Lie Groups. Representation Theory and Symmetric Spaces

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Fundamentals of Lie Groups

In this Chapter we discuss elementary properties of Lie groups, Lie algebras and their relationship. We will assume a good knowledge of manifolds, vector fields, Lie brackets, and the Frobenius theorem, see e.g. [Wa],[Sp] or [Le], Ch. 1-8 and 17-19, and covering space theory, see e.g. [Ha] Ch. 1 or [Mu] Ch. 9 and 12.

Although our presentation is sometimes somewhat different and shorter, there are a number of good books on the basics in this Chapter, see e.g. [Wa],[Sp] or [Le], Ch 20.

Lie groups and Lie algebras

It will always be understood without saying that all manifolds and vector spaces are finite dimensional.

Definition 1.1 *A* **Lie group** G *is an abstract group and a smooth* n*-dimensional manifold so that multiplication* $G \times G \to G\colon (a, b) \to ab$ *and inverse* $G \to G\colon a \to a^{-1}$ *are smooth.*

We will also occasionally consider *complex Lie groups* where the underlying manifold is complex and multiplication and inverse are holomorphic.

This innocent combination of two seemingly unrelated properties has amazing consequences. As we will see, a Lie group is classified, up to coverings, by a linear object, called a Lie algebra. Many of the questions about Lie groups can be quickly converted into a Linear Algebra problems (though those may be difficult) on the corresponding Lie algebra. Nevertheless, the translation back to the Lie group is not always obvious and so we will emphasize the Lie group aspect as well.

Definition 1.2 *A* **Lie algebra** *over* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *is a vector space* V *over* $\mathbb{K}$ *with a skew-symmetric* $\mathbb{K}$*-bilinear form (the Lie bracket)* $[\ ,\ ] : V \times V \to V$ *which satisfies the Jacobi identity*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0 \tag{1.3}$$ Jacobi

for all $X, Y, Z \in V$.

We relate the two via so called *left invariant* vector fields. We use the standard notation

$$L_g \colon G \to G,\ \ h \to gh \text{ and } R_g \colon G \to G,\ \ h \to hg$$

and define

Definition 1.4 *A vector field* X *on a Lie group* G *is called left invariant if* $d(L_g)_h(X(h)) = X(gh)$ *for all* $g, h \in G$*, or for short* $(L_g)_*(X) = X$.

We then have

Proposition 1.5 *If we denote by* $\mathfrak{g}$ *the set of all left invariant vector fields, then the linear map* $L \colon \mathfrak{g} \to T_eG$ *with* $L(X) = X(e)$ *is an isomorphism.*

Proof A left invariant vector field must have the property that $X(g) = d(L_g)_e(v)$, i.e. is determined by its value at e. Conversely, given $v \in T_eG$ the vector field defined by $X(g) = d(L_g)_e(v)$ is left invariant: $d(L_g)_h(X(h)) = d(L_g)_h(d(L_h)_e(v))) = d(L_{gh})_e(v) = X(gh)$. All that remains is to show that X is smooth. But if $m \colon G \times G \to G$ is multiplication, then $dm \colon TG \otimes TG \to TG$ is smooth as well and $X(g) = dm_{(g,e)}(0,v)$. Indeed, if s is a curve in G with $s'(0) = v$, then $dm_{(g,e)}(0,v) = \frac{d}{dt}_{|t=0}(m(g,s(t)) = \frac{d}{dt}_{|t=0}(gs(t)) = d(L_g)_e(s'(0)) = d(L_g)_e(v)$. Thus X is smooth. $\square$

Notice that this in particular implies that a Lie group is parallelizable, i.e., the tangent bundle is trivial.

Since diffeomorphisms respect Lie brackets, the Lie bracket of two left invariant vector fields is again left invariant: $(L_g)_*[X,Y] = [(L_g)_*X, (L_g)_*Y] = [X,Y]$. This induces a Lie bracket on $\mathfrak{g} \simeq T_eG$. We call this the Lie algebra of G. In general we will denote, without saying, the Lie algebra of a Lie group with the corresponding German script letter. Thus, e.g., $\mathfrak{h}$ is the Lie algebra of H, and $\mathfrak{k}$ is the Lie algebra of K.

matrix **Example 1.6** The most basic examples are matrix groups. Let V be a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathrm{End}(V)$ the set of all $\mathbb{K}$-linear maps from V to V.

Furthermore, $\mathrm{GL}(V) \subset \mathrm{End}(V)$ is the subset of invertible linear maps. Then $\mathrm{GL}(V)$ is a Lie group under composition of maps and $e = \mathrm{Id}$ is the identity element. Indeed, $\mathrm{GL}(V)$ is an open subset of $\mathrm{End}(V)$ and hence a manifold. To see that multiplication and inverse are smooth, it is easiest to identify with matrices. In terms of a basis of V, $\mathrm{End}(V) \simeq M(n,n,\mathbb{K})$, the set of $n \times n$ matrices with coefficients in $\mathbb{K}$. Matrix multiplication then becomes polynomial and inverses rational in the natural coordinates $y_{ij}(A) = A_{ij}$, and hence they are smooth. We denote its Lie algebra by $\mathfrak{gl}(V)$ or by $\mathrm{End}(V)$ interchangeably. We will also use $\mathrm{GL}(n,\mathbb{K})$ for $\mathrm{GL}(\mathbb{K}^n)$ as well as $\mathfrak{gl}(n,\mathbb{K})$ for its Lie algebra. For $\mathbb{K} = \mathbb{R}$ we also have the subgroup $\mathrm{GL}^+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A > 0\}$.

We now claim that the Lie algebra structure is given by $[X,Y] = XY - YX$ for $X, Y \in \mathfrak{gl}(V) \simeq M(n,n,\mathbb{K})$. To see this, observe that the left invariant vector field $\bar{X}$ with $\bar{X}(e) = X \in M(n,n,\mathbb{K})$ is given by $\bar{X}_A := \bar{X}(A) = AX$ since left translation is linear. Hence $\bar{X}_A(y_{ij}) = d(y_{ij})_A(\bar{X}_A) = (AX)_{ij}$ since y_{ij} is linear. Now $[\bar{X},\bar{Y}]_e(y_{ij}) = \bar{X}_e(\bar{Y}(y_{ij})) - \bar{Y}_e(\bar{X}(y_{ij}))$. But $\bar{X}_e(\bar{Y}(y_{ij})) = \bar{X}_e(A \to (AY)_{ij}) = (XY)_{ij}$ and hence $[\bar{X},\bar{Y}]_e(y_{ij}) = (XY - YX)_{ij}$, which proves our claim. Indeed, for a manifold with coordinates x_i we have $v = \sum v(x_i)\frac{\partial}{\partial x_i}$.

Exercises 1.7

(1) Show that $(\mathbb{R}^n,+)$, $\mathbb{R}^n/\mathbb{Z}_n = \mathrm{T}^n$, and $\mathbb{R}^n \times \mathrm{T}^m$ are Lie groups with "trivial" Lie algebra, i.e. all brackets are 0.

(2) Show that $\mathrm{SL}(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid \det A = 1\}$ is a Lie group and compute its Lie algebra.

(3) Classify all two dimensional Lie algebras.

(4) If X, Y are the left invariant vector fields with $X(e) = v, Y(e) = w$ and $\bar{X}, \bar{Y}$ the right invariant vector fields with $\bar{X}(e) = v, \bar{Y}(e) = w$, show that $[\bar{X},\bar{Y}] = -[X,Y]$.

(4) If G is a Lie group show that the identity component G_o is open, closed and normal in G.

