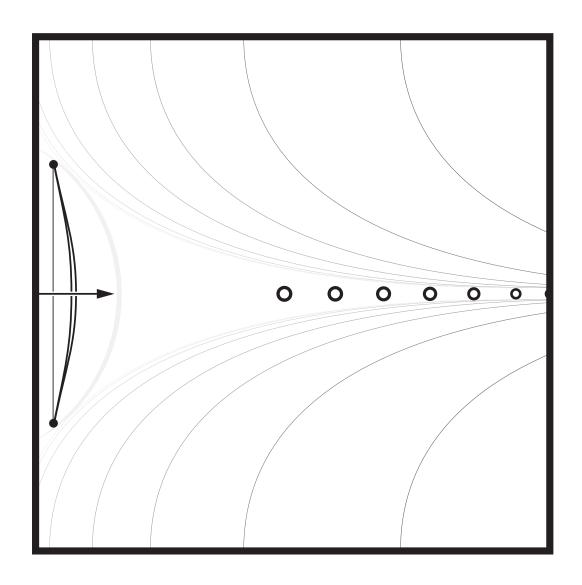
## Chapter 4

# Homology



ecomposing a space into cells, counting, and canceling according to parity of dimension, is efficacious for defining one topological invariant:  $\chi$ . More refined invariants arise upon lifting this enumerative data to the domain of linear algebra. This leads to the notion of **homology**.

## 4.1 Simplicial and cellular homology

The crucial construction of this chapter is that which converts a decomposition of a space in terms of simple pieces into a collection of vector spaces (or modules) and linear transformations (or homomorphisms): an algebraic version of a cell complex. The initial discussion proceeds using the language of linear algebra and passes to more general algebraic constructs. The reader needing review in either linear or abstract algebra is encouraged to consult the Appendix before proceeding.



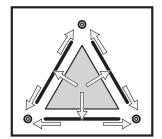
For simplicity, mod-2 arithmetic is used initially. Recall that  $\mathbb{F}_2 = \{0,1\}$  is the field with two elements. Counting in this field is analogous to flipping a light switch. To build intuition, consider a cell complex X, each cell of which is outfitted with a metaphorical light switch. The building blocks of a rudimentary homology for X are as follows.

- 1. Define k-chains  $C_k$  as the vector space over the field  $\mathbb{F}_2$  with basis the k-cells of X.
- 2. Consider the **boundary maps**: the linear transformations  $\partial_k : C_k \to C_{k-1}$  which send a basis k-cell to the abstract sum of basis (k-1)-cell faces.

The collection of chains and boundary maps is assembled into a **chain complex**:

$$\cdots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 . \tag{4.1}$$

The chain complex is **graded**, in this case by the dimension of the cells. It is beneficial to denote the chain complex as a single object  $\mathcal{C}=(C_{\bullet},\partial)$  and to write  $\partial$  for the boundary operator acting on any chain of unspecified grading. Chain complexes are a representation of a cell complex within linear algebra. It seems at first foolish to algebraicize the problem in this manner – why bother with vector spaces which simply record whether a cell is present (1) or absent (0)? Why express the boundary of a cell in terms of linear transformations, when the geometric meaning of a boundary



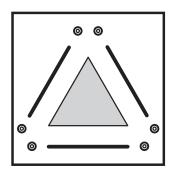
is clear? By the end of this chapter, probably, and the next, certainly, this objection will have been forgotten.

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#### Example 4.1 (Simplices and cubes)

Consider a single n-simplex  $\Delta^n$ . The resulting chain complex has  $C_k$  of dimension  $\binom{n+1}{k+1}$ . The top-dimensional boundary map  $\partial_n$  is a 1-by-n matrix with all entries 1. In contrast, for  $I^n = [0,1]^n$ , the n-dimensional cube with cell structure inherited from the interval I = [0,1] with two endpoints, the chain complex has  $C_k$  of dimension  $2^{n-k}\binom{n}{k}$ .

The following lemma is deeper than it appears.



**Lemma 4.2.** The boundary of a boundary is null:

$$\partial^2 = \partial_{k-1} \circ \partial_k = 0, \tag{4.2}$$

for all k.

**Proof.** For simplicity, consider the case of an abstract simplicial complex on a vertex set  $V = \{v_i\}$ . The **face map**  $D_i$  acts on a simplex by removing the  $i^{\text{th}}$  vertex  $v_i$ 

from the simplex's list, if present; else, do nothing. The graded boundary operator  $\partial \colon C_{\bullet} \to C_{\bullet}$  is thus a formal sum of face maps  $\partial = \bigoplus_i D_i$ . It suffices to show that  $\partial^2 = 0$  on each basis simplex  $\sigma$ . Computing the composition in terms of face maps, one obtains:

$$\partial^2 \sigma = \sum_{i \neq j} D_j D_i \sigma. \tag{4.3}$$

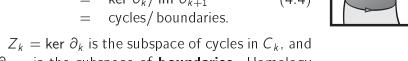
Each (k-2)-face of the k-simplex  $\sigma$  is represented exactly twice in the image of  $D_j D_i$  over all  $i \neq j$ . Thanks to  $\mathbb{F}_2$  coefficients, the sum over this pair is zero.<sup>1</sup>  $\odot$ 

The homology of  $\mathcal{C}$ ,  $H_{\bullet}(\mathcal{C})$ , is a sequence of  $\mathbb{F}_2$ -vector spaces built from the following subspaces of chains.

**Corollary 4.3.** For all k, im  $\partial_{k+1}$  is a subspace of ker  $\partial_k$ .

A **cycle** of  $\mathcal{C}$  is a chain with empty boundary, *i.e.*, an element of ker  $\partial$ . Homology is an equivalence relation on cycles of  $\mathcal{C}$ . Two cycles in  $Z_k = \ker \partial_k$  are said to be **homologous** if they differ by an element of  $B_k = \operatorname{im} \partial_{k+1}$ . The **homology** of X is the sequence of quotient vector spaces  $H_k(X)$  over  $\mathbb{F}_2$ , for  $k \in \mathbb{N}$ , given by:

$$H_k(X) := Z_k/B_k$$
  
=  $\ker \partial_k/\operatorname{im} \partial_{k+1}$  (4.4)  
= cycles/ boundaries.



To repeat:  $Z_k = \ker \partial_k$  is the subspace of cycles in  $C_k$ , and  $B_k = \operatorname{im} \partial_{k+1}$  is the subspace of **boundaries**. Homology inherits the grading of  $\mathbb C$  and will be therefore denoted  $H_{\bullet}(X)$  when a dimension is

<sup>&</sup>lt;sup>1</sup>If the reader is reminded by the notation of certain calculus results that rely on mixed partial derivatives commuting and cancelling, then the notation has done its job.

not specified. Elements of  $H_{\bullet}(X)$  are **homology classes** and are denoted  $[\alpha]$ , where  $\alpha \in Z_k$  is a k-cycle and  $[\cdot]$  denotes the equivalence class modulo elements of  $B_k$ .

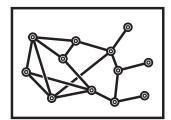
One often distinguishes between **simplicial homology**,  $H_{\bullet}^{\text{simp}}$ , and **cellular homology**,  $H_{\bullet}^{\text{cell}}$ . Note that only simplicial homology has been given a proper definition above, since Lemma 4.2 was proved only in the simplicial setting. The details of cellular homology will appear in §4.3 and §5.5. Before this chapter is done, several more homology theories will be introduced, including some that do not depend on any cellular structure. It is asserted (with justification in §5.4) that these various homology theories agree (when well-defined) except under unusual circumstances not often seen in applications. The notation  $H_{\bullet}$  will therefore be used to denote whatever homology theory is most convenient or appropriate. In the examples to follow, explicit cellular structures may not be given: let the reader take on faith now what will be explicitly detailed in Chapter 5 – homology is a homotopy invariant of spaces.

## 4.2 Homology examples

## Example 4.4 (Graphs)

Consider X a topological graph – a compact 1-d cell complex. As  $\partial_0 = 0$ , all 0-chains are 0-cycles:  $Z_0 = C_0$ . The boundary subspace  $B_0 \subset Z_0$  consists of finite unions of vertices, with an even number in each connected component of the graph. If one chooses the vertex set of X as a basis for  $Z_0$ , it follows that any two basis elements are homologous if and only if they are the endpoints of an end-to-end sequence of edges in X – that is, if and only if they lie in the same connected path-component of X. The dimension of  $H_0(X)$  is therefore the number of such path components.

