (or more general constructible functions).

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