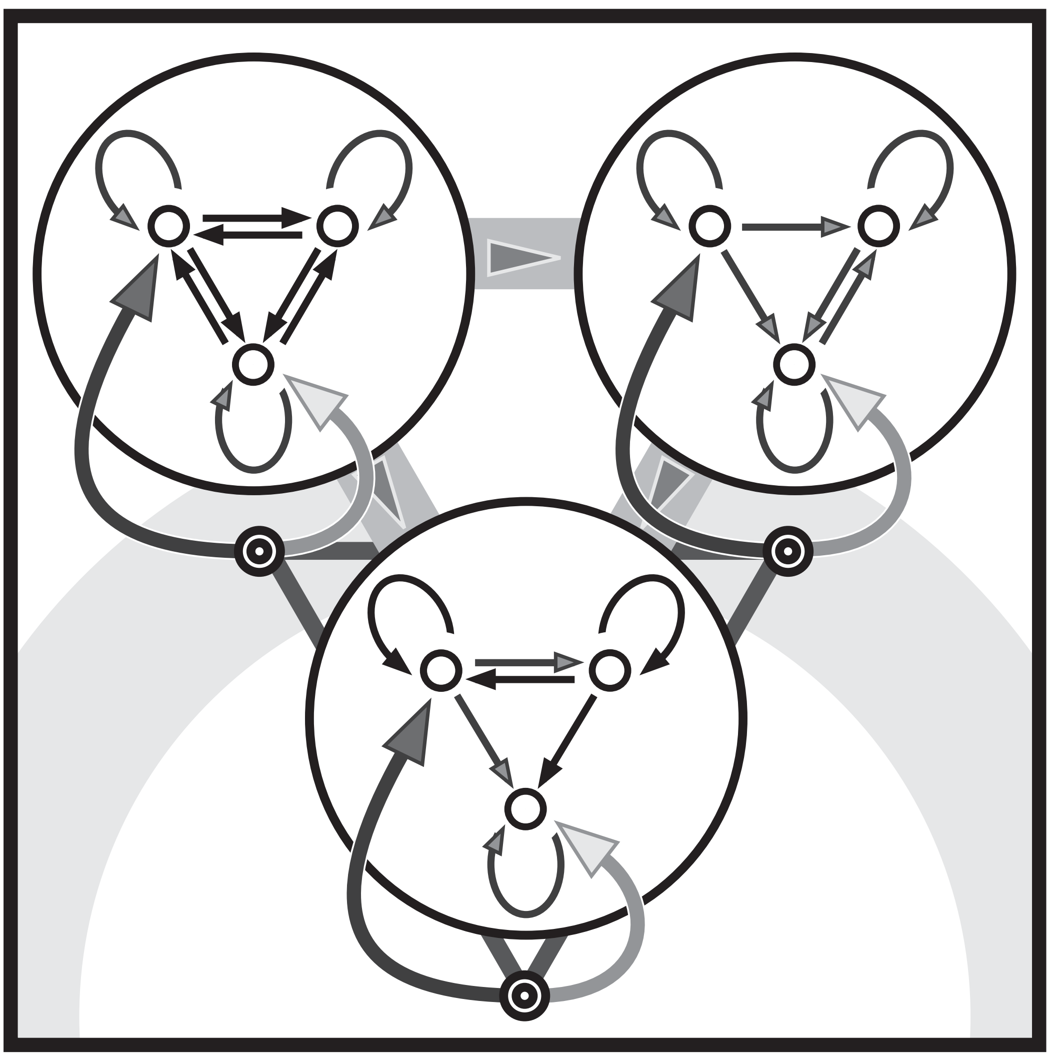
Chapter 10

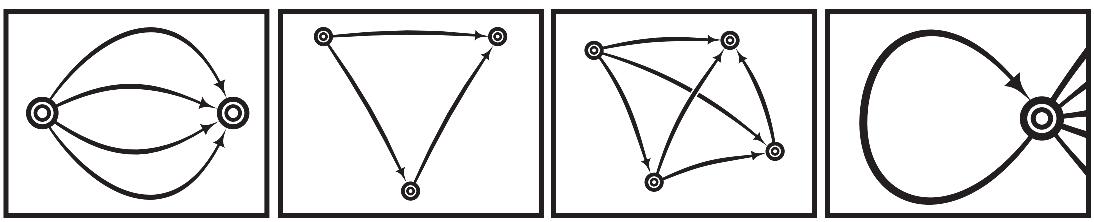
Categorification



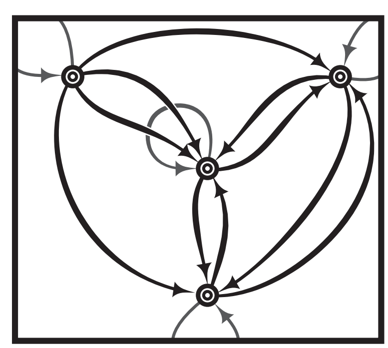
Mathematics is the science of patterns. As mathematicians have unraveled the patterns comprising algebra, analysis, topology, and more, certain meta-patterns have emerged. Notions of equivalence, limits, duality, and transformation have taken shape and precipitated a unified theory. It has been noted repeatedly that the power of topological invariants lies not only in their ability to characterize *spaces*, but *maps* as well. The oft-invoked *functoriality* and *naturality* of co/homology and homotopy groups are crucial ingredients of topology, pure and applied. The study of functoriality and its generalizations leads to **category theory**.

10.1 Categories

A **category** C consists of: (1) a collection[1] of **objects** $\mathcal{O}$; (2) for each ordered pair (a, b) in $\mathcal{O}$, a set of **morphisms** $\mathcal{M}(a, b)$; and (3) for each ordered triple (a, b, c) in $\mathcal{O}$ a **composition** operation $\circ\colon \mathcal{M}(a, b) \times \mathcal{M}(b, c) \to \mathcal{M}(a, c)$. In addition, these satisfy the following:



1. **Associativity:** Composition of morphisms is associative.
2. **Identity:** For each $a \in \mathcal{O}$, there exists an identity morphism $\mathsf{Id} \in \mathcal{M}(a, a)$ with $f \circ \mathsf{Id} = f$ for all $f \in \mathcal{M}(a, b)$ and $\mathsf{Id} \circ g = g$ for all $g \in \mathcal{M}(b, a)$.



The word *category* is so generic and ubiquitous as to be uninformative. The above definition is unrelated to the *LS category* of Chapter 7; this precise word was chosen to evoke an Aristotelian organization. This definition is, like all else Aristotelian, deceptively unexciting. A more transparently beautiful definition is possible when the category is sufficiently small, say, when objects and morphisms are countable. It is possible to represent such a category as a diagram of points (objects) and arrows (morphisms). A category C is visualized as a directed graph of vertices ($\mathcal{O}$) and, for each oriented pair of vertices a, b, a set $\mathcal{M}(a, b)$ of arrows **from** a **to** b (the direction is important). To each vertex is attached a loop-like identity arrow. Composition of arrows can be visualized by a 2-simplex whose boundary is the commutative triangle. The associativity of composition likewise has an arrow diagram best represented as the wireframe of a 3-simplex. This encapsulates an *Apollonian*[2] approach to cate-

[1] A **class** as opposed to a set, but the initiate should not worry about such things.
[2] This deity connotes reason and order, and he keeps a full quiver of arrows.

gories. Very small categories are easily visualized (*cf.* the chain complex of a finite cell complex); categories with large sets of objects and morphisms (*e.g.*, singular chain complexes) demand too much of the Apollonian seer. Viewing the algebraic laws of composition and associativity in their simplicial guise is this subject's first hint at its relevance to topology.

The most common examples of categories are not small enough to be so illustrated; these include:

1. Vector spaces (over a field $\mathbb{F}$) and linear transformations Vect;
2. Groups and homomorphisms Grp;
3. Abelian groups and homomorphisms Ab;
4. Graded abelian groups and graded homomorphisms GrAb;
5. Topological spaces and continuous maps Top;
6. Topological spaces and homotopy classes of maps hTop.
7. Manifolds and smooth maps Man;
8. Sets and functions Set;
9. Finite sets and functions FinSet;
10. Chain complexes and chain maps ChCo; and
11. Posets and order-preserving functions Pos.

Other examples are less sweeping, though still useful:

1. Given a topological space X, the category Op_X has objects equal to the open sets of X (including the empty set!), with inclusions $V \subset U$ defining morphisms $V \to U$.
2. A regular cell complex X is a category Face_X whose objects are cells with morphisms $\sigma \to \tau$ iff σ is a face of τ, $\sigma \trianglelefteq \tau$.
3. Any poset (partially ordered set) $(P, \trianglelefteq)$ is a category with objects elements of P and morphisms $a \to b$ iff $a \trianglelefteq b$. The previous two examples are special cases of a poset category.
4. A group **G** can be defined as a category with one object $\star$ and with morphisms $\star \to \star$ corresponding to elements of **G**, the composition being the group operation. The identity of **G** is the identity morphism, and each morphism must be invertible (where invertible hopefully means what you think it means: see §10.2).
5. A **groupoid** is, despite the grotesque name, easily defined: it is a category with invertible morphisms (a group being a single-object groupoid). A **monoid** is a category with a single object (a group being a monoid-with-inverses).

The reasons for wanting to use categorical language take time and space to fully unfurl. It is perhaps best to internalize some examples of categories that have appeared implicitly in other portions of this text.

Example 10.1 (Simplices) ⊚

The definition of the standard n-simplex in Chapter 2 was explicitly geometric. A categorical n-simplex is the category

$$[n] \quad := \quad 0 \longrightarrow 1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n-1 \longrightarrow n\,,$$

whose objects are the ordinals through n and whose morphisms are induced by the total order $\trianglelefteq$. Of course, there are more morphisms than the *generators* displayed above, as composition must be applied: for example, drawing the picture associated to [3] (without drawing the identity morphisms) should reveal a familiar picture. In §10.3 it will be shown how to build complexes (in a category) from such simplices by building a larger category, Simp, whose objects are the n-simplices above. ⊚

Example 10.2 (Temporal dynamics) ⊚

Dynamics bifurcates into continuous-time and discrete-time. Continuous-time dynamics (such as, solutions to differential equations) involves a flow on a space X – an action of $\mathbb{R}$ on X via homeomorphisms. In contrast, discrete-time dynamics involves an iterated homeomorphism $h : X \to X$, interpretable as an action of $\mathbb{Z}$ on X. Both settings are interpretable as a space X acted on by a category – in these cases, the totally ordered groups $\mathbb{R}$ and $\mathbb{Z}$, with objects interpretable as *time* and morphisms determined by the total ordering. With a categorical perspective, it becomes clear how to investigate dynamics with more subtle temporal features, such as irreversibility, multi-dimensional time, locally-orderable but not globally-orderable time, and branching timelines: one simply replaces *time* with a different category. ⊚

Example 10.3 (Boolean logics) ⊚

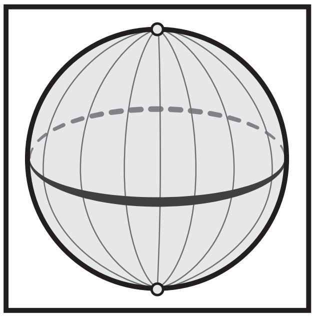
Logic gates and circuits are built on a Boolean foundation, expressed as a ring $\{\bot, \top\}$ (or, often, $\{0, 1\}$) connoting false/true, along with the operations of disjunction ($\wedge$, *i.e.*, AND), conjunction ($\vee$, *i.e.*, OR), and negation ($\neg$, *i.e.*, NOT). These are greatly generalizable. A **Boolean algebra** is a set B with a pair of distinguished members: 0, the minimum; and 1, the maximum; having commutative, associative, and distributive binary operations $\wedge$, $\vee$, as well as a unary operation $\neg$ which relate to the min/max values as follows:

$$b \vee 0 = b = b \wedge 1 \quad ; \quad b \vee \neg b = 1 \quad ; \quad b \wedge \neg b = 0.$$

A Boolean algebra forms the objects of a category whose morphisms are $a \to b$ iff $a \vee b = b$ (or, equivalently, $a \wedge b = a$). It helps to read this out loud using logical terminology. Boolean algebras form the objects of a category, Bool, whose morphisms are functions preserving 0, 1, $\wedge$, $\vee$, and $\neg$: $f(a \wedge b) = f(a) \wedge f(b)$, *etc.* Boolean algebras are a first hint at the utility of categories in logic, as commutative diagrams in Bool yield equations inside each Boolean algebra. ⊚

Example 10.4 (Flow category) ⊚

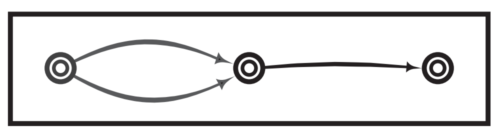
It is not the case that morphisms need to encode something explicitly algebraic or set-theoretic: dynamics is another source of morphisms. Consider the following category C_h associated to a smooth function $h\colon M \to \mathbb{R}$ on a manifold M [69]. The objects of C_h are the critical points of h. Morphisms are flowlines of the gradient flow of h, up to time-reparametrization.