Any element of  $Z_1(X)$  consists of a finite union of cyclic end-to-end sequences of edges. As a graph is a 1-dimensional complex, there are no higher dimensional chains with which to form homology classes: for graphs, 1-cycles are homologous if and only if they are identical. Correspondingly,  $H_1(X)$  collates the linearly independent cyclic chains of edges, and  $H_k(X) = 0$  for k > 1. Note that homology gives the collection of 1-cycles in X the structure of a vector space, complete with the notions of sums, spans, and bases.



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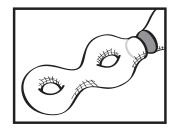
#### Example 4.5 (Surfaces)

For  $S_g$  a cell complex homeomorphic to a connected compact orientable surface of genus g, the homology is nonzero only in dimensions less than three, since all cells are of dimension at most two. The relation to genus is as follows:

$$\dim H_k(S_g) = \begin{cases} 1 & : & k = 0 \\ 2g & : & k = 1 \\ 1 & : & k = 2 \\ 0 & : & k > 2 \end{cases}.$$

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It is worth stressing that  $H_1(X)$  does not measure whether any cycle in X is contractible – whether it can be shrunk to a point in Xcontinuously. For example, on an orientable surface, any simple closed curve which divides the surface into two connected components is nullhomologous as a 1-cycle, even though it may not be contractible within the surface. A few facts may help the beginning reader build up an intuition for what it is that homology measures. Justifications will be filled in over subsequent chapters.



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- 1. For X contractible,  $H_k(X) = 0$  for all k > 0.
- 2. The number of path components of X equals dim  $H_0(X)$ .
- 3. For  $D_n^2$  a disc in the plane with n disjoint discs removed from the interior, dim  $H_0(\mathcal{D}_n^2)=1$ , dim  $H_1(\mathcal{D}_n^2)=n$ , and dim  $H_k(\mathcal{D}_n^2)=0$  for k>1.
- 4. The sphere  $\mathbb{S}^n$  has  $H_k = 0$  for all k except  $H_0$  and  $H_n$ , both of dimension one.
- 5. For a disjoint union of spaces  $X = A \sqcup B$ ,  $H_k(X) = H_k(A) \oplus H_k(B)$ .
- 6. For a **wedge sum** of path-connected spaces  $X = A \vee B$  (obtained by identifying a single point of A with a single point of B),  $H_k(X) = H_k(A) \oplus H_k(B)$  for all k > 0 and dim  $H_0(X) = 1$ .

## Example 4.6 (Products and Künneth)

A torus  $\mathbb{T}^n = (\mathbb{S}^1)^n$  has homology satisfying

$$\dim H_k(\mathbb{T}^n) = \binom{n}{k}.$$

The reader will rightly suspect a relationship with polynomial algebra, via the the coefficients of  $(1+t)^n$ : indeed, this foreshadows an algebraic result for products of general spaces. For X a space with finite-dimensional homology, consider the **Poincaré polynomial** of X,  $P_t(X) := \sum_k \beta_k t^k$ , with coefficients  $\beta_k := \dim H_k(X)$  the **Betti numbers** of X. For example, the Poincaré polynomial of  $\mathbb{S}^1$  is 1+t. The following is a simplistic reduction of the classical **Künneth Theorem**:

Theorem 4.7 (Künneth Formula). 
$$P_t(X \times Y) = P_t(X)P_t(Y)$$
.

The computation for dim  $H_k(\mathbb{T}^n)$  above follows. The simplicity of this statement comes from the linear-algebraic approach used. Complications arise when using coefficients not in a field.

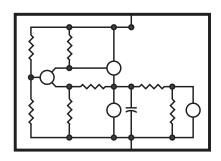
#### 4.3 Coefficients

The homology defined in §4.1 is more properly called the simplicial or cellular homology of X with  $\mathbb{F}_2$  coefficients, denoted  $H_{\bullet}(X; \mathbb{F}_2)$ . This additional parameter hints at other coefficients. For example, instead of the  $C_k$  being vector spaces over a field  $\mathbb{F}_2$ , other fields may be used. For any field  $\mathbb{F}$ , one may construct chains  $C_k$  for a cell complex X

as  $\mathbb{F}$ -vector spaces. Such coefficients permit expressiveness:  $\mathbb{R}$  is effective in describing simplices' *intensities*, in contrast to  $\mathbb{F}_2$ 's involutive switch. This entails a little more care with directions, but is a well-motivated generalization.

## **Example 4.8 (Kirchhoff's current rule)**

Consider an electric circuit as a 1-d cell complex (or graph), with circuit elements (resistors, capacitors, etc.) located at certain vertices. **Kirchhoff's current rule** states that the current flowing through the edges of the circuit satisfies a conservation principle: at each node, the sum of the incoming currents equals the sum of the outgoing currents, with directionality encoded in the  $\pm$  sign. In the language of this chapter: *current is a 1-cycle*. Note the unambiguous need for  $\mathbb R$  coefficients, since current is measured as a real quantity.



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A crucial feature of using coefficients other than  $\mathbb{F}_2$  is this:  $-1 \neq 1$ . This algebraic unfolding necessitates an orientation associated to cells. For electric circuits, graph cycles, and 1-chains in general, this is naturally encoded as a direction. In calculus class, one uses a twist of the right hand for orientations on curves and surfaces: a more principled approach is needed for the present setting. Simplicial complexes yield the most explicit case: recall that an abstract simplicial complex on a vertex set V has each k-simplex specified as  $\sigma = \{v_0, v_1, \ldots, v_k\} \subset V$ . To define an *oriented* k-simplex, one begins with an (arbitrary) ordering of vertices, denoted  $[v_0, v_1, \ldots, v_k]$ .



An **orientation** on  $\sigma$  is a choice of equivalence class of orderings up to *even* permutations; thus,  $[v_0, v_1, v_2] = -[v_0, v_2, v_1] = [v_2, v_0, v_1]$ . This yields a multiplicative action of  $\{-1, +1\}$  on oriented chains: multiplication by -1 is the chain analogue of reversing orientation. In the algebra of chains, as with contour integrals, adding two simplices of opposite orientation cancel. For homology with coefficients in a field  $\mathbb{F}$ , one builds  $C_n$  on a basis of oriented n-simplices. The boundary operator  $\partial: C_n \to C_{n-1}$  tracks orientation as follows, using the simplicial face map nota-

tion from the proof of Lemma 4.2:

$$\partial \sigma = \sum_{i=0}^{k} (-1)^{i} D_{i} \sigma. \tag{4.5}$$

The reader may check that the use of  $(-1)^{i}$ 's suffices to keep Lemma 4.2 in effect. The resulting simplicial homology with  $\mathbb{F}$  coefficients is denoted  $H^{\text{simp}}_{\bullet}(X;\mathbb{F})$ . That it is independent of the initial choice of orientations on the simplices is by no means obvious. Chapter 5 will provide a reason for this invariance.

The definition of orientation for abstract simplices is a mechanical index: cellular homology requires more machinery. In the simple setting of a regular cell complex,

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choose an "orientation" for each cell. For the moment, this requires a little imagination – a direction for an edge, a sense of rotation for a 2-cell, a handedness for a 3-cell, etc. The boundary of a cell inherits an orientation from the interior, in a manner that is used (if not understood) in calculus courses. Just as in calculus, one compares orientations on a domain and its boundary. For  $\sigma \triangleleft \tau$ , with  $\sigma$  a codimension-1 face of  $\tau$ , define the **incidence number**  $[\sigma \colon \tau]$ , to be +1 if the attaching map of  $\tau$  preserves the induced orientation from  $\partial \tau \to \sigma$ ; for orientation reversal,  $[\sigma \colon \tau] = -1$ . The boundary map is then defined as

$$\partial \tau = \sum_{\sigma \leqslant \tau} [\sigma \colon \tau] \sigma. \tag{4.6}$$

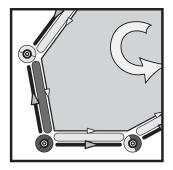
Defining  $[\rho:\tau]=0$  for all faces  $\rho$  of codimension greater than one allows the above sum to range over all faces. This bookkeeping yields a chain complex  $\mathfrak{C}^{\text{cell}}=(C_{\bullet},\partial)$  with homology  $H^{\text{cell}}_{\bullet}(X;\mathbb{F})$  once it is shown that  $\partial\circ\partial=0$ ; namely, that the signs cancel:

$$\sum_{\rho \leqslant \sigma \leqslant \tau} [\rho \colon \sigma] [\sigma \colon \tau] = 0. \tag{4.7}$$

The serendipitous notation of an abstract simplicial complex leads to a simplification in that  $[\sigma:\tau]=(-1)^i$ , where  $\sigma$  is the  $i^{\text{th}}$  face of  $\tau$ : these accidents and subsequent formulae such as (4.5) obscure the meaning of the boundary operator  $\partial$  – signs are really about orientation, as will be explained in §5.5.

