ON THE CONTACT TOPOLOGY AND GEOMETRY OF IDEAL FLUIDS

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ABSTRACT. We survey certain topological methods for problems in inviscid fluid dynamics in dimension three. The tools come from the topology of *contact structures*, or nowhere-integrable plane fields. The applications are most robust in the setting of fluids on Riemannian three-manifolds which are not necessarily Euclidean. For example, these methods can be used to construct surprising examples of inviscid flows in non-Euclidean geometries. Because of their topological basis, these methods point one toward a theory of "generic" fluids, where the geometry of the underlying domain is the genericity parameter.

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1. Ideal fluids on Riemannian manifolds

1.1. **The Euler equation.** The equation of motion of an unforced, incompressible, inviscid fluid with velocity field $\mathbf{u}(\mathbf{x}, t)$ is

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p, \quad ; \quad \mathrm{div}(\mathbf{u}) = 0, \tag{1.1}$$

where p denotes a real-valued pressure function. In most instances in the literature, the domain in which the fluid resides is a Euclidean domain. On those occasions when compactness is desired and the complexities of boundary conditions are not, the fluid domain is usually taken to be a Euclidean torus T^3 given by quotienting out Euclidean space by the action of three mutually orthogonal translations. Although such a situation is of dubious physical relevance, it is nevertheless a fairly common domain in the literature on mathematical fluids.

There are several unavoidable problems in the attempt to analyze Euler flows in dimension three, not the least of which is that the fundamental starting point, the global existence of solutions to the Euler equations, is unknown in dimension three and perhaps not true. And although fluid dynamics can lay claim to having inspired many ingenious and fundamental contributions to analysis and PDEs, a casual reading of the relevant literature shows that fluid dynamics in general (both viscous and inviscid fluids) is "hard" in dimension three without some type of symmetry or other reduction to a lower-dimensional setting. Despite what we teach our undergraduate students about general principles for fluid flows — e.g., Kelvin's Circulation Theorem — there is not an abundance of theorems which hold for ideal fluid flows in dimension three.

Since so little is known about the rigorous behavior of fluid flows, any methods which can be brought to bear to prove theorems about their behavior are of interest and potential use. Following the pioneering work of Arnold [3, 4], Moffatt [85], and others, we propose that a more geometric and topological view of fluid dynamics can provide new tools and insights which lead to very general results. Neither geometric nor topological approaches to fluid dynamics are novel: Tait's initial foray into knot theory was inspired by Kelvin's interests in knotted vortex tubes in the æther [101]. The contact topological tools which we survey in this paper are of rather recent relevance to fluid dynamics.

1.2. **An analogy.** The precise topological tools which this article discusses are those which come from a particular branch of geometric topology concerning *contact structures*. In a three-dimensional domain, a contact structure is a field of tangent planes which varies smoothly point-to-point and which is "nowhere integrable", meaning that these planes, unlike scales on a fish, do not fit together to yield two-dimensional sheets. Such plane fields possess a wealth of wonderful local and global properties: see §3 for detailed information. In this article, contact structures will arise naturally as plane fields orthogonal to certain vector fields which solve the steady Euler equations. This yields a variety of novel topological methods and results which apply to arbitrary three-dimensional domains.

This connection between the topology of a steady Euler field and its orthogonal plane field is not so foreign as might at first appear. Indeed, the study of steady (time-independent) inviscid fluids on two-dimensional domains has a very topological feel to it (cf. texts such as [20]) which is not unrelated to the notion of a dual hyperplane field. We sketch out a simple analogy in dimension two which illustrates the naturality of contact topological perspectives in dimension three.

Consider a steady Euler field $\mathbf{u}(x, y) = (u(x, y), v(x, y))$ on the Euclidean plane $\mathbb{R}^2$. Incompressibility implies that $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ and so, as any student of fluid dynamics knows, there exists a *stream function* $\Psi : \mathbb{R}^2 \to \mathbb{R}$ which is constant along integral curves of $\mathbf{u}$. The coarse qualitative features of a steady Euler field $\mathbf{u}$ on $\mathbb{R}$ are perfectly encoded in this stream function. For example, the critical points of Ψ correspond to the stagnation points of the fluid. For an incompressible fluid in 2-d, there are two types of (nondegenerate) stagnation points: an *elliptic* point (or *center*) and a *hyperbolic* point (or *saddle*), as illustrated in Figure 1.

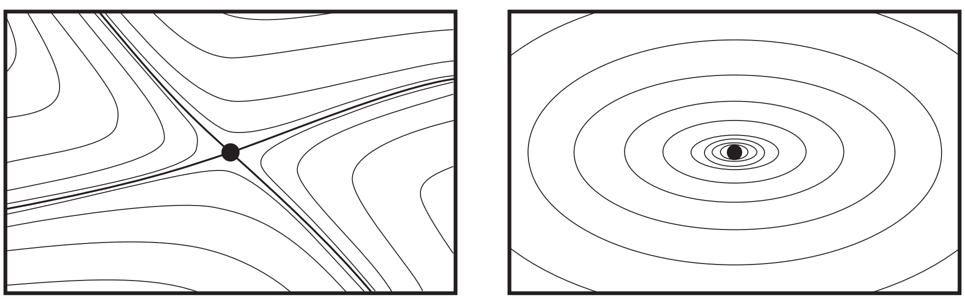

FIGURE 1. Two types of stagnation points in a planar flow: hyperbolic [left] and elliptic [right]

There is a dynamical means of extracting this information from Ψ: consider the gradient field $\nabla\Psi$. This auxiliary dynamical system tells one what is happening in the direction *orthogonal to the fluid flow*. The fixed points of $\nabla\Psi$ are precisely the stagnation points of $\mathbf{u}$ and the *Morse index* of a fixed point for Ψ (the dimension of its unstable manifold) says whether the critical point of Ψ is a local max or min (an elliptic stagnation point) or is a saddle (a hyperbolic stagnation point).

Upon moving to dimension three, the Bernoulli Theorem states that there is a real-valued function H which, like Ψ above, is constant along flowlines. The first general theorem for 3-d flows we survey, due to Arnold, states that as long as H is not a constant, almost all flowlines are constrained to tori which fill up the 3-d domain. For this case, the gradient of the function H gives dynamical information in the direction orthogonal to the level sets of H and likewise allows one to recapture the rough features of the flow.

But three-dimensional ideal fluids admit the possibility of fully nonintegrable flowlines in certain cases (the eigenfields of the curl operator). The theme of this survey is that instead of trying to generalize the idea of a stream function, the appropriate generalization is to consider the topological structure of what is happening orthogonal to the velocity field. In 2-d, this orthogonal structure is a line field, which corresponds to the gradient field of the stream function. In 3-d, the analogous orthogonal structure takes the form of a *plane field*. For those steady Euler flows with nonintegrable dynamics, this plane field likewise exhibits its own form of nonintegrability: it is a *contact structure*. A careful analysis of the topology and dynamics of contact structures can yield global information about the velocity field, including the existence of periodic orbits and the types of periodic orbits which arise (elliptic and hyperbolic).

This analogy is not strict, but rather points to the fact that there is a relationship between the dynamics of a steady ideal fluid and the geometry of the field orthogonal to the fluid.

1.3. **A geometric formulation.** All of the results of the previous section about the stream function in 2-d are dependent only on the fact that the domain is $\mathbb{R}^2$: the precise Euclidean features of the domain geometry are not necessary for the existence of a stream function. Such is the case in higher-dimensional fluids as well.

We begin by interpreting the Euler equations on more arbitrary geometric domains: see, e.g. [1, §8.2]. Let M denote a sufficiently smooth, connected, oriented differential manifold. In order to make sense of operations such as directional derivative and divergence, it behooves us to assume an underlying geometry and volume on the flow domain M. We therefore take g to be a Riemannian metric on M: a symmetric 2-tensor which defines an inner product on tangent spaces of M. We also choose a volume form μ on M, a top-dimensional form which is pointwise nonvanishing. One can of course choose the precise volume form μ_g induced by the metric; however, for the sake of generality, we allow for arbitrary μ. This has physical significance to compressible *isentropic* fluids, as noted in [6]. See, e.g., [1] for a wealth of background material on the tools and language of global geometry and analysis on manifolds.

The form which the Euler equations take on an oriented Riemannian manifold is the following:

$$\frac{\partial \mathbf{u}}{\partial t} + \nabla_{\mathbf{u}}\mathbf{u} = -\nabla p \quad ; \quad \mathcal{L}_{\mathbf{u}}\mu = 0, \tag{1.2}$$

where $\nabla_{\mathbf{u}}$ is the covariant derivative along $\mathbf{u}$ defined by the metric g, and $\mathcal{L}_{\mathbf{u}}$ is the Lie derivative along $\mathbf{u}$.

We give an explicit example on the *3-sphere*, S^3, the points in $\mathbb{R}^4$ a unit distance from the origin. Via stereographic projection, this 3-manifold is equivalent to $\mathbb{R}^3$ with an added "point at infinity." The *round* metric on S^3 is that inherited by it as a subset of Euclidean $\mathbb{R}^4$, much in the same manner as we would describe the geometry of the round 2-sphere $S^2 \subset \mathbb{R}^3$. The simplest example of a steady Euler field on the round S^3 is that given by the *Hopf field*, pictured in Figure 2. One may realize this field in Euclidean coordinates on $\mathbb{R}^4$ via

$$\mathbf{u} = \begin{cases} \dot{x_1} = -x_2 \quad ; \quad \dot{x_3} = -x_4 \\ \dot{x_2} = x_1 \quad ; \quad \dot{x_4} = x_3 \end{cases} \tag{1.3}$$

These equations correspond to the Hamiltonian equations of two identical uncoupled simple harmonic oscillators at a fixed energy level; it therefore preserves the natural volume form on the round 3-sphere. Also, because the oscillators are identical, every orbit of the flow is periodic. This exhibits a very special type of solution to the Euler equations: $\nabla_{\mathbf{u}}\mathbf{u}$ is identically zero, meaning that every flowline is a geodesic on the round S^3.

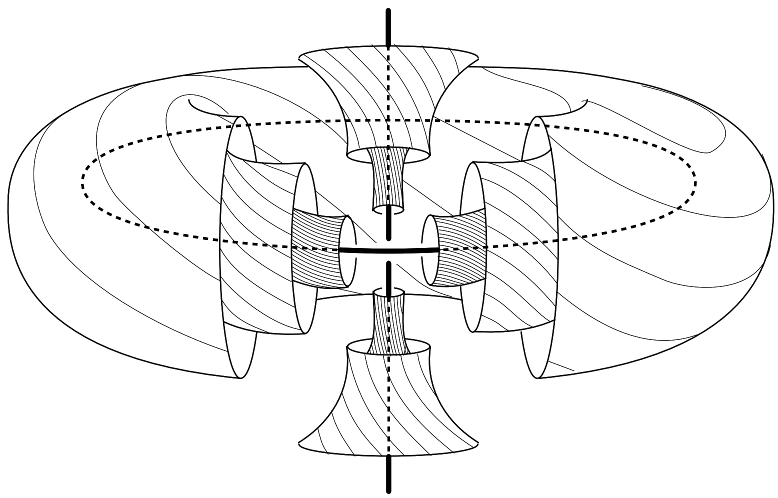

FIGURE 2. The Hopf flow on $S^3 = \mathbb{R}^3 \cup \{\infty\}$. Every flowline is a simple closed curve, each pair of which is linked.