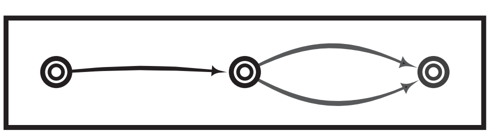
To allow for composition, flowlines are interpreted as **broken flowlines** with critical points in the interior allowed. This forms a category with some additional structure – each set of morphisms $\mathcal{M}(p, q)$ can be given the structure of a topological space. For example, the round sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the simple linear height function yields a category with two objects (the two critical points a and b). The morphisms are spaces: $\mathcal{M}(a, a)$ and $\mathcal{M}(b, b)$ are each a single point (the identity, corresponding to the invariant fixed point as a flowline); $\mathcal{M}(a, b)$ is homeomorphic to $\mathbb{S}^1$ and represents all possible flowlines from top-to-bottom; $\mathcal{M}(b, a)$ is empty. ⊚

10.2 Morphisms

The beauty of category theory is the ability to work with mathematical operations in a platform-independent manner. As a sample of what is possible with these very basic definitions, consider the following constructs, interpretable as lifting basic ideas from the algebraic to the categorical. All of the following demonstrate the difficulties and opportunities implicit in working with morphisms.

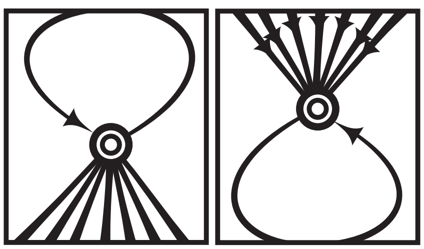


Monic: What does it mean to have a morphism in a category C that is *injective*? One is not permitted to discuss the inverse image of an arrow; kernels, images, and all other linear-algebraic thinking requires a reformation. The categorical analogue of an injection is a **monic** morphism: a morphism f is monic if it is left-cancellative, *i.e.*, whenever $f \circ g = f \circ h$, then $g = h$. This is perhaps best digested in diagrammatic form. It is a delightful exercise to convince oneself that this is, indeed, the proper definition of injective when one has no recourse to sets, but only to morphisms, identities, and composition.



Epic: The associated notion of an onto morphism – **epic** – is pleasantly symmetric to the monic case. A morphism f is epic if whenever $g \circ f = h \circ f$, then $g = h$ (right-cancellative). The symmetry between epic and monic is manifest and is not the last time that initially disconnected notions (into, onto) are revealed as dual under the appropriate categorical lift.

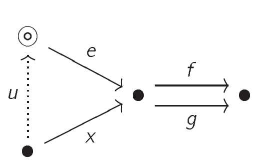
Iso: The reader might guess that an iso-morphism would be any morphism that is both epic and monic. This is *not* equivalent to the true definition of an isomorphism – a morphism f that has an inverse $\overline{f}$ such that $f \circ \overline{f}$ and $\overline{f} \circ f$ are both identities (on potentially different objects). For example, the inclusion $\mathbb{N} \hookrightarrow \mathbb{Z}$ is epic and monic in the category of monoids[3], but is not an iso. In sufficiently nice categories, such as Set, iso *is* equivalent to epic-plus-monic. Isomorphic *objects* in a category are ones which have an isomorphism between them. In Set, isomorphic objects are equicardinal (via a relabelling of elements); in Top, isomorphic objects are homeomorphic spaces; in hTop, isomorphic objects are homotopic spaces; in Grp, isomorphic means isomorphic.



Initials and terminals: The simplex category $[n]$ has two distinguished objects: 0 and n, at the *beginning* and *end* of the category. In a general category, an **initial** object is one with a *unique* morphism from it to *every* object in the category (*cf.* 0). A **terminal** object is one with a unique morphism from *every* object in the category to it (*cf.* n). Initials and terminals are easily shown to be unique up to isomorphism. Not all categories possess initial or terminal objects; even those derived from a total order (like $(\mathbb{Z}, \leq)$) do not necessarily contain their *limits* in this sense. Examples of categories with initials or terminals include the following:

1. Op_X has $\varnothing$ as initial and X as terminal objects.
2. The empty set $\varnothing$ is the initial object of Set: any 1-point set is a terminal object (unique up to isomorphism).
3. In Grp and Vect, the singleton object is both initial and terminal.
4. In a given Boolean algebra, 0 is initial and 1 is terminal.
5. In Bool, the 2-element algebra $\{0, 1\}$ is initial, and the single-element algebra $\{0\}$ is terminal.

Equalizers: In linear algebra, one cares about the kernel of a transformation; in calculus, one defines level sets as kernel-like objects. The category-theoretic analogue is an **equalizer**.

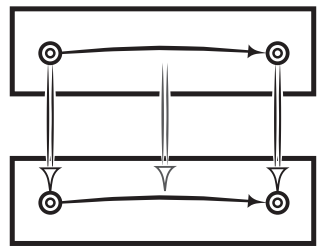


Given a pair of morphisms $f, g \in \mathcal{M}(a, b)$, the equalizer is defined to be the **universal** object-morphism pair $e : \odot \to a$ such that $f \circ e = g \circ e$. That is, given any morphism x with $f \circ x = g \circ x$, there exists a *unique* morphism u by which x factors through e. Note how everything is defined in terms of morphisms – the objects are implicit. The connection to kernels and level sets is discernable when working in the appropriate categories. The diagram implies that equalizer is the solution to the equation "$f - g = 0$" when such is sensible.

[3]A category whose objects are monoids and whose morphisms are monoid homomorphisms that respect multiplication and identities.

10.3 Functors

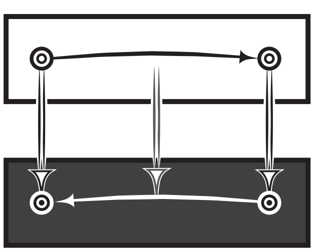
By this point, the reader will not be surprised to learn that in category theory, it is not the *objects* that matter so much as the *morphisms*. However, the delimited purview of a single category is too weak: vim resides in structure-respecting transformations between categories. A **functor** is an assignment $\mathcal{F}\colon \mathsf{C} \to \mathsf{C}'$ taking objects to objects and morphisms to morphisms in a manner that respects composition and identities: identity morphisms are sent to identity morphisms and likewise with composed morphisms. Some readers will find it helpful to write out all the equations implicit in this formulation.



The simplest functors are the **forgetful functors**, which simply remove structure from a category. For example, the removal of a (topological, group, differential, order) structure from a (space, group, manifold, poset) is a forgetful functor from the category (Top, Grp, Man, Pos) to Set. Homology and homotopy groups are at the heart of algebraic topology: *these are functors* and convert topological data to algebraic data in a manner that is *functorial*. The reason for that particular word choice should now be clear. Homology $H_\bullet$ is a functor from Top to GrAb, since a map $f\colon X \to Y$ has induced homomorphism $H(f)\colon H_\bullet(X) \to H_\bullet(Y)$. Cohomology $H^\bullet$ is not *quite* a functor from Top $\to$ GrAb, since the induced homomorphism goes from $H^\bullet(Y)$ to $H^\bullet(X)$: however, the induced maps behave as a functor would, but backwards. The language of an older dispensation distinguished **covariant** and **contravariant** functors – homology is covariant; cohomology, contravariant. Contemporary fashion keeps functors face-forward and expresses cohomology as a functor on a flipped category.

Example 10.5 (Duality & opposites) ⊚

Duality is seen to live in many forms throughout mathematics, hinting at a general construct. Given a category C, the **opposite category** C° has the same objects, but reverses the direction of all arrows: $\mathcal{M}^\circ(A,B) := \mathcal{M}(B,A)$.



For example, given a poset $(P, \trianglelefteq)$, thought of as a category with a unique morphism $a \to b$ iff $a \trianglelefteq b$, the dual category is the poset $(P, \trianglerighteq)$ which reverses the partial order. Opposites can also be used to set aright those functors that do things *in reverse*. While homology $H_k(\cdot)$ gives a functor from Top $\to$ Ab, cohomology $H^k(\cdot)$ yields a functor from $\mathsf{Top}^\circ \to \mathsf{Ab}$. The operation of passing to the opposite category is, like other forms of duality, involutive: $(\mathsf{C}^\circ)^\circ = \mathsf{C}$. ⊚

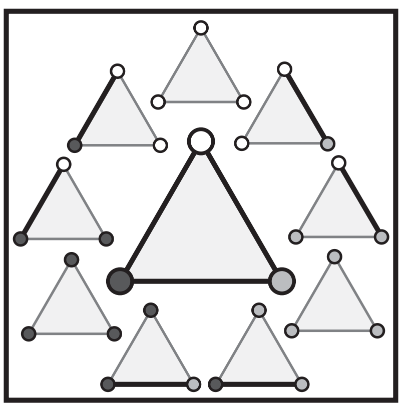
Example 10.6 (Simplicial sets) ⊚

The treatment of simplicial complexes in Chapter 2 conflated geometric things (spaces, gluings) with algebraic data (simplices, orderings, faces). This mixture of the geometric and the algebraic is best viewed via functors. Recall the definition of an n-simplex

$[n]$ from Example 10.1. One can build a larger category, the **simplex category**, Simp, whose objects are $[n]$ for $n \in \mathbb{N}$ and whose morphisms are functors $[n] \to [m]$, that is, order-preserving functions. These morphisms are generated by the **face maps**, $D_k \colon [n] \to [n-1]$, and **degeneracy maps**, $S_k \colon [n] \to [n+1]$, which respectively skip or repeat the k^{th} index:

$$D_k(0 \to 1 \to \ldots \to n) = (0 \to 1 \to \ldots \to k{-}1 \to k{+}1 \to \ldots \to n)$$
$$S_k(0 \to 1 \to \ldots \to n) = (0 \to 1 \to \ldots \to k{-}1 \to k \to k \to k{+}1 \to \ldots \to n)$$

These do not freely generate the category – there is a list of relations that these morphisms must satisfy in order to mimic the network of faces of simplices [158, 220]. As indicated by the terminology, there are **degenerate simplices** in this theory. For example, a morphism $[3] \to [1]$ given by $(0, 1, 2, 3) \to (0, 0, 1, 1)$ resembles a degenerate 3-simplex with projected image a 1-simplex.



