The Forster/Swan theorem

§1. Notation and definition

In this note R is a commutative ring with 1, $Y := \mathrm{Spec}(R)$ is the spectrum of R, $X := J(R)$ is the J-spectrum of R.

(1.1) DEFINITION Let M be a finitely generated R-module. For every prime ideal $\mathfrak{p} \in Y$, define

$$\mu(\mathfrak{p}, M) = \dim_{\kappa(\mathfrak{p})}(M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}})$$

to be the minimum number of generators for the $R_{\mathfrak{p}}$-module $M_{\mathfrak{p}}$. It is easy to see that $\mathfrak{p} \mapsto \mu(\mathfrak{p}, M)$ is an upper semi-continuous function on $\mathrm{Spec}(R)$; i.e. the subset

$$Y_{n,M} := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid \mu(\mathfrak{p}, M) \geq n\} \subseteq Y$$

is closed for every $n \in \mathbb{N}$. Let

$$X_{n,M} := X \cap Y_{n,M} \qquad \text{for all } n \in \mathbb{N}.$$

(1.2) LEMMA *Let R be any commutative ring, let M be a finitely generated R-module, and let S be a finite set of prime ideals of R such that $M_{\mathfrak{p}} \neq 0$ for every $\mathfrak{p} \in S$. There exists an element $m \in M$ such that the image of m in $M_{\mathfrak{p}}/\mathfrak{p}M_{\mathfrak{p}}$ is not zero for all $\mathfrak{p} \in S$. In other words*

$$\mu(\mathfrak{p}, M/R\cdot m) = \mu(\mathfrak{p}, M) - 1 \qquad \forall \mathfrak{p} \in S.$$

PROOF. Induction on $s = \mathrm{card}(S)$. The statement obviously holds for $s = 0$. Suppose that the statement holds for $s-1$, $s \geq 1$. Pick a minimal element in S and call it $\mathfrak{p}_s$. Let $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_{s-1}\}$ be the rest of the elements in S. Then $\mathfrak{p}_1 \cdots \mathfrak{p}_{s-1} \not\subseteq \mathfrak{p}_s$. Pick an element $x_s \in \mathfrak{p}_1 \cdots \mathfrak{p}_{s-1} - \mathfrak{p}_s$. According to the induction hypothesis, there is an element $m_1 \in M$ such that the image of m_1 in $M_{\mathfrak{p}_i}/\mathfrak{p}_i M_{\mathfrak{p}_i}$ is non-zero for $i = 1, \ldots, s-1$. Pick an element $m_2 \in M$ whose image in $M_{\mathfrak{p}_s}/\mathfrak{p}_s M_{\mathfrak{p}_s}$ is non-zero. The element

$$m = \begin{cases} m_1 & \text{if the image of } m_1 \text{ in } M_{\mathfrak{p}_s}/\mathfrak{p}_s M_{\mathfrak{p}_s} \text{ is non-zero} \\ m_1 + x_s m_2 & \text{otherwise} \end{cases}$$

of M has the required property. □

(1.3) In the rest of this note we *assume that X is a noetherian topological space and* $\dim(X) < \infty$.

For any finitely generated R-module M, define a function $b(\mathfrak{p}, M)$ on X by

$$b(\mathfrak{p}, M) = \begin{cases} 0 & \text{if } M_{\mathfrak{p}} = 0 \\ \mu(\mathfrak{p}, M) + \dim(V(\mathfrak{p}) \cap X) & \text{otherwise.} \end{cases}$$

Here $\dim(V(\mathfrak{p}) \cap X)$ is the combinatorial dimension of the topological space $V(\mathfrak{p}) \cap X$, which is equal to the combinatorial dimension of $V(\mathfrak{p}) \cap \mathrm{Max}(R)$. Clearly the closed subset $X_{n,M}$ of X is empty for $n \gg 0$ and the function $\mathfrak{p} \mapsto b(\mathfrak{p}, M)$ is bounded on X, because $\dim(X)$ is assumed to be finite.

§2. The theorem

(2.1) THEOREM *Let R be a commutative ring such that the $X = J(R)$ is noetherian as above. Let M be a finitely generated R-module. Let k be a natural number such that $k \geq \max_{\mathfrak{p}\in X} b(\mathfrak{p}, M)$. Then M can be generated over R by k elements.*

PROOF. Induction on k; the case $k = 0$ is obvious. Suppose that the statement holds for $k-1$, $k \geq 1$. Let S be the subset of X consisting of all prime ideals $\mathfrak{p} \in X$ such that $b(\mathfrak{p}, M) = k$.

Consider an element $\mathfrak{q} \in S$, and let $n = \mu(\mathfrak{q}, M)$. Let Z be the closure of $\{\mathfrak{q}\}$. Then Z is an irreducible component of $X_{n,M}$; otherwise there exists an element $\mathfrak{p} \in X_{n,M}$ such that $\mathfrak{p} \subsetneqq \mathfrak{q}$ and

$$b(\mathfrak{p}, M) = \mu(\mathfrak{p}, M) + \dim(V(\mathfrak{p} \cap X)) \geq n + \dim(V(\mathfrak{p} \cap X)) > n + \dim(V(\mathfrak{q} \cap X)) = k,$$

a contradiction. So S is a finite set, contained in the finite set consisting of all generic points of irreducible subsets of the finitely many non-empty closed subsets $X_{n,M} \subset X$.

By Lemma 1.2, there exists an element $m \in M$ such that $\mu(\mathfrak{q}, M/R\cdot m) = \mu(\mathfrak{q}, M) - 1$ for all $\mathfrak{q} \in S$. Then $b(\mathfrak{p}, M/R\cdot m) \leq k-1$ for all $p \in X$, so $M/R\cdot m$ can be generated by $k-1$ elements by the induction hypothesis. □