The vector-calculus identity for the directional derivative in dimension three,

$$(\mathbf{u}\cdot\nabla)\mathbf{u} = (\nabla\times\mathbf{u})\times\mathbf{u} + \frac{1}{2}\nabla\left(\mathbf{u}\cdot\mathbf{u}\right),$$

has a more general geometric formulation [1, p. 588] by using the $^\flat$ operation to transform vector fields to dual 1-forms:

$$(\nabla_{\mathbf{u}}\mathbf{u})^\flat = \mathcal{L}_{\mathbf{u}}\mathbf{u}^\flat - \frac{1}{2}d\left(\mathbf{u}^\flat(\mathbf{u})\right).$$

This allows one to transform (1.2) into an equation on differential forms:

$$\frac{\partial(\iota_{\mathbf{u}}g)}{\partial t} + \mathcal{L}_{\mathbf{u}}\iota_{\mathbf{u}}g - d\left(\frac{1}{2}\iota_{\mathbf{u}}\iota_{\mathbf{u}}g\right) = -dp \quad ; \quad \mathcal{L}_{\mathbf{u}}\mu = 0.$$

Here, $\mathbf{u}^\flat = \iota_{\mathbf{u}}g = g(\mathbf{u},\cdot)$ denotes the one-form obtained from $\mathbf{u}$ via contraction into the first slot of the metric. One uses the Cartan formula $\mathcal{L}_{\mathbf{u}} = d\iota_{\mathbf{u}} + \iota_{\mathbf{u}}d$ and absorb the energy terms into the pressure function to obtain

$$\frac{\partial(\iota_{\mathbf{u}}g)}{\partial t} + \iota_{\mathbf{u}} d\iota_{\mathbf{u}} g = -dH \quad ; \quad \mathcal{L}_{\mathbf{u}}\mu = 0, \tag{1.4}$$

where $H = p + \frac{1}{2}\iota_{\mathbf{u}}\iota_{\mathbf{u}}g$. The dependence of H upon $\mathbf{u}$ is of no consequence — one says that $\mathbf{u}$(t) is an Euler field if it satisfies the Euler equations for *some* function $H(t)$.

The vector calculus form of (1.4) is

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \times (\nabla \times \mathbf{u}) = -\nabla H,$$

where $\nabla \times \mathbf{u}$ is the *vorticity* of the fluid. In terms of differential forms, the vorticity corresponds to the 2-form $d\iota_{\mathbf{u}}g$. One correspondingly defines the *curl operator* for a Riemannian 3-manifold with volume (M, g, μ) to be the linear operator $\nabla\times$ on divergence-free vector fields which takes a vector field $\mathbf{u}$ to the field $\nabla \times \mathbf{u}$ satisfying

$$\iota_{\nabla\times\mathbf{u}}\mu = d\iota_{\mathbf{u}}g. \tag{1.5}$$

This operator is well-defined since μ is a nonvanishing 3-form. In the language of dual 1-forms, the curl operator is $*d$, where $*$ is the Hodge star operator [1].

With this notion of curl, the Euler equations take the particularly clean form

$$\frac{\partial(\iota_{\mathbf{u}}g)}{\partial t} + \iota_{\mathbf{u}}\iota_{\nabla\times\mathbf{u}}\mu = -dH \quad ; \quad \mathcal{L}_{\mathbf{u}}\mu = 0. \tag{1.6}$$

By taking the exterior derivative of (1.6), one obtains the *Helmholtz* equation for the evolution of the vorticity 2-form $\omega = d\iota_{\mathbf{u}}g$:

$$\frac{\partial\omega}{\partial t} + \mathcal{L}_{\mathbf{u}}\omega = 0. \tag{1.7}$$

This equation has the advantage of discarding the exact pressure-related term dH from (1.6).

For more information on the equations of motion for fluids on Riemannian manifolds, see [7, 1].

2. THE GEOMETRY OF STEADY SOLUTIONS

There are numerous deep open problems about the solutions to (1.6), the most famous being the existence of finite-time blow-ups. We circumvent this and other delicate questions by focusing on a much simpler problem, namely, the problem of existence and classification of steady solutions to (1.6). This approach is not without precedence in the broader context of dynamical systems. It is an argument going back to the work of Poincaré and elucidated by Conley, Hale, and others, that in any dynamical system it is the *bounded* solutions which are most important and which should be investigated first. These, then, are stratified according to dimensions of invariant sets: first are the fixed points, then connecting orbits between fixed points along with periodic orbits, etc. We therefore consider most carefully the fixed points of the dynamical system that the Euler equation induces on the space of volume-preserving vector fields on a fixed Riemannian manifold.

A classical result of Arnold's on integrable Hamiltonian systems [5] has applications to the geometry and topology of fluids. It has an unfortunately strong smoothness requirement: all fields, metrics, and volumes are assumed to be C^ω (real-analytic).

Theorem 2.1 (Arnold [7]). *Let $\mathbf{u}$ be a C^ω nonvanishing Euler field on a closed oriented Riemannian three-manifold M. Then, at least one of the following is true:*

1. *There exists a compact analytic subset $\Sigma \subset M$ of codimension at least one which splits M into a finite collection of cells diffeomorphic to $T^2 \times \mathbb{R}$. Each torus $T^2 \times \{x\}$ is an invariant set for $\mathbf{u}$ having flow conjugate to linear flow.*
2. *The field $\mathbf{u}$ is a curl-eigenfield: $\nabla \times \mathbf{u} = \lambda \mathbf{u}$ for some $\lambda \in \mathbb{R}$.*

Proof: The Bernoulli Theorem states that H is an integral for $\mathbf{u}$, namely,

$$\mathcal{L}_{\mathbf{u}} H = \iota_{\mathbf{u}} dH = \iota_{\mathbf{u}} \iota_{\mathbf{u}} \iota_{\nabla\times\mathbf{u}} \mu = 0. \tag{2.1}$$

The Inverse Function Theorem implies that, when c is not a critical value of $H : M \to \mathbb{R}$, $H^{-1}(c)$ is a closed 2-manifold which has a nonvanishing vector field $\mathbf{u}$ tangent to it. Thus it has Euler characteristic zero. Since $dH \neq 0$ here, dH orients the surface and thus $H^{-1}(c)$ is a disjoint union of tori for all regular values c. Since this is a steady Euler field, the vorticity is also time-independent, and Equation (1.7) implies that the velocity and vorticity fields commute. This implies that the flow on each such torus is conjugate to linear flow, since the flowlines of the velocity field are transported by the vorticity field.

Define $\Sigma \subset M$ to be the inverse image of the critical values of H. As all the data in the equation is assumed real-analytic, H must be C^ω. If H is non-constant, real-analyticity implies that Σ is a C^ω subset of M of codimension at least one, and the complement of Σ is composed of invariant tori with linear flow.

The only other possibility is that H is constant and $\Sigma = M$; namely, that the vorticity and velocity fields are collinear at each point. Assume $\nabla \times \mathbf{u} = h\mathbf{u}$ for some $h : M \to \mathbb{R}$. In this case h is an integral of $\mathbf{u}$:

$$0 = d(d\iota_{\mathbf{u}} g) = d(h\iota_{\mathbf{u}}\mu) = dh \wedge \iota_{\mathbf{u}}\mu + hd\iota_{\mathbf{u}}\mu = dh \wedge \iota_{\mathbf{u}}\mu, \tag{2.2}$$

which implies that $\iota_{\mathbf{u}} dh = 0$. Since h is an integral for $\mathbf{u}$, the same argument as that for H yields the appropriate set $\sigma \subset M$ off of which the flow consists of invariant tori.

The only instance in which $\mathbf{u}$ is not integrable is thus when h is constant and $\mathbf{u}$ is a curl eigenfield. □

There is a surprising corollary to this proof: 'most' 3-manifolds do not admit a nonvanishing (analytic) integrable vector field. Any 3-manifold which admits such a decomposition into blocks of the form $T^2 \times \mathbb{R}$ glued together appropriately can be outfitted with a *round handle decomposition* [14]. Those 3-manifolds which admits such a decomposition have been classified and are among the so-called *graph 3-manifolds*. These are relatively rare [104, 105], e.g., no 3-manifold which admits a metric of constant negative curvature (a *hyperbolic* 3-manifold) can ever appear. This is a very powerful conclusion, and leaves us with two choices: for sufficiently smooth fluid flows on a typical closed 3-manifold, the only steady ideal fluids either (1) always have stagnation points, or (2) are curl eigenfields.

Arnold's Theorem leads one naturally to examine carefully the class of fields for which the integral H degenerates. A vector field $\mathbf{u}$ is said to be a *Beltrami field* if it is parallel to its curl: *i.e.*, $\nabla \times \mathbf{u} = h\mathbf{u}$ for some function h on M. A *rotational* Beltrami field is one for which $h \neq 0$; that is, it has nonzero curl.[1]

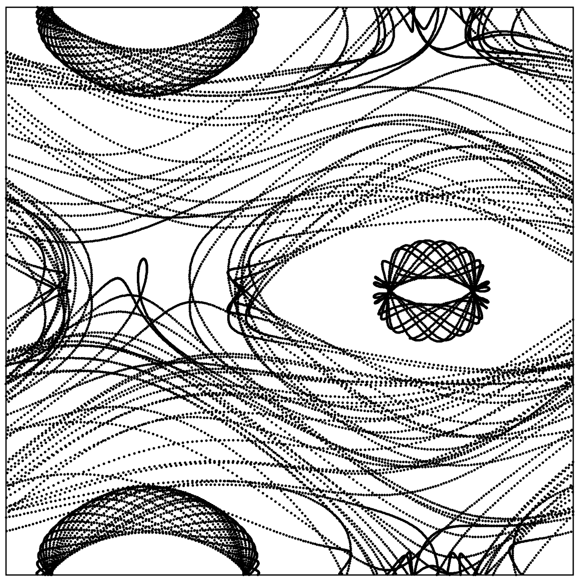

FIGURE 3. A 2-d projection of some flowlines of the ABC fields on a periodic $\mathbb{R}^3$. Some orbits of this nonintegrable field are not constrained to 2-d surfaces.

Curl eigenfields possess several interesting geometric, analytic, and dynamical features: see [7, 59, 22, 47] for more information. A key example of a curl eigenfield is the class of *ABC fields*:

$$\begin{aligned} \dot{x} &= A\sin z + C\cos y \\ \dot{y} &= B\sin x + A\cos z \\ \dot{z} &= C\sin y + B\cos x \end{aligned} \quad . \tag{2.3}$$

Here, $A, B, C \geq 0$ are constants, and the vector field is defined on the three-torus T^3. By symmetry in the variables and parameters, we may assume without loss

[1]There is some variation in the literature over whether *Beltrami* means that vorticity is parallel to velocity or, more restrictively, a scalar multiple. For purposes of this article, we use *Beltrami* to mean parallel and *curl eigenfield* to mean a scalar multiple. In [7], the term *"Beltrami"* means that vorticity is a scalar multiple of velocity.

of generality that $1 = A \geq B \geq C \geq 0$. Under this convention, the vector field is nonvanishing if and only if $B^2 + C^2 < 1$ (see [22]). At 'most' parameter values, the ABC fields exhibit a so-called *Lagrangian turbulence* — there are apparently large regions of nonintegrability and mixing within the flow. Though this has nothing whatsoever to do with genuine turbulence in fluids, it remains a fascinating phenomenon.

Although the ABC flows have been repeatedly analyzed [3, 22, 47, 54, 70, 107], few rigorous results are known, other than for near-integrable examples. Beltrami fields in general are even less well understood, due to the fact that nonintegrable dynamics is both prevalent and difficult. We survey some topological tools which are especially suited to global problem in nonintegrable dynamics, with a focus on curl eigenfields. In particular, we consider the following problems:

(1) Given a steady Euler flow on a Riemannian 3-manifold, are there any periodic flowlines?
(2) In the case where there are periodic orbits, which knot and link types may/must occur?
(3) For a 'generic' steady Euler flow, is this solution hydrodynamically stable or unstable?