Field coefficients and linear algebra are, still, not the full story. The generalization to  $\mathbb{Z}$  coefficients — in which each  $C_k$  is a  $\mathbb{Z}$ -**module** — is particularly relevant. One visualizes chains  $C_{\bullet}(X;\mathbb{Z})$  over a cell complex X with integer coefficients as recording a finite collection of simplices with orientation and multiplicity. Ultimately, one works with a chain complex over an  $\mathbf{R}$ -module (for  $\mathbf{R}$  a ring) with the boundary maps module homomorphisms.

The following examples are not explicitly justified (such computations will require better methods from Chapter 5), but are nonetheless useful for building intuition.



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#### **Example 4.9 (Coefficients)**

Using different coefficients can lead to genuinely different homology groups. For example, the non-orientable Klein bottle  $K^2$  satisfies:

$$H_{k}(K^{2}; \mathbb{F}_{2}) = \begin{cases} \mathbb{F}_{2} & : & k = 0 \\ \mathbb{F}_{2} \oplus \mathbb{F}_{2} & : & k = 1 \\ \mathbb{F}_{2} & : & k = 2 \\ 0 & : & k > 2 \end{cases} ; \quad H_{k}(K^{2}; \mathbb{Z}) = \begin{cases} \mathbb{Z} & : & k = 0 \\ \mathbb{Z} \oplus \mathbb{Z}_{2} & : & k = 1 \\ 0 & : & k > 1 \end{cases}$$

In this example,  $\mathbb{Z}$  coefficients yields a different rank of homology groups at gradings one and two. The Klein bottle has no 2-cycle (a boundaryless nonzero 2-chain) in  $\mathbb{Z}$  coefficients, but it does in  $\mathbb{F}_2$ : thus,  $K^2$  is a non-orientable surface. Note

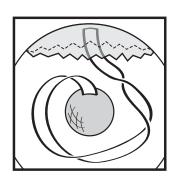
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also the presence of **torsion** in  $H_1(K^2; \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}_2$ , where  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$  is the quotient of  $\mathbb{Z}$  by the subgroup of evens. This indicates that the meridian and longitudinal curves on  $K^2$ , unlike those of the torus  $\mathbb{T}^2$ , are qualitatively different: sliding a meridional curve along a longitude reverses orientation. In general, the presence of a torsional element in  $H_{\bullet}(X; \mathbb{Z})$  is indicative of some type of twisting in X which, in surface examples, manifests itself in non-orientability.

#### Example 4.10 (Rotations and projections)

Rotations in  $\mathbb{R}^3$  offer a fascinating, simple, and useful example of torsional phenomena where coefficients matter. Consider  $SO_3$ , the group of real 3-by-3 orthogonal matrices with determinant 1. These are precisely the orientation-preserving rotations of Euclidean 3-space. As one can demonstrate using a variety of physical devices (plates, belts, fermions, etc.), there is a torsional core writhing within:  $H_1(SO_3; \mathbb{Z}) \cong \mathbb{Z}_2$ . Since  $SO_3 \cong \mathbb{P}^3$ , this is an example of the homology of projective spaces. In general, dim  $H_k(\mathbb{P}^n; \mathbb{F}_2) = 1$  for all  $0 \le k \le n$ . The story for integer coefficients is quite different:

$$H_k(\mathbb{P}^n; \mathbb{Z}) = \left\{ egin{array}{ll} \mathbb{Z} & : & k = 0 \text{ or } k = n, \text{ odd} \\ \mathbb{Z}_2 & : & 0 < k \text{ odd } < n \\ 0 & : & \text{else} \end{array} \right..$$



## 4.4 Singular homology

A definition with sufficient strength has emerged: a chain complex  $\mathcal{C} = (\mathcal{C}_{\bullet}, \partial)$  is any sequence of  $\mathbf{R}$ -modules  $\mathcal{C}_k$  with homomorphisms  $\partial_k \colon \mathcal{C}_k \to \mathcal{C}_{k-1}$  satisfying  $\partial_k \circ \partial_{k+1} = 0$ . Though the generators of  $\mathcal{C}$  may be simplices or cells of a space X, there are many different objects worthy of being counted and compared. To be noted are the commonalities (e.g., notions of dimension associated to the objects counted) and the distinctions (e.g., auxiliary structures imposed). The next several sections delineate a few major homology theories. The first example is singular homology – the most common and generally useful theory.

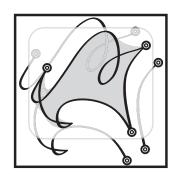
The reader will have observed that cellular homology is, like Euler characteristic, independent of how the space is decomposed into cells. The best method of proof comes not from serendipitous cancellative combinatorial refinements of cell structures, but rather by limits to uncomputable abundance.

One of the first to take such an approach was Vietoris, who introduced his homology theory for metric spaces [296]. In this theory, one fixes an  $\epsilon>0$  and considers the chain complex generated by k-tuples of distinct points in X of pairwise distance  $\leq \epsilon$  (something like the Vietrois-Rips complex of all points in X). These simplices do not fit together to form a triangulation of X: there are far too many. However, for reasonable spaces (e.g., metric finite cell complexes), the homology of

the Vietoris chain complex stabilizes as  $\epsilon \to 0^+$ , and this agrees with the cellular homology of a cell complex. The reason that Vietoris' theory is mostly forgotten lies in the efficacy of **singular homology**, which requires neither a metric space nor the explicit limiting process.

The **singular chain complex** of a topological space X is the complex  $\mathbb{C}^{\text{sing}} = (C_{\bullet}, \partial)$ , where the generators of  $C_k$  are continuous maps  $\sigma \colon \Delta^k \to X$  from the standard k-simplex  $\Delta^k$  to X. The boundary of a singular simplex is the formal sum of restrictions of the map to faces, using the orientation convention of Equation (4.5):

$$\partial \sigma = \sum_{i=0}^k (-1)^i D_i \sigma = \sum_{i=0}^k (-1)^i \sigma \big|_{D_i \Delta^k}.$$



Notice that there is a decoupling between the grading and the dimension of the *image* of the singular simplex. Indeed, the image of a singular simplex may be very convoluted, with no sensible notion of dimension other than in its preimage. The resulting chain complex  $\mathcal{C}^{\text{sing}}$  is enormous – certainly of uncountably infinite dimension except in the most trivial cases. However, in this lies its flexibility. For two spaces that are homeomorphic, there is an equivalence between their singular chain complexes that guarantees equivalent homologies. The yet more useful and general homotopy invariance of singular homology in §5.1 is the true reward for the unwieldy bulk of the singular complex.

## 4.5 Reduced homology

It is sometimes convenient to augment a homology theory beyond grading zero. Given a chain complex  $\mathfrak C$  over an **R**-module, define the **reduced** chain complex  $\tilde{\mathfrak C}$ ,

$$\cdots \longrightarrow C_k \xrightarrow{\partial_k} C_{k-1} \xrightarrow{\partial_{k-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} \mathbf{R} \longrightarrow 0 , \quad (4.8)$$

where  $\epsilon \colon C_0 \to \mathbf{R}$  sends each basis 0-chain to the sum of the coefficients.<sup>2</sup> It is advantageous to write  $\tilde{C}_k$  for the chain groups, where  $k \in \mathbb{Z}$  and:

$$\tilde{C}_k = \begin{cases} C_k & : k \ge 0 \\ \mathbf{R} & : k = -1 \\ 0 & : k < -1 \end{cases},$$

with all boundary maps as above.