5) Let

$$G = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

be a group under matrix multiplication. G is called the Heisenberg group. Show that G is a Lie group. If we regard x, y, z as coordinates in $\mathbb{R}^3$, this makes $\mathbb{R}^3$ into a Lie group. Compute explicitly the left invariant vector fields in these coordinates and determine the Lie brackets directly.

Lie subgroups and homomorphisms

The analogue of algebra homomorphisms is

Definition 1.8 *Let H and G be Lie groups.*

(a) $\phi\colon H \to G$ is called a **Lie group homomorphism** *if it is a group homomorphism and smooth.*

(b) ϕ is called a **Lie group isomorphism** *if it is a group isomorphism and ϕ and ϕ^{-1} are smooth.*

Similarly, we define Lie algebra homomorphism and isomorphisms.

Note that ϕ is a group homomorphism iff $\phi \circ L_g = L_{\phi(g)} \circ \phi$. A homomorphism $\phi\colon G \to \mathrm{GL}(n, \mathbb{R})$ resp. $\mathrm{GL}(n, \mathbb{C})$ is called a real resp. complex representation.

dphi

Proposition 1.9 *If $\phi\colon H \to G$ is a Lie group homomorphism, then $d\phi_e\colon T_eH \to T_eG$ is a Lie algebra homomorphism*

Proof Recall that for any smooth map f, the (smooth) vector fields X_i are called f-related to Y_i if $(df)_p(X_i(p)) = Y_i(f(p))$ for all p and that in that case $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$. Thus, if we denote by X_i the left invariant vector field on H with $X_i(e) = v_i \in \mathfrak{h}$, $i = 1, 2$, and by Y_i the left invariant vector field on G with $Y_i(e) = d\phi_e(v_i)$, all we need to show is that X_i and Y_i are ϕ related. Indeed, it will then follow that $d\phi_e([X_1, X_2]_e) = [Y_1, Y_2]_e$. They are ϕ related since

$$\begin{aligned} d(\phi)_g(X(g)) = d(\phi)_g d(L_g)_e(v) = d(\phi \circ L_g)_e(v) &= d(L_{\phi(g)} \circ \phi)_e(v) \\ &= d(L_{\phi(g)}) d(\phi)_e(v) = Y(\phi(g)). \end{aligned}$$

□

If $\phi\colon H \to G$ is a Lie group homomorphism, we simply denote by $d\phi\colon \mathfrak{h} \to \mathfrak{g}$ the above Lie algebra homomorphism. We can now apply this to subgroups of Lie groups.

Definition 1.10 *Let G be a Lie group.*

(a) H is called a **Lie subgroup** *of G if $H \subset G$ is an abstract subgroup, and H is a Lie group such that the inclusion is a smooth immersion.*
(b) $\mathfrak{h}$ is a **Lie subalgebra** *of $\mathfrak{g}$ if $[X, Y] \in \mathfrak{h}$ for all $X, Y \in \mathfrak{h}$.*

The relationship between the two is again very simple.

subgroup **Proposition 1.11** *Let G be a Lie group.*

(a) If H is a Lie subgroup of G, then $\mathfrak{h} \simeq T_eH \subset T_eG \simeq \mathfrak{g}$ is a Lie subalgebra.
(b) If $\mathfrak{h} \subset \mathfrak{g}$ a Lie subalgebra, there exists a unique connected Lie subgroup $H \subset G$ with Lie algebra $\mathfrak{h}$.

Proof Part (a) follows immediately from Proposition 1.9 applied to the inclusion. For part (b), define a distribution on G by $\Delta_g = d(L_g)_e(\mathfrak{h}) \subset T_gG$. This distribution is integrable since $\mathfrak{h}$ is a subalgebra. Let H be the maximal leaf through $e \in G$, which is by definition a one-to-one immersed submanifold. Since $(L_g)_*\Delta = \Delta$, the left translation L_g permutes leafs. Hence $L_{h^{-1}}(H) = H$ for all $h \in H$ since both contain e, i.e. H is a subgroup. Multiplication and inverse is smooth, since this is so in G, and restrictions of smooth maps to leafs of a distribution are smooth. Uniqueness of H follows since a subgroup H with $T_eH = \mathfrak{h}$ is a leaf of the distribution Δ since $T_gH = d(L_g)_e(\mathfrak{h}) = \Delta_g$ for $g \in H$. □

When clear from context, we will often simply say subgroup instead of Lie subgroup, and subalgebra instead of Lie subalgebra. The reason why we allow Lie subgroups to be immersed, instead of just embedded, is so that Proposition 1.11 (b) holds for all subalgebras $\mathfrak{h} \subset \mathfrak{g}$. Indeed, a line through the origin in the Lie group $(\mathbb{R}^2, +)$ with irrational slope is a Lie subgroup, and its image in the Lie group $\mathbb{R}^2/\mathbb{Z}^2$ is an immersed submanifold but not embedded.

uniquedph **Corollary 1.12** *Let H, G be connected Lie groups. If ϕ, $\psi \colon H \to G$ are Lie group homomorphisms with $d\phi = d\psi$, then $\phi = \psi$.*

Proof Clearly $H \times G$ is a Lie group (multiplication is defined componentwise) with Lie algebra $\mathfrak{h} \oplus \mathfrak{g}$ (brackets again defined componentwise). ϕ is a homomorphism iff its graph $\mathrm{Graph}(\phi) = \{(h, \phi(h)) \mid h \in H\} \subset H \times G$ is a Lie subgroup. Since its Lie algebra is clearly $\mathrm{Graph}(d\phi) = \{(v, d\phi(v)) \mid v \in \mathfrak{h}\}$, the assumption $d\phi = d\psi$ implies $\mathrm{Graph}(d\phi) = \mathrm{Graph}(d\psi)$ and the claim follows from the uniqueness in Proposition 1.11 (b). □

The following is a very useful and surprising fact. The proof is somewhat technical, and we will therefore omit it for now.

closedsubgroup

Theorem 1.13 *Let G be a Lie group.*

(a) *A Lie subgroup $H \subset G$ is embedded iff if it is closed.*
(b) *If $H \subset G$ is an abstract subgroup and if it is closed, then H is a Lie subgroup.*

As we saw, to every Lie group we can associate its Lie algebra, and it is a natural question wether the converse holds. The answer is yes, but the proof is non-trivial. It follows from Ado's theorem:

Ado

Theorem 1.14 *Every Lie algebra $(V, [\ ,\ ])$ is isomorphic to a subalgebra of $\mathfrak{gl}(n, \mathbb{R})$ for some n.*

Combining this with Proposition 1.11 (b), we get

existenceG

Corollary 1.15 *For every Lie algebra $(V, [\ ,\ ])$ there exists a Lie group G with $\mathfrak{g}$ isomorphic to V.*

A further natural question is wether every Lie group is isomorphic to a subgroup of $\mathrm{GL}(n, \mathbb{R})$. As we will see, this is not the case.

Exercises 1.16

1) If $\phi\colon H \to G$ is a Lie group homomorphism with $d\phi_e$ an isomorphism, show that $d\phi_g$ an isomorphism for all $g \in G$.
2) Show that $\det\colon \mathrm{GL}(n, \mathbb{R}) \to (\mathbb{R}\backslash\{0\}, \cdot\ )$ is a Lie group homomorphism with $d\det = \mathrm{tr}$.
3) Let H, G be Lie groups and $K \subset G$ a Lie subgroup. If $\phi\colon H \to G$ is a Lie group homomorphism with $\phi(H) \subset K$, show that $\phi\colon H \to K$ is a Lie group homomorphism (the issue is smoothness).