Curl eigenfields occupy an important place not only within hydrodynamics [7, 22, 43, 84], but also within the study of magnetic fields and plasmas (where they are known as *force-free fields*) [7, 16, 81] and in the stability of matter [78]. It is not unreasonable to hope that the techniques surveyed here have implications in these fields as well.

3. Basic contact topology

This rather lengthy excursion into contact topology is justified by the later use of nearly every definition, example, and theorem in the remainder of this survey.

We restrict to the setting of three-dimensional manifolds, although contact structures are defined in any odd dimension. Roughly speaking, a contact structure on a three-manifold is a distribution of smoothly varying tangent plane fields which "twist" so as to be maximally nonintegrable. Recall that any nonvanishing smooth vector field on a manifold can be *integrated*, or stitched together so as to fill up the space by curves tangent to the vector field. An integrable plane field likewise comes from a foliation of the space by two-dimensional sheets tangent to the plane field. The subject of whether a field of k-dimensional subspaces of the tangent space to a manifold is integrable is classical. Interestingly enough, the subject of *contact transformations* was long ago very closely related to topics in partial differential equations related to the method of characteristics: see, e.g., the text of Goursat from 1891 [53]. The subject of contact structures finds its origin in this time period, when Lie considered "contact elements" of curves were studied. We present a modern formulation.

3.1. **Definitions.** The following definitions come from the proper interpretation of the Frobenius condition on the integrability of a distribution, and provides a way of saying that a plane distribution is maximally nonintegrable via the calculus of differential forms [1]. Let M denote a three-dimensional manifold. A *contact form* on M is a differential 1-form α satisfying

$$\alpha \wedge d\alpha \neq 0 \tag{3.1}$$

pointwise on M. A *contact structure* is a smooth plane field ξ on M which is (locally) the kernel of a contact form, namely

$$\xi = \ker(\alpha) \quad \Leftrightarrow \quad \alpha(v) = 0 \quad \forall\, v \in \xi. \tag{3.2}$$

The standard contact structure on $\mathbb{R}^3$ is the kernel of the contact form $dz + x\,dy$. We see that its kernel consists of tangent planes in $\mathbb{R}^3$ which (conflating local and global coordinates) satisfy

$$\frac{dz}{dy} = -x,$$

and thus correspond to the plane field illustrated in Figure 4. Along the $y - z$ plane, all the contact planes have slope zero. As one walks along the x direction, the planes twist in a counterclockwise fashion. The reader should convince himself that such a plane field cannot arise as the tangent planes to a 2-dimensional foliation of $\mathbb{R}^3$.

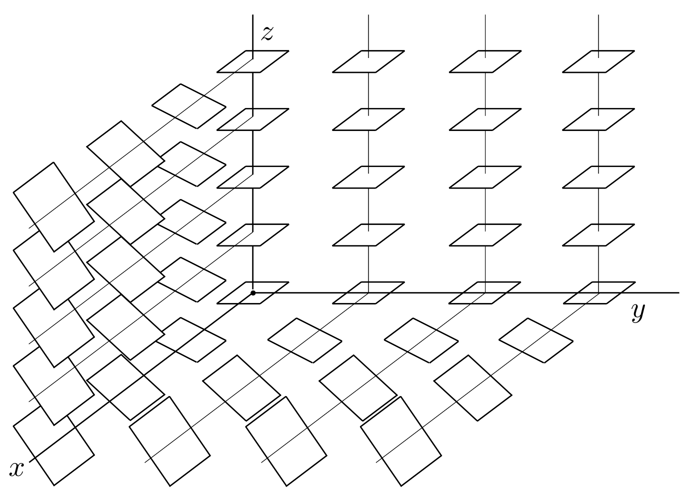

FIGURE 4. The standard contact structure on $\mathbb{R}^3$.

More interesting examples are abundant. The standard contact structure on the 3-sphere $S^3 \subset \mathbb{R}^4$ is given by the kernel of the 1-form

$$\alpha_H = \frac{1}{2}\left(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3\right). \tag{3.3}$$

The contact structure $\xi = \ker(\alpha_H)$ is the plane field orthogonal to the Hopf field on the round sphere in (1.3).

Both of the above examples are *coorientable* contact structures — they are the kernel of a globally defined 1-form. There exist non-coorientable contact structures, just as there exist line fields on a surface which are not coorientable. However, since all the contact structures which arise in fluid dynamics come with a globally-defined contact form, we will assume for the remainder of this chapter that all relevant contact structures are cooriented.

3.2. **Local contact topology.** The interesting (and difficult) problems in contact geometry are all of a global nature, thanks to the following results. These results give local classifications for contact structures in terms of the natural equivalence relation. This relation is *contactomorphism*, or, diffeomorphism which takes one contact structure to another.

Theorem 3.1 (The Darboux Theorem)**.** *All contact structures on a 3-manifold are locally contactomorphic to the standard structure on* $\mathbb{R}^3$*.*

See, e.g., [83, 2] for a proof. In polar coordinates, the standard contact structure is contactomorphic to the kernel of $dz + r^2 d\theta$: see Figure 5[left]

There is a broad generalization of the Darboux Theorem due to Moser and Weinstein that characterizes a contact structure in a neighborhood of a surface as opposed to a neighborhood of a point. Given a surface Σ embedded in M, the contact structure ξ on M "slices" Σ along a (possibly singular) line field. This induces the *characteristic foliation*, Σ_ξ. The Moser-Weinstein Theorem implies that studying characteristic foliations is an effective means of analyzing the contact structure.

Theorem 3.2 (The Moser-Weinstein Theorem)**.** *The contactomorphism type of a neighborhood of* Σ *in* M *is determined by the characteristic foliation* Σ_ξ*.*

One final local result in contact topology says that deforming a contact structure on M is really the same thing as fixing the contact structure and deforming M. More precisely:

Theorem 3.3 (Gray's Theorem)**.** *If* α_t *is a smooth family of contact forms, then there exists a smooth family of diffeomorphisms* ϕ_t *such that* $\alpha_t = f_t \phi_t^*(\alpha_0)$ *for some functions* $f_t : M \to \mathbb{R}$*.*

3.3. **Global contact topology.** Contact structures form a rich class of plane fields for 3-manifolds. We begin with an existence result.

Theorem 3.4. *Every 3-manifold possesses a contact structure.*

The proof of this result for compact manifolds comes from work of Lutz [79] and Martinet [82], who used a construction called the "Lutz twist" to perform Dehn surgery on a closed loop transverse to a given contact structure. More specifically, given such a closed loop on a contact manifold (M, ξ), one chooses a tubular neighborhood N which is homeomorphic to $D^2 \times S^1$. Using a parameterized Darboux Theorem, this N can be chosen so that the characteristic foliation $(\partial N)_\xi$ consists of closed curves of a fixed (small) slope on the boundary torus. To obtain a new

3-manifold, one removes N from M and replaces it with a solid torus $N' = D^2 \times S^1$ sewn in by a map $\Phi : \partial N' \to \partial N$ which is not homotopic to the identity (see, e.g., [97] for an introduction to Dehn surgery). One can explicitly construct a contact structure on N' that matches that of M when glued via Φ by controlling the characteristic foliation on $\partial N'$ and using the Moser-Weinstein Theorem to show that the contact structures match if the foliations match. The Fundamental Theorem of Surgery on 3-manifolds [97] states that any closed 3-manifold can be obtained by performing Dehn surgery along some link in S^3; hence, all 3-manifolds possess a contact structure.

There are several other means of proving this result, just as there are several means of constructing all compact 3-manifolds: open book decompositions [102], branched covers [52], etc. The proof for non-compact 3-manifolds is more subtle, following from Gromov's *h-principle* [56, 28].

Contact structures on closed 3-manifolds are implicitly global objects whose properties depend crucially upon a dichotomy first explored by Bennequin [10] and Eliashberg [24]. A contact structure ξ is *overtwisted* if there exists an embedded disc D in M whose characteristic foliation D_ξ contains a limit cycle. If ξ is not overtwisted then it is called *tight*. The rationale behind the definition of tight and overtwisted is difficult to comprehend unless one is familiar with some of the intricacies in classifying codimension-1 foliations on 3-manifolds [29]. It is perhaps best to point out that there is a coupling between dynamical features of characteristic foliations of surfaces in (M, ξ) [limit cycles] and the global topological features of the plane field.

An example helps justify the name. The standard contact structure in polar coordinates, $\ker(dz + r^2 d\theta)$, is tight [10]. The contact structure defined by the 1-form $\cos r\, dz + r \sin r\, d\theta$ is in fact overtwisted. A disc of the form $\{z = r^2 \,;\, 0 \leq r \leq \pi/2\}$ has a limit cycle in its characteristic foliation at the boundary. What this limit cycle picks up is the fact that the slope of the contact planes $dz/d\theta = -r \tan r$ becomes vertical and "twists over" in a periodic manner: see Figure 5[right]. Note that, in a neighborhood of the z-axis, this overtwisted contact structure is tight (Taylor expand the form about $z = 0$): this is consistent with the Darboux Theorem, which implies that all contact structures are locally tight. Hence, being overtwisted is a global property. Gray's theorem reinforces this dichotomy by implying that tight and overtwisted structures are stable up to deformation — you cannot have a smooth family of contact structures change from tight to overtwisted or vice versa.

The rationale for the term "tight" has to do with their lack of flexibility as compared to the overtwisted structures. For example, Eliashberg [24] has completely classified overtwisted contact structures on closed 3-manifolds. In short, on a compact 3-manifold, two overtwisted contact structures are isotopic through contact structures if and only if they are homotopic as plane fields. This means that algebraic topological invariants (characteristic classes) suffice to distinguish overtwisted contact structures. Such insight into tight contact structures is slow in

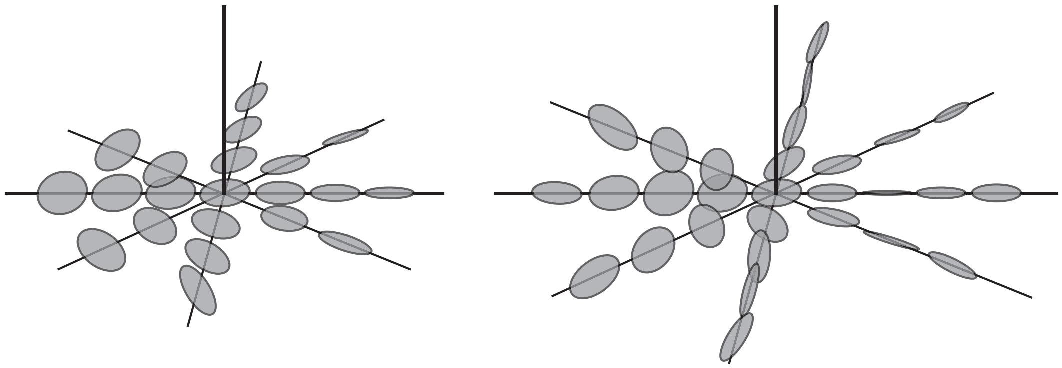

FIGURE 5. Tight [left] and overtwisted [right] contact structures in $\mathbb{R}^3$ with cylindrical symmetry. Pictures are the contact planes at a slice $\{z = 0\}$: the full structure translates these planes along the vertical axis.

coming. Some 3-manifolds (such as S^3, [24]) admit a unique tight contact structure; some 3-manifolds (such as T^3, [51, 71]) admit an infinite number of tight contact structures; and some 3-manifolds admit no tight contact structures at all [37]

This completes our brief introduction to contact topology. For a more comprehensive treatment, see [25, 26, 30].

4. CONTACT STRUCTURES AND STEADY EULER FIELDS IN 3-D

We now attend to our stated goals. First, we give existence and classification results for steady Euler fields on three-dimensional Riemannian manifolds using a contact-topological approach.