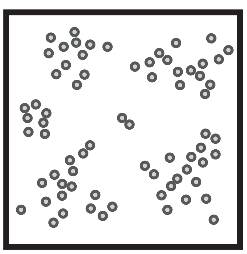
It is easy to use such a structure as a *model* on which to build representations that capture the combinatorics of oriented simplices. For example, a **simplicial set** is a functor $X \colon \mathsf{Simp}^{\circ} \to \mathsf{Set}$. This functor associates a set to each object in Simp and *glues* them together along all faces. Simplicial sets are ideal for keeping track of the combinatorics of an oriented simplicial complex in a unified package, and are an especially nice class of structures on which to do homotopy theory: one has the freedom of working in infinite dimensions (note that Simp contains simplices of all dimensions) while maintaining an efficient bookkeeping. One thinks of a simplicial set as a single infinite-dimensional Platonic simplex outfitted with a list of folding instructions sufficient to produce the (potentially finite) output. To complete the intuition back to the topological, any simplicial set can be converted into a topological space by means of another functor, the **geometric realization** functor, $| \bullet |$, which, in one instantiation, results in a CW complex with one n-cell for each nondegenerate n-simplex [158, 220]. These are not the best spaces for computation, since the data structure is prodigal: each nondegenerate 2-simplex comes with 9 degenerate 2-simplices and an infinite number of higher-dimensional degenerate cousins. ⊚

Example 10.7 (Nerves, redux) ⊚

The complexity implicit in the definition of simplicial sets induces elegance elsewhere. Nerves of covers can be lifted to more general settings, to great effect. The **nerve** of a small category, $\mathcal{N}(\mathsf{C})$, is the simplicial set whose n-simplices are ordered sequences of n composable morphisms in C – a chain of n arrows. Thus, the objects of C are the vertices of $\mathcal{N}$; the arrows of C form its edges; pairs $\to\to$ of incident arrows are 2-simplices, *etc.* Such a formal 2-simplex really does *look* like a 2-simplex, since the two arrows can be composed to obtain a third *edge*. This hints at how to specify the face and degeneracy maps, as must be done for a simplicial set. The degeneracy map D_k removes the $(k+1)^{\text{st}}$ arrow and replaces the k^{th} arrow with the composition (or eliminates the first/last arrow if $k = 0, n$ resp.). The face map S_k inserts an additional arrow by using the identity arrow at that object in C. One checks that the relations hold and that the result is a simplicial set. In the case of $\mathsf{C}(\mathcal{U})$ the poset induced by intersections of elements of a finite cover $\mathcal{U} = \{U_i\}$ of a space, the nondegenerate simplices of $\mathcal{N}(\mathsf{C}(\mathcal{U}))$ are precisely the simplices of the classical nerve complex $\mathcal{N}(\mathcal{U})$ as in §2.6. The categorical nerve, being a simplicial set, has a great many more degenerate simplices. These all collapse out, and the geometric realization of the categorical nerve is homotopic to the classical nerve. ⊙

10.4 Clustering functors

Categorical language has found its way into a few disciplines outside of Mathematics, with Computer Science being chief among them. At first glance, such applications might seem like a translation to a foreign language: intricate, symbol-ridden, and unreadable. The following application to statistics should convince an otherwise sceptical reader of the utility of categorical thinking.



Sections §2.2, §2.5, and §5.14 have discussed methods for approximating the topology of a cloud of data points via homology. The first-order term of this sequence is the computation of the number of connected components. Though this problem of **clustering** is easily stated, its importance in statistics and the natural and engineering sciences is immense. The subtlety of partitioning a discrete set $\mathcal{Q} \subset \mathbb{R}^n$ into clusters is evidenced both by the enormous literature and by results like the following.

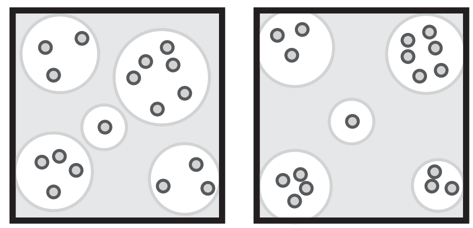
Consider a clustering algorithm as a function which takes as input a finite metric space $\mathcal{Q}$ (thus, pairwise distances between points are known, but placement up to rigid Euclidean motions is irrelevant) and returns a partition into **clusters**. Desirable properties for a clustering algorithm to possess would seem to include the following:

1. **Scale-invariance:** Clusters are invariant under rigid rescaling of the metric.
2. **Surjectivity:** For any partition of a finite set $\mathcal{Q}$, there exists a metric on $\mathcal{Q}$ which yields that partition as the clusters.
3. **Consistency:** Given an input $\mathcal{Q}$ with resulting cluster, move the points of $\mathcal{Q}$ so that within clusters, distances between points do not increase, and between clusters, distances between points do not decrease. The resulting input $\mathcal{Q}'$ has clustering identical to that of $\mathcal{Q}$.

The consistency property, though hardest to state, is no less desirable than the others: indeed, it seems vital. The following theorem of Kleinberg asserts the mutual incompatibility of all three conditions, in a manner not unlike the Arrow impossibility theorem in voting.

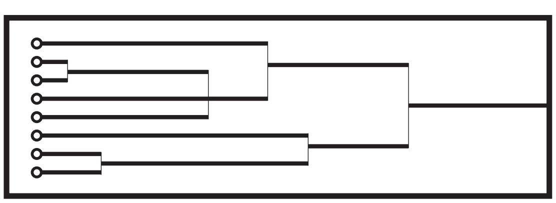
Theorem 10.8 ([194]). *There does not exist a clustering algorithm which is scale-invariant, surjective, and consistent.*

The critical observation of Carlsson-Mémoli is this: *clustering can be functorial.* A classical clustering algorithm takes an object from a category of finite metric spaces and returns an object in the **cluster category**, Clust, whose objects are pairs $(\mathcal{Q}, P^{\mathcal{Q}})$ consisting of a finite set $\mathcal{Q}$ and a partition $P^{\mathcal{Q}}$ thereof. The morphisms in Clust consist of functions $f\colon \mathcal{Q} \to \mathcal{Q}'$ that send points-to-points and partition elements to partition elements in such a manner that $f^{-1}(P^{\mathcal{Q}'})$ is a *refinement* of $P^{\mathcal{Q}}$ – a cluster morphism can coalesce clusters but not break them up.



A **clustering functor** is a functor from a category of finite metric spaces to Clust. It not only assigns clusters to a point-cloud, it converts morphisms between point-clouds into correspondences between and refinements of the resulting clusters. The desired properties for a clustering algorithm – *e.g.*, consistency or scale-invariance – should be built into the morphisms of the categories chosen. Consider the category $\mathsf{FinMet}^{\leq}$ whose objects are finite metric spaces $\mathcal{Q}$ and whose morphisms are *distance non-increasing*. That is $f\colon \mathcal{Q} \to \mathcal{Q}'$ with $d(f(x), f(x')) \leq d(x, x')$. With this structure for $\mathsf{FinMet}^{\leq}$, a clustering functor to Clust must of necessity satisfy a property like consistency. The other conditions of Theorem 10.8 can be likewise programmed into the categories in such a manner that the proper interpretation is that there exist no nontrivial onto functors from $\mathsf{FinMet}^{\leq}$ to Clust: see [60] for details.

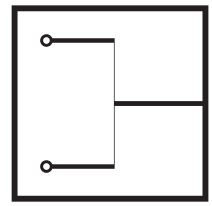
What is the good of this? Category theory is criticized as an esoteric language: formal and fruitless for conversation. *This is not so.* The virtue of reformulating (the negative) Theorem 10.8 functorially is a clearer path to a positive statement. If the goal is to have a theory of clustering; if clustering is, properly, a nontrivial functor; if no nontrivial functors between the proposed categories exist; then, naturally, the solution is to alter the domain or codomain categories and classify the ensuing functors. One such modification is to consider a category of persistent clusters.



Define PClust, the category of persistent clusters, to be the category whose objects are pairs $(\mathcal{Q}, P_t^{\mathcal{Q}})$, where $P_t^{\mathcal{Q}}$ is a persistent partition: a family of partitions of $\mathcal{Q}$ depending on $t \in [0, \infty)$ such that $P_t^{\mathcal{Q}}$ is a refinement of $P_{t'}^{\mathcal{Q}}$ for $t \leq t'$. The morphisms are t-dependent morphisms from Clust – a t-dependent family of refinements of clusters. Such a persistent clustering is related to the notion of a **dendrogram**: one thinks of t as something like a decreasing *resolution* of the clustering.

Theorem 10.9 ([60]). *There exists a unique functor* $\mathsf{FinMet}^{\leq} \to \mathsf{PClust}$ *which takes the input* $\{\bullet - \bullet\}$ *consisting of two points at distance* R *to the persistent cluster having* *one cluster for $t \geq R$ and two clusters for $0 \leq t < R$.*

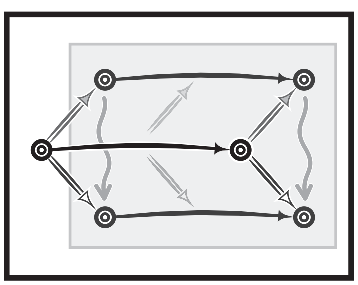
This provides a resolution to the conundrum of Theorem 10.8: it is surjective and consistent, and the persistent clustering scales with metric scaling in a clear manner. Not surprisingly, this clustering method is well-known: it is called **single linkage** clustering and is equivalent to saying that the clusters are given by $\pi_0(\mathsf{VR}_t(\mathcal{Q}))$ – the connected components of the distance-t Vietoris-Rips complex of $\mathcal{Q}$. The appearance of the Vietoris-Rips complex here is not unexpected, but pleasant nonetheless. Functoriality of the clusters is captured in the appearance of the functor π_0.



10.5 Natural transformations

The reader for whom this material is an introduction may suspect that a joke is being played when it is asserted that the *truly* interesting objects of study in category theory are neither morphisms nor functors, but rather correspondences between functors. Nevertheless, this is asserted in all seriousness: such a construction was indeed the true impetus for the creation of category theory [211]. A **natural transformation**, $\eta\colon \mathcal{F} \Rightarrow \mathcal{F}'$ connects a pair of functors $\mathcal{F}, \mathcal{F}'\colon \mathsf{C} \to \mathsf{C}'$ by sending each object $a \in \mathsf{C}$ to a morphism $\eta(a)\colon \mathcal{F}(a) \to \mathcal{F}'(a)$ so that for each morphism $h\colon a \to b$ in C, there is a commutative square connecting $\mathcal{F}(h)$ and $\mathcal{F}'(h)$:

$$\begin{array}{ccc} \mathcal{F}(a) & \xrightarrow{\mathcal{F}(h)} & \mathcal{F}(b) \\ {\scriptstyle \eta(a)}\downarrow & \Downarrow & \downarrow{\scriptstyle \eta(b)} \\ \mathcal{F}'(a) & \xrightarrow[\mathcal{F}'(h)]{} & \mathcal{F}'(b) \end{array}$$



Said better, a natural transformation is a functor in the Category of categories. This is a deep idea – that categories themselves form the Objects of a Category whose Morphisms are functors (under functor composition and with the identity functor playing the obvious role). The Functors of this Category relate a pair of Morphisms: in the original category, this is precisely what a natural transformation is.