Coverings of Lie groups

The theory of covering spaces is greatly simplified if restricted to Lie groups. Although not necessary, we will use covering theory within the realm of manifolds, i.e. coverings are local diffeomorphisms.

covering

Proposition 1.17 *Let G be a connected Lie group.*

(a) *If $\tilde G$ is a connected manifold and $\pi\colon \tilde G \to G$ is a covering, then $\tilde G$ has a unique structure of a Lie group such that π is a homomorphism.*
(b) *A homomorphism $\phi\colon \tilde G \to G$ of Lie groups is a covering iff $d\phi$ is an isomorphism.*

Proof For part (a), choose an element $\tilde e \in \pi^{-1}(e)$. Covering space theory implies that $\tilde G \times \tilde G \xrightarrow{\pi\times\pi} G \times G \xrightarrow{m} G$ has a lift $\tilde m\colon \tilde G \times \tilde G \to \tilde G$, uniquely defined by $\tilde m(\tilde e, \tilde e) = \tilde e$. Similarly, the inversion $I(g) = g^{-1}$ has a unique lift $\tilde I$ with $\tilde I(\tilde e) = \tilde e$. $\tilde m$ defines a multiplication on $\tilde G$ and $\tilde I$ an inverse. The group law properties for $\tilde G$ easily follow from those for G by using the uniqueness properties of lifts under coverings. The map π is now by definition a homomorphism and the uniqueness of the above lifts shows that any two Lie group structures on $\tilde G$, such that π is a homomorphism, must be isomorphic.

One direction in (b) is clear since coverings are local diffeomorphism. For the other direction assume that $d\phi$ is an isomorphism. We need to show that every point in G has an evenly covered neighbor hood. By the inverse function theorem there exists a neighborhood U of $e \in \tilde G$ such that $\phi\colon U \to \pi(U)$ is a diffeomorphism. If $\Gamma = \ker\phi$, this implies that $\Gamma \cap U = \{e\}$. Since multiplication and inverse are continuous we can choose a neighborhood $V \subset U$ such that $V \cdot V^{-1} \subset U$. Then the open sets γV, $\gamma \in \Gamma$, are all disjoint since $\gamma u = \gamma' u'$ implies that $\gamma'^{-1}\gamma = u' u^{-1} \in \Gamma \cap V \cdot V^{-1} \subset \Gamma \cap U$ and thus $\gamma = \gamma'$. Furthermore, $\phi^{-1}(\phi(V)) = \cup_\gamma \gamma V$ since $\phi(a) = \phi(v)$, $v \in V$ implies that $\phi(av^{-1}) = e$ and hence $a = \gamma v$ for some $\gamma \in \Gamma$. Finally, since ϕ is a homomorphism, $\phi\colon \gamma U \to \pi(U)$ is a diffeomorphism for all $\gamma \in \Gamma$. Hence $\pi(V)$ is an evenly covered neighborhood of e which easily implies that $\phi(gV)$ is an evenly covered neighborhood of $\phi(g) \in G$.

It remains to show that π is onto. This immediately follows from the following Lemma, which we will use frequently.

generate

Lemma 1.18 *A connected Lie group is generated by any neighborhood of the identity.*

Proof Let U be a neighborhood of $e \in G$ and $V \subset U$ an open set with $V \cdot V^{-1} \subset U$. If we define $H = \cup_{n=-\infty}^{\infty} V^n$, then H is clearly open and a subgroup. It is also closed since its complement, being the union of all cosets of H different from H, is open. Since G is connected, $H = G$. $\square$

This finishes the proof of part (b). □

We say that $\pi\colon \tilde{G} \to G$ is a covering of Lie groups, or simply a covering, if π is a covering and a homomorphism. Notice that Proposition 1.17 (a) says that the assumption that ϕ be a homomorphism is actually not restrictive. We can now state the classification of coverings of Lie groups.

Liecoverings

Proposition 1.19 *Let $G, \tilde{G}$ be connected Lie groups.*

(a) If $\phi\colon \tilde{G} \to G$ is a covering of Lie groups, then $\ker\phi$ is a discrete subgroup of $Z(\tilde{G})$, the center of $\tilde{G}$.
(b) If $\Gamma \subset G$ is a discrete subgroup of $Z(G)$, then G/Γ is a Lie group and the projection $\phi\colon G \to G/\Gamma$ is a (normal) covering with deck group $\{L_\gamma \mid \gamma \in \Gamma\}$.

Proof For part (a) we observe that, since ϕ is a local diffeomorphism, there exists a neighborhood U of $e \in \tilde{G}$ such that $U \cap \Gamma = \{e\}$, where $\Gamma = \ker\phi$. Thus we also have $\gamma U \cap \Gamma = \{\gamma\}$ since $\gamma u = \gamma'$ implies that $u = \gamma^{-1}\gamma'$. Hence Γ is a discrete normal subgroup of $\tilde{G}$. But a discrete normal subgroup lies in the center. Indeed, if we fix $g \in \tilde{G}$ and $\gamma \in \Gamma$, and let g_t be a path with $g_0 = e$ and $g_1 = g$, then $g_t\gamma g_t^{-1} \in \Gamma$ which starts at γ and by discreteness is equal to γ for all t.

Next we prove (b). Recall that an action of Γ on a manifold M is called properly discontinuous if it satisfies the following two properties:

(1) For any $p \in M$ there exists a neighborhood U of p such that the open sets L_gU are all disjoint.

(2) For any $p, q \in M$ with $p \notin \Gamma q$ there exist neighborhoods U of p and V of q such that $\gamma U \cap \gamma' V = \emptyset$ for all $\gamma, \gamma' \in \Gamma$.

Part (1) guarantees that $M \to M/\Gamma$ is a covering since the image of U is an evenly covered neighborhood. Part(2) guarantees that the quotient is Hausdorff, which is part of the definition of a manifold. Since most books on coverings only talk about coverings of topological spaces, part (2) is sometimes deleted in the definition. One easily gives examples which satisfy (1) but not (2).

In summary, if Γ acts properly discontinuously on M, then M/Γ is a manifold and $M \to M/\Gamma$ a covering with deck group Γ.

In our case, let Γ be a discrete subgroup of the center. For part (1) let U be a neighborhood of $e \in G$ such that $\Gamma \cap U = \{e\}$, which is possible since Γ is discrete. Furthermore, choose V such that $e \in V \subset U$ and $V \cdot V^{-1} \subset U$.

Then we claim that L_gV are all disjoint. Indeed, if $g_1u = g_2v$, for some $u, v \in V$, then $g_2^{-1}g_1 = vu^{-1} \in \Gamma \cap U$ which implies $g_1 = g_2$.

For part (2), fix $g_1,\ g_2 \in G$ with $g_1 \notin \Gamma g_2$. Let $V \subset U$ be neighborhoods of e as above, which in addition satisfy $g_2^{-1}\Gamma g_1 \cap U = \emptyset$, which is possible since $g_2\Gamma g_1$ is discrete and does not contain e by assumption. Then we claim that g_1V and g_2V are the desired neighborhoods of g_1 and g_2. Indeed, if $\gamma_1 g_1 u = \gamma_2 g_2 v$ for some $\gamma_1, \gamma_2 \in \Gamma$ and $u, v \in V$, then $g_2^{-1}\gamma_2^{-1}\gamma_1 g_1 = vu^{-1} \in g_2^{-1}\Gamma g_1 \cap U$ which is not possible. Thus the projection $G \to G/\Gamma$ is a covering. Since Γ lies in the center, G/Γ is a group a since ϕ is a covering, it is a manifold as well. Since ϕ is a local diffeomorphism, multiplication and inverse is smooth. Furthermore, the deck group is $\{L_\gamma = R_\gamma \mid \gamma \in \Gamma\} \simeq \Gamma$ since $\phi(a) = \phi(b)$ implies $\phi(ab^{-1}) = e$, i.e. $a = \gamma b$ for some $\gamma \in \Gamma$. In other words, Γ acts transitively of the fibers of ϕ, which is the definition of a normal cover. □

In particular, the universal cover of a Lie group is again a Lie group.