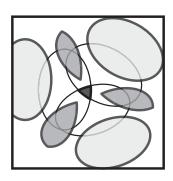
**Proposition 4.11.** For a nonempty chain complex  $\mathbb{C}$  with  $\mathbf{R}$ -module coefficients, the reduced homology  $\tilde{H}_{\bullet}(\mathbb{C})$  satisfies  $H_k(\mathbb{C}) \cong \tilde{H}_k(\mathbb{C})$  for k > 0 and  $H_0(\mathbb{C}) \cong \tilde{H}_0(\mathbb{C}) \oplus \mathbf{R}$ .

<sup>&</sup>lt;sup>2</sup>For  $\mathbb{F}$ -vector spaces,  $\epsilon: C_0 \to \mathbb{F}$ . So, in the case of  $\mathbb{F}_2$ ,  $\epsilon$  records the parity of vertices in a 0-chain.

The reader may think of the reduced complex as having a single basis element at grading -1. Reduced homology simplifies certain results: it is convenient to have a nonempty contractible space have  $\tilde{H}_{\bullet} = 0$ , as such an **acyclic** space plays the role of a *zero* in homological algebra.

## 4.6 Čech homology

A homology theory counts objects with grading and cancelation. While the objects and gradings are usually indicative of cells and dimensions, the correspondence may be disguised. Consider the case of the **Čech homology** of a cover. Let X be a topological space and  $\mathcal{U} = \{U_{\alpha}\}$  a locally-finite collection of open sets whose union is X. The **Čech complex** of  $\mathcal{U}$  is the chain complex  $\check{\mathcal{C}}(\mathcal{U}) = (\check{\mathcal{C}}_{\bullet}, \check{\partial})$  generated by nonempty intersections of the cover  $\mathcal{U}$ . That is, the 0-chains  $\check{\mathcal{C}}_0$  have as basis the elements  $U_{\alpha}$  of  $\mathcal{U}$ . The k-chains  $\check{\mathcal{C}}_k$  have as basis non-empty intersections of k+1 distinct elements of  $\mathcal{U}$ , denoted  $U_J$ , where  $J=(\alpha_0,\alpha_1,\ldots,\alpha_k)$  is a multi-index, ordered up to even permutations as per the ordered simplices of §4.3. The resulting chain complex is outfitted with a boundary operator of familiar form:



$$\delta(U_J) = \sum_{i=0}^k (-1)^i U_{D_i J},$$
(4.9)

where  $D_i$  is the face map of §4.1 that deletes the  $i^{\text{th}}$  entry from the multi-index J. One checks that  $\check{\partial}^2 = 0$  and thus the Čech homology of the cover,  $\check{H}_{\bullet}\mathcal{U} = H_{\bullet}(\check{\mathbb{C}}(\mathcal{U}))$ , is well-defined. The reader will be reminded of the nerve complex of §2.6; the Čech complex  $\check{\mathbb{C}}(\mathcal{U})$  is the simplicial chain complex associated to the nerve  $\mathcal{N}(\mathcal{U})$  of the cover. A homological version of the Nerve Lemma (Theorem 2.4)

holds:

**Theorem 4.12.** If all nonempty intersections of elements of  $\mathcal{U}$  are acyclic  $(\tilde{H}_{\bullet}(U_J) = 0)$  for all nonempty  $U_J$ , then the Čech homology of  $\mathcal{U}$  agrees with the singular homology of  $X: \check{H}_{\bullet}(\mathcal{U}) \cong H_{\bullet}(X)$ .

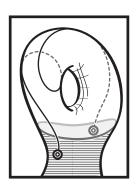
In the same manner that the cellular homology of a (reasonably nice) space is independent of the cell structure used, the Čech homology of a (reasonably nice) space X is well-defined, independent of the (acyclic) cover used to compute it.

## 4.7 Relative homology

Homology makes sense not only for spaces, but for subspaces as well. Let  $A \subset X$  be a subset (or subcomplex, as appropriate) and  $\mathcal{C}(X)$  a chain complex (singular, cellular, etc.) on X. There are two natural chain complexes implicating A. The first,  $\mathcal{C}(A)$ , consists of subgroups  $C_k(A) < C_k(X)$  with the obvious restriction of  $\partial$ . The homology

of this chain (sub-)complex gives, as expected, the homology of A as a space in its own right.

There is, however, a more subtle construct involving the collapse of the subcomplex that yields an important homology complementary to that of A. Let  $\mathcal{C}(X,A)$  denote the quotient chain complex  $(C_k(X,A),\overline{\partial})$ , where  $C_k(X,A):=C_k(X)/C_k(A)$  consists of chains on X modulo chains on A, with the induced boundary maps  $\overline{\partial}$  on the quotients. These relative chains do not vanish on A; rather, they are equivalence classes of chains in X which are identical off of A. The resulting **relative homology**  $H_{\bullet}(X,A)$  collates homology classes of relative cycles (chains in X whose boundaries vanish or lie in A). A relative 1-cycle may therefore be either a genuine 1-cycle in X or may implicate a chain in X whose boundary points are in A.



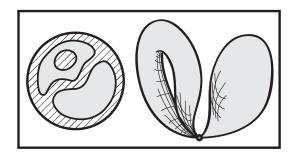
#### Example 4.13 (Reduced homology)

The reduced homology of a space X is isomorphic to the homology of X relative to a basepoint  $p \in X$ :  $\tilde{H}_{\bullet}(X) \cong H_{\bullet}(X,p)$ . The only nontrivial chain in X whose boundary is a nonzero multiple of p is, precisely, a multiple of p and a zero-cycle. Thus, by definition,  $H_k(X,p) \cong H_k(X)$  for k>0 and  $H_0(X,p)$  has rank one less than  $H_0(X)$ , since the subspace corresponding to the homology class  $[\{p\}]$  has been quotiented out.

One of the foundational theorems about homology concerns relative homology. The following is stated for the singular theory: for other (cellular, *etc.*) homology theories, the hypotheses must include the relevant structures present.

**Theorem 4.14 (Excision Theorem).** Let  $U \subset A \subset X$  with the closure of U contained in the interior of A. Then  $H_{\bullet}(X-U,A-U) \cong H_{\bullet}(X,A)$ .

Excision implies that for the pair (X, A), what happens inside of A is irrelevant.



**Corollary 4.15.** For  $A \subset X$  a closed subcomplex of a cell complex X,  $H_{\bullet}(X, A) \cong \tilde{H}_{\bullet}(X/A)$ .

The hypothesis of *A* a subcomplex is usually too restrictive. In the singular setting, it suffices to have *A* closed and possessing an open neighborhood which deformation retracts to *A*. This provides a good way to *visualize* relative homology

and to compute the homology of quotient spaces.

The homology of a closed disc  $\mathbb{D}^n$  relative to its boundary  $\partial \mathbb{D}^n$  has as a nontrivial generator  $\mathbb{D}^n$  itself, as a relative *n*-cycle. From Corollary 4.15 and the fact that  $\mathbb{D}^n/\partial \mathbb{D}^n \cong \mathbb{S}^n$ , one sees  $H_{\bullet}(\mathbb{D}^n, \partial \mathbb{D}^n) \cong \tilde{H}_{\bullet}(\mathbb{S}^n)$ .

## 4.8 Local homology

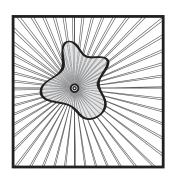
The **local homology** of  $p \in X$  is the (singular) homology  $H_{\bullet}(X, X-p)$ . This is a little hard to visualize, given that Corollary 4.15 is inapplicable, but a moment's thought yields the intuition that it measures something of the features of X in an arbitrarily small neighborhood of p. The correctness of this intuition follows from excision: local homology is local.

**Corollary 4.17.** Let  $p \in X$  be a (closed) point and V an open neighborhood of p. Then  $H_{\bullet}(X, X-p) \cong H_{\bullet}(V, V-p)$ : that is, local homology can be computed from an arbitrarily small neighborhood of p.

