Math 620 Home Work Assignment 1

1. Let K be a finite separable extension field of k.
   (a) Show that the character group of $\mathrm{Res}_{K/k}(\mathbb{G}_\mathrm{m})$ is naturally isomorphic to the free $\mathbb{Z}$-module generated by the set of embeddings of K into the separable clousre k^{sep} of k.
   (b) Describe the action of $\mathrm{Gal}(k^{\mathrm{sep}}/K)$ on the character group of $\mathrm{Res}_{K/k}(\mathbb{G}_\mathrm{m})$.
2. Let T be a torus over $\mathbb{R}$. Prove that $T(\mathbb{R})$ is compact if and only if the non-trivial element $\tau \in \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ operates on $X^*(T)$ as -1.
3. Prove that $T(\mathbb{R})$ is Zariski dense in T for any torus T over $\mathbb{R}$.
4. Let T be a torus over $\mathbb{R}$.
   (i) Prove that for any homomorphism $\nu : \mathbb{G}_m \to T$ over $\mathbb{C}$, there exists a homomorphism $h : \mathbb{S} \to T$ over $\mathbb{R}$ such that $h \circ \mu = \nu$.
   (ii) If $h_1, h_2 : \mathbb{S} \to T$ are two $\mathbb{R}$-homomorphisms of tori over $\mathbb{R}$ such that $h_1 \circ \mu = h_2 \circ \mu$, then $h_1 = h_2$.
5. Let T be a torus over $\mathbb{R}$. Prove that for any homomorphism $\chi : T \to \mathbb{G}_m$, there exists exactly one $\mathbb{R}$ homomorphism $h : T \to \mathbb{S}$ such that $\chi_z \circ h = \chi$.
6. Prove that the homomorphism $\mathrm{Nm} \circ \beta : \mathbb{G}_m \times \underline{\mathbb{C}_1^\times} \to \mathbb{G}_m$ is equal to the square morphism on the first factor of the source and is trivial on the second factor.
7. (i) Find all $\mathbb{R}$-endomorphisms of the torus $\mathbb{S}$.
   (ii) Find all $\mathbb{R}$-homomorphisms from $\mathbb{S}$ to $\underline{\mathbb{C}_1^\times}$.
8. Let V be a finite dimensional vector space over $\mathbb{Q}$. Let $h : \mathbb{S} \to \mathrm{GL}(V_\mathbb{R})$ be an $\mathbb{R}$-homomorphism, giving $V_\mathbb{R}$ the stucture of a real Hodge structure. Then (V, h) is a $\mathbb{Q}$-Hodge structure if and only if the weight cocharacter
   $$w_h := h \circ w : \mathbb{G}_{\mathrm{m}\mathbb{R}} \to \mathrm{GL}(V_\mathbb{R})$$
   of $\mathrm{GL}(V_\mathbb{R})$ is defined over $\mathbb{Q}$.
9. Let V be an irreducible $\mathbb{R}$-Hodge structure, and $\dim_\mathbb{Q}(V) = 2$ of weight n. Prove that the homomorphism $h_V : \mathbb{S} \to \mathrm{GL}(V)$ has the form $h_V = h_1 \circ [n]_\mathbb{S}$, where $h_1 : \mathbb{S} \to \mathrm{GL}(V)$ is an $\mathbb{R}$-homomophism giving V the structure of a real Hodge structure of weight 1.
10. Prove that every one-dimensional $\mathbb{Q}$-Hodge structure is isomorphic to $\mathbb{Q}(i)$ for some $i \in \mathbb{Z}$.

11. Give an example of a subgroup S of $\mathrm{GL}_n(\mathbb{C})$ such that the smallest $\mathbb{Q}$-subvariety Z of the $\mathbb{Q}$-algebraic group GL_n which contains S is not a stable under multiplication. In other words, Z is not a $\mathbb{Q}$-algebraic subgroup.