As we saw in Corollary 3.10, a homomorphism ϕ is uniquely determined by $d\phi$. For the converse we have

dphexistence

Proposition 1.20 *If H and G are Lie groups with H simply connected, then for any Lie algebra homomorphism $\psi\colon \mathfrak{h} \to \mathfrak{g}$ there exists a unique Lie group homomorphism $\phi\colon H \to G$ with $d\phi = \psi$.*

Proof Recall that $\mathrm{Graph}(\psi) = (v, \psi(v)) \subset \mathfrak{h}\oplus\mathfrak{g}$ is a Lie subalgebra and hence by Proposition 1.11 there exists a connected subgroup $A \subset H \times G$ with Lie algebra $\mathrm{Graph}(\psi)$. Let π_1 and π_2 be the projections from $H \times G$ to the first and second factor. They are clearly homomorphisms and $\pi_1\colon A \to G$ is a covering since $d(\pi_1)_{|\mathfrak{a}}$ is clearly an isomorphism. Since H is simply connected, A is isomorphic to H. Thus we get a homomorphism $\pi_2\colon G \simeq A \to G$ which by construction has derivative ψ. □

If, on the other hand, H is not simply connected, it follows that there exists a homomorphism $\phi\colon \tilde{H} \to G$ where $\pi\colon \tilde{H} \to H$ is the universal cover. Clearly ϕ descends to $H \to G$ iff $\ker \pi \subset \ker \phi$.

isoLie

Corollary 1.21

(a) Two simply connected Lie groups with isomorphic Lie algebras are isomorphic.
(b) For every Lie algebra V, there exists a unique simply connected Lie group G with $\mathfrak{g} \simeq V$.
(c) Any Lie group with Lie algebra $\mathfrak{g}$ is isomorphic to G/Γ where G is the simply connected Lie group with Lie algebra $\mathfrak{g}$, and Γ is a discrete subgroup of $Z(G)$.

Proof (a) Let $\psi_1 \colon \mathfrak{h} \to \mathfrak{g}$ be a Lie algebra isomorphism with inverse ψ_2. Let ϕ_i be the Lie group homomorphism with $d\phi_i = \psi_i$. Since $\psi_1 \circ \psi_2 = \mathrm{Id}$, it follows that $\phi_1 \circ \phi_2 = \mathrm{Id}$ by the uniqueness part of Proposition 1.20.

(b) by Theorem 1.14, there exists some Lie group $G^* \subset \mathrm{GL}(n, \mathbb{R})$ with Lie algebra V, and hence the universal cover of G^* is the desired G. Its uniqueness follows from (a).

(c) If G^* is a Lie group with Lie algebra $\mathfrak{g}$, let $G \to G^*$ be the universal cover. The claim then follows from Proposition 1.19 (a). $\square$

Exercises 1.22

1) In the proof of Proposition 1.17, the astute reader may have noticed that a step is missing. Namely, covering theory tells us that in order to obtain the lifts $\tilde{m}$ and $\tilde{I}$, we need to show that $(I \circ \pi)_*(\pi_1(\tilde{G})) \subset \pi_*(\pi_1(\tilde{G}))$ and similarly for $((\pi \times \pi) \circ m)_*$. Fill in the details why this is true by showing that multiplication and inverse of loops in G becomes multiplication and inverse in the group structure of $\pi_1(G)$.
2) Let $\phi \colon \tilde{G} \to G$ be a covering with $\tilde{G}$ and G connected. Show that $\phi(Z(\tilde{G})) = Z(G)$ and $Z(\tilde{G}) = \phi^{-1}(Z(G))$. Furthermore, $Z(\tilde{G})$ is discrete iff $Z(G)$ is discrete.
3) Show that the fundamental group of a connected Lie group is abelian.
3) Classify all 2-dimensional Lie groups.
4) Give an example of two Lie groups H, G and a Lie algebra homomorphism $\psi \colon \mathfrak{h} \to \mathfrak{g}$ such that there exists no Lie group homomorphism $\phi \colon H \to G$ with $d\phi = \psi$.

Exponential Map

We start with the concept of a one parameter group.

onepar **Definition 1.23** *A homomorphism* $\phi\colon (\mathbb{R},+) \to G$ *is called a* **one parameter group**. *Equivalently,* $\phi(t) \in G$ *for all* t *with* $\phi(t+s) = \phi(t)\phi(s)$ *for all* t, s.

It follows from Proposition 1.20 that for each $X \in T_eG$ there exists a one parameter group ϕ_X with $d\phi_X(t) = tX$, or in other words $\phi'_X(0) = X$. We thus define

expdef **Definition 1.24** *If* G *is a Lie group with Lie algebra* $\mathfrak{g}$*, then the* **exponential map** *is defined as:*

$$\exp\colon \mathfrak{g} \to G \text{ where } \exp(X) = \phi_X(1) \text{ with } \phi'_X(0) = X$$

We now collect the basic properties of the exponential map.

exp **Proposition 1.25** *The exponential map* $\exp\colon \mathfrak{g} \to G$ *satisfies:*

(a) *For each* $X \in \mathfrak{g}$, $\phi(t) = \exp(tX)$ *is a one parameter group with* $\phi'(0) = X$.
(b) *The integral curve* c *of the left invariant vector field* $X \in \mathfrak{g}$ *with* $c(0) = g$ *is* $c(t) = g\exp(tX)$.
(c) $\exp$ *is smooth with* $d(\exp)_0 = \mathrm{Id}$.
(d) *If* $\phi\colon H \to G$ *is a Lie group homomorphism, then* $\phi(\exp_H(X)) = \exp_G(d\phi(X))$ *for* $X \in \mathfrak{h}$.
(e) *If* $H \subset G$ *is a Lie subgroup then*

$$\mathfrak{h} = \{X \in \mathfrak{g} \mid \exp_G(tX) \in H \text{ for } |t| < \epsilon \text{ for some } \epsilon > 0\}.$$

Proof First observe that ϕ_X is an integral curve of X through e since

$$\begin{aligned}\phi'_X(s) &= \frac{d}{dt}(\phi_X(s+t))_{|t=0} = \frac{d}{dt}(\phi_X(s)\phi_X(t))_{|t=0}\\ &= d(L_{\phi_X(s)})_e(\phi'_X(0)) = d(L_{\phi_X(s)})_e(X) = X(\phi_X(s)).\end{aligned}$$

Thus $\phi_{tX}(s) = \phi_X(ts)$, since, for fixed t, both are integral curves of tX through e. To see this for the right hand side, observe that in general if $\gamma(s)$ is an integral curve of a vector field X, then $\gamma(ts)$ is an integral curve of tX. Hence $\exp(tX) = \phi_{tX}(1) = \phi_X(t)$, which implies (a). Since L_g takes integral curves to integral curves, (b) follows as well.

To see that exp is smooth, define a vector field Z on $G \times \mathfrak{g}$ by $Z(g, X) = (X(g), 0)$. Z is clearly smooth and by part (b), its flow is $\psi_t(g, X) = (g \exp(tX), X)$. Thus $\psi_1(e, X) = (\exp(X), X)$ is smooth in X and hence exp is smooth as well. Finally, $d(\exp)_0(X) = \frac{d}{dt}(\exp(tX))_{|t=0} = X$, which proves the second claim in (c).

To prove (d), observe that a homomorphism takes one parameter groups to one parameter groups. Thus $\psi(t) = \phi(\exp(tX))$ is a one parameter group with $\psi'(0) = d\phi(d(\exp)_0(X) = \phi(X)$ and hence $\psi(t) = \exp(td\phi(X))$, which proves our claim by setting $t = 1$.