**Proof.** Apply Excision, with U = X - V and A = X - p.

## **Example 4.18 (Local orientation)**

Consider the local homology of an n-manifold M at p: this is  $H_{\bullet}(M, M-p)$ . By excision and the definition of a manifold as locally Euclidean, this is isomorphic to  $H_{\bullet}(\mathbb{D}^n, \mathbb{D}^n-0)$ . As the punctured disc deformation retracts to the boundary, it follows from Example 4.16 that the local homology is isomorphic to  $\tilde{H}_{\bullet}(\mathbb{S}^n)$  and is thus nontrivial of rank one in dimension n. This leads to a painless definition of orientation, a concept that usually requires recourse to a dextrous sleight of hand. A **local orientation** at  $p \in M$  is a choice of generator for the n-dimensional local homology at p (in  $\mathbb{Z}$  coefficients). A global **orientation** on a compact



 $\odot$ 

(0)

*n*-manifold M is a choice of generator for  $H_n(M; \mathbb{Z})$ . A global orientation, in keeping with one's experience from calculus class, is a consistent choice of local orientations.

This definition of orientation illuminates the cellular homology of §4.3. The orientation of an n-dimensional cell  $\sigma$  is defined to be a choice of generator for  $H_n(\sigma,\partial\sigma;\mathbb{Z})\cong\mathbb{Z}$ . The **incidence number**  $[\sigma:\tau]$  of a face pair  $\sigma \triangleleft \tau$  is  $\pm 1$ , depending on whether the orientations of  $\sigma$  and  $\partial \tau$  agree or disagree. What remains undefined (until §5.5) is how the orientation on  $\tau$  induces an orientation on  $\partial \tau$ .  $\odot$ 

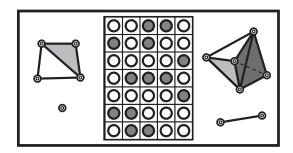
## 4.9 Homology of a relation

A **relation** between two sets X and Y is a subset  $\mathcal{R} \subset X \times Y$  of their product. A point  $x \in X$  is related to a point  $y \in Y$  if and only if  $(x, y) \in \mathcal{R}$ . For example, the sensing modality used for target enumeration in §3.7 is a relation between sensors

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and targets collating which sensors detect which targets. Other relations are prevalent: in marketing, the *puchase* relation between customers and products; on *Twitter*, the *follows* relation between individuals; or in manufacturing, the *assignment* relation between workers and tasks.

It is possible and profitable to build homologies for relations as follows. Given  $\mathcal{R} \subset X \times Y$ , define the chain complex  $\mathcal{C}(X;\mathcal{R})$  with  $\mathbb{F}_2$  coefficients as follows:  $C_k(X;\mathcal{R})$  has as basis unordered (k+1)-tuples of points in X related to some  $y \in Y$ . One builds a dual complex  $\mathcal{C}(Y;\mathcal{R})$  from columns of  $\mathcal{R}$  – unordered tuples of points of Y related to some fixed  $x \in X$ . The boundary operators for  $\mathcal{C}(X;\mathcal{R})$  and



 $\mathcal{C}(Y;\mathcal{R})$  forget points in the tuples, mimicking the boundary operator of a simplicial complex (the property of being related to a common element is closed under subsets). The homology of the relation,  $H_{\bullet}(\mathcal{R})$ , is defined as follows:

## Theorem 4.19 (Dowker's Theorem [100]). $H_{\bullet}(\mathbb{R}) := H_{\bullet}(X; \mathbb{R}) \cong H_{\bullet}(Y; \mathbb{R})$ .

This result was originally used to prove an equivalence between Čech and Vietoris homology theories, in the case where X is a metric space, Y is a dense net of points in X, and  $\mathcal R$  records which points in X are within  $\epsilon > 0$  of points in Y. The nerve of the cover of X by metric balls about Y has homology  $H_{\bullet}(Y;\mathcal R)$ , and the Vietoris homology of X is captured by  $H_{\bullet}(X;\mathcal R)$ .

Instead of passing to homology, one can take the intermediate step of assigning to  $\mathcal{R}$  a pair of simplicial complexes – the **Dowker complexes**,  $\mathcal{R}_X$  and  $\mathcal{R}_Y$  – given by the following process. Given  $\mathcal{R}$ , let  $\mathcal{R}_X$  be the nerve complex of the cover of Y by columns of  $\mathcal{R}$ , and let  $\mathcal{R}_Y$  be the nerve complex of the cover of X by rows of  $\mathcal{R}$ . Otherwise said, X is the vertex set of  $\mathcal{R}_X$  and a simplex of  $\mathcal{R}_X$  corresponds to a collection of points in X witnessed by some common  $y \in Y$  via the relation  $\mathcal{R}$ . The full version of Dowker's Theorem states that these nerves are homotopic (and, hence, have the same homology). For example, the witness complex of §2.3 is an example of a Dowker complex where Y (the landmarks) is a subset of X (the point cloud) and the relation is via witnessing. There is no reason why, in general, one need constrain landmarks in that manner: they may live in an entirely different space.

#### **Example 4.20 (Transmitters and receivers)**

Consider a finite set X of transmitters in some domain  $\mathcal{D}$  which broadcast their identities (assumed unique) through an unspecified modality (pings, continuous signals, etc.). A distinct finite collection of receivers Y can read transmissions and discern identities of transmitters that are within range. The resulting system of transmitters and receivers can be encoded in a relation  $\mathcal{R} \subset X \times Y$ , where  $(x_i, y_j) \in \mathcal{R}$  if and only if signals from device  $x_i$  are heard by receiver  $y_j$ . One may wish or assume or impose that  $\mathcal{R}$  has geometric constraints (receivers hear only reasonably nearby transmitters);

such a case permits approximating the common domain  $\mathcal{D}$  as a simplicial complex via the Dowker complexes [152]. The duality theorem can assist in computation when there is a disparity in the sizes of X and Y. For example, if there are relatively few transmitters as compared to receivers, then the complex  $\mathcal{R}_Y$  has Y as its 0-skeleton, and will be a lower-dimensional complex than  $\mathcal{R}_X$ , whose 0-skeleton is X. This leads to an *a priori* bound on the possible dimensions for nonzero homology classes.

## 4.10 Functoriality

Algebraic topology concerns not only features of spaces but features of *maps* between spaces. This has been hinted at in §1.6 (transversality of *maps* being a crucial tool), §2.9 (where a decision task induces a *map* between input and output complexes), and §3.8 (where *maps* induce actions on Euler integrals). These hints come to fruition in homology, which gives a natural characterization of the qualitative features not only of spaces but also of maps between spaces. This property, at the heart of algebraic topology, is **functoriality**, and it is from this principle the power of the subject emerges.

Consider, for simplicity, a cellular map  $f: X \to Y$  between cell complexes. In the same manner that X unfolds cell-by-cell into a chain complex  $C_{\bullet}(X)$ , one can unfold f to a graded sequence  $f_{\bullet}$  of homomorphisms from  $C_k(X) \to C_k(Y)$ , generated by basis cells of X being sent to basis cells of Y. If a k-cell  $\sigma$  of X is sent by f to a cell of dimension less than k, then the algebraic effect is to send the basis chain in  $C_k(X)$  to  $0 \in C_k(Y)$ . The continuity of the map f induces a **chain map**  $f_{\bullet}$  that fits together with the boundary maps of  $C_{\bullet}(X)$  and  $C_{\bullet}(Y)$  to form a **commutative diagram**:

Commutativity means the chain map respects the boundary operation,  $f_{\bullet}\partial = \partial' f_{\bullet}$ . Because of this, f acts not only on chains but on cycles and boundaries to yield the **induced homomorphism**  $H(f) \colon H_{\bullet}X \to H_{\bullet}Y$  on homology. For  $\alpha$  a cycle in X, define  $H(f)[\alpha] := [f_{\bullet}\alpha] = [f \circ \alpha]$ . This is well-defined: if  $[\alpha] = [\alpha']$ , then, as chains,  $\alpha' = \alpha + \partial \beta$  for some  $\beta$ , and,

$$f_{\bullet}\alpha' = f \circ \alpha' = f \circ (\alpha + \partial \beta) = f \circ \alpha + f \circ \partial' \beta = f_{\bullet}\alpha + \partial'(f_{\bullet}\beta),$$

so that  $[f_{\bullet}\alpha'] = [f_{\bullet}\alpha]$  in  $H_{\bullet}(Y)$ . Homological functoriality means that the induced homomorphisms on homology are an algebraic reflection of the properties of continuous maps between spaces.