4.1. Reeb fields.

To every contact form α is associated a unique vector field, called the *Reeb field*, which captures the geometry of the 1-form in the directions transverse to the contact structure. The Reeb field of α, denoted X, is defined implicitly via the two conditions:

$$\iota_X d\alpha = 0 \quad ; \quad \iota_X \alpha = 1. \tag{4.1}$$

These conditions are sensible. The kernel of the contact 1-form α at a point is a plane in the tangent space. The kernel of the 2-form $d\alpha$ at a point is a line in the tangent space. The contact condition $\alpha \wedge d\alpha \neq 0$ is equivalent to saying that these two subspaces are transverse. Hence, the Reeb field captures the geometry of the contact form in the "orthonormal" direction to ξ. Of course, since a contact structure is defined independent of a Riemannian metric, there is no rigid notion of orthonormality other than that induced by the geometry of the contact 1-form. We say that a vector field is *Reeb-like* for α if $\iota_X d\alpha = 0$ and $\iota_X \alpha > 0$. A Reeb-like field has the same dynamics as the Reeb field, but with a different parametrization.

Note that a fixed contact structure ξ possesses many different defining 1-forms, possessing Reeb fields which are potentially very different. The dynamics of the Reeb field, together with the geometry of the contact structure, suffice to reconstruct the contact 1-form.

In the case of the standard tight contact form on S^3 given in (3.3), the Reeb field is precisely the Hopf field of (1.3). Thus, in the round metric, this contact structure is orthogonal to its Reeb field. On the other hand, for the standard tight contact structure on $\mathbb{R}^3$, $dz+x\,dy$, the Reeb field is $\partial/\partial z$. This is not orthogonal with respect to the Euclidean metric.

4.2. **A correspondence.** It has been observed at various points in the literature [18, 100, 50] that a nonvanishing curl eigenfield is dual to a contact form. A more exact correspondence between Reeb-like fields for contact forms and Beltrami fields follows by adapting metrics to contact forms [17]. The following theorem gives a precise formulation.

Theorem 4.1 ([32]). *Any rotational Beltrami field on a Riemannian 3-manifold is a Reeb-like field for some contact form. Conversely, any Reeb-like field associated to a contact form on a 3-manifold is a rotational Beltrami field for some Riemannian structure.*

Proof: Assume that $\mathbf{u}$ is a Beltrami field where $\nabla \times \mathbf{u} = f\mathbf{u}$ for some $f > 0$. Let g denote the metric and μ a volume form on M with respect to which $\mathbf{u}$ is divergence-free. Let α denote the one-form $\iota_{\mathbf{u}} g$.

The condition $\nabla \times \mathbf{u} = f\mathbf{u}$ translates to $d\alpha = f\iota_{\mathbf{u}}\mu$. Hence, α is a contact form since

$$\alpha \wedge d\alpha = f\iota_{\mathbf{u}} g \wedge \iota_{\mathbf{u}}\mu \neq 0. \tag{4.2}$$

Finally, $\mathbf{u}$ is Reeb-like with respect to α since

$$\iota_{\mathbf{u}} d\alpha = f\iota_{\mathbf{u}}\iota_{\mathbf{u}}\mu = 0. \tag{4.3}$$

Conversely, assume further that α is a contact form for M having Reeb field X. Assume that $Y = hX$ for some $h > 0$. There is a natural geometry making Y an eigenfield of the curl operator. Let

$$g(v,w) = \frac{1}{h}\left(\alpha(v) \otimes \alpha(w)\right) + d\alpha(v, Jw), \tag{4.4}$$

where J is any almost-complex structure on $\xi = \ker\alpha$ (a bundle isomorphism $J : \xi \to \xi$ satisfying $J^2 = -\mathrm{ID}$) which is adapted to $d\alpha$ so that $d\alpha(\cdot, J\cdot)$ is positive definite. Such a J is known to exist [83] since $d\alpha$ is nondegenerate on ξ.

Then, by definition of the Reeb field, $\iota_Y g = \alpha$. Thus, $d\iota_Y g = d\alpha$. Let μ be the volume form $h^{-1}\alpha\wedge d\alpha$ on M. Then $\iota_Y\mu = d\alpha$ and Y is a divergence-free eigenform of curl in this geometry. □

We leave it as an exercise to show that Y is divergence-free under the particular g-induced volume form if and only if the scaling function h is an integral for the flow, $\mathcal{L}_Y h = 0$, as one would expect.

4.3. **Existence on 3-manifolds.** Knowing the basic properties of the curl operator tells us that every Riemannian 3-manifold admits a volume-preserving curl eigenfield and thus a steady solution to the Euler equations. It seems at first unclear how one would construct a *nonvanishing* curl eigenfield solution on an arbitrary three-manifold.

Corollary 4.2. *Every closed oriented three-manifold has a nonvanishing steady solution to the Euler equations under some geometry.*

Proof: Theorem 3.4 implies the existence of a contact structure. Choose any defining 1-form, consider its Reeb field, and outfit the manifold with a geometry as in Theorem 4.1. □

It remains an open question whether every *Riemannian* 3-manifold admits admits a nonvanishing curl eigenfield, or whether there are certain geometries which are hostile to contact geometry. Another way to think of this problem is in terms of the space of contact 1-forms in the set of divergence-free 1-forms on (M, g). The eigenforms of the curl operator (or, if you like, the Laplacian) yield a basis for this space. Question: is the set of contact forms "fat" or "thin" with respect to the spectral geometry of this operator? Does the eigenbasis always pierce the set of contact forms for any geometry?

5. Knots and links in three-dimensional flows

Any exploration of topological features of 3-dimensional flows prompts a discussion of knotting and linking. Any periodic flowline forms an embedded loop, and, in a 3-dimensional domain, there are a countably infinite number of distinct embedding classes for loops. More formally, a *knot* is an embedded loop in a 3-manifold, and a *link* is a collection of disjoint knots. Two knots or links are said to be equivalent or *(ambient) isotopic* if there exists a deformation (1-parameter family of homeomorphisms starting at the identity map) of the manifold taking the one knot/link to the other. The *trivial knot* or *unknot* is any loop which bounds an embedded disc.

In 3-dimensional continuous dynamics, there is an intriguing relationship between the dynamics of a flow and the types of knots and links that are present. For example,

(1) Any 3-d flow which is dynamically complicated (has positive topological entropy on a bounded invariant set) must necessarily possess infinite number of distinct knot types as periodic orbits [40].
(2) One can use linking data to order and infer information about bifurcations in certain classes of flows and in suspensions of 2-d maps [66, 64, 65].
(3) The existence of certain knot types as periodic orbits is incompatible with a nondegenerate integrable Hamiltonian system [14, 39] and hence can imply that a given Hamiltonian system is nonintegrable.
(4) There is a type of *renormalization theory* for links of periodic orbits which mirrors the dynamical renormalization theory in that the most complex

FIGURE 6. Examples of a nontrivial knot [left], an unknot [center], and a link [right].

types of periodic orbit links can be *self-similar* on the level of knot types [49].

With regards to fluid flows, there have been numerous investigations into the knotting and linking of flowlines [86, 87, 88, 94, 11, 41]. Perhaps the best known topological feature of fluid flows is *helicity*, a quantity which measures the average asymptotic linking of flowlines [4, 85, 7] and has applications to energy bounds.

5.1. **Unknots.** Recall the example of the Hopf field (1.3) on the round 3-sphere. This nonvanishing field satisfies the steady Euler equations and fills the round 3-sphere with periodic orbits. Topologists are intimately familiar with this filling up of S^3 by circles: it is the well-known *Hopf fibration* of S^3. The reader may verify that each of these periodic orbits is in fact an unknot. Furthermore, any two distinct orbits of the Hopf field are linked with linking number one, giving rise to the so-called *Hopf link*.

To what extent do these knot-theoretic features hold for more general steady Euler fields on the 3-sphere with different geometries? There is a *rigidity* for topological fluids on spheres which is metric-independent. Again, as in the case with Arnold's theorem, we require a sufficient degree of smoothness.

Theorem 5.1 ([36]). *Any nonvanishing solution to the C^ω steady Euler equations on a Riemannian S^3 must possess an unknotted closed flowline.*

It is significant both that there is a periodic orbit and that it is unknotted. The existence of a periodic orbit in a nonvanishing vector field on S^3 was the content of the Seifert Conjecture, counterexamples to which have been ingeniously constructed in the C^ω class by K. Kuperberg [75] and by G. Kuperberg [74] in the C^1 volume-preserving class. (It is as yet not known if a sufficiently smooth nonvanishing volume-preserving field on S^3 must possess a periodic flowline, but the suspected answer is 'no'.) It is known [74] that there exist arbitrarily smooth nonvanishing volume-preserving fields on S^3 with a finite number of periodic orbits, all of which have nontrivial knot types. Hence, Theorem 5.1 implies that there is something peculiar to the topology of inviscid flowlines which is not shared by more general flows which are merely volume-preserving.

There are two components to the proof of Theorem 5.1, corresponding precisely to the dichotomy implicit in Arnold's theorem. Given the smoothness requirements, it is either the case that a steady nonvanishing Euler field $\mathbf{u}$ has a nontrivial integral, or that $\mathbf{u}$ is a curl eigenfield. In the former case, the following result gives a slightly stronger conclusion.

Theorem 5.2 ([36])**.** *Any nonvanishing vector field on S^3 having a C^ω integral of motion must possess a pair of unknotted closed orbits.*

We do not give the proof of this result here, as it involves both a very detailed examination of *round handle decompositions* of 3-manifolds [8, 89, 103] as well as a classification of all possible critical sets Σ for said integral. It is perhaps best to think of this theorem via a 2-dimensional analogue. Given a 2-sphere S^2 and a flow on it that possesses a nontrivial C^ω integral (a stream function), there must exist at least two stagnation points of elliptic type corresponding to a maximum and a minimum of the stream function.

The second component to the proof of Theorem 5.1 concerns the curl-eigenfield case. If there is no nonconstant integral for $\mathbf{u}$, then we know that $\nabla \times \mathbf{u} = \lambda \mathbf{u}$ for some $\lambda \in \mathbb{R}$. It cannot be the case that $\lambda = 0$, for this would mean that $\mathbf{u}$ is dual to a closed 1-form on S^3. This closed form is not exact, since it is nonvanishing, and thus violates the fact that the first cohomology group of S^3 vanishes.

Hence, we are left with the case where $\lambda \neq 0$ and $\mathbf{u}$ is, by Theorem 4.1, a Reeb-like field for the contact form $\alpha = \iota_{\mathbf{u}} g$. This is the point at which tools from contact topology show their strength. An extremely strong result of Hofer, Wysocki, and Zehnder provides the desired result.

Theorem 5.3 (Hofer, Wysocki, and Zehnder [62])**.** *Any Reeb field on S^3 possesses an unknotted periodic orbit.*

We can provide neither a proof nor a careful exposition of this proof. Instead we give an introduction to a related theorem, as this plays a role in many current ideas in contact topology and will be very important in our discussion of hydrodynamic instability in §6.

Theorem 5.4 (Hofer [61])**.** *Any Reeb field associated to an overtwisted contact structure on a closed orientable 3-manifold possesses a periodic orbit.*

The proof of this result contains the key idea for Theorem 5.3, and itself draws on the work of Gromov on pseudoholomorphic curves in symplectic manifolds [9]. The counterintuitive idea is that to analyze the *three*-dimensional contact manifold (M, α) with a Reeb field X, it is helpful to construct a *four*-dimensional symplectic manifold (W, ω) built from (M, α) and use *two*-dimensional techniques from complex analysis. This is perhaps not too foreign an idea to fluid dynamicists, for whom complex analysis is already an invaluable tool for planar flows. It is in the application to higher dimension fluid flows, however, that more sophisticated contact and symplectic topology arises.