Part (e) follows easily by applying (d) to the inclusion of H in G. $\square$

In particular, the exponential map of a Lie subgroup is simply the restriction of the exponential map of G and we will therefore not distinguish them from now on.

As we will see, part (d) is surprisingly powerful and part (e) often enables one to compute the Lie algebra of a Lie subgroup.

`matrixexp` **Example 1.26** As we saw in Example 2.22, $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, the set of invertible matrices, are Lie groups. For these groups we claim that $\exp(A) = e^A$, which explains the name exponential map. Indeed, from the power series definition of e^A it easily follows that $e^{(t+s)A} = e^{tA}e^{sA}$, i.e. $\phi(t) = e^{tA}$ is a one parameter group. Furthermore $\phi'(0) = A$ and hence $\exp(A) = \phi(1) = e^A$.

Exercises 1.27

(1) Show that $\exp(X)^{-1} = \exp(-X)$.

(2) Show that the flow of a left invariant vector field X is $R_{\exp(tX)}$.

(3) If $\phi \colon H \to G$ is a Lie group homomorphism, then $\ker \phi$ is a Lie subgroup of H with Lie algebra $\ker(d\phi)$, and $\operatorname{Im} \phi$ is a Lie subgroup of G with Lie algebra $\operatorname{Im}(d\phi)$. Furthermore, if ϕ is onto, $H/\ker \phi$ is a Lie group, which is isomorphic to G.

(4) Let $\phi \colon H \to G$ is a Lie group homomorphism. If ϕ is injective, show it is an immersion and thus $\phi(H)$ a Lie subgroup of G. If ϕ is a group isomorphism, show it is a Lie group isomorphism, i.e. ϕ smooth implies ϕ^{-1} smooth.

(4) Carry out the details in Example 1.26

Adjoint representation

For $g \in G$ consider the conjugation map $C_g = L_g \circ R_{g^{-1}}$. Since C_g is a homomorphism

$$\mathrm{Ad}(g) := d(C_g)_e \colon \mathfrak{g} \to \mathfrak{g}$$

is a Lie algebra homomorphism. Since $C_{gh} = C_g C_h$, it follows that $\mathrm{Ad}(gh) = \mathrm{Ad}(g)\,\mathrm{Ad}(h)$ and thus

$$\mathrm{Ad} \colon G \to GL(\mathfrak{g})$$

is a Lie group homomorphism, also called the adjoint representation. Before we collect its basic properties, we make one more definition.

For $X \in \mathfrak{g}$ let $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ be defined by $\mathrm{ad}_X(Y) = [X, Y]$. The Jacobi identity is equivalent to saying that $\mathrm{ad}_{[X,Y]} = \mathrm{ad}_X \mathrm{ad}_Y - \mathrm{ad}_Y \mathrm{ad}_X = [\mathrm{ad}_x, \mathrm{ad}_Y]$ i.e.

$$\mathrm{ad} \colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g}) \simeq \mathfrak{gl}(\mathfrak{g})$$

is a Lie algebra homomorphism.

Ad

Proposition 1.28 *The adjoint representation satisfies:*

(a) $d(\mathrm{Ad})_e(X) = \mathrm{ad}_X$, *or simply* $d\,\mathrm{Ad} = \mathrm{ad}$.
(b) $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}_X}$,
(c) $\exp(\mathrm{Ad}(g)(X)) = g \exp(X) g^{-1}$,
(d) If G *is connected,* $\ker(\mathrm{Ad}) = Z(G)$.

Proof For part (a) we see that for any $Y \in \mathfrak{g}$

$$\begin{aligned}
d(\mathrm{Ad})_e(X)(Y) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \mathrm{Ad}(\exp(tX))(Y) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} d(R_{\exp(-tX)}) \circ d(L_{\exp(tX)})(Y) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} d(R_{\exp(-tX)})(Y(\exp(tX))) \\
&= L_X Y = [X, Y],
\end{aligned}$$

where L_X is the Lie derivative. In the last passage, we used the definition of Lie derivative, and the fact that $R_{\exp(tX)}$ is the flow of X.

Part (b) and (c) follow from Proposition 1.25 (d) using the homomorphism Ad and C_g resp.

One direction of (d) is clear, if $g \in Z(G)$, then $C_g = \mathrm{Id}$ and hence $\mathrm{Ad}(g) =$

Id. Conversely, if $\mathrm{Ad}(g) = \mathrm{Id}$ then C_g and Id are two homomorphisms with the same derivative, and hence by Corollary 3.10 are equal. □

matrixAd **Example 1.29** In the case of $\mathrm{GL}(n, \mathbb{R})$ and $\mathrm{GL}(n, \mathbb{C})$, we have $\mathrm{Ad}(A)(B) = ABA^{-1}$ since conjugation is linear. Hence the fact that Lie brackets in $\mathfrak{gl}(n, \mathbb{R})$ and $\mathfrak{gl}(n, \mathbb{C})$ are commutators (see Example 2.22) is an immediate consequence of part (a).

Part(b) and (c) are particularly powerful, as the reader may see in the following exercises.

exeAd **Exercises 1.30** We now have the tools to prove a number of important facts.

(1) A connected Lie group is abelian iff its Lie algebra is abelian.
(2) A connected abelian Lie group is isomorphic to $\mathrm{T}^n \times \mathbb{R}^m$ where $\mathrm{T} = \mathbb{R}/\mathbb{Z}$.
(3) If H is a Lie subgroup of G, and both are connected, then H is normal in G iff $\mathfrak{h}$ is an ideal in $\mathfrak{g}$, i.e. $[\mathfrak{g}, \mathfrak{h}] \subset \mathfrak{h}$.
(4) If $[X, Y] = 0$ then $\exp(X + Y) = \exp(X) \exp(Y)$.
(5) $Z(G)$ is a Lie subgroup of G with Lie algebra $\mathfrak{z}(\mathfrak{g}) = \{X \mid \mathrm{ad}_X = 0\}$.
(6) Complete the argument that $\mathrm{Ad}\colon G \to GL(\mathfrak{g})$ is a Lie group homomorphism by showing it is smooth.
(7) If $H \subset G$ is a Lie subgroup show that the adjoint representation for H is the restriction of the one for G.
(8) Show that $\exp\colon \mathfrak{gl}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ is onto, but $\exp\colon \mathfrak{gl}(n, \mathbb{R}) \to \mathrm{GL}^+(n, \mathbb{R})$ is not. Determine the image of $\exp\colon \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$.

Automorphisms

We start with the following Definition.

Definition 1.31 *Let $\mathfrak{g}$ be a Lie algebra.*

(a) A linear isomorphism $A\colon \mathfrak{g} \to \mathfrak{g}$ is an **automorphism** *if it is a Lie algebra homomorphism. Let $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$ be the set of automorphisms of $\mathfrak{g}$.*

(b) A linear map $A\colon \mathfrak{g} \to \mathfrak{g}$ is a **derivation** *if*

$$A[X, Y] = [AX, Y] + [X, AY], \ \forall X, Y \in \mathfrak{g}.$$

Let $\mathfrak{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g})$ be the set of derivations of $\mathfrak{g}$.

They are of course related:

Aut **Proposition 1.32** $\mathrm{Aut}(\mathfrak{g})$ *is a closed Lie subgroup of* $\mathrm{GL}(\mathfrak{g})$ *with Lie algebra* $\mathfrak{Der}(\mathfrak{g})$.