This definition was framed with cellular homology and cellular maps for concreteness. Everything carries over naturally to the setting of singular chains and singular homology. The other homology theories of this chapter (relative, reduced, Čech, etc.) likewise posses well-defined induced homomorphisms for the appropriate class of maps. The following are simple properties of induced homomorphisms, easily shown from the definitions above:

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- 1. Given  $f: X \to Y$ ,  $H(f): H_{\bullet}X \to H_{\bullet}Y$  is a (graded) homomorphism.
- 2. The identity map  $\operatorname{Id}: X \to X$  induces the isomorphism  $\operatorname{Id}: H_{\bullet}X \to H_{\bullet}X$ .
- 3. Given  $f: X \to Y$  and  $g: Y \to Z$ ,  $H(g \circ f) = H(g) \circ H(f)$ .

$$H_{\bullet}Y_{1} \xrightarrow{H(f_{1})} H_{\bullet}Y_{2}$$

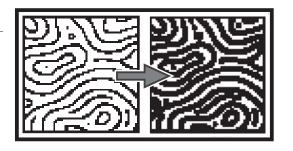
$$H_{\bullet}(f_{1}) \xrightarrow{H(f_{2})} H_{\bullet}X$$

There is hardly a more important feature of homology than this functoriality. One implication in the sciences is to  $H_{\bullet}Y_1 \xrightarrow{H(f_1)} H_{\bullet}Y_2$  than this functionality. One implication in the sciences is a inference. It is sometimes the case that what is desired is knowledge of the homology of an important but unobserved space X; the observed data comprises the homology of a pair of spaces  $Y_1$ ,  $Y_2$ , which are related by a map  $f: Y_1 \to Y_2$ 

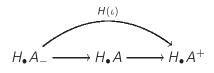
that factors through a map to X, so that  $f = f_2 \circ f_1$  with  $f_1: Y_1 \to X$  and  $f_2: X \to Y_2$ . If the induced homomorphism H(f) is known, then, although  $H_{\bullet}(X)$  is hidden from view, inferences can be made.

## Example 4.21 (Experimental imaging data)

The problem of measuring topological features of experimental data by means of imaging is particularly sensitive to threshold effects. Consider, e.g., an open tank of fluid whose surface waves are experimentally measured and imaged. Perhaps the region of interest is the portion of the fluid surface above the ambient height h=0; the topology of the set  $A=\{h>$ 0} must be discerned, but can only be ap-



proximated by imprecise pixellated images of  $\{h \geq 0\}$ . Similar situations arise in MRI data, where the structure of a tissue of interest can be imaged as an approximation. Given a reasonably close image, is an observed topological feature (say, a hole, or lack of connectivity) true?

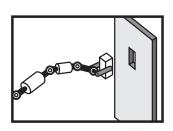


If the goal is to capture the topological features of an exact but unmeasurable set, functoriality may assist. Let, e.g.,  $A = \{h > 0\}$  be the desired but unseen set. If one can measure approximants  $A_{-} \subset A \subset A^{+}$  from below and above and then match common features of these images, then one has a simple commutative diagram where the map

 $H(\iota)$  is the induced map on the inclusion  $\iota: A_- \hookrightarrow A^+$  that itself factors through inclusion to the invisible desideratum A. Any nonzero element in the image of  $H(\iota)$ must factor through a nonzero homology class in A: one can discern the presence of a true hole with two imprecise observations and a map between them. This simple observation is greatly extendable to the concept of persistence, as will be seen in §5.13-5.15. (0)

## 4.11 Inverse kinematics

Other applications of homology are *obstructive*. Consider the idealized **robot arm** of Example 1.5 consisting of rotational joints and rigid rods, grasping a part at the end of the arm. In many applications in manufacturing, one must perform *part placement* – manoeuver the arm so as to locate the part in the correct location and/or orientation. If the arm has, say, N rotational joints, then its configuration space is  $\mathbb{T}^N$  (as is common when mathematicians study robot arms, the mechanisms are considered insubstantial, and no thought is wasted on the problem of self-intersection). The critical issue is that of understanding effect of the arm on the part grasped at the end. Consider, *e.g.*, the orientation problem: given a desired part orientation, is there a sequence of rotations to realize it?

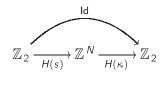


One topologizes the problem [165]: consider the **kinematic map**  $\kappa \colon \mathbb{T}^N \to \mathsf{SO}_3$  taking the ordered sequence of rotations to the net orientation of the part at the end of the arm. The critical issue is inverting the kinematic map. Given a part orientation in  $\mathsf{SO}_3$ , is there a sequence of rotations to realize it? Certainly,  $\kappa$  is onto for N not-too-small and a choice of joint axes which span  $\mathbb{R}^3$ . It is the inverse kinematic map which is problematic. Given a fixed part ori-

entation, are all nearby part orientations realizable via small changes in the rotations required? This local problem seems to be solvable; the global version is not.

**Proposition 4.22.** There is no continuous section to  $\kappa$ . That is, there is no map  $s : SO_3 \to \mathbb{T}^N$  satisfying  $\kappa \circ s = Id$ .

**Proof.** Consider a putative section  $s \colon SO_3 \to \mathbb{T}^N$  with  $\kappa \circ s = \operatorname{Id}$ . Then on  $H_1$  in  $\mathbb{Z}$  coefficients one has, from Examples 4.6 and 4.10,  $H_1(\mathbb{T}^N) \cong \mathbb{Z}^N$  and  $H_1(SO_3) \cong \mathbb{Z}_2$ . These fit together in a commutative diagram with induced homomorphisms  $H(\kappa)$  and H(s) such that  $H(\kappa) \circ H(s) = H(\kappa \circ s) = \operatorname{Id}$ , an identity map, thanks to functoriality. This yields a contradiction: since  $\mathbb{Z}^N$  has no nonzero elements of finite order, H(s) = 0 and  $H(\kappa) \circ H(s) = 0$ .

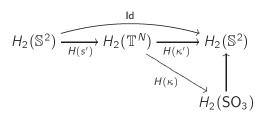


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This result means that a *continuous* assignment of robot arm rotation angles as a function of part orientation is impossible. This decidedly non-intuitive result is a direct consequence of the algebraic topology of configuration spaces, as revealed by  $H_1$ . Note the role of luck in the appearance of  $\mathbb{Z}_2$  in the homology with  $\mathbb{Z}$ -coefficients: torsion is not always available. Consider the case where the part grasped by the robot arm is rotationally symmetric about the last axis, such as a bolt or pin to be inserted. The desideratum is no longer an orientation in  $SO_3$  but rather a point in  $\mathbb{S}^2$ , the direction in which the last axis of the part points.

Repeating the argument of Proposition 4.22 with the modified kinematic map  $\kappa' \colon \mathbb{T}^N \to \mathbb{S}^2$  leads to frustration, as  $H_1(\mathbb{S}^2) = 0$ . There is no analogous contradic-

tion in assuming that the map  $H(\kappa') \circ H(s')$  is the identity, since it is the identity homomorphism on  $H_1(\mathbb{S}^2) = 0$ . Of course, one does not conclude that the inverse kinematic map  $s': \mathbb{S}^2 \to \mathbb{T}^N$  necessarily exists. In fact, s' does not exist.



To see this, build the commutative diagram on  $H_2$ , using H(s') and  $H(\kappa')$ . Here, one notes that  $\kappa'$  factors through the original kinematic map  $\mathbb{T}^n \stackrel{\kappa}{\to} SO_3$ , since the joint angles give the part a true orientation in  $SO_3$ . Ignoring all but the last axis in the frame gives a projection  $\mathbb{T}^n$ The top-row composition  $H(\kappa') \circ H(s')$  can-

not be the identity map on  $H_2(\mathbb{S}^2)$ , since  $H_2(SO_3) \cong 0$  while  $H_2(\mathbb{S}^2) \neq 0$ . Again, functoriality reveals what individual homology groups do not. There is no continuous assignment of rotation angles to a directional axis for a robot arm manipulating a part.

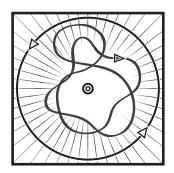
#### 4.12 Winding number and degree

It is helpful to think of an induced map on homology as akin to a winding or linking: to what extent are the holes of X wound about the holes of Y via  $f: X \to Y$ ? The full complexity of homomorphisms between abelian groups gives an algebraic picture of the wrapping and winding that maps can execute. Indeed, induced homomorphisms are the right way to express the classical notion of winding numbers.