Given a 3-manifold M with contact form α, the *symplectization* of M is the 4-manifold $W = M \times \mathbb{R}$ outfitted with the 2-form $\omega = d(e^t\alpha)$, where t denotes the $\mathbb{R}$-factor in W. This 2-form is a *symplectic* form, as it is closed ($d\omega = 0$) and nondegenerate ($\omega \wedge \omega \neq 0$). Nondegeneracy follows from the fact that α is contact:

$$\omega^2 = [e^t(d\alpha + dt \wedge \alpha)]^2 = 2e^{2t}\alpha \wedge d\alpha \wedge dt \neq 0.$$

Indeed, contact and symplectic structures are intimately related, the former being an odd-dimensional cousin of the latter. See, e.g., [83] for a more complete exploration of these ideas.

The symplectic manifold W therefore captures the contact geometry of M. To utilize complex methods, we outfit W with an *almost complex structure* (a linear map $J : TW \to TW$ satisfying $J^2 = -\mathrm{ID}$) that respects the geometry of W. Think of each tangent space T_xW as being spanned by the Reeb field X, the t-direction $\partial/\partial t$, and the contact plane ξ. Choose J so that it rotates the plane spanned by the X and $\partial/\partial t$ directions by a quarter-turn (like multiplication by i in $\mathbb{C}$). In the plane of ξ, J is chosen so as to be adapted to $d\alpha$, as in Equation 4.4. This choice of J decouples the contact directions from the Reeb field X and entwines the dynamics of this Reeb field with the t-direction: see Figure 7.

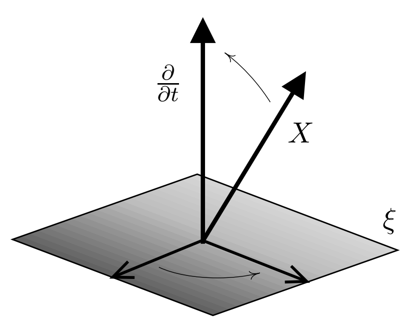

FIGURE 7. The effect of the almost-complex multiplication J on TW is to rotate in the contact plane ξ according to a symplectic basis for $d\alpha$ and to rotate the Reeb direction into the symplectization direction $\partial/\partial t$.

Fixing such a J, one defines *J-holomorphic curves* in W as maps $\varphi : \Sigma \to W$ from a Riemann surface (Σ, j) to W such that $d\varphi \circ j = J \circ d\varphi$. It follows easily from Stokes' Theorem that there are no compact Riemann surfaces in W; one must introduce punctures [61]. If Σ is a punctured Riemann surface, the *energy* of Σ is defined to be $\int_\Sigma \varphi^*(d\alpha)$. This energy measures how much area of the surface is "visible" to the contact planes (on which $d\alpha$ is an area form).

The crucial lemma of Hofer's: if a J-holomorphic curve $\varphi = (w, h) : \Sigma \to M \times \mathbb{R}$ has finite energy then any (non-removable) puncture can be shown to possess a neighborhood parametrized by $\{(\theta, \tau) : \theta \in S^1$ and $\tau \in [0, \infty)\}$ such that $\lim_{\tau\to\infty} h$ approaches $\pm\infty$ and $\lim_{\tau\to\infty} w(\theta, \tau)$ approaches a parametrization of a periodic orbit γ for X. The intuition behind this is that if a surface has finite energy, then in the limit as $t \to \pm\infty$, the surface must be orthogonal to the contact planes, and thus tangent to the $(X, \frac{\partial}{\partial t})$ planes: see Figure 8. For more information on finite energy holomorphic curves and their asymptotics see [9].

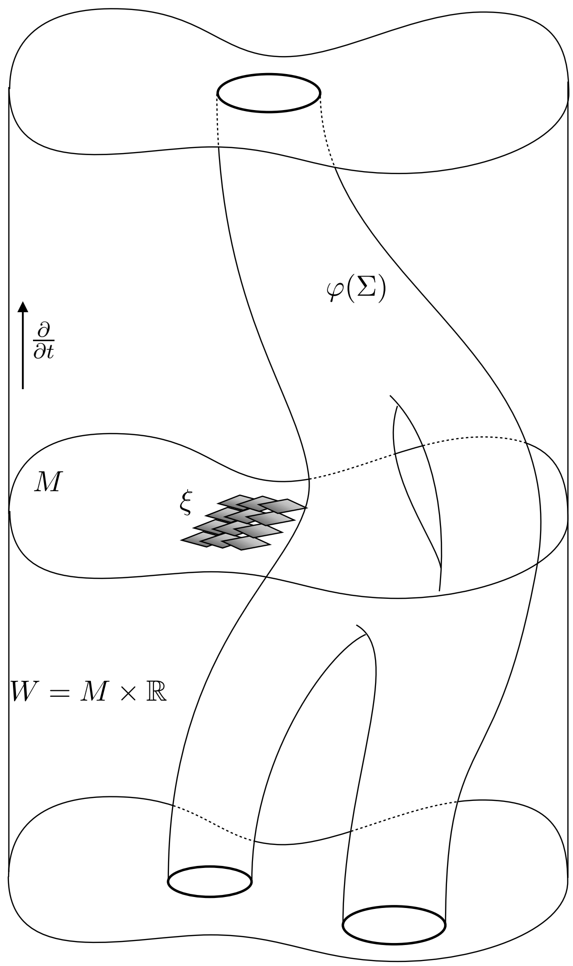

FIGURE 8. A finite energy J-holomorphic map φ from a punctured surface Σ into the symplectization $W = \mathcal{M} \times \mathbb{R}$ has punctures limiting to cylinders over periodic orbits of the Reeb field as $t \to \pm\infty$.

Now the goal is to find finite energy J-holomorphic curves in W. The main result of [61] is that whenever ξ is an overtwisted contact structure on M, then there exists a finite energy J-holomorphic curve in the symplectization. (The process for constructing this is derived from a *Bishop filling*). Thus, any Reeb field for any overtwisted contact structure on a closed orientable 3-manifold possesses a periodic orbit (solving the *Weinstein Conjecture* in these cases). On S^3, since there is a unique tight contact structure up to contactomorphism [24], one need merely augment the above argument with a variational method that works for the specific tight case [61]. With that, one has that all Reeb fields on S^3 possess a periodic flowline.

This (strenuous) effort yields a single periodic orbit, with no information about its knot type. To prove the unknotting result of Theorem 5.3 requires the construction of parametrized families of finite-energy curves in the symplectization W which project down in M to a foliation branched over certain singular loops in M. See [62] for the full details.

These sophisticated methods give information about fluids which are of a purely topological nature. No matter what geometry is placed on a 3-sphere, any (smooth enough) steady Euler flow is guaranteed to possess an unknotted flowline. The results are completely implicit: one has no idea where the periodic orbit lies.

5.2. **Knots.** The results on unknots above provide a “lower bound” on the size and structure of the periodic orbit link for a fluid flow on the 3-sphere. One naturally wonders if there is a corresponding “upper bound” for the periodic orbit link, or, indeed, any other types of restrictions on what is and is not possible.

We begin by noting that integrable fields have a great deal of restrictions associated to the periodic orbit knot types. This should come as no surprise, given the constraints imposed by Arnold’s Theorem — the manifold is filled with invariant tori. It is known precisely which types of knots can appear as closed flowlines in a nonvanishing integrable field on a 3-sphere: these are the so-called *zero-entropy knots* [49, 36]. This class consists of those knots obtainable from the unknot by iterating the operations of cabling (wrapping around a core curve) and connected sum (splicing two knots together), as in Figure 9. Such knots are a very small subclass of knots, excluding, e.g., the hyperbolic knots (knots whose complement in S^3 supports a hyperbolic geometry).

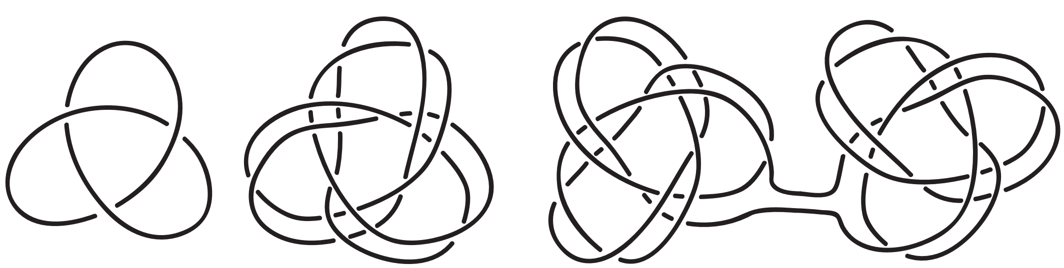

FIGURE 9. Examples of zero-entropy knots.

Hence we conclude that the only types of (sufficiently smooth) fluid flows which may support more complex periodic orbit links are the curl eigenfields, and thus, Reeb fields.

In a series of papers [86, 87], Moffatt discusses knots and links in Euler flows "with arbitrarily complex topology." What is meant by this is the construction of steady solutions to the Euler equations on Euclidean $\mathbb{R}^3$ which realize the same orbit topology as any given volume-preserving flow on the space. These results have the advantage of staying within the class of Euclidean metrics. However, there are two caveats associated to this work: (1) the techniques do not guarantee a continuous solution — so-called "vortex sheet" discontinuities may develop; (2) the proof itself relies crucially upon the global-time existence of solutions to the Navier-Stokes equations (with a modified viscosity term). Such an existence theorem is to date unknown.

We circumvent these problems by allowing for flexibility in the geometry of the flow domain. With contact-topological techniques, it is relatively simple to use surgery (or *cut-and-paste*) methods on contact forms: cf. the discussion after Theorem 3.4. Using such techniques one can produce all manner of complicated knotting and linking in a Reeb field, and, hence, in a steady fluid flow. In fact, the most complex forms of knotting and linking imaginable are realizable within the category of steady ideal fluids.

Theorem 5.5 ([34])**.** *For some C^ω Riemannian structure on S^3 (or $\mathbb{R}^3$ if preferred), there exists a C^ω nonvanishing steady Euler field possessing periodic flowlines of all knot and link types.*

Note that this does not merely say that any given knot or link type may be realized in some steady Euler flow; rather, *all* of them are realized together in a single fluid flow. It is not a priori obvious that such any vector field in 3-d can have this property, much less one which is a volume-preserving Euler field: it was shown in [48] that such vector fields do exist in general.

The types of flows which admit such periodic orbits links must of necessity be very complex and chaotic. Those chaotic flows which can be best analyzed possess *hyperbolic invariant sets* of dimension one. Roughly speaking, this means that one can find a one-dimensional invariant set Λ and a flow-invariant splitting of the tangent bundle $T\Lambda = E^s \oplus E^c \oplus E^u$ so that E^c is spanned by the vector field and the flow is exponentially expanding (resp. contracting) on E^u (resp. E^s). To analyze knotting and linking of orbits within Λ, one collapses the E^s direction in a neighborhood of Λ to obtain a branched surface with a semiflow — a *template*, as in Figure 10 — which captures the topology of a 1-dimensional hyperbolic invariant set in a 3-d vector field, much as a 2-dimensional approximation to the Lorenz attractor does [58, 12].

To construct such a flow for an ideal fluid in 3-d is possible, thanks to Theorem 4.1.