Proof Since $\mathrm{Aut}(\mathfrak{g})$ is defined by the equation $A[X,Y] = [AX,AY]$, it is closed in $\mathrm{GL}(\mathfrak{g})$ and by Theorem 1.13 is a Lie subgroup of $\mathrm{GL}(\mathfrak{g})$. If $A(t)[X,Y] = [A(t)X, A(t)Y]$ with $A(0) = e$, then by differentiation we see $A'(0) \in \mathfrak{Der}$. If $A \in \mathfrak{Der}$, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} e^{-tA}[e^{tA}X, e^{tA}Y] &= -e^{-tA}A[e^{tA}X, e^{tA}Y] + e^{-tA}[e^{tA}AX, e^{tA}Y] \\ &+ e^{-tA}[e^{tA}X, e^{tA}AY] = -e^{-tA}\left(A[Z,W] - [AZ,W] - [Z,AW]\right) = 0\end{aligned}$$

where $Z = e^{tA}X,\ W = e^{tA}Y$. Thus $e^{-tA}[e^{tA}X, e^{tA}Y] = [X,Y]$ which shows that $e^{tA} \in \mathrm{Aut}(\mathfrak{g})$ for all t and the claim follows from Proposition 1.25 (e). □

Notice that thus:

$$\mathrm{Ad}\colon G \to \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g})$$

is a Lie group homomorphism (why is it smooth?).

These Lie groups have further subgroups. Notice that the Jacobi identity implies that $\mathrm{ad}_X([Y,Z]) = [\mathrm{ad}_X(Y), Z] + [Y, \mathrm{ad}_X(Z)$, i.e. ad_X is a derivation.

Definition 1.33 *Let* $\mathfrak{g}$ *be a Lie algebra.*

(a) A derivation $A \in \mathfrak{Der}(\mathfrak{g})$ *is called an* **inner derivation** *if* $A = \mathrm{ad}_X$ *for some* $X \in \mathfrak{g}$. *Set* $\mathfrak{Int}(\mathfrak{g}) = \{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$. .

(b) Let $\mathrm{Int}(\mathfrak{g})$ *be the connected Lie subgroup of* $\mathrm{Aut}(\mathfrak{g})$ *with Lie algebra* $\mathfrak{Int}(\mathfrak{g})$. *Elements of* $\mathrm{Int}(\mathfrak{g})$ *are called* **inner automorphism**.

If G is a connected Lie group with Lie algebra $\mathfrak{g}$, then $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}_X}$ implies that $\mathrm{Int}(\mathfrak{g}) = \mathrm{Ad}(G)$ since they agree in a neighborhood of the identity. Thus $\mathrm{Ad}\colon G \to \mathrm{Int}(\mathfrak{g})$ is a Lie group homomorphism which is onto with kernel $Z(G)$. Hence, for any connected Lie group with Lie algebra $\mathfrak{g}$, we have:

$$\mathrm{Int}(\mathfrak{g}) \simeq \mathrm{Im}\,\mathrm{Ad} \simeq G/Z(G) \text{ and } \mathfrak{Int}(\mathfrak{g}) \simeq \mathfrak{g}/\mathfrak{z}(\mathfrak{g}).$$

Summing up, we have a chain of Lie groups

$$\mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g}) \subset \mathrm{GL}(\mathfrak{g}),$$

which induces a chain of Lie algebras

$$\mathfrak{Int}(\mathfrak{g}) \subset \mathfrak{Der}(\mathfrak{g}) \subset \mathrm{End}(\mathfrak{g}).$$

One more property of this chain is that:

normalInt **Proposition 1.34** *The Lie group* $\mathrm{Int}(\mathfrak{g})$ *is normal in* $\mathrm{Aut}(\mathfrak{g})$.

Proof Since $\mathrm{Int}(\mathfrak{g})$ is by definition connected, it is, by Excercise 1.30 (3), normal in $\mathrm{Aut}(\mathfrak{g})$ iff $\mathfrak{Int}(\mathfrak{g})$ is an ideal in $\mathfrak{Der}(\mathfrak{g})$. One easily show that if L is a derivation, then $[L, ad_x] = L \circ \mathrm{ad}_x - \mathrm{ad}_X \circ L = \mathrm{ad}_{LX}$ (in fact this is equivalent to being a derivation), which proves our claim. □

Thus, if $\mathrm{Aut}(\mathfrak{g})/\,\mathrm{Int}(\mathfrak{g})$ is a Lie group, its Lie algebra is $\mathfrak{Der}(\mathfrak{g})/\mathfrak{Int}(\mathfrak{g})$. In general though, $\mathrm{Int}(\mathfrak{g})$ may not be closed in $\mathrm{Aut}(\mathfrak{g})$, and hence the quotient is not always a Lie group.

We can also consider $\mathrm{Aut}(G)$ as the set of Lie group isomorphisms. By Proposition 1.20, $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(\mathfrak{g})$ if G is a simply connected Lie group. One of the exercises below shows that $\mathrm{Aut}(G)$ is a closed Lie subgroup of $\mathrm{Aut}(\mathfrak{g})$.

Another important algebraic object is the **Killing form** defined by:

$$B\colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{R} \ \ (\text{or } \mathbb{C}) \quad , \quad B(X,Y) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) \tag{1.35}$$ Killing

Clearly, B is a symmetric bilinear form. Its behavior under automorphisms is:

KillingAut **Proposition 1.36** *Let* $\mathfrak{g}$ *be a real or complex Lie algebra with Killing form* B.

(a) If $A \in \mathrm{Aut}(\mathfrak{g})$, *then* $B(AX, AY) = B(X,Y)$.
(b) If $L \in \mathfrak{Der}(\mathfrak{g})$, *then* $B(LX,Y) + B(X,LY) = 0$.

Proof One easily show that if A is an automorphism, then $\mathrm{ad}_{AX} = A \circ \mathrm{ad}_X \circ A^{-1}$. Thus

$$B(AX,AY) = \mathrm{tr}(\mathrm{ad}_{AX} \circ \mathrm{ad}_{AY}) = \mathrm{tr}(A \circ \mathrm{ad}_X \circ \mathrm{ad}_Y \circ A^{-1}) = \mathrm{tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y)$$

which proves our first claim. If L is a derivation, e^{tL} is an automorphism and thus $B(e^{tL}X, e^{tL}Y) = B(X,Y)$. Differentiating at $t = 0$ proves our second claim. □

The Killing form does not have to be non-degenerate (although it is for the important class of semisimple Lie algebras), it can even be 0. But we have

Killingkernel

Proposition 1.37 *Let B be the Killing form of $\mathfrak{g}$ with kernel* $\ker B := \{X \in \mathfrak{g} \mid B(X,Y) = 0 \ \text{ for all } Y \in \mathfrak{g}\}$. *Then* $\ker B$ *is an ideal.*

Proof If $X \in \ker(B)$ and $Y \in \mathfrak{g}$, then, since ad_y is a derivation, $B([X,Y],Z) = -B(\mathrm{ad}_Y(X),Z) = B(X,\mathrm{ad}_Y(Z)) = 0$. for all $Z \in \mathfrak{g}$. Thus $[X,Y] \in \ker B$, which proves our claim. □

Exercises 1.38

(1) Show that if G is a connected Lie group and $\tilde{G}$ is the universal cover of G, then $\mathrm{Aut}(G)$ is the closed subgroup of $\mathrm{Aut}(\tilde{G}) \simeq \mathrm{Aut}(\mathfrak{g})$ (and thus a Lie subgroup) which normalizes the deck group of the universal cover $\tilde{G} \to G$.

(2) Show that $\phi\colon \mathfrak{gl}(n,\mathbb{C}) \to \mathfrak{gl}(n,\mathbb{C})$, $\phi(X) = \bar{X}$ is an automorphism which is not inner.

(3) If $\mathfrak{z}(\mathfrak{g}) = 0$, then the center of $\mathrm{Int}(\mathfrak{g})$ is trivial as well.