A continuous simple closed curve  $\gamma \colon \mathbb{S}^1 \to \mathbb{R}^2$  in the plane separates the plane into two connected regions:

dim  $H_0(\mathbb{R}^2 - \gamma(\mathbb{S}^1)) = 2$ . The **mod-2 winding number** of  $\gamma$  about a point  $p \in \mathbb{R}^2$ not in the image of  $\gamma$  is, intuitively, the number which represents whether p is inside (1) or outside (0) the image of  $\gamma$ . Of the many definitions the reader may have seen (either involving miraculous integrals or the clever counting of intersections), the best is via homology. For  $\gamma: \mathbb{S}^1 \to \mathbb{R}^2 - p$ , consider the induced homomorphism:



$$H(\gamma): H_1(\mathbb{S}^1) \to H_1(\mathbb{R}^2 - p).$$

Both the domain and codomain of  $H(\gamma)$  are of rank one, and the map  $H(\gamma)$  is therefore multiplication by a constant: that constant  $deg(\gamma)$  is the winding number of  $\gamma$  about p (in  $\mathbb{F}_2$  or  $\mathbb{Z}$  depending on the coefficients used). There is no need to restrict to smooth or non-selfintersecting curves: any map  $\gamma \colon \mathbb{S}^1 \to \mathbb{R}^2 - p$  determines a homology class. The winding number measures the algebraic number of times the image of  $\gamma$  wraps about p, with

a choice of orientations (both of  $\gamma$  and  $\mathbb{R}^2 - p$ ) determining the sign.

From the fact that  $H_n\mathbb{S}^n\cong\mathbb{Z}$ , any self-map of  $\mathbb{S}^n$  induces a homomorphism on  $H_n$  which is multiplication by an integer: this is the **degree** of the map. The resulting degree theory is an important classical topic, some of the important points of which are as follows: for maps  $\mathbb{S}^n\to\mathbb{S}^n$ ,

- 1. deg is a homotopy invariant;
- 2.  $deg(f \circ g) = (deg f)(deg g)$ ;
- 3. deg Id = 1;
- 4. deg  $a = (-1)^n$  for the antipodal map a on  $\mathbb{S}^n$ ; and
- 5. deg  $f = \deg g$  if and only if  $f \simeq g$  [the **Hopf Theorem**].

## Example 4.23 (Index theory and vector fields)

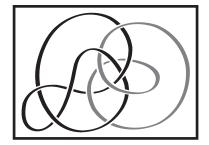
The index theory for vector fields introduced in §3.3 used line integrals and Green's Theorem to obtain a definition for  $\mathbb{R}^2$ . This recourse to vector calculus is appropriate only for beginners. Let V be a vector field on an n-manifold M with isolated fixed point p, and let  $B_p$  be a sufficiently small ball about p. The **index** of V at p,  $\Im_V(p)$ , is defined to be the degree of the map  $V|_{\partial B_p}: \partial B_p \cong \mathbb{S}^{n-1} \to \mathbb{R}^n - 0 \simeq \mathbb{S}^{n-1}$ , where V is represented in local coordinates on  $B_p$ . This index is well-defined, since p is an isolated fixed point. The reader may easily show by the properties of degree that the index is independent of the neighborhood  $B_p$  chosen, so long as  $B_p \cap \operatorname{Fix}(V) = p$ .  $\odot$ 

## **Example 4.24 (Linking number)**

One can define a degree for any map  $f: M \to N$  between oriented compact n-manifolds via  $H(f): H_n(M) \to H_n(N)$ , since, for M and N,  $H_n \cong \mathbb{Z}$ . One classical application of this type of degree is in *knot theory*. A **knot** is an embedding (a smooth injective map)  $\mathbb{S}^1 \hookrightarrow \mathbb{S}^3$ . It is clear that there are many inequivalent ways to tie a simple closed curve in  $\mathbb{R}^3$ : §8.3 will show how to distinguish some of them. Degree can be used to characterize how two disjoint oriented knots entwine or link one another. To define the **linking number** of two disjoint oriented knots,  $K_1$  and  $K_2$ , one can parameterize each knot in Euclidean coordinates as  $\gamma_1, \gamma_2 \colon \mathbb{S}^1 \to \mathbb{R}^3 \subset \mathbb{S}^3$  and set  $\ell k(K_1, K_2)$  to be the degree of the map  $\phi \colon \mathbb{T}^2 \to \mathbb{S}^2$  given by:

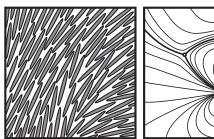
$$\phi \colon \theta_1, \theta_2 \mapsto \frac{\gamma_2(\theta_2) - \gamma_1(\theta_1)}{\|\gamma_2(\theta_2) - \gamma_1(\theta_1)\|}. \tag{4.11}$$

The reader may observe that this quantity is invariant of the parametrization chosen, so long as orientation is respected. Linking number has been found useful in a variety of contexts, from flowlines in fluid- and magnetohydro-dynamics [16] to chemistry [121] and DNA strands [285]: see §8.3.

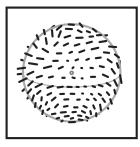


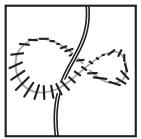
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Liquid crystals are a mesophase of matter which interpolate between ordered and disordered structure. Besides being of great commercial interest, liquid crystals offer fascinating observable defects whose classification is inherently topological. For concreteness, consider the **nematic** liquid crystals, composed of axisymmetric rod-like molecules whose alignment (or director field), under a continuum assumption based on average behavior, is a continuous function of position apart from certain defects or disclinations [6]. A topological classification of such defects is intricate, but an initial step uses degree as follows. Motivated by Example 4.23, one sets up a map whose induced action on homology yields an appropriate degree.









In a nematic liquid crystal that is taken so thin as to be approximated by a plane region to which the molecules align (a so-called **Schlieren texture**), the director field  $\xi$  assigns to points in  $\mathbb{R}^2$  a planar direction in  $\mathbb{P}^1$ , well-defined and continuous off the set of singular point-defects. Note that, although  $\mathbb{P}^1$  is homeomorphic to  $\mathbb{S}^1$ , there is no "arrow" and a half-turn suffices to match directors. To assign an index to a defect at  $x \in \mathbb{R}^2$ , choose an oriented small loop  $\gamma$  encircling x and consider the restriction  $\xi \colon \gamma \to \mathbb{P}^1 \cong \mathbb{S}^1$ . The resulting index  $\mathfrak{I}_{\xi}(x) \in \mathbb{Z}$  is an undirected version of that used for vector fields. In the literature, one often divides this integer quantity by 2 so as to obtain agreement with the vector field index when the director field is orientable.

In the case of a nematic in  $\mathbb{R}^3$ , the director field  $\xi$  takes values in  $\mathbb{P}^2$ . Following the planar case, choose a sufficiently small ball B around an isolated point-defect xand consider the restriction of  $\xi \colon \partial B \cong \mathbb{S}^2 \to \mathbb{P}^2$  and its induced map  $H(\xi) \colon H_2(\mathbb{S}^2) \to \mathbb{P}^2$  $H_2(\mathbb{P}^2)$ . From Example 4.10,  $H_2(\mathbb{P}^2; \mathbb{Z}) = 0$ , and the (integer-valued) degree is trivial; however, passing to  $\mathbb{F}_2$  coefficients for  $H_2$  yields a well-defined index  $J_{\xi}(x) \in \mathbb{F}_2$ , since  $H_2(\mathbb{P}^2; \mathbb{F}_2) \cong \mathbb{F}_2.$ 

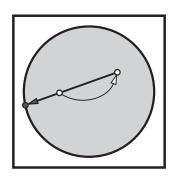
For singularities occurring along a disclination curve, index is measured by a loop  $\gamma \cong \mathbb{S}^1$  locally linking the disclination. The resulting map on homology  $H(\xi): H_1(\mathbb{S}^1) \to \mathbb{S}^1$  $H_1(\mathbb{P}^2)$  is again going to return a degree  $\mathcal{I}_{\mathcal{E}} \in \mathbb{F}_2$ , which, like that of a point-singularity, is sensitive to orientability of the director field about the disclination.

#### 4.13 Fixed points and prices

The following fixed point theorem was one of the earliest triumphs of algebraic topology.

**Theorem 4.26 (Brouwer Fixed Point Theorem).** Every self-map of the closed disc  $\mathbb{D}^n$  has a fixed point.

As might be expected for so primal a result, there are numerous proofs. The following is of the classical variety.