Natural transformations are, like functors, composable. Given $\eta\colon \mathcal{F} \Rightarrow \mathcal{G}$ and $\eta'\colon \mathcal{F}' \Rightarrow \mathcal{G}'$, there is the obvious natural transformation $\eta \circ \eta'\colon \mathcal{F} \cdot \mathcal{F}' \to \mathcal{G} \cdot \mathcal{G}'$. However, it is also possible to compose a natural transformation η with a functor $\mathcal{H}$, either on the left, $\mathcal{H} \cdot \eta$, or on the right, $\eta \cdot \mathcal{H}$, obtaining a modified natural transformation.

Example 10.10 (Translation) ⊚

Consider the reals $(\mathbb{R}, \leq)$ with the total order as a category: there is one morphism $a \to b$ whenever $a \leq b$. Translation by ϵ is a functor $T_\epsilon\colon \mathbb{R} \to \mathbb{R}$ that sends $a \mapsto a + \epsilon$ and preserves the order (hence morphisms). The translation functor does very little: just a shift in objects. Thus, it comes as no surprise that there is a natural transformation $\tau_\epsilon\colon \mathsf{Id} \Rightarrow T_\epsilon$ from the identity that takes $(a \leq b)$ to $(a + \epsilon \leq b + \epsilon)$.

One can interpret the *naturality* in this setting as the indifference to whether one $\leq$-compares objects before or after the translation. ⊚

Example 10.11 (Snakes) ⊚

The Snake Lemma (Lemma 5.5) asserts the existence of the connecting homomorphism $\delta\colon H_\bullet(\mathcal{C}) \to H_{\bullet-1}(\mathcal{A})$, given a short exact sequence on $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$. What makes the result so powerful is the so-called *naturality* of the resulting long-exact sequence, as per Equation (5.5). This is equivalent to saying that δ is a natural transformation as follows. Consider the category **ChCoSES** of short exact sequences of chain complexes. The Snake Lemma asserts that δ is a natural transformation on homology functors to **GrAb**:

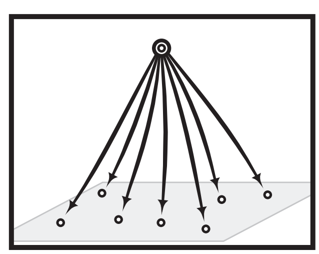
$$\begin{pmatrix} 0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \\ \downarrow \\ H_\bullet(\mathcal{C}) \end{pmatrix} \overset{\delta}{\Rightarrow} \begin{pmatrix} 0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0 \\ \downarrow \\ H_{\bullet-1}(\mathcal{A}) \end{pmatrix}.$$

⊚

Example 10.12 (Co/homology equivalences) ⊚

One of the great advantages of the multiple homology theories developed in Chapter 4 is that they are all isomorphic *and* that these isomorphisms are *natural*. This means that not only do the cellular and singular homologies of a cell complex agree, but maps between cell complexes induce the "same" homomorphism on homologies, as noted in §5.4. Of course, this naturality really means that there are natural transformations between the various homology functors: cellular, singular, Morse, Čech, *etc.*, restricted to the subcategory of spaces/maps on which both theories are defined. These are, specifically, **natural isomorphisms** between homology functors, meaning that, *e.g.*, between $H_\bullet^{\text{sing}}$ and $H_\bullet^{\text{cell}}$, there are an inverse pair of natural transformations whose compositions (both ways) yield the identity natural transformation on each homology functor. Natural isomorphisms are fundamental. ⊚

Example 10.13 (Retraction to a cone point) ⊚



Given any category **C**, there is a unique functor $\mathcal{R}\colon \mathsf{C} \to \{\bullet\}$ to the category with one object and one (identity) morphism. This $\mathcal{R}$ collapses all objects to $\bullet$ and all morphisms to the identity. For any fixed object $a \in \mathsf{C}$, the obvious functor $\mathcal{I}_a\colon \{\bullet\} \to \mathsf{C}$ that sends $\{\bullet\} \to a$ acts like an inclusion that satisfies $\mathcal{R}\cdot\mathcal{I}_a = \mathsf{Id}_\bullet$. In what sense could $\mathcal{I}_a\cdot\mathcal{R}$ be compared to the identity functor Id_C? In the case that **C** has an initial object $0 \in \mathsf{C}$, then there is a natural transformation $\eta\colon \mathcal{I}_0\cdot\mathcal{R} \Rightarrow \mathsf{Id}_\mathsf{C}$ given by sending $(a \to \bullet \to 0) \to a$, using the unique morphism from the initial. This hints at the role played by an initial: it acts as the apex of a cone along which the category is, metaphorically, contractible.[4] ⊚

[4]More than metaphor: the nerve of any small category with an initial object has contractible geometric realization.

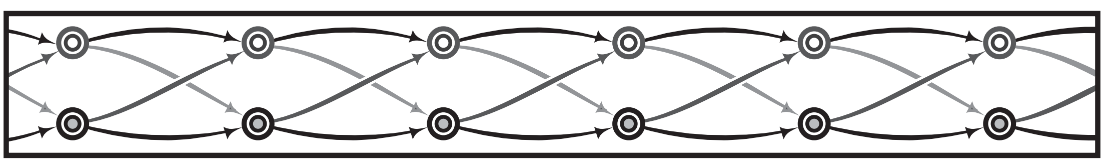
One says that categories are *isomorphic*, $\mathsf{C} \cong \mathsf{D}$, when there are functors $\mathcal{F}\colon \mathsf{C} \to \mathsf{D}$ and $\mathcal{G}\colon \mathsf{D} \to \mathsf{C}$ which are inverses in that $\mathcal{G}\cdot\mathcal{F} = \mathsf{Id}_{\mathsf{C}}$ and $\mathcal{F}\cdot\mathcal{G} = \mathsf{Id}_{\mathsf{D}}$. This is rarely satisfied, even for categories that seem very closely related. More common (and useful) is the case where there are not equalities but rather natural transformations $\mathcal{G}\cdot\mathcal{F} \Rightarrow \mathsf{Id}_{\mathsf{C}}$ and $\mathcal{F}\cdot\mathcal{G} \Rightarrow \mathsf{Id}_{\mathsf{D}}$ to the identities. One says that such categories $\mathcal{C} \approx \mathcal{D}$ are **equivalent categories**. This is analogous to the way in which homotopic spaces are topologically equivalent, though not necessarily homeomorphic.

Example 10.14 (Duals and isomorphisms) ⊚

Duality can be subtle. On the category FinVect of finite-dimensional vector spaces, there is the dual-space functor $\mathcal{D}\colon \mathsf{FinVect} \to \mathsf{FinVect}$ that sends $V \to V^\vee$. Although $V \approx V^\vee$ are isomorphic as vector spaces, this isomorphism is *not* natural. What this means *precisely* is that there is no natural isomorphism $\mathcal{D} \Rightarrow \mathsf{Id}$. It is true, however, that any V in FinVect is *naturally* isomorphic to its double-dual $(V^\vee)^\vee$: *i.e.*, there is a natural isomorphism $\eta\colon \mathcal{D}^2 \Rightarrow \mathsf{Id}$. ⊚

10.6 Interleaving and stability in persistence

One of the more widely-cited results in topological data analysis is the Stability Theorem for persistent homology [70], alluded to in §7.2 for sublevel set persistence. The treatment of persistence in §5.13 and §7.2 usedc a discretization of the parameter line: in practice, one may want to use a real parameter. Consider, therefore, the setting of persistence over $\mathbb{R}$, in which $\{X_t : t \in \mathbb{R}\}$ is a family of spaces with inclusion maps $X_a \subset X_b$ for $a \leq b$, thought of as (lower) excursion sets $X_t = \{h \leq t\}$ of a height function $h\colon X \to \mathbb{R}$.



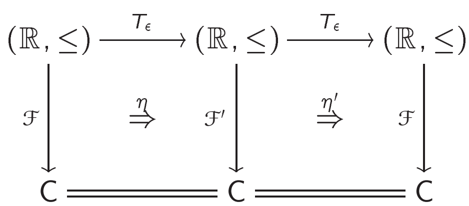
One question of stability is this: for h' close to h, how much can the topology of the excursion sets X_t change? Any individual X_t can change dramatically with a small perturbation; the content of the Stability Theorem is that the impact on persistent homology is small. This requires some notion of proximity for persistent homology. Current practice uses the following definition. Given two $\mathbb{R}$-indexed homology sequences, say, $H_k(X_t)$ and $H_k(X'_t)$, they are said to be ϵ-**interleaved** if there exist homomorphisms $\phi_t\colon H_k(X_t) \to H_k(X'_{t+\epsilon})$ and $\phi'_t\colon H_k(X'_t) \to H_k(X_{t+\epsilon})$ such that $\phi'_t \circ \phi_t$ and $\phi_t \circ \phi'_t$ are each the inclusion maps on homology induced by the shift $+2\epsilon$.

An ϵ-interleaving implies that the homologies of the two sequences can only differ substantially over short parameter intervals. This motivates defining a pseudometric on persistent homologies by declaring the distance $d(H_k(X_t), H_k(X'_t))$ to be the infimum of ϵ over all ϵ-interleavings. This is not quite a metric, since it may take on the value $+\infty$ (if no interleaving exists) and two persistence diagrams with interleaving distance zero are not necessarily identical (since the infimum may be 0 without a 0-interleaving). Nevertheless, reflexivity and the triangle inequality do hold for this pseudo-metric. In this language, the relevant result is:

Theorem 10.15 ([70]). *Given $h, h' : X \to \mathbb{R}$, the interleaving distance between the sublevel set persistent homology sequences is bounded by the L_∞ norm of the difference of the height functions:*

$$d(H_k(X_t), H_k(X'_t)) \leq \|h - h'\|_\infty .$$

This result can be translated into categorical language. Consider a function $h : X \to \mathbb{R}$. Then for $a \leq b$, the inclusion $X_a \to X_b$ of sublevel sets of h defines an excursion-set functor $\mathcal{E}_h : (\mathbb{R}, \leq) \to \mathsf{Top}$. The resulting sublevel set persistent homology is the composition of this functor with $H_\bullet$.



The prevalence of functors motivates the question: what is the interleaving distance of arbitrary functors $(\mathbb{R}, \leq) \to \mathsf{C}$ for C a category? The answer involves natural transformations [51]. By Example 10.10, translation in $(\mathbb{R}, \leq)$ is a functor T_ϵ isomorphic to the identity functor via the natural transformation τ_ϵ. An ϵ-**interleaving** of two functors $\mathcal{F}, \mathcal{F}' : \mathbb{R} \to \mathsf{C}$ is a pair of natural transformations $\eta : \mathcal{F} \Rightarrow \mathcal{F}' \cdot T_\epsilon$ and $\eta' : \mathcal{F}' \Rightarrow \mathcal{F} \cdot T_\epsilon$ such that

$$(\eta' \cdot T_\epsilon) \circ \eta = \mathcal{F} \cdot \tau_{2\epsilon} \quad \text{and} \quad (\eta \cdot T_\epsilon) \circ \eta' = \mathcal{F}' \cdot \tau_{2\epsilon}.$$

The interleaving distance on functors, $d(\mathcal{F}, \mathcal{F}')$, is the infimal ϵ for an ϵ-interleaving. As before, this is not a metric, but rather a pseudo-metric than can take on the value ∞ for non-interleavable functors.