(4) Show that the Killing form of $\mathfrak{gl}(n,\mathbb{R})$ and $\mathfrak{gl}(n,\mathbb{C})$ is given by $B(X,X) = 2n \operatorname{tr} X^2 - 2(\operatorname{tr} X)^2$.

(5) Let $\mathfrak{h}$ be an ideal in $\mathfrak{g}$. Show that the Killing form of $\mathfrak{h}$ is the restriction of the Killing form of $\mathfrak{g}$. Thus the Killing form of $\mathfrak{sl}(n,\mathbb{C})$ or $\mathfrak{sl}(n,\mathbb{R})$ is $B(X,X) = 2n \operatorname{tr} X^2$.

Complexification

Linear Algebra over $\mathbb{C}$ is much simpler than over $\mathbb{R}$, so it often helps to complexify. The same is true for Lie algebras. In particular, the classification and representation theory of semisimple Lie algebras in Chapter ? and ? will be done first over $\mathbb{C}$, which will then be used to interpret the results over $\mathbb{R}$ as well.

If $\mathfrak{g}$ is a real Lie algebra, we define a complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ by making the Lie brackets complex linear, i.e.

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C} \text{ and } [u+iv, x+iy] = [u,x] - [v,y] + i([u,y] + [v,x]).$$

We call $\mathfrak{g}_{\mathbb{C}}$ the **complexification** of $\mathfrak{g}$. For example, $\mathfrak{gl}(n,\mathbb{R})_{\mathbb{C}}$ is clearly isomorphic to $\mathfrak{gl}(n,\mathbb{C})$.

We can also start with a complex Lie algebra $\mathfrak{g}$ and define a real Lie algebra by forgetting the complex structure. We call this (real) Lie algebra $\mathfrak{g}_{\mathbb{R}}$ the **realification** of $\mathfrak{g}$. Notice that $\mathfrak{gl}(n,\mathbb{C})_{\mathbb{R}}$ is not $\mathfrak{gl}(n,\mathbb{R})$.

It may sometimes be helpful to think of a complex Lie algebra as a pair $(\mathfrak{g}, I)$, where $\mathfrak{g}$ is a real Lie algebra and I a complex structure, i.e. $I^2 = -\operatorname{Id}$, with $[Iu, v] = [u, Iv] = I[u, v]$. It becomes a complex vector space by declaring $(a + ib)(u) = au + Iu$ for $a + ib \in \mathbb{C},\ u \in \mathfrak{g}$. We can associate to the complex Lie algebra $(\mathfrak{g}, I)$ its **complex conjugate** $(\mathfrak{g}, -I)$ which we denote by $\bar{\mathfrak{g}}$.

complexreal

Proposition 1.39 *If $\mathfrak{g}$ is a complex Lie algebra, then $(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$ is isomorphic to $\mathfrak{g} \oplus \bar{\mathfrak{g}}$.*

Proof Let $(\mathfrak{g}, I)$ be the complex Lie algebra and J the complex multiplication due to the complexification of $\mathfrak{g}_{\mathbb{R}}$. For simplicity we identify $u + Jv \in (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$ with $(u, v) \in \mathfrak{g} \oplus \mathfrak{g}$ and thus $J(u, v) = (-v, u)$ and $I(u, v) = (Iu, Iv)$. Since I and J commute, the composition $L = IJ$ satisfies $L^2 = \operatorname{Id}$. Let $V_{\pm} \subset (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$ be the -1 and $+1$ eigenspaces of L. Notice that $V_{\pm} = \{(u, \pm Iu) \mid u \in \mathfrak{g}\}$ are complementary J invariant subspaces and since $L[u, v] = [Lu, v] = [u, Lv]$ they are also ideals. Hence $(\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}} \simeq V_- \oplus V_+$. The linear maps $f_- \colon u \in \mathfrak{g} \to (u, -Iu) \in V_-$ and $f_+ \colon u \in \mathfrak{g} \to (u, Iu) \in V_+$ clearly respects Lie brackets. Since $f_-(Iu) = (Iu, u) = J(u, -Iu) = J(f_-(u))$ and $f_+(Iu) = (Iu, -u) = -J(u, Iu) = -J(f_+(u))$ the map f_- is complex linear and f_+ complex antilinear. Thus $(f_-, f_+) \colon (\mathfrak{g}, I) \oplus (\mathfrak{g}, -I) \to V_- \oplus V_+ \simeq (\mathfrak{g}_{\mathbb{R}})_{\mathbb{C}}$ is a complex linear isomorphism of Lie algebras.. □

The complex Lie algebra $\bar{\mathfrak{g}}$ is often isomorphic to $\mathfrak{g}$, in fact this is the case iff there exists a complex conjugate linear isomorphism $f \colon \mathfrak{g} \to \mathfrak{g}$, i.e. $f(\lambda u) = \bar{\lambda} u$. This is for example the case for $\mathfrak{gl}(n, \mathbb{C})$ since $f(A) = \bar{A}$ is conjugate linear.

If $\mathfrak{h}$ is a real Lie algebra with $\mathfrak{h}_{\mathbb{C}}$ isomorphic to $\mathfrak{g}$, we call $\mathfrak{h}$ a **real form** of $\mathfrak{g}$. Equivalently, a real subalgebra $\mathfrak{h} \subset \mathfrak{g}_{\mathbb{R}}$ is a real form of $\mathfrak{g}$ if $\mathfrak{g} = \{u + iv \mid u, v \in \mathfrak{h}\}$. As we will see, not every complex Lie algebra has a real form, and if it does, it can have many real forms.

Exercises 1.40

(1) Show that $\bar{\mathfrak{g}} \simeq \mathfrak{g}$ iff $\mathfrak{g}$ has a real form.
(2) Let $\mathfrak{g}$ be the 3 dimensional complex Lie algebra spanned by X, Y, Z with $[X, Y] = 0,\ [X, Z] = X,\ [Y, Z] = aY$. Show that $\mathfrak{g}$ has a real form iff $a \in \mathbb{R}$ or $a \in \mathbb{C}$ with $|a| = 1$.
(3) If $\mathfrak{g}$ is real, the real Lie algebra $(\mathfrak{g}_{\mathbb{C}})_{\mathbb{R}}$ is "unrelated" to $\mathfrak{g}$, e.g. not isomorphic to $\mathfrak{g} \oplus \mathfrak{g}$. For example, for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{R})$ show that $\mathfrak{sl}(n, \mathbb{C})_{\mathbb{R}}$ has no non-trivial ideals.

(4) If $B_{\mathfrak{g}}$ is the Killing form of a Lie algebra $\mathfrak{g}$, show that $B_{\mathfrak{g}} = B_{\mathfrak{g}\otimes\mathbb{C}}$ if $\mathfrak{g}$ is real, and $B_{\mathfrak{g}_{\mathbb{R}}} = 2\,\mathrm{Re}(B_{\mathfrak{g}})$ if $\mathfrak{g}$ is complex.

2

A Potpourri of Examples

As is always the case, examples are the heart of a subject. There are many matrix groups, i.e. subgroups of $\mathrm{GL}(n,\mathbb{R})$ or $\mathrm{GL}(n,\mathbb{C})$, that we will later come back to frequently. We therefore devote an extra chapter to studying these Lie groups in detail.

Recall that for $G = \mathrm{GL}(n,\mathbb{R})$ and $\mathrm{GL}(n,\mathbb{C})$ we have the following:

$$[A,B] = AB - BA, \quad \exp(A) = e^A, \quad Ad(g)B = ABA^{-1} \tag{2.1}$$ MatrixAll

where $A, B \in \mathfrak{g}$ and $g \in G$. Hence, by Proposition 1.11 (a), Proposition 1.25 (d) and Excercise 1.30 (7), the same holds for any of their Lie subgroups. In most cases the subgroup is defined by equations. Since it is thus closed, Theorem 1.13 implies that it is a Lie subgroup. We will use Proposition 1.25 (e) to identify the Lie algebra. We will use all of these results from now on without repeating them.