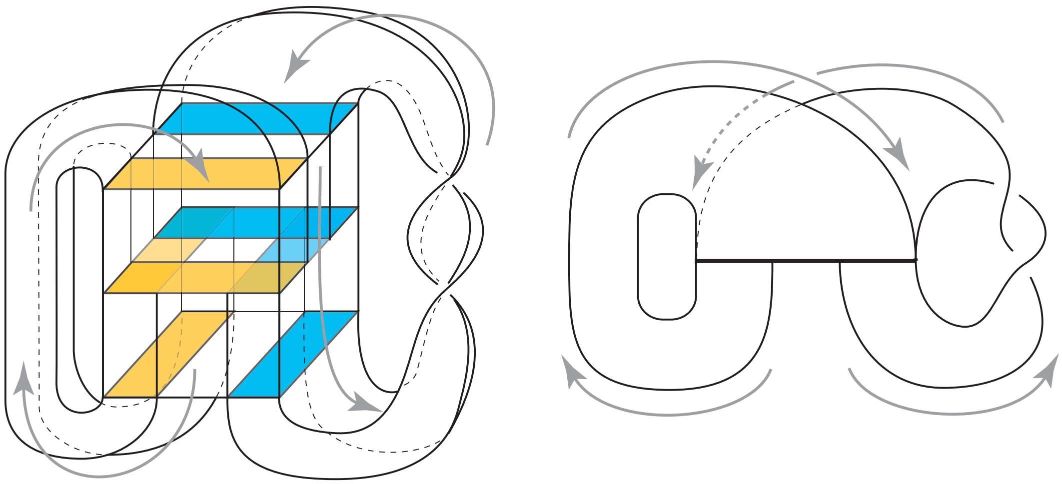

FIGURE 10. An example of a flow defined by stretching, squeezing, and twisting of tubes [left] which possesses a 1-d hyperbolic invariant set and collapses to a branched surface [right]. This branched surface is a *universal template*: the semiflow on this object possesses all knot and link types as closed orbits.

Lemma 5.6. *There exists a Reeb field on the 3-torus possessing a nontrivial one-dimensional hyperbolic invariant set.*

Proof: Consider the ABC equations of §2. From Equations (2.3) and (4.1) it follows that the ABC fields lie within the kernel of the derivative of the 1-form

$$(A\sin z + C\cos y)dx + (B\sin x + A\cos z)dy + (C\sin y + B\cos x)dz, \tag{5.1}$$

and that this is a contact form when the vector field is nonvanishing. In the limit where $A = 1, B = 1/2$, and $C = 0$, the vector field is nonvanishing and has the form

$$\begin{array}{rl} \dot{x} = & \sin z \\ \dot{y} = & \frac{1}{2}\sin x + \cos z \\ \dot{z} = & \frac{1}{2}\cos x \end{array} . \tag{5.2}$$

For these values, there exists a pair of periodic orbits whose stable and unstable invariant manifolds intersect each other nontransversally (see, *e.g.*, [22]). Upon perturbing C to a small nonzero value, this connection may become transverse, inducing chaotic dynamics. Indeed, a Melnikov perturbation analysis establishes this fact [19, 43, 47, 54, 70, 107]. It thus follows from the Birkhoff-Smale Homoclinic Theorem [57, 95] that nearby parameters force Equation (2.3) to possess a nontrivial 1-d hyperbolic invariant set as a solution. □

Sketch of proof of Theorem 5.5: It follows from the Birkhoff-Smale Theorem that the 1-d hyperbolic invariant set of Lemma 5.6 is a suspended Smale horseshoe (see [57, 95] for background and explanation). As such, this invariant set lies within a neighborhood N, a solid handlebody of genus two. It is an unfortunate consequence that the image of the embedding of N in T^3 unknown: it may be a very thin and tangled subset.

To construct a vector field on S^3 with all knots and links, N must be unwound. It suffices, via [48] to embed N in S^3 so that the handles of N are (1) unknotted, (2) unlinked, and (3) twisted in an appropriate manner as in Figure 10[right]. See [106, 49] for an exposition of this result. By choosing the handles of N to be sufficiently thin (which is possible by restricting to a smaller hyperbolic invariant set), one can use an argument as in the Darboux Theorem (Theorem 3.1) to determine the characteristic foliation of the contact structure on ∂N. A surgery argument then suffices, as follows. Choose two curves, γ_1 and γ_2, which are transverse to the characteristic foliation on ∂N and wrap around each handle of N once with an appropriate number of twists. One then glues two thickened discs of the form $D^2 \times (-\epsilon, \epsilon)$ to N by attaching the annuli $\partial D^2 \times (-\epsilon, \epsilon)$ to neighborhoods of γ_1 and γ_2 on ∂N. By outfitting these balls with the standard contact structure for $\mathbb{R}^3$ and matching the characteristic foliations along the gluing, the Moser-Weinstein Theorem (Theorem 3.2) gives a well-defined contact structure on the union, which completes N to a contact 3-ball. Contact structures on the 3-ball are classified [24] and it is known that any such structure can be completed to a contact structure on S^3. Thus, there exists a contact form α on S^3 which agrees with the 1-form of (5.2) at the chosen parameter values on $N \subset T^3$ and whose Reeb field has a 1-d hyperbolic invariant set embedded so as to exhibit all knot and link types.

This method of proof yields only a C^∞ Reeb field, since cutting and pasting arguments are used. To obtain a C^ω Reeb field, one can use a classification argument in [34] to show that the contact form for this Reeb field is equivalent to $f\alpha_H$, where α_H is the C^ω form associated to the standard contact form on S^3 in Equation (3.3). A small perturbation to the coefficient function f yields $\tilde{f}\alpha_H$, a C^ω form with Reeb field of the same smoothness class. Since 1-d hyperbolic invariant sets in 3-d flows are structurally stable, this perturbation does not change the link of periodic orbits in this set: hence, the all-knots property still holds. □

6. Instability

Recall that the theme of our investigations into the Euler equations is to begin with the dynamically simplest elements — the steady solutions — and then build up a repertoire of invariant objects. The steady solutions comprise the fixed points of the Euler equations in the space of volume-preserving vector fields. The next natural phenomenon to investigate is the local dynamics of the time-dependent Euler equations near these fixed points. This corresponds to the classical question of *hydrodynamic stability*. The problem of hydrodynamic stability and instability for steady Euler flows on three-dimensional domains is an important subject with a rich history. See, e.g., [15, 23, 77].

There are numerous notions of stability and instability for fluids, the full extent of which we do not here survey. A steady velocity field $\mathbf{u}$ is said to be (nonlinearly) hydrodynamically stable if, given any neighborhood of $\mathbf{u}$ in the space of volume-preserving vector fields, then any arbitrarily small divergence-free perturbation $\mathbf{u}+\mathbf{v}$ evolves under the time-dependent Euler equations in such a way that it does not leave the assigned neighborhood. Otherwise, $\mathbf{u}$ is said to be hydrodynamically unstable (some authors require an exponential divergence of a perturbed field from $\mathbf{u}$). In all this, it is necessary to carefully specify the norm via which "smallness" is measured. We will use the L^2 or *energy norm* exclusively:

$$\|\mathbf{u}\| = \int_M \mathbf{u} \cdot \mathbf{u} \, d\mu.$$

Examples of stable flows are hard to come by in dimension higher than two, since long-time existence of solutions is unknown. One of the common themes to be found in the literature on hydrodynamic stability is that "almost every" steady Euler field is hydrodynamically unstable in dimension three. This begs the question of what is meant by "almost every" in the context of steady Euler fields, when, as we have seen in Theorem 2.1, steady Euler fields have some topological constraints and may or may not appear in parameterized families.

We therefore turn to questions of generic fluid behavior in dimension three. The small literature on generic properties of fluid flows (primarily [38, 98]) focuses on the Navier-Stokes setting and uses external forcing or Dirichlet data as a parameter, see [1, 3.6A]

In accordance with the theme of this article, we use the geometry of the domain as the genericity parameter. We outline a clear formulation of the problem and present a generic instability theorem for the curl eigenfields with respect to the underlying Riemannian metric. This approach skirts the difficulty of dealing with having steady solutions which are isolated in the space of volume-preserving vector fields (up to normalization). This approach also allows for some very fascinating topological methods to be applicable.

6.1. **Instability criteria.** The literature on hydrodynamic stability is full of various criteria, which vary in scope and effectiveness. See [15, 23, 77] for a classical introduction and [42, 46] for more recent surveys. It suffices for our purposes to note that stability and instability criteria for 3-d Euler flows, when they exist at all, tend to be very limited in applicability. We therefore turn to a weaker notion of stability: linear stability. A steady Euler field $\mathbf{u}$ is said to be *linearly stable* if, for every sufficiently small divergence-free field $\mathbf{v}(0)$, the evolution of $\mathbf{v}(t)$ under the linearized Euler equation about $\mathbf{u}$,

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{u} = -\nabla p, \tag{6.1}$$

is bounded in some predetermined norm. We will use exclusively, following [42, 45], the energy or L^2 norm on vector fields. The solution $\mathbf{u}$ is thus said to be *linearly unstable* if, for some arbitrarily small $\mathbf{v}(0)$, the solution $\mathbf{v}(t)$ has exponential growth in energy norm. We note that, unlike in finite-dimensional dynamics, it is not necessarily the case that linear instability implies nonlinear instability.

Arnold suggested that the underlying dynamics of the flowlines of the steady solution **u** can force instability. We now know this to be the case following the contributions of several researchers, including Bayly, Friedlander, Hameiri, Lifshitz, and Vishik (see, e.g., [42, 44, 46] and references therein). The criterion we will use comes from the work of Friedlander-Vishik [45], who used techniques from geometric optics developed by Lifshitz-Hameiri [76]. This instability criterion requires some expanding dynamics within the flow, the simplest examples of which are fixed points and periodic orbits which are *nondegenerate* and of saddle-type. A nondegenerate fixed point is one whose eigenvalues are all nonzero. A nondegenerate periodic orbit for a volume-preserving flow is defined to be one whose Floquet multipliers (eigenvalues of the linearized return map to a cross-section of the orbit) are not equal to one. Nondegerate periodic orbits of a 3-d volume-preserving flow are either of *hyperbolic* or *elliptic* type: see Figure 11 for an illustration, and compare with the 2-d case of Fig. 1. (It is not the case that elliptic orbits are necessarily surrounded by invariant tori — this classification records only linear behavior of the return map.)

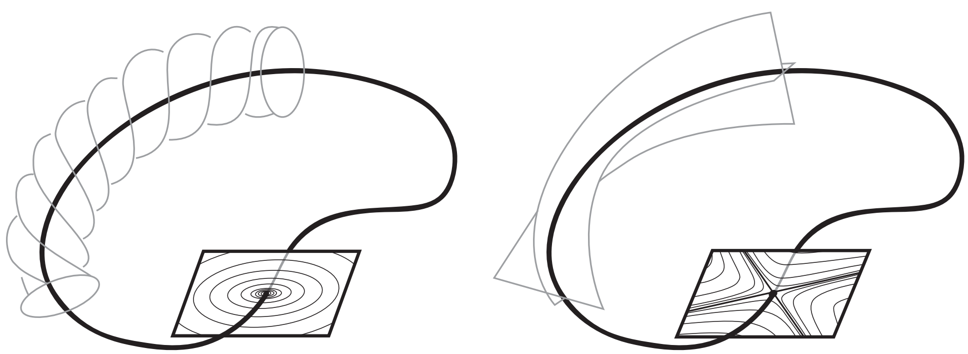

FIGURE 11. Elliptic [left] and hyperbolic [right] periodic orbits in a volume-preserving 3-d flow are characterized by the linearization of the cross-sectional return map.

Instability Criterion: [45] *The presence of a nondegenerate hyperbolic fixed point or periodic orbit in a steady Euler flow induces linear instability in the energy norm.*

6.2. **Generic curl eigenfields.** The genericity result we survey here uses the Riemannian metric as a parameter space. Theorem 2.1 gives an indication of how to approach the set of steady Euler fields. Surprisingly, it appears much easier to examine the generic behavior of curl eigenfields than that of integrable solutions, even though the dynamics of curl eigenfields can be so much more complex.