**Proof.** Assume that  $f: \mathbb{D}^n \to \mathbb{D}^n$  is continuous and fixed-point-free. Then, since  $f(x) \neq x$ , for any x, there is a well-defined point  $r(x) \in \partial \mathbb{D}^n$  given by following the ray starting at f(x) and passing through x unto the boundary  $\partial \mathbb{D}^n$ . This implicitly defines  $r: \mathbb{D}^n \to \partial \mathbb{D}^n$ . Note that r is continuous (since f is) and that it is a retraction: r(x) = x for all  $x \in \partial \mathbb{D}^n$ . Thus, denoting by  $\iota: \partial \mathbb{D}^n \hookrightarrow \mathbb{D}^n$  the inclusion, the diagram on the left commutes:

$$\partial \mathbb{D}^n \xrightarrow{\iota} \mathbb{D}^n \xrightarrow{r} \partial \mathbb{D}^n \qquad \mathbb{Z} \xrightarrow{\iota} 0 \xrightarrow{r} \mathbb{Z} .$$

Passing to homology in degree n-1 (on the right) reveals the contradiction, since  $H_{n-1}(\partial \mathbb{D}^n) \cong \mathbb{Z}$  but  $H_{n-1}(\mathbb{D}^n) = 0$ : contradiction.

The reader will likely have seen applications of fixed-point theorems (the efficacy of "YOU ARE HERE" signs, perhaps). Infinite-dimensional versions (e.g., the Schauder Fixed Point Theorem for Banach spaces) are important in proving the existence of solutions to differential equations or the termination of certain algorithms. Fixed point theorems for multi-valued mappings (e.g., the Kakutani Fixed Point Theorem) are important in proving existence of Nash equilibria in game theory (see §5.11). Perhaps the most well-known applications are in Economics.

## Example 4.27 (Prices)

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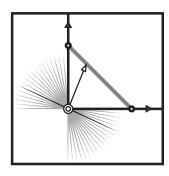
Equilibrium theory – a signature achievement of 20th century economics – asserts that market prices exist. For a simple, explicit example, consider the following result on the existence of equilibrium price distributions in an economy, following Arrow and Debreu [18]. Consider an economy of N items for sale. The space of possible prices is  $[0,\infty)^N$ . As a way of working with price ratios (and imposing compactness), assume that not all prices are zero (!) and normalize the price space to the closed unit (N-1)-simplex  $\Delta \subset [0,\infty)^N$  by dividing each vector by the sum of its components. Thus, any point of of  $\Delta$  represents a complete set of price ratios in the economy.

Assume a finite set of customers, each with a fixed demand function  $D_{\alpha} \colon \Delta \to \mathbb{R}^N$  and a finite set of suppliers, each with a fixed supply function  $S_{\beta} \colon \Delta \to \mathbb{R}^N$ . Both supply and demand have ranges which can be positive or negative. For each choice of price ratio  $p \in \Delta$ , consider the *excess demand* function  $Z \colon \Delta \to \mathbb{R}^N$  given by:

$$Z(p) := \sum_{\alpha} D_{\alpha}(p) - \sum_{\beta} S_{\beta}(p).$$

Positive components of Z connote an excess of demand, while negative values connote an excess of supply. Assume (for the sake of Mathematics) that Z is continuous and (for the sake of Economics) that  $p \cdot Z(p) = 0$ : this is known as Walras' Law and it represents a balance between net income and net expenditure in the system. Then, it can be shown that the system possesses a **price equilibrium**, meaning a point  $p \in \Delta$  such that  $Z(p) \leq 0$ , meaning that all items with nonzero price have demand equal to supply (since  $p \cdot Z(p) = 0$  and  $p \geq 0$ ).

This follows from Theorem 4.26. Consider the map  $f\colon \Delta \to \Delta$  given by sending p to  $\max\{0,p+Z(p)\}/C$ , where the denominator  $C:=\sum_i \max\{0,p+Z(p)\}$  is chosen to project to  $\Delta$ . Since  $p\cdot Z(p)=0$ , one has  $p\cdot (p+Z(p))=\|p\|^2>0$ , making C>0 and f well-defined. There is a fixed point,  $p^*$ , of f, and this satisfies  $p^*=\max\{0,p^*+Z(p^*)\}/C$ . Assume the case where  $p^*>0$  (all components are positive); then, since  $p^*=(p^*+Z(p^*))/C$  and  $Z(p^*)$  is orthogonal to  $p^*$ , it follows that C=1 and, hence,  $Z(p^*)=0$ . On the other hand, if some items have equilibrium price  $p_i^*=0$ , then, since  $p_i^*+Z_i(p^*)\leq 0$ , it



follows that  $Z_i(p^*) \leq 0$ , and one projects to the subspace of items with nonzero price.

This argument is inelegant (the dot product condition is suspiciously rigid). A more general and beautiful result on equilibria in multi-agent systems — the Nash equilibrium theorem — will be attainable in §5.11 after learning better tools.

## Notes

- 1. Linear algebra concerns itself very little with vectors and vector spaces: it is with linear transformations that the subject comes to life. The homological algebra hinted at in this chapter is the natural evolution in linear transformations from the single to the sequential.
- 2. The reader will note, perhaps with displeasure, that computational issued have not been discussed: it is too large and fluid a topic. See [104, 186, 305] for an introduction to the many techniques available for fast computation of homology, some of which are distributable (a near-necessity for realistic scientific applications). The short version seems to be that cellular homology is computable in time near-linear in the number of cells. The curse of dimensionality (high-dimensional complexes have many, many cells) remains a challenge in some contexts, as does the problem of storage (space complexity).
- 3. Cellular homology has been treated lightly, with an implicit assumption of a regular cell complex, such as with simplicial or cubical complexes. For a general cell complex, the definition of incidence number  $[\sigma:\tau]$  is a bit more involved, requiring the tools of §5.5.
- 4. The Künneth formula of Example 4.6 holds for singular or cellular homology in field coefficients, for spaces with finite dimensional homology groups. There is a more general version which accommodates general coefficients and infinite spaces. In the case of  $\mathbb{Z}$  coefficients and for spaces with torsion, it is exact but algebraically delicate: see, e.g., [176].

- 5. §4.5 is the first hint that taking advantage of *negative*-dimensional chains and homology is permissible. There are homology theories in which  $H_{\bullet}$  takes on significant meaning for negative gradings: see Example 7.18.
- 6. The Excision Theorem 4.14 is more foundational than the brief treatment of this chapter might indicate. Eilenberg and Steenrod, in their axiomatization of homology, have the excision property as one of the five axioms of a homology theory.
- 7. This text's treatment of winding numbers begins with an invocation of the classic Jordan Curve Theorem: Let A be a subset of  $\mathbb{S}^n$  homeomorphic to  $\mathbb{S}^k$ . Then  $\tilde{H}_p(\mathbb{S}^n-A)\cong\mathbb{Z}$  for p=n-k-1 and 0 else. Note that there is no assumption of tameness in this statement.
- 8. The linking numbers of Example 4.24 are just the beginning of the algebraic topology of knots and links a vast and beautiful subject. Knot theory has found several applications, notably in physics, biology, fluid dynamics, and differential equations: see Examples 6.25, 8.10, and 8.11.
- 9. The treatment of singularities in nematic liquid crystals in Example 4.25 is incomplete. Very interesting phenomena occur as the field is continuously changed and disclination lines are enticed to collide or entangle, or as point-defects move so as to encircle a disclination line [6]. This will be revisited in §8.5.
- 10. Baryshnikov has announced applications of homology and linking numbers to problems of **caging** in robotics, where, given a fixed geometric object in  $\mathbb{R}^2$ , one wishes to choose the smallest discrete set  $D \subset \mathbb{R}^2$  that prevents the object from being able to be moved 'to infinity' by means of Euclidean motions (translations and rotations) in  $\mathbb{R}^2 D$  [259]. The related problem of performing motion-planning in robotics about obstacles by means of specifying homology classes leads to computable optimization methods [35].
- 11. The applications of fixed point theorems to economics is worthy of a text in itself. Example 4.27 is of limited value as many important considerations (time-dependence, absolute-versus-relative price, zero-price demand, etc.) have been ignored. Nor is it to be denied that equilibria can be proved (and in some cases computed) via combinatorial means. The power of the fixed point method is the ability to analyze large collections of agents with diverse goals.