Proof. *(of Theorem 10.15)* [51] Given $h, h' : X \to \mathbb{R}$, these define excursion set functors $\mathcal{E}_h, \mathcal{E}_{h'} : \mathbb{R} \to \mathsf{Top}$. Note that for $\epsilon = \|h - h'\|_\infty = \sup_x |h(x) - h'(x)|$,

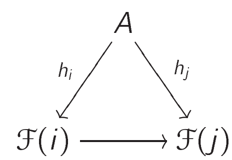
$$\mathcal{E}_h(t) = \{h \leq t\} \subset \{h' \leq t + \epsilon\} = \mathcal{E}_{h'}(t + \epsilon)$$
$$\mathcal{E}_{h'}(t) = \{h' \leq t\} \subset \{h \leq t + \epsilon\} = \mathcal{E}_h(t + \epsilon).$$

This implies an ϵ-interleaving η, η' of $\mathcal{E}_h$ and $\mathcal{E}_{h'}$. Note that for any $\mathcal{F}, \mathcal{F}'$ that are ϵ-interleaved, so are $\mathcal{G} \cdot \mathcal{F}$ and $\mathcal{G} \cdot \mathcal{F}'$, for any functor $\mathcal{G}$, by functoriality. Applying the homology functor H_k to $\mathcal{E}_h$ and $\mathcal{E}_{h'}$ yields an ϵ-interleaving on homology for $\epsilon = \|h - h'\|_\infty$. ⊙

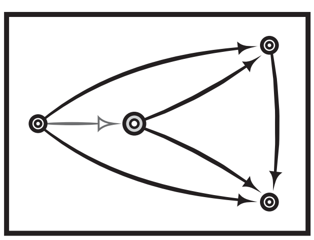
10.7 Limits

Let J be a (small) **index category**. A **diagram** is a functor $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ from an index category to a representation category. A diagram can sometimes be thought of as a "picture" of J in C [20, 160]. Another interpretation is that a diagram is something akin to a "sequence" in C (as is the case when $\mathsf{J} = \mathbb{N}$). This second interpretation prompts the notion of a limit of a diagram.

Every student of Mathematics eventually grasps that limits are as subtle as they are useful. A limiting process has two inputs: that which is converging to a limit, and the indexing family over which the convergence occurs. The most familiar examples of limits – a limit of a sequence of points in a metric space, or an intersection of nested open sets in a topological space – limit over $\mathbb{N}$ as a poset. For a categorical limit, the converging objects reside in a category C and the indexing family is an index category J. The limit of a diagram $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ is a distinguished object $\lim_{\mathsf{J}} \mathcal{F} \in \mathsf{C}$ that is thought of as a *terminus* of $\mathcal{F}$.



The definition is facilitated by an auxiliary construct. Fix $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ a diagram. A **cone** over the diagram $\mathcal{F}$ is a J-indexed family of morphisms $h_j : A \to \mathcal{F}(j)$ from a fixed object A in C to the image of J in C that respects composition (as per the diagram) for each morphism $i \to j$ in J. The collection of cones (A, h_{J}) over $\mathcal{F}$ forms the objects of the **cone category**, $\mathsf{Cone}_{\mathcal{F}}$, where a morphism between cones $(A, h_{\mathsf{J}}) \to (A', h'_{\mathsf{J}})$ means that there is a morphism $A \to A'$ that makes the triangles with all h_j and h'_j pairs commute. One visualizes the cones over $\mathcal{F}$ as pyramid-like structures balanced atop the base image of $\mathcal{F}$.



The **limit** of a diagram $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ is the terminal object in the cone category $\mathsf{Cone}_{\mathcal{F}}$. This has an interpretation as a universal property: such a limit gives a distinguished object $\lim_{\mathsf{J}} \mathcal{F} = \lim_j \mathcal{F}(j) \in \mathsf{C}$ and J-respecting morphisms $h_j : \lim_{\mathsf{J}} \mathcal{F} \to \mathcal{F}(j)$ such that any other cone must factor through the limit. As terminals, limits carry a connotation that blends intersection, restriction, GCD, preimage, and gluing.

Example 10.16 (Limit examples) ⊚

As in the case of calculus, limits may or may not exist. When a limit exists, it usually corresponds to something important. The following examples of limits reinforce the interpretation of a limit as being restrictive in nature.

[intersection] Consider an index category of the form $\mathsf{J} = \bullet\ \bullet$ with no non-identity morphisms, and a diagram $\mathcal{F} : \mathsf{J} \to \mathsf{Op}_X$. This is, simply, a pair of open sets in a space X. The cone category $\mathsf{Cone}_{\mathcal{F}}$ has as objects open sets in X with inclusions into the two open sets defined by $\mathcal{F}$. The limit is the largest such open set (any other factors through it via inclusion): this is the intersection of the two open sets.

[products] The same index category $\mathsf{J} = \bullet\ \bullet$ when sent via $\mathcal{F}$ to Top yields a different style of limit: the cartesian product of spaces. Let $\mathcal{F}$ have image objects topological spaces X and Y. A cone over $\mathcal{F}$ is a space Z and maps $Z \to X$, $Z \to Y$ such that any other space Z' with maps to X and Y must factor through Z: this is the cartesian product $Z = X \times Y$. The same construction works to give the familiar products in

Vect and Grp as well: all are limits of this simple $\mathsf{J} = \bullet\ \bullet$.

[AND] Using again the same index category, but taking a diagram into a poset $(P, \trianglelefteq)$ gives as limit the meet, $\wedge$, of the image of $\mathcal{F}$ in P: the $\trianglelefteq$-largest object of P that is $\trianglelefteq$-smaller that both terms in the image of $\mathcal{F}$. In a Boolean algebra, the limit corresponds to the logical AND of the two image objects.

[equalizers] Consider the index category $\mathsf{J} = \bullet \rightrightarrows \bullet$ with two objects and a pair of morphisms between them. For any diagram $\mathcal{F}$ of J in C, one notes that a cone over $\mathcal{F}$ is precisely an object in C that factors through both morphisms of the image of $\mathcal{F}$. The limit is therefore the universal such object: the equalizer. Limits therefore encompass kernels in linear algebra.

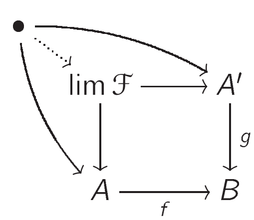
[cohomology] This implies, in particular, that the simplest cohomology group is a limit. The zero$^{\text{th}}$ cohomology $H^0(\mathcal{C})$ of a cochain complex $\mathcal{C}$ is simply the kernel of $d : C^0 \to C^1$. Since a limit is a generalized kernel, H^0 should be expressible as a limit: it is. For a discrete example, let X be a cell complex with face poset given by $\trianglelefteq$. Then

$$H^0(X; \mathbf{G}) = \lim_{\sigma \in X} \mathbf{G}, \tag{10.1}$$

where the limit is over the constant diagram that sends each simplex in the face poset of $(X, \trianglelefteq)$ to the group $\mathbf{G}$.

[terminals] The empty category is a valid choice for J. The only diagram of this J in C is the trivial diagram. By definition, a cone is simply an object of C, and the limit, if it exists, is precisely a terminal object in C: thus limits generalize terminals.

[pullbacks]



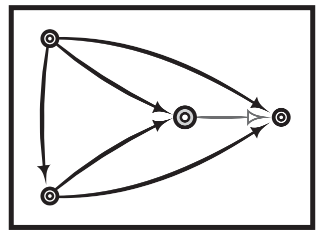
The diagram $\mathsf{J} = \bullet \to \bullet \leftarrow \bullet$ leads to an interesting type of intersection in the limit. Assume a diagram $\mathcal{F}$ that embeds J into C as $A \to B \leftarrow A'$. Then, a cone over $\mathcal{F}$ and the terminal limit is determined by commutative squares in the appropriate diagram. The colimit is called the **pullback** of the diagram and is sometimes denoted $A \times_B A'$, co-opting notation from Top. Examples of pullbacks in this category include: (1) the pullback of a fiber bundle $\pi\colon E \to B$ to a bundle over X via a map $f : X \to B$; and (2) the preimage of a subset $A \subset Y$ (with inclusion map $\iota\colon A \to Y$) under the map $f\colon X \to Y$. ⊚

There are more interesting limits to be had with the use of complex, non-finite index categories, including the limits used in calculus. The terminology is intentionally suggestive: in the same way that a limit of a sequence in calculus is a single point that best approximates the $\mathbb{N}$-indexed sequence of points, $\lim \mathcal{F}$ is a single object in C that best approximates the image of the diagram $\mathcal{F}$. What distinguishes a categorical limit is its implicit uniqueness and its attendant morphisms to the diagram.

10.8 Colimits

The astute reader will note that the above examples possess parallel or dual notions which should likewise have a categorical formulation: direct sums, unions, disjoint unions, free products, and the like. Each is a **colimit** obtained by dualizing the definition as follows.

Given a diagram $\mathcal{F}\colon \mathsf{J} \to \mathsf{C}$, a **cocone** is a is an J-indexed family of morphisms $h_j\colon \mathcal{F}(j) \to A$ to a fixed object A in C from the image of J in C that respects composition in J. These are objects in the corresponding **cocone category**, $\mathsf{CoCone}_{\mathcal{F}}$, where a morphism between cones $(A, h_{\mathsf{J}}) \to (A', h'_{\mathsf{J}})$ means that there is a morphism $A \to A'$ that makes the triangles with all h_j and h'_j pairs commute. The **colimit** of a diagram $\mathcal{F} : \mathsf{J} \to \mathsf{C}$ is the *initial* object in the cocone category of $\mathcal{F}$. This, too, has an interpretation as a universal property: the colimit gives a distinguished object $\operatorname{colim}_{\mathsf{J}} \mathcal{F} = \operatorname{colim}_j \mathcal{F}(j) \in \mathsf{C}$ along with J-respecting morphisms $h_j\colon \operatorname{colim}_{\mathsf{J}} \mathcal{F} \to \mathcal{F}(j)$ that factors through any other cocone.



Example 10.17 (Colimit examples) ◎

The colimit generalizes the initial (and indeed is the initial if J is empty). Other examples of colimits build an interpretation of a colimit as being agglomerative or disjunctive in nature.

[union] For the simple index category $\mathsf{J} = \bullet\ \bullet$ and a diagram $\mathcal{F}\colon \mathsf{J} \to \mathsf{Op}_X$, $\mathsf{CoCone}_{\mathcal{F}}$ has as objects open sets in X containing the two open sets defined by $\mathcal{F}$. The colimit is the *smallest* such open set (it factors through any other via inclusion): this is the union.

[coproducts] A diagram of the same index category to Top has image spaces X and Y. The colimit of $\mathcal{F}$ is a space $\operatorname{colim}_{\mathcal{F}}$ and maps from X and Y such that any other space with maps from X and Y must factor through the colimit: this is precisely the **coproduct** (or disjoint union) $X \sqcup Y$. The same colimit in algebraic categories like Vect or Grp is the **direct sum**, $\oplus$, of the objects. All of these colimits express a union and a disjunction.

[OR] In a poset $(P, \trianglelefteq)$, the colimit is the join, $\vee$, of the $\mathcal{F}$-image: the $\trianglelefteq$-smallest object of P that is $\trianglelefteq$-larger that both terms in the image of $\mathcal{F}$. In a Boolean algebra, the colimit corresponds to the logical OR of the two image objects.

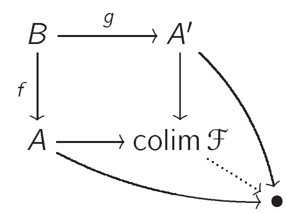
[coequalizers] The index category $\mathsf{J} = \bullet \rightrightarrows \bullet$ leads to a colimit that is dual to an equalizer. This is called the **coequalizer** of the diagram, and, in Vect, expresses the cokernel. This emphasizes that colimits are more like quotient objects than subobjects.