In order to analyze the "typical" behavior of curl eigenfields, we use an Implicit Function Theorem argument similar to that Uhlenbeck [108] and Henry [60] used for eigenfunctions of the Laplacian. We switch to eigenforms for simplicity, under which the curl operator translates to $*d$. We work on the space of divergence-free 1-forms on M with respect to the metric g as a parameter. Denote the space of Riemannian metrics on M by $\mathcal{G}$ and let

$$E_0 = \{(g, \alpha) \in \mathcal{G} \times \Omega^1(M) : \text{div}\alpha = 0\}, \tag{6.2}$$

and

$$E = \ker(*d|_{E_0})^{\perp}. \tag{6.3}$$

Note E is a bundle over $\mathcal{G}$ and the curl operator $*d$ is a fibrewise map. From the Hodge theorem we know that $*d : E \to E$ is a bundle isomorphism. Normalize E to $S = \{(g, \alpha) \in E : \|\alpha\|_2 = 1\}$ and consider the operator

$$\phi : S \times \mathbb{R} \to E \quad ; \quad \phi(g, \alpha, \lambda) = (g, *d\alpha - \lambda\alpha)\,. \tag{6.4}$$

The ϕ-inverse image of the zero section gives the curl eigenforms. Thanks to the normalization, this is a fibrewise index zero Fredholm operator to which the transversality theory detailed in [108] applies.

The following technical result is extremely illustrative, as it demonstrates that the curl eigenfield solutions are very well-behaved for a "typical" Riemannian geometry. We sketch enough of the proof to give an idea of what is involved.

Lemma 6.1. *For each $r \geq 1$, there exists a residual set in the space of C^r metrics on a closed M^3 such that the eigenspaces of the curl operator (with non-zero eigenvalue) are 1-dimensional and vary smoothly with the metric.*

To prove this, one shows that the zero-section 0 of E is a regular value of ϕ; then, following [108], $Q = \phi^{-1}(0)$ is a manifold that fibers over $\mathcal{G}$ with projection π. A G_δ-dense set of metrics will be regular values of π and, for these values, $Q_g = \pi^{-1}(g) = \phi_g^{-1}(0)$ is a 0-dimensional manifold (here $\phi_g = \phi|_{\pi^{-1}(g)}$). For each point (α, λ) in Q_g we have $*d\alpha = \lambda\alpha$. This λ is a simple eigenvalue of curl since 0 is a regular value of ϕ (cf. [108, Lemma 2.3]). The eigendecompositions vary smoothly since Q is a manifold. The smoothness condition is required for the application of the Sard-Smale theorem. Showing that the zero-section of E is a regular value of ϕ is a fairly straightforward computation of the derivative of ϕ with respect to the Riemannian metric: see [35] for details.

Although this result seems technical, it is really quite straightforward — one wishes to use an Implicit Function Theorem style of argument, and to do so always requires care in computing derivatives and checking smoothness. What this work gains us is the following. For such a generic metric, one can unambiguously designate the i^{th} eigenfield of curl, for $i \in \mathbb{N}$. These eigenfields vary smoothly with the metric parameter and allow one to work "one eigenfield at a time." Since we are concerned with topological genericity (countable intersections of open, dense sets), we may prove a genericity result for the i^{th} curl eigenfield and then take the intersection over all $i \in \mathbb{N}$. This is the strategy behind the proof of the following results:

Lemma 6.2. *There is a G_δ dense subset of C^r metrics in $\mathcal{G}$ ($r \geq 2$) for which all curl eigenfields with non-zero eigenvalues have all fixed points nondegenerate.*

The idea behind this proof is to consider

$$\psi : Q \times M \to T^*M \quad ; \quad \psi(g, \alpha, \lambda, x) = \alpha(x). \tag{6.5}$$

and show that the zero-section 0 is a regular value of ψ.

Lemma 6.3. *For each $i \in \mathbb{Z} - \{0\}$ and each positive integer T, there exists an open dense set of metrics in $\mathcal{G}$ so that, if the i^{th} eigenfield of curl has no fixed points, then all of the periodic orbits of period less than T are nondegenerate.*

This result is more involved, and requires a few tools from contact geometry. We sketch the main steps. The idea is to consider an open set U_α of contact 1-forms near a given contact eigenform α. The goal is to argue for the existence of a dense open set in U_α of 1-forms whose Reeb fields are nondegenerate (i.e., all periodic orbits are nondegenerate).

Let α' be a contact 1-form in U_α. Gray's Theorem (Theorem 3.3) says that any nearby contact form α' can be deformed through a contact isotopy to the contact form $f\alpha$, for some near-identity scalar function f. From the proof of Gray's Theorem, this isotopy is smooth with respect to α' — the entire neighborhood of 1-forms near α can be contact-isotoped to near-identity rescalings of α. From this, one proceeds as in [63, Prop. 6.1] to examine how the Reeb of $\alpha' = f\alpha$ behaves for generic choice of f.

This is best seen by using the symplectization of (M, α). Recall from the proof of Theorem 5.4 that this is the 4-manifold $M \times \mathbb{R}$ with the symplectic form $\omega = d(e^t\alpha)$. The contact form α is that induced on M by ω via the embedding of M to $M \times \{0\} \subset M \times \mathbb{R}$ and the Reeb field is the Hamiltonian dynamics on an energy level in the 4-manifold $M \times \mathbb{R}$ defined by the Hamiltonian function f. A genericity result of C. Robinson [96, Thm. 1.B.iv] meant for Hamiltonian dynamics now implies that there is a generic set of near-identity functions f such that the Reeb field for $f\alpha$ has all periodic orbits nondegenerate.

From Lemmas 6.1 through 6.3, we have that, for a generic geometry on M, all the curl eigenfields have either all fixed points nondegenerate, or, in the case of no fixed points, all the periodic orbits are nondegenerate. If there are any fixed points, then volume-conservation implies that they are of saddle type and hence induce linear instability. The obvious question is, then, what to do about the case where there are no fixed points? We have a guarantee that all periodic orbits must be nondegenerate, but we do not know that any periodic orbits necessarily *exist*. Indeed, problems associated with the existence of periodic orbits in nonvanishing vector fields on 3-manifolds are exceedingly subtle.

Fortunately, we do have Theorem 5.4 for the case when the contact structure is overtwisted. However, even in that case, there is the added complication of worrying about elliptic versus hyperbolic periodic orbits. Although both types are nondegenerate, only the hyperbolic orbits yield information about hydrodynamic instability. Fortunately, this is easily overcome for overtwisted structures (see the proof of Theorem 6.5 for the argument). The question of what to do with the case of tight structures is, as always, more challenging. To address this last, most delicate, question, we turn to a very powerful tool in contact topology.

6.3. **Contact homology.** One of the central problems in the topology of contact structures is the classification problem: given contact structures ξ and ξ' on M, are they equivalent? This problem was greatly clarified by the tight versus overtwisted dichotomy and the theorem of Eliashberg [24] that the overtwisted structures are classified by the homotopy type of the plane field. That is, the plane fields are isotopic if there is a continuous 1-parameter family of smooth plane fields connecting the two. This is fairly simple to determine with standard tools from algebraic topology [characteristic classes]. This classification is much more subtle in the case for the tight structures. See [30, 69] for recent progress in this area.

With regards to classification, Eliashberg, Givental, and Hofer have constructed a powerful new homology theory for contact structures. Homology is a tool from algebraic topology which, in the usual category, provides an invariant of topological spaces up to homotopy type. There are numerous flavors of homology theory, each of which involves 'counting' special objects in a particular manner. E.g., the Euler characteristic of a triangulated surface is a particular way of counting simplices which yields a topological invariant: this invariant is homological in nature.

A more relevant example is *Morse homology* for a finite dimensional oriented manifold M [99]. One begins with a choice of function $f : M \to \mathbb{R}$. The Morse homology counts the number and type of fixed points of the gradient vector field $X = -\nabla f$. It is necessary to assume that f is chosen so that these fixed points are all nondegenerate (i.e., that f is *Morse*). One defines the grading on these fixed points to be the *Morse index*, the dimension of the unstable manifold. Next define the *n-chains* to be the real vector space $C_n(M, f)$ which has basis all the fixed points of Morse index n. The tricky part of this homology theory is to define boundary operators: linear transformations $\partial : C_n \to C_{n-1}$ which satisfies $\partial \circ \partial = 0$. For any fixed point $p \in C_n$, $\partial(p)$ is a linear combination of points $q_i \in C_{n-1}$ whose stable manifolds intersect the unstable manifold of p in a heteroclinic connection. By assigning the proper orientation to the stable and unstable manifolds, one gets $\partial \circ \partial = 0$. The homology of the resulting chain complex, $MH_*(M, f)$, is well-defined and can be shown to be independent of f; in fact, the Morse homology is isomorphic to the homology of M. See Figure 12 for an example on a 2-sphere, where the relevant portion of the chain complex is:

$$\mathbb{Z}_a \oplus \mathbb{Z}_{a'} \overset{\left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right]}{\longrightarrow} \mathbb{Z}_b \oplus \mathbb{Z}_{b'} \overset{\left[\begin{smallmatrix} 0 & 1 \\ 0 & 1 \end{smallmatrix}\right]}{\longrightarrow} \mathbb{Z}_c \oplus \mathbb{Z}_{c'} \tag{6.6}$$

This chain complex has $MH_2 \cong \mathbb{Z}$, $MH_1 \cong 0$, and $MH_0 \cong \mathbb{Z}$, as expected from $H_*(S^2)$.

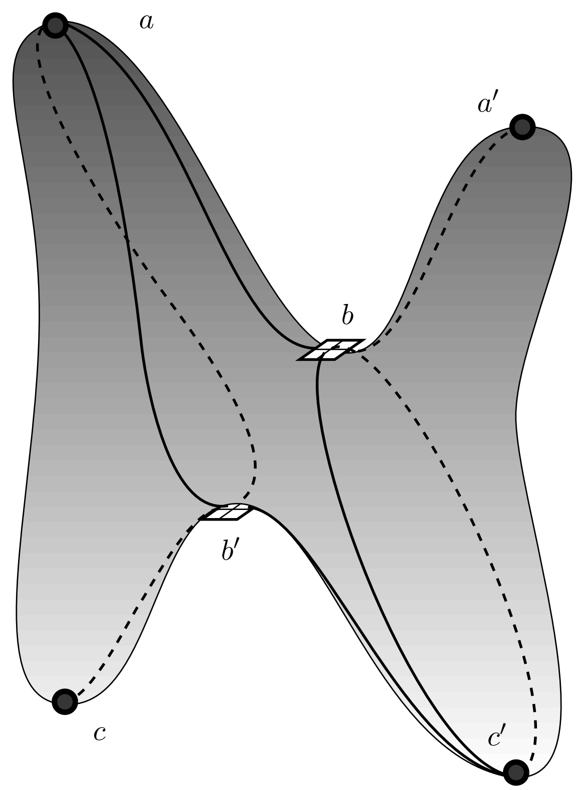

FIGURE 12. An example of Morse homology on a 2-sphere with height function $f : S^2 \to \mathbb{R}$ as shown. The critical points and connecting orbits generate chain groups and boundary maps.

Contact homology shares many similarities with Morse homology, though it is an infinite-dimensional theory which exploits contact topology. Briefly stated, contact homology counts periodic orbits of a Reeb field modulo the boundary map which counts pseudoholomorphic curves in the symplectization of the contact manifold.

The details are as follows. Given the contact structure ξ, one begins with a choice of a contact form α. Any two such 1-forms are related by some $f : M \to \mathbb{R}$; hence the 'choice,' as in Morse homology, is an appropriate f. Given α, the Reeb field X is the dynamical object corresponding to the gradient field in Morse homology. It's natural invariant sets, the periodic orbits, are what one counts. Let $\mathcal{C}$ be the set of periodic orbits for X. To each element $c \in \mathcal{C}$, a grading, $|c|$, can be assigned using a shifted Conley-Zehnder index — an integer which is approximately equal to the number of half-twists the linearized flow performs along one cycle of the periodic orbit. We do not give a precise definition as the only feature of the grading of concern here is the following: *Any nondegenerate orbit c with $|c|$ odd is hyperbolic* [61].