    (Hint: Here is one example. Take an integer $m \geq 2$. Consider the following subsubgroup

    $$S = \left\{ Ad\left(\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\right) \cdot \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix} : A \in \mathrm{GL}_m(\mathbb{C}) \right\}$$

    of GL_{2m}, where $b = (b_{ij}$ is an element of $\mathrm{M}_m(\mathbb{C})$ such that tr. deg $\mathbb{Q}(n_{ij}) = m^2$. Then it is not difficult to show that the $\mathbb{Q}$-Zariski closure of S is equal to

    $$\left\{ \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} : \mathrm{tr}(A^i\, B) = 0, i = 0, 1, \ldots, m-1 \right\}$$

    On the other hand, one can show that the smallest $\mathbb{Q}$-subgroup containing S is equal to

    $$\left\{ \begin{bmatrix} A & B \\ 0 & A \end{bmatrix} \right\}$$

    Notice that this group is not reductive.)

12. Let F be a totally real number field. Let V be a two-dimensional vector space over F, with a $\mathbb{Q}$-Hodge structure of Hodge type $\{(0,-1),(-1,0)\}$ on V, such that the subset $F \subset \mathrm{End}_{\mathbb{Q}}(V)$ consists of Hodge cycles. In other words, the action of F on V preserves the Hodge filtration. Prove that the Mumford-Tate group $\mathrm{MT}(V)$ is either $\mathrm{GL}_F(V)$, or is isomorphic to the inverse image of $\mathbb{G}_{\mathrm{m}}$ under

    $$\mathrm{Nm}_{K/F} : \mathrm{Res}_{K/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}}) \to \mathrm{Res}_{F/\mathbb{Q}}(\mathbb{G}_{\mathrm{m}}),$$

    where K is a totally imaginary quadratic extension field of F.

13. Let V be a finite dimensional vector space over $\mathbb{C}$, which is a direct sum of two nontrivial subspaces V_0 and V_1 such that the dimensions n_0, n_1 of V_0, V_1 are relatively prime. Let G be a connected reductive algebraic subgroup of $\mathrm{GL}(V)$ such that the commutant of G in $\mathrm{GL}(V)$ is equal to the group of homotheties on V. Assume moreover that G contains the subgroup H of $\mathrm{GL}(V)$ consisting of all elements which act as homotheties on V_0 and on V_1.

    (a) Let Z be the center of $\mathbb{G}$ and let G^{der} be the derived group of G. Show that Z is equal to the group of homotheties on V, and the action of G^{der} is irreducible.

    (b) For any finite dimensional linear representation W of the torus H, denote by $\chi_H(W)$ the character of V; $\chi_H(W)$ is a linear combination of elements of the character group $X^*(H)$ with coefficients in $\mathbb{Z}_{\geq 0}$. Show that if W_1, W_2 are H-modules such that $\chi_H(V) = \chi_H(W_1) \cdot \chi_H(W_2)$, then either W_1 or W_2 is one-dimensional.

(c) Deduce from (b) that G^{der} is an almost simple semisimple algebraic subgroup of $\mathrm{SL}(V)$.

(d) Use the fact that the center of G^{der} contains a cyclic subgroup of order $n = \dim(V)$ and the classification of simply connected semisimple algebraic groups to prove that G^{der} is equal to $\mathrm{SL}(V)$.

14. Let V be a $\mathbb{Q}$-Hodge structure of type $\{(0,-1),(-1,0)\}$, such that $\mathrm{End}_{\mathbb{Q}-\mathrm{Hdg}}(V)$ contains a imaginary quadratic field K. Decompose $V^{-1,0}$ as

$$V^{-1,0} = V_0^{-1,0} \oplus V_1^{-1,0}$$

such that K operates on $V_0^{-1,0}$ via an embedding $\sigma : K \hookrightarrow \mathbb{C}$, and K operates on $V_1^{-1,0}$ via the complex conjugate of σ. Prove that if the dimensions n_0, n_1 of $V_0^{-1,0}, V_1^{-1,0}$ are relatively prime, then the Mumford-Tate group $\mathrm{MT}(V)$ is isomorphic to $\mathrm{GL}(V)$. (Note: Problems 13 and 14 are taken from *Serre, Proc. Cof. Local Fields, Driebergen, 118–131, 1966.*)