[homology] As with cohomology and limits, the zero$^{\text{th}}$ homology, $H_0(\mathcal{C})$ of a chain complex $\mathcal{C}$ is a colimit. For example, if $(X, \trianglelefteq)$ is a cell complex with face poset, then

$$H_0(X; \mathbf{G}) = \operatorname*{colim}_{\sigma \in X} \mathbf{G}, \tag{10.2}$$

where the limit is over the constant diagram $(X, \trianglelefteq) \to \mathbf{G}$.

[pushouts] The diagram $\mathsf{J} = \bullet \leftarrow \bullet \rightarrow \bullet$ leads to an amalgamation in the colimit. The colimit is called the **pushout** of the diagram and is sometimes denoted $A \cup_B A'$, co-opting notation from Top. An example of a pushout in this category is the wedge sum of pointed spaces $A \vee A'$, where B is a singleton and f and g are inclusions. An algebraic example of a pushout in Grp is in the form of the Van Kampen Theorem (8.4), the statement of which is simplified greatly with categorical language: $\pi_1(U \cup V)$ is the pushout of the diagram



$$\pi_1(U) \leftarrow \pi_1(U \cap V) \rightarrow \pi_1(V).$$

[infinite-dimensional spaces] Throughout the text, invocations of certain infinite-dimensional cell complexes – such as $\mathbb{S}^\infty$ or $\mathbb{P}^\infty$ – have been made in ignorance of their precise definition. Colimits assist with this. Consider the diagram of $\mathbb{N}$ in Top given by the standard embeddings $\mathbb{R}^0 \hookrightarrow \mathbb{R}^1 \hookrightarrow \mathbb{R}^2 \hookrightarrow \cdots$. One can define $\mathbb{R}^\infty$ to be the colimit of this diagram. The reader can easily adapt this to define $\mathbb{S}^\infty$, $\mathbb{P}^\infty$, $\mathbb{T}^\infty$, and inductively built CW complexes. Fortunately, these colimits in Top exist. ⊚

In applications, one must work to show that [co]limits exist and to show how they behave under a given functor. A category for which [co]limits of *all* diagrams exist is called a **[co]complete** category. Functors that preserve [co]limits are called, of course, **[co]continuous** . A visceral comprehension of limits and colimits is essential to applications of category theory. Contemporary problems in data, networks, and sensing all involve localization of data and integration of local data into global: *limits and colimits*.

10.9 Sheaves, redux

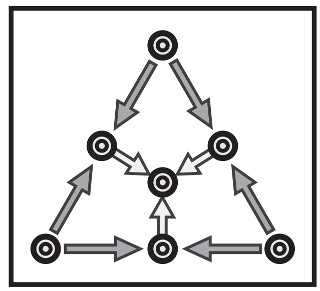
The language of categories mirrors, expands, and simplifies greatly the many definitions of Chapter 9.

Cellular Sheaves: Let X be a regular cell complex and Face_X be the poset category of the cells of X under the face relation $\trianglelefteq$, so that objects are cells and there is a unique morphism $\sigma \to \tau$ for every face $\sigma \trianglelefteq \tau$ (with the identity face giving the identity morphism). By now the reader has probably observed that a cellular sheaf over X is neither more nor less than a functor $\mathcal{F}\colon \mathsf{Face}_X \to \mathsf{C}$, where, in Chapter 9, Vect and Ab were used extensively for C. The first consequence of this extended language is that one may easily interpret what is meant by a sheaf of sets, or a sheaf of monoids, or any kind of categorical data: it is the composition that is key. One thinks of a cellular sheaf as being a *representation* of Face_X in C.

The wide variety of categories available as data types greatly expands the vector spaces used in Chapter 9 and yields a number of interesting objects. For example, a **complex of groups** is a sheaf on a cell complex X taking values in Grp, a **complex**

of spaces is a Top-valued sheaf on X, *etc.* The important construct comes in gluing together local data over cells into a global object. The process for doing this gluing is specified in Equation (9.2): it is a choice of data on each cell that agrees according to faces and restriction maps. As an exercise in understanding concepts, the reader should show that the value of the sheaf $\mathcal{F}$ on all of X is precisely the limit over the face poset category:

$$\mathcal{F}(X) = \lim_{\sigma \in X} \mathcal{F}(\sigma) = H^0(X; \mathcal{F}). \tag{10.3}$$



This equality is a slight abuse of notation. To wit: the explicit definition of $\mathcal{F}(X)$ from Equation (9.2) specifies local data on each cell. This is precisely a cone over the diagram $\mathcal{F}\colon \mathsf{Face}_X \to \mathsf{C}$. By the definition of a limit, there is thus a unique map from $\mathcal{F}(X)$ to the limit. By equality is meant that this unique map is a natural isomorphism, as can be shown by means of the compatibility condition for the assignment of local data. Note that for data taking values in more general categories than Vect or Ab, one must become concerned with the existence of limits. Fortunately, finite limits tend to be uncomplicated things. The same cannot be said for infinite limits.

Topological Sheaves: The subtleties of sheaves over a tooplogy demand the refinements of categorical language. A presheaf on a space X taking values in a category C is, precisely, a functor $\mathcal{F}\colon \mathsf{Op}_X^{\circ} \to \mathsf{C}$, where the preservation of composition of morphisms corresponds to the respecting of restriction maps. The stalk of a (pre)sheaf was defined via Equation (9.8) as an awkward sort of limiting equivalence. In truth, it is a colimit,

$$\mathcal{F}_x = \operatorname*{colim}_{U \ni x} \mathcal{F}(U),$$

where the U are open sets containing x, partially-ordered by reverse inclusion ($U \to V$ for $V \subset U$) to provide a diagram over which the colimit is computed. It is an exercise to show that this categorical colimit gives the same answer as the more explicit mechanical process of (9.8).

As noted in Chapter 9, a presheaf alone does not make a sheaf. For this, an additional condition must be satisfied. This *gluing axiom* has several equivalent formulations in the language of this chapter. The most direct is a reinterpretation of the exact sequence in Equation (9.9). Namely, for any open cover $\mathcal{U} = \{U_i\}$ of U, the value of $\mathcal{F}$ on U is an equalizer:

$$\mathcal{F}(U) \xrightarrow{\mathcal{F}(U \leqslant U_i)} \prod_i \mathcal{F}(U_i) \underset{\mathcal{F}(U_j \leqslant U_{ij})}{\overset{\mathcal{F}(U_i \leqslant U_{ij})}{\rightrightarrows}} \prod_{i,j} \mathcal{F}(U_{ij}) \ . \tag{10.4}$$

While this formulation is correct and canonical, it is perhaps not optimal in its reliance on a mechanistic collation of pairwise gluings. A more elegant reformulation uses the full nerve complex $\mathcal{N}(\mathcal{U})$ of the cover $\mathcal{U}$ of U and recapitulates the approach in §9.6

of approaching sheaves over a topology via nerves. The gluing axiom is equivalent to saying that

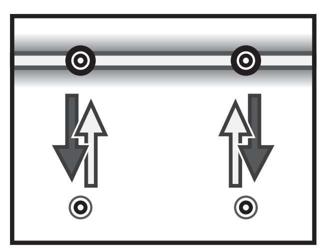
$$\mathcal{F}(U) = \lim_{U_J \in \mathcal{N}} \mathcal{F}(U_J), \tag{10.5}$$

independent of the cover $\mathcal{U}$. As in Equation (10.3), the equality is an abuse of notation, meaning that $\mathcal{F}(U)$ is naturally isomorphic to the limit of $\mathcal{F}$ over the cover. This limit is computed over the face poset of the nerve $\mathcal{N}$ of the cover $\mathcal{U}$, as in the cellular case above. A cell U_J in the nerve $\mathcal{N}$ is indexed by a multi-index J that determines a nonempty intersection of the open sets U_i, $i \in J$, making $\mathcal{F}(U_J)$, and the limit, well-defined. This condition is worth repeating: *a sheaf converts open covers into isomorphic limits*. Depending on one's predilection and mood, the equalizer or limit formulations of the gluing axiom will come more readily.

Sheaf Operations: The various manipulations of sheaves outlined in §9.7 provide excellent instances of categorical constructs. For example, a sheaf morphism is neither more nor less than a natural transformation $\eta \colon \mathcal{F} \Rightarrow \mathcal{F}'$ between sheaves-as-functors. In fact, depending one which subject one learns first – sheaves or categories – this example may assist in making concrete an otherwise opaque definition. Thinking of a natural transformation as a mapping between data structures over a category can be illuminating.

Sheaves provide a topological and data-centric view of categories that may assist in making the subject more visceral. One can think of any functor $\mathcal{F} \colon \mathsf{C} \to \mathsf{C}'$ as an assignment of data from the target category C' to the source category C in a sheaf-like manner. Natural transformations $\eta \colon \mathcal{F} \to \mathcal{F}'$ provide the means of transforming from one data structure to another. When the target category is sufficiently algebraic (say, Ab), then it is possible to define kernels and cokernels of data, leading to the cohomology of $\mathcal{F}$. This foreshadows one of the most powerful set of topological tools in category theory: **derived functors**, of which sheaf cohomology is a motivating precursor.

10.10 The genius of categorification

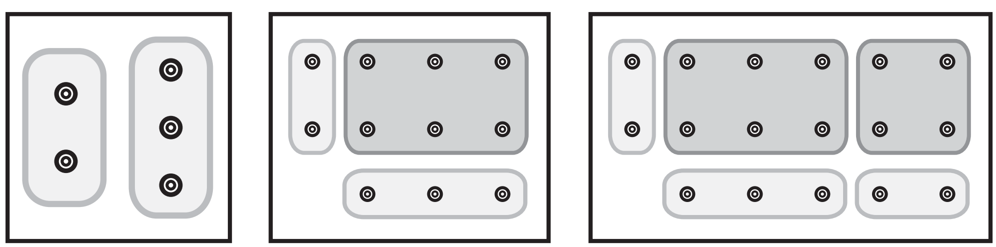


The goal of this chapter is to point the reader to **categorification**: the systematic lifting of, say, numerical data to a higher categorical structure, with a concomitant functoriality. This functoriality is key, and permits a lifting of numerical equality (*"I wonder why these two numbers happen to be equal?"*) to an algebraic equivalence (*"These two structures are isomorphic."*), with the ability to make additional high-level connections. A subsequent **decategorification** back to the numerical can, in the best cases, provide explanatory power and intuition for deeper results.

Categorification – like category theory – is not itself a branch of topology; however, it has been so influenced by and effective in topology that it is fitting to end this text with a gentle invocation. The focus will be on the sprit of the subject rather than on rigorous results.