At this stage, the difficulty arises in defining the chains at grading n and the boundary maps between chains. A fair amount of algebraic machinery is required. One defines the graded algebra $\mathcal{A}$ as the free super-commutative unital algebra over $\mathbb{Z}_2$ with generating set $\mathcal{C}$. This algebra $\mathcal{A}$ will be the set of all chains for contact homology.

To define the boundary operator, we recall the relationship between periodic orbits of the Reeb field and finite-energy pseudoholomorphic curves, from the proof of Theorem 5.4. Unlike in the case of Theorem 5.4, we care about J-holomorphic curves which have punctures which go to $+\infty$ and $-\infty$, each such puncture asymptoting to a periodic orbit of the Reeb field.

Given periodic orbits $a, b_1, \ldots, b_k \in \mathcal{C}$, let $\mathcal{M}^a_{b_1 \ldots b_k}$ denote the set of finite energy holomorphic curves in the symplectization W with one positive puncture asymptotic to a and k negative punctures asymptotic to $b_1, \ldots, b_k$, (modulo holomorphic reparametrization). Since J is $\mathbb{R}$-invariant on W, there is an $\mathbb{R}$-action on $\mathcal{M}$. One now defines

$$\partial a = \sum \# \left\{ \mathcal{M}^a_{b_1 \ldots b_k} / \mathbb{R} \right\} b_1 \ldots b_k, \tag{6.7}$$

where the sum is taken over all $b_1, \ldots b_k$ such that the dimension of $\mathcal{M}^a_{b_1 \ldots b_k}$ is 1.

It is a deep and difficult theorem [27, 13] that the differential ∂ lowers the grading by 1, and, for a generic contact 1-form (and almost complex structure) $\partial^2 = 0$ and the homology of $(\mathcal{A}, \partial)$ is independent of the contact form chosen for ξ (and the almost complex structure).

The resulting homology of $(\mathcal{A}, \partial)$ is called the *contact homology* of (M, ξ) and is denoted $CH(M, \xi)$. Contact homology comes in a number of specialized flavors (by restricting the types of surfaces used) which are useful in computations: see [13]. In addition, there is an extremely broad generalization of contact homology called *symplectic field theory* [27] which we do not here discuss.

We now have the tools available to prove a genericity result for hydrodynamic instability. For concreteness, we restrict to the case of periodic flows on $\mathbb{R}^3$, or, flows on a Riemannian 3-torus $T^3 = \mathbb{R}^3/\mathbb{Z}^3$. Our aim is to show that, for a generic geometry, all of the curl eigenfields (except perhaps that with eigenvalue zero) are hydrodynamically unstable.

Proving the hydrodynamic instability theorem requires knowing the existence of a hyperbolic periodic orbit for all non-degenerate Reeb fields on T^3. The proof of Theorem 5.4 guarantees that any Reeb field for an overtwisted contact structure possesses a closed orbit of grading $+1$. In the nondegenerate case, such an orbit is of hyperbolic type. Thus, we need merely cover the case of the tight contact structures. For T^3, these are classified [51, 71]: there is an infinite family of isomorphism classes represented by

$$\xi_k = \ker \left(\sin(kz)dx + \cos(kz)dy \right), \tag{6.8}$$

for $k \in \mathbb{Z} - \{0\}$. A relatively simple contact homology computation is the crucial step in the following result:

Lemma 6.4 ([35]). *For a nondegenerate Reeb field associated to any tight contact structure on T^3, there is always a hyperbolic periodic orbit.*

The idea of this proof is, of course, to compute the contact homology of each ξ_k above and show that there must (either because of a nontrivial homology class or through clever counting) exist a nonzero chain in an odd grading; hence a saddle-type hyperbolic periodic orbit. See [35] for the computation.

6.4. **Generic instability.** These ingredients combine to prove the following result.

Theorem 6.5. *For generic choice of C^r metric ($2 \leq r < \infty$), all of the curl-eigenfields on a three-torus T^3 (with nonzero eigenvalue) are hydrodynamically unstable [linear, L^2 norm].*

Proof: Lemma 6.1 implies that we can work one eigenfield at a time. First, use Lemmas 6.2 and 6.3 to reduce everything to either nondegenerate fixed points or periodic orbits. Given such a field $\mathbf{u}$, if it possesses a fixed point, then it is immediately of saddle type due to volume conservation and satisfies the Instability Criterion. If the field is free of fixed points, then it is (after a suitable rescaling which preserves the topology of the flowlines) a Reeb field for the contact form $\alpha = \iota_u g$. If the contact structure $\xi = \ker \alpha$ is overtwisted, Theorem 5.4 implies the existence of a periodic orbit with grading $+1$. The nondegeneracy implies that the orbit is of saddle type and hence forces hydrodynamic instability. In the final case where ξ is tight, the contact homology computation of Lemma 6.4 implies instability. □

The advantage of a result such as Theorem 6.5 is that it is precise — there is no guessing what 'generic' means. That is uses some very deep ideas from contact topology is an advantage from the point of view of a topologist, but perhaps only to a topologist.

To a fluid dynamicist, however, the result is by no means optimal. Genericity in the geometry means that the Euclidean geometry (a highly non-generic metric for which fluid dynamicists have a preference) is not covered by this result. In addition, the use of the Instability Criterion provides information only about linear instability, and a fairly localized high-frequency instability at that: information about nonlinear instability would be greatly preferable. It is likely that there are other approaches to a 'generic instability' theorem which would yield stronger results. Our goal was to demonstrate rigorous results about fully 3-d ideal flow using a minimal set of restrictions or assumptions.

7. Concluding unscientific postscript

Flows are inherently topological. Many of the most basic and general results in inviscid fluids have a topological spirit to them — the Bernoulli theorem, the Kelvin theorem, and the Helmholtz theorem being prominent examples that all spring from very basic results of differential topology. Results which depend upon more sophisticated topological machinery abound in certain fragments of the literature. The best place to start is the fairly recent monograph of Arnold and Khesin [7] which contains a wealth of information that is complementary to the tools described in this survey.

Fluid dynamics (mathematical or physical) encompasses far more than topological methods, as a brief perusal of this volume will demonstrate. However, we do argue that there is merit in exploring very basic open questions about the topological features of ideal flow to which the most current ideas in topology and geometry are applicable. This article has outlined the small role played by methods in contact topology alone — itself a very small but rapidly developing branch of topology. We neither assert nor believe that these techniques compete with current analytic methods for understanding physical fluids.

7.1. **Generic fluids.** However, we do argue that this perspective on fluids — which is 'unscientific' in so far as it is of limited use to a scientist for whom fluids are wet — is not without benefit. Throughout the article we have seen theorems that either did not depend on the Riemannian structure or which applied to generic Riemannian metrics. This leads one to the question, *"Which features of a fluid are generic with respect to the domain geometry?"*

We have already seen certain deleterious effects of Euclidean geometry on steady solutions to the Euler equations. For example:

(1) Any attempt to analyze the knots and links which can arise as periodic flowlines of a steady Euler field on a Euclidean domain is doomed to languish in numerical simulations. As Theorem 5.5 demonstrates, it is relatively easy to prove that anything is possible, so long as one is not confined to the Euclidean setting.
(2) We showed in Lemma 6.1 that for a generic Riemannian metric, the curl operator has a simple eigendecomposition with one-dimensional eigenspaces and smooth continuation with respect to the metric. This is certainly false in the Euclidean case: witness the ABC fields, which form a three-dimensional eigenspace of curl, reflecting the symmetry of the Euclidean 3-torus.
(3) In Lemmas 6.2 and 6.3, we further demonstrated that for a generic set of metrics, all of the curl eigenfields have nondegenerate fixed points or periodic orbits, which permits the use of Morse-theoretic tools. This is certainly not the case in the Euclidean geometry.
(4) Such results were helpful in proving the generic linear instability result — again, a result that does not apply to the Euclidean case.

Could it be the case that a number of difficulties in global results about fully 3-d inviscid flow are the result of degeneracies in the Euclidean geometry? Perhaps the biggest open problem in mathematical fluid dynamics is that of finite-time blowups. Several controversial papers have attempted to detect blow-ups numerically [72, 90, 91]. Though these results are not conclusive, they do indicate the possibility of a blow-up. Furthermore, some of these very clever configurations are leveraging symmetry in order to try and force a blow-up [91].

Given the examples above in which degenerate phenomena vanish when restricting to a generic set of Riemannian geometries, one is led to the highly speculative idea that Eulerian finite-time blow-ups, if they exist, may be a function of the degeneracies implicit in Euclidean geometry. Whether this is the case or not, it might help remove some difficulties in the analysis to restrict attention to a generic Riemannian geometry on a compact manifold and use global analysis to develop a theory of "generic" inviscid fluids.

7.2. **Closing questions.** Turning again to the contact topological methods surveyed in this article, there is an abundance of open directions for future work.

(1) This article has not covered domains with boundary, for which one must modify the Euler equations to keep the boundary an invariant set for the velocity field. Such settings are very relevant in, e.g., accelerators in plasma dynamics and MHD. The methods we use are applicable to such domains. See [33, 31] for examples of contact topological methods on domains with boundary. One of the possible improvements in the generic instability result of Theorem 6.5 would be to work on a solid torus $M = D^2 \times S^1$ embedded in Euclidean $\mathbb{R}^3$ and use as a parameter space the space of all embeddings of the boundary ∂M into $\mathbb{R}^3$. Instead of varying the metric, one varies the shape of the boundary. It is an open problem to determine if the genericity results hold for this parameter space. The recent monograph of Henry [60] gives a careful analysis of the generic behavior of the scalar Laplacian with respect to this parameter space.
(2) A related generalization arises in consider flows on non-compact domains. Proofs involving hydrodynamic instability, in particular, may change their character substantially as the spectral geometry of the curl operator differs on a noncompact domain.
(3) The initial reaction to the statement of the Arnold Theorem is that the integrable solutions are the 'typical' steady Euler fields and the curl eigenfields are the 'exceptional' solutions. Likewise, one is tempted to say that among the class of Beltrami fields, where $\nabla \times \mathbf{u} = f\mathbf{u}$, the case of a pure curl eigenfield where f is a constant is the exceptional case. To what extent are these intuitions true? Lemma 6.1 implies that the curl eigenfields are very robust with respect to perturbations of the metric — most eigenfields can be continued smoothly. It is not clear to us that the same can be said for integrable solutions to the Euler equations.
(4) The tight-overtwisted dichotomy in contact topology is one which has a great amount of power: in general, any problem in 3-d contact topology is 'easy' in the overtwisted category and 'hard' in the tight category, due to the rigidity of the tight structures. A prime example of this is the question of existence of periodic orbits in the Reeb field of a contact form: for overtwisted structures, this is guaranteed by Theorem 5.4; for tight structures,

one needs to use case-specific methods. Does this dichotomy have a place in fluid dynamics?

(5) The topology of contact structures has been intensely investigated for the past twenty years with amazing success. What has been slower in evolving is the role that geometry plays in understanding contact structures. In particular, the tight-overtwisted dichotomy has not impacted nor been effected by geometric considerations. Is there a geometric approach to tightness? It would be sublime if perspectives from fluid dynamics [an intrinsically geometric field] can return the favor and give some insight into the geometry of contact structures. It was conjectured in [32] that any curl eigenfield with smallest nonzero eigenvalue (the *principal eigenfield*) is, if nonvanishing, always dual to a tight contact structure. This is not the case, as shown recently by Komendarczyk [73]. His method of proof establishes some intriguing relationships between nodal curves of Laplacians on surfaces and characteristic foliations of overtwisted contact structures. Hopefully, this will lead to a better understanding of geometric tightness.

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