Example 10.18 (Arithmetic, simple) ⊚

Counting is the primal decategorification, so internal as to have been sublimated as such. To each finite collection of objects (apples, oranges, coins, cats, or czars), one associates an element of $\mathbb{N}$ – the cardinality of the set. This abstraction permits referencing a generic set of N objects, without having to specify the membership thereof. To categorify this, the reader might first try the category of finite sets, **FinSet**. The *cardinality* function $|\cdot|$, sends objects of this category to $\mathbb{N}$ and descends to a bijection from isomorphism classes of objects in FinSet to $\mathbb{N}$. Note that FinSet has more structure than $\mathbb{N}$: an isomorphism between objects specifies identities as well as preserving cardinalities. Because FinSet is a category, one can apply categorical constructs and decategorify to see the numerical impact. For example, the colimit of a pair of sets (the disjoint union) decategorifies to addition; the limit of a pair (the cartesian product) decategorifies to multiplication. Certain basic laws – commutativity, associativity, distributivity – have higher equivalents, and the order relation $\leq$ on $\mathbb{N}$ is enriched to the language of injectives and projectives. Interesting though this may be, a reformulation of arithmetic in terms of category theory provides no new insight; rather, it is an elementary example of how the most primal bits of mathematics are the shadows of emanations from higher-up the hierarchy of structure. ⊚



Example 10.19 (Arithmetic, complex) ⊚

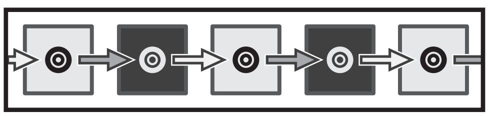
The categorification of $\mathbb{N}$ to FinSet is not optimal: it leaves unclear how to recover negative numbers, for example. Fortunately, there are multiple possible categorifications, some of which are more generalizable. Consider the subcategory FinVect of Vect consisting of finite-dimensional vector spaces over a field $\mathbb{F}$. Then one can lift $\mathbb{N}$ to this category by means of the decategorification dim: an n-dimensional vector space is the lift of $n \in \mathbb{N}$. Isomorphic objects in FinVect have the same dimension. In this categorification, the lift of addition is the direct sum, $\oplus$, and the lift of multiplication is to the tensor product (over $\mathbb{F}$), $\otimes$. There are (unique) identity objects for these operations: the 0-dimensional vector space, for $\oplus$, and the 1-dimensional vector space $\mathbb{F}$, for $\otimes$. By categorifying to a linear-algebraic setting, the morphisms between (even isomorphic) objects are richer and can store more data about relationships.

Other categorifications are more enlightening still. Consider FinChCo, the category of finite, finite-dimensional chain complexes (over a field $\mathbb{F}$), with chain maps as morphisms. The previous categorification to **FinVect** embeds in **FinChCo** as a sequence with one nonzero term. With complexes, one has a categorification of $\mathbb{Z}$ as follows. Given a two-term sequence (extended by zeros),

$$\cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow V \xrightarrow{A} W \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots,$$

one can decategorify to $\mathbb{Z}$ by a difference of dimensions: $\dim V - \dim W$. The reader will recognize once again the appearance of the Euler characteristic χ: **FinChCo** $\to \mathbb{Z}$. By lifting $\mathbb{Z}$ to all of **FinChCo**, arbitrary (finite) sequences of vector spaces collapse via alternating sums of dimensions, providing a rich structure of sequences and chain maps comparing them. ⊚

Example 10.20 (Co/Homology) ⊚



The two primal examples of decategorification – dimension and Euler characteristic – have appeared repeatedly in this text. By Theorem 3.7, they are together *complete* invariants of definable sets up to definable homeomorphism. Lifts of these two types of numerical invariants are central to algebraic topology. For a cell complex X, the combinatorics of adding together the number of cells, weighted by the parity of the cell dimension *seemed* in Chapter 3 to give a serendipitous topological invariant in χ. Likewise, in Chapter 4, computing dimensions of simplicial homology groups to count cycles up to equivalence *seemed* to not depend on the simplicial structure. Mathematics knows no such generous deity, and coincidence hints at deeper reasons. The explanation given in Chapter 5 used the language of homology and exact sequences. More vocabulary is now available: *co/homology is a categorification*.

By converting the cells and assembly instructions into a chain complex $\mathcal{C} = (C_\bullet, \partial)$, one obtains a homology $H_\bullet$ that is functorial (maps between spaces yield chain maps that descend to homomorphisms on homology) and thus explains why topological invariance holds. Better still, natural equivalences between homology theories permit lifting χ and $H_\bullet$ to singular or non-cellular settings. The final ingredients are the corresponding decategorifications which send homology groups back to $\mathbb{N}$ (via dimension – a Betti number) or to $\mathbb{Z}$ (via the Euler characteristic). With the addition of cohomology, one picks up a multiplicative structure and a wealth of new dualities and relationships.

Nearly every tool in algebraic topology – long exact sequences, duality, naturality, excision, connecting homomorphisms, cup products, bundles, fibrations, sheaves – is built on the functoriality that categorification enables. Nearly every application in this text – to data, dynamics, games, networks, sensing, signals, clustering, coloring, motion planning, material defects, and more – rises from this functoriality. ⊚

Example 10.21 (Co/Sheaves) ⊚

Sheaves and sheaf cohomology (and their duals) provide categorifications of numerical data distributed over a space. One simple example from this text is a flow sheaf from

§9.4-9.5. Recall that for a given flow on a directed acyclic graph X with capacity constraints, one cares about the net flow value at the source/target. The numerical capacities were lifted to a flow sheaf $\mathcal{F}$ by means of dimension – stalks over edges are vector spaces whose dimension equals the flow capacity at the edge. Restriction maps for the sheaf encode flow routing (or coding). The benefits of this categorification include (1) the ability to use sheaf cohomology to characterize information flows and flow values; (2) the ability to relate flow- and cut-values by means of long exact sequences; (3) the presence of duality in sheaf cohomology as the analogue of decomposition of flows into loops; and (4) the ability to use Euler characteristic to see obstructions to max-flow and min-cut values.

A better example of sheaves-as-categorification in this text is the Euler calculus of Chapter 3. As the Euler characteristic is a decategorification of co/homology, the Euler integral is a decategorification of compactly supported sheaf cohomology with coefficients in sheaves associated to constructible functions. As per §9.9, any constructible function $h \in \mathsf{CF}(X)$ on a tame set X lifts to a complex of cellular sheaves $\mathcal{F}_h^\bullet$ compatible with the triangulation of X induced by h. The decategorification of this complex by means of Euler characteristic *is* the Euler integral $\int_X \cdot d\chi \colon \mathsf{CF}(X) \to \mathbb{Z}$. The advantage of this perspective is an immediate access to functoriality, yielding the Fubini Theorem (Theorem 3.11), the integral transforms of §3.9, and more [95, 191, 272]. ⊙

10.11 "Bring out number"

With reflection, one suspects that many questions in applied mathematics are at heart functorial in nature and are profitably viewed through the lens of categorification. A few speculations appear below. Some of these may be realized as proper categorifications; others are a loose lifting that invoke the insubstantial ghost of functoriality. This text closes with a hope of provoking the reader into finding and using functoriality.

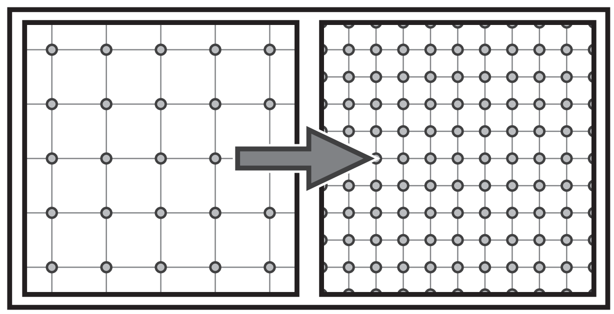
Example 10.22 (Data analysis) ⊙

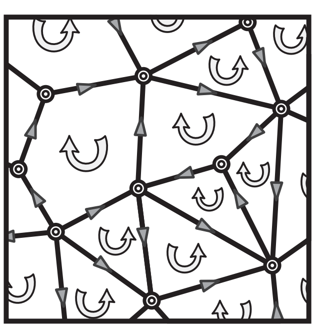
Topological data analysis has generated a substantial body of evidence that topology facilitates the management and interpretation of large, unwieldy point clouds. How is this accomplished? One ingredient is the dextrous application of co/homology functors in order to capture features of a data set that are *global* (large-scale as opposed to fine detail), *computable* (thanks to Mayer-Vietoris and other exact sequences), and *robust* (topological invariants have good properties with respect to noise). The fact that homology is a functor from the category of topological spaces grants to topological data analysis a degree of nonchalance with respect to coordinates.

Data analysis begins with clustering, in the same way that homology begins with H_0. The clustering functors of §10.4 provide an immediate and satisfying categorification, but the problem of classifying and using novel clustering schemes based on functoriality is as yet incomplete [60]. The (persistent) homologies H_k provide the higher-order terms in the Taylor series of shape for data. This is limited primarily in the restriction to a single parameter – multi-parameter persistence cannot access a suitably simple Structure Theorem as in 5.21. Perhaps a solution lies in the reformulation of persistent homology as a cosheaf over a 1-dimensional base space, as in §9.12: the classification of constructible cosheaves over the plane is much more complex that over the line, but not unimaginably so. Whether this, or more advanced representation-theoretic ideas, or deeper categorical methods arise to resolve the problem, it seems clear that the solution to multiparameter topological analysis of data lies in more abstraction, not less. ⊙

Example 10.23 (Numerical analysis) ⊙

When solving a partial differential equation [PDE] by means of a numerical method, one typically applies the method and examines the large-scale qualitative features of the solution. To validate that these results are not artifacts of the numerical scheme, it is common to run the method again on a refined grid or with a shorter time step, noting again the qualitative features of the solution. If there is a match (or, one might say, "equivalence") between these two solutions, then one infers that the solution is genuine. This is a physical instance of the maxim that morphisms and functors are more useful than objects. This example, though a bit cartoonish, illustrates the difficulty in building a careful categorification: what, exactly, is meant by the qualitative features of a numerical solution and an equivalence thereof?





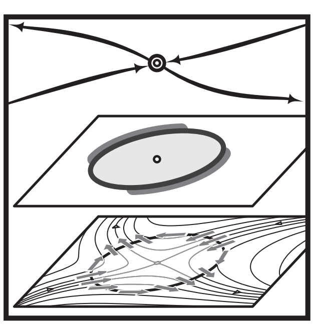
One way to proceed is to enrich a discretization with structure inherited from the PDE. Any numerical scheme works with functions on a discrete spatial domain with discrete time axis. At each such discretization point in space-time, one has a numerical value. The next time-value is given as a function of nearby space-time values, according to the relevant numerical scheme coming from the differential equation. The simplest such numerical schemes are agnostic as to the precise form of the differential equation. Recently, however, it has been shown that one can get better numerical results when the discretization is enriched and forced to preserve some structure inherited from the differential equation. For example, in the case where a PDE is conservative, such as Hamiltonian systems (*cf.* Examples 6.16 and 7.19), the dynamics must preserve an invariant differential form (say, a volume or symplectic form). Discretizing (decategorifying) a solution removes this structure. By modifying the discretization to retain a shadow of the appropriate invariant form, one can work to ensure that the numerical scheme preserves this structure, yielding more accurate simulations. This is precisely the motivating idea behind the **discrete exterior calculus** [12, 13, 32, 94]. Though often presented as a structure-enriched variant of finite-element methods, it is perhaps better to think of it as a decategorification of differential forms and Hodge calculus that *forgets less* so as to retain enough functoriality to perform operations (wedge products, exterior derivatives, Hodge-stars, *etc.*). The specific decategorification is critical.

Example 10.24 (Dynamics and index) ⊚

Applications of algebraic topology in dynamics and differential equations rest on a foundation of index theory. In its first appearance in §3.4, the fixed point index $\mathcal{I}$ was described as an integer invariant for equilibria of planar vector fields, computable via a line integral and robust thanks to Green's Theorem. The Poincaré-Hopf Theorem (3.5) asserted that this index is *additive* and yields the Euler characteristic when *"integrated"* over the domain. In Example 4.23, $\mathcal{I}$ was revealed to be a degree, and therefore applicable to fixed points of self-maps as well as equilibria of vector fields, as per Example 7.20. This culminated in the Lefschetz index of §5.10, interpreted first as something like an Euler characteristic of degrees, then, in §7.7, revealed to be an integral with respect to Euler characteristic. In the language of Chapter 9, the fixed point index lifts to a complex of constructible sheaves, the Euler characteristic of which *is* the Lefschetz index.

V	dim
$A\colon V \to V$	trace
$V_\bullet$	χ
$A_\bullet\colon V_\bullet \to V_\bullet$	τ

This progression of indices exemplifies categorification nicely. It has been seen on the level of co/homology that the two simplest types of decategorification – dimension and Euler characteristic – work well with vector spaces and sequences of vector spaces respectively. When lifting to dynamics, these have analogues. The dimension of a vector space generalizes to the trace of a self-map; the Euler characteristic of a chain complex generalizes to the Lefschetz index of a chain map. Applying these to the identity map recovers dimension and Euler characteristic. Thus, trace and Lefschetz index are the appropriate numerical invariants of dynamics. To what do these categorify? In the same way that counting cells and faces lifts to co/homology and functoriality, counting fixed points and their indices lifts to the action of dynamics on co/homology.



The story of categorification in dynamics is richer than it at first appears. Consider the gradient fields of Morse theory in Chapter 7. Here, the fixed point index is nearly mute, in that a nondegenerate critical point has fixed point index ± 1. This lifts, however, to an $\mathbb{N}$-valued Morse index, μ, which characterizes the local dynamics. Furthermore, with this richer index, one can stack the critical points into the Morse polynomial: the Morse inequalities of (7.1) are an instance of a richer algebra unveiled with slight added structure. This continues with the Morse homology of §7.3. By converting critical points into chains, the dynamics is shown to recover the homology of the underlying manifold (Theorem 7.3). Corollary 7.6 asserts that the Morse homology decategorifies to give the Euler characteristic.

As one ascends to higher types of structure, categorical language is essential. The flow category of Example 10.4 converts gradient dynamics into a category, with critical points as objects and morphisms as flowlines. This category, like Morse homology, remembers the underlying manifold, via the nerve construction of Example 10.7 and the geometric realization of simplicial sets as in Example 10.6: the flow category C_h of a Morse function $h\colon M \to \mathbb{R}$ has nerve $\mathcal{N}(\mathsf{C}_h)$ whose geometric realization is homotopic to M [69].

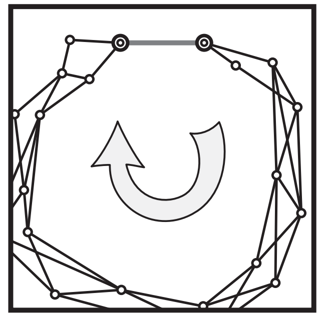
One of the lessons of Chapter 7 is that the restrictions of classical Morse theory – manifolds, smoothness, nondegeneracy, gradients – are largely ignorable, given an appropriate ascension in technique. For example, the categorification of critical points in a nondegenerate smooth gradient field to Morse homology in §7.1 returns the homology of the base manifold, as in Theorem 7.3. This is mirrored in the degenerate setting of a discrete gradient field on a cell complex using discrete Morse homology, in Theorem 7.23. Better still is the ability to jettison the gradient assumption and work with dynamics, as occurs in the Conley index of §7.6. This, in its homotopic and homological variants, is the vanguard of efforts to categorify dynamics. The prototypical argument for existence of connecting orbits in Examples 7.15-7.16 hints at a functoriality of the Conley index which exceeds that of the index of Morse.

This story, however, is not completed. There are a multiplicity of extensions of the Conley index for different settings [128, 129, 266, 286], including, most notably, the setting of discrete dynamics based on the index pairs as in §7.7. Some recent work has focused on properties of the Conley index over a large parameter domain [11, 140], work that hints at sheaf-like properties of the index. This lies within the purview of the categorification of dynamics, but does not exhaust the possibilities. Space and time prevent an explanation of zeta-functions for counting periodic orbits [264], model-category structures for dynamical systems [184], the categorification of Floer homology to the **Fukaya category** [19, 275] and its applications [96], and the interaction of Lagrangian submanifolds with sheaves [170]. There is much more to be done. ⊚

Example 10.25 (Optimization) ⊚

There are numerous instances of *minimax theorems* in applied mathematics: in game theory, in optimization and linear programming, in differential equations, and more. The multiplicity of incarnations of a *minimax* nature lead to a suspicion of a deeper principle in action. Given the fact the minima and maxima lift to lattice notions of meet/join or categorical notions of limits/colimits, it may be conjectured that all minimax theorems are expressions of relationships between limits and colimits in an appropriate category.

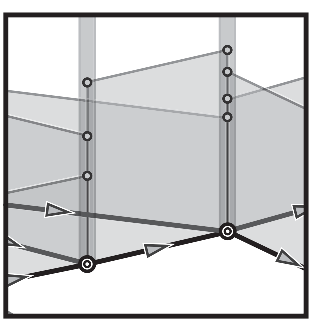
One instance of this is the Max-Flow/Min-Cut Theorem, alluded to previously in Example 6.5 and §9.4. Recent results of Krishnan [201] provide a dramatic categorification. Recall that one begins with a directed (acyclic) graph X from source to target nodes, $\mathsf{s} \to \mathsf{t}$, with edges having capacities in $\mathbb{R}^+$. The goal is to place a conservative flow on X with maximal throughput at the source/target nodes. As a means of keeping flow conservation at the source and target, append to X a feedback edge e from target-to-source. Then, as noted in Example 6.5, a flow evokes a 1-cycle in homology, and a cut a dual cocycle. When framed this way, it seems clear that the max-flow min-cut theorem is a topological theorem. It is, and is best seen as such through the lens of categorification.



The first step is to categorify the capacity constraints as a cellular **capacity sheaf** over X. In the classical setting of numerical capacities $\mathsf{cap}(e) \in \mathbb{R}^+$ assigned to an edge e of X, the capacity sheaf $\mathcal{F}$ over e takes values in the interval $[0, \mathsf{cap}(e)]$ under addition, where the addition operation is (1) *non-invertible* – subtraction is forbidden; and (2) *partial* – not all pairs of numbers are allowed to be added. This partial addition encodes the constraint, since addition must not exceed the capacity. This kind of algebraic structure is a **partial commutative monoid**. It is a brilliant observation that one can categorify the constraints of the optimization problem as a sheaf, albeit over the algebraically intricate category of partial commutative monoids.

All other ingredients of the max-flow min-cut theorem lift likewise. Flow conservation mimics the homological cycle condition. However, to respect the directedness of the underlying graph, it is essential to build a homology theory that remembers and respects the directions of edges. This is encoded in an **orientation sheaf**, $\mathcal{O}$, taking values in $\mathbb{N}$-modules (in contrast to the un-oriented $\mathbb{Z}$-modules); this has the property that the stalks of $\mathcal{O}$ are copies of $\mathbb{N}$ summands, where each summand represents an independent directed local path. A directed homology theory H_1 taking values in sheaves of partial commutative monoids is constructed via an equalizer diagram (kernels being not well-defined), and the partial commutative monoid of flows on X respecting the directions and constraints of $\mathcal{F}$ is, precisely,

$$H_1(X; \mathcal{F}) := H^0(X; \mathcal{F} \otimes \mathcal{O}) = \lim_{C \subset X} (\mathcal{F} \otimes \mathcal{O})(C). \tag{10.6}$$



This definition/theorem has bound up within it a version of Poincaré duality for directed spaces proved by Krishnan [200] that both foretells and enables the sheaf-theoretic max-flow min-cut theorem. The left term of this equation becomes a categorification of global flows on X, and the right term becomes a categorification of local flow capacities. The subtle work is how to partially decategorify the left and right sides to flow-values and cut-values comparable in a common framework. This is done by mapping both sides above into a directed homology $H_0(X; \mathcal{F})$, expressible as something like a colimit. In the classical setting, this partial decategorification translates to the following equality of partial commutative monoids:

$$\bigcup_{\text{flow } \varphi} [0, \mathsf{val}(\varphi)] = \bigcap_{\text{cut } C} [0, \mathsf{val}(C)] \tag{10.7}$$

Thus, the maximal flow value equals the minimal cut value: but more than this, these coincident numbers descend from isomorphic structures in a categorification. The machinery required to complete the details is formidable and does not fit in this text, but the benefits are compelling. Because the duality holds for a very general category of capacity sheaves taking values in partial commutative monoids, other non-classical optimal flow problems (of commodities, signals, and logics) over networks can be solved via duality. This is a stellar example of categorification in applied mathematics. ⊙

Notes

1. This text has worked with **small** categories whose objects and morphisms form sets. There are several unpleasant technicalities associated with categories that are not small.
2. There are relatively few computational tools for working with categories; one hopes that with increased awareness of applications that this will improve.
3. The interpretation of clustering as a functor in §10.4 is a very simple example to demonstrate the benefit of categorical thinking. Not all clustering methods used in practice are functors. One of the more popular methods, ***k*-means** clustering, takes as its input the point cloud $\mathcal{Q}$ along with a number $k \in \mathbb{N}$ and an initial k-tuple of points in $\mathcal{Q}$. It returns, via an iterative process, a partition of $\mathcal{Q}$ into k clusters. It seems impossible to render such a method functorial, since the number of clusters is fixed *a priori*.
4. The interleaving approach to stability of persistent homology in §10.6 is a rapidly-developing subject. In the initial works on stability [70] a metric (called the **bottleneck distance**) on multi-sets of points in the birth-death plane was used. This has been shown to be isometric to the interleaving distance. Bubenik and Scott [51] use the language of natural transformations; Lesnick [207] uses the language of modules and structure theorems. The most recent work (at the time of publication) is a broad generalization of the categorical approach in §10.6 [50].
5. The terminology of limits and colimits in this text is not universal. Many sources use **direct limit**, **inductive limit**, or $\varinjlim$ to denote the *colimit*; the corresponding terms for a *limit* are **inverse limit**, **projective limit**, and $\varprojlim$. Yes, this *is* confusing.
6. Derived functors, of which sheaf cohomology is the precursor, are the true, unrealised, goal of this chapter. The interested reader should with all haste master the basics so as to ascend to this cornice.
7. Yes, dear reader, the progression from objects to morphism to functors does not end with natural transformations; nor, indeed, does it end at all. Mathematics is currently brimming with A_∞ structures, N-categories, and other progressions which iterate the notions of ever-higher morphisms between ever-higher objects to dizzying heights. The use of A_∞ structures seems to be particularly potent [31]– that this potency will descend to applications is to be hoped and remains to be seen.
8. Like the previous, this chapter is a shocking reduction of a vast intricate theory into a cartoonish sketch. The reader whose sense of adventure is aroused will find an unending country to explore. The book by Awodey [20] is particularly friendly to readers from Computer Science. Students of topology may wish to begin further exploration with adjunctions, exponentials, Kan extensions, homotopy co/limits, nerves, the Yoneda Lemma, and the other basic tools of category theory. The classic, complete reference

is [211]. For a serious high-level treatment with applications to sheaves, [192] is a valuable resource; for applications to logic, [160, 212] are the appropriate references. The most interesting new books on the subject are by Leinster [205] (extremely well-written) and Spivak [282] (extremely creative at making connections to data structures).

9. The author, in his youth, mocked category theory as so much unapplicable generalized nonsense. May the reader learn from his errors. *Miserere mihi peccatori.*