Orthogonal Groups

Let V be an n-dimensional real vector space with a positive definite symmetric bilinear form $\langle \cdot\,,\,\cdot \rangle$, also called an inner product. We define the **orthogonal group**

$$\mathrm{O}(V) = \{A \in \mathrm{GL}(V) \mid \langle Au, Av \rangle = \langle u, v \rangle \text{ for all } u, v \in V\}$$

consisting of **isometries** A, and the **special orthogonal group**

$$\mathrm{SO}(V) = \{A \in \mathrm{O}(V) \mid \det A = 1\}$$

The normal form for orthogonal matrices can be described as follows.

Let $R(\theta) = \begin{pmatrix} \cos(\theta) & -sin(\theta) \\ sin(\theta) & \cos(\theta) \end{pmatrix}$ be a rotation by angle θ. Then there exists an orthonormal basis such that the matrix representing $A \in \mathrm{O}(V)$ has the form $A = \mathrm{diag}(R(\theta_1), \ldots R(\theta_m), \pm1, \pm1)$ if $n = 2m+2$ or $A = \mathrm{diag}(R(\theta_1), \ldots, R(\theta_m), \epsilon)$, $\epsilon = \pm1$, if $n = 2m+1$. In the first case we can assume that the lower 2×2 block takes on the form $\mathrm{diag}(1, \epsilon)$ since otherwise it can be written as a rotation. In either case, by continuously changing the angles, we can connect A to the matrix $\mathrm{diag}(1, \ldots, 1, \epsilon)$ with $\det = \epsilon$. This shows that $\mathrm{O}(V)$ has two components, and that $\mathrm{SO}(V)$ is connected. In particular they have the same Lie algebra, which is given by

$$\mathfrak{o}(V) \simeq \mathfrak{so}(V) \simeq \{A \in \mathfrak{gl}(V) \mid \langle Au, v\rangle + \langle u, Av\rangle = 0 \text{ for all } u, v \in V\}$$

so called **skew adjoint** endomorphism. Indeed, differentiating $\langle A(t)u, A(t)v\rangle = \langle u, v\rangle$ along a curve $A(t) \in \mathrm{SO}(V)$ with $A(0) = \mathrm{Id}$, we see that $A'(0)$ is skew adjoint. Conversely, if A is skew adjoint, one shows that e^{tA} is orthogonal by differentiating $\langle e^{tA}u, e^{tA}v\rangle$. This method works for all of the examples in the next few sections, and we will therefore not repeat it.

It is often convenient to choose an orthonormal basis u_i of V and identify $V \simeq \mathbb{R}^n$ by sending u_i to the standard orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$. The endomorphism is then identified with a matrix and $\mathrm{O}(V), \mathrm{SO}(V)$ are isomorphic to:

$$\mathrm{O}(n) = \{A \in M(n, n, \mathbb{R}) \mid A^T A = \mathrm{Id}\}\ ,\ \mathrm{SO}(n) = \{A \in \mathrm{O}(n) \mid \det A = 1\}$$

consisting of **orthogonal** matrices A. This easily follows by writing the inner product in $\mathbb{R}^n$ as $\langle u, v\rangle = u^T v$. The Lie algebra is now

$$\mathfrak{so}(n) = \{A \in M(n, n, \mathbb{R}) \mid A + A^T = 0\}$$

consisting of **skew symmetric** matrices A. Thus $\dim \mathrm{SO}(n) = n(n-1)/2$. Since $A^T A = \mathrm{Id}$ is equivalent to the condition that the rows (or the columns) form an orthonormal basis, it follows that $\mathrm{O}(n)$ is compact.

More generally, we consider a non-degenerate symmetric bilinear form of signature (p, q) on V, where an orthonormal basis u_i satisfies $\langle u, v\rangle = \sum_{i=1}^{p} a_i b_i - \sum_{i=p+1}^{p+q} a_i b_i$ for $u = \sum a_i u_i$, $v = \sum b_i u_i$. Isometries are defined as before as subgroups of $\mathrm{GL}(V)$. After choosing an orthonormal basis, we can write the inner product as $\langle u, v\rangle = u^T I_{p,q} v$ where $I_{p,q} = \mathrm{diag}(I_p, -I_q)$ and I_k is the $k \times k$ identity matrix. Thus $\langle Au, Av\rangle = \langle u, v\rangle$ iff $u^T A^t I_{p,q} A v = u^T I_{p,q} v$

for all u, v and hence in terms of matrices the Lie group is isomorphic to

$$\mathrm{O}(p,q) = \{A \in M(n,n,\mathbb{R}) \mid A^T I_{p,q} A = I_{p,q}\}$$

with Lie algebra

$$\mathfrak{o}(p,q) = \{A \in M(n,n,\mathbb{R}) \mid A^T I_{p,q} + I_{p,q} A = 0\},$$

If in addition $\det = 1$, we denote it by $\mathrm{SO}(p,q)$, and by $\mathrm{SO}^+(p,q)$ the identity component. An important example is the **Lorenz group** $O(p,1)$.

We also have the complexification of $\mathfrak{so}(n)$:

$$\mathfrak{so}(n,\mathbb{C}) = \{A \in M(n,n,\mathbb{C}) \mid A + A^T = 0\}.$$

Notice that $\mathfrak{o}(p,q) \otimes \mathbb{C} \simeq \mathfrak{so}(n,\mathbb{C})$ for all p, q with $n = p + q$ since in the basis u_i we can change $u_{p+1}, \dots, u_{p+q}$ to $iu_{p+1}, \dots, iu_{p+q}$. Thus $\mathfrak{so}(n,\mathbb{C})$ has many different real forms.

We finally discuss applications of the **polar decomposition** of matrices. We will from now one denote by $Sym_n(\mathbb{R})$ the set of $n \times n$ real symmetric matrices and by $Sym_n^+(\mathbb{R})$ the positive definite ones.

polarreal

Proposition 2.2 *Given $A \in \mathrm{GL}(n,\mathbb{R})$, there exists a unique decomposition $A = R\, e^S$ with $R \in \mathrm{O}(n)$ and $S \in Sym_n(\mathbb{R})$.*

Proof We first claim that we can uniquely write A as $A = RL$ with $A \in \mathrm{O}(n)$ and L symmetric $L = L^t$ and positive definite $L > 0$. Recall that L is called positive definite if $Lu \cdot u > 0$ for all $u \neq 0$, or equivalently all eigenvalues of L are positive. Indeed, if this is possible, then $A^tA = L^tR^tRL = L^2$. Now A^tA is clearly symmetric, and positive definite since $A^tAu \cdot u = Au \cdot Au > 0$. Thus A^tA has a unique (positive definite) square root. We therefore define $L = \sqrt{A^tA}$ and set $R = AL^{-1}$. Then $R \in \mathrm{O}(n)$ since $R^tR = L^{-1}A^tAL^{-1} = L^{-1}L^2L^{-1} = \mathrm{Id}$. This proves existence and uniqueness. Next we claim that we have a diffeomorphism from $Sym_n(\mathbb{R})$ to $Sym_n^+(\mathbb{R})$ given by $L \to e^L$. Clearly $e^L \in Sym_n^+(\mathbb{R})$ since $(e^L)^t = e^{L^t}$ and if $Lv = \lambda v$, then $e^Lv = e^\lambda v$ which one sees by using the power series definition of e^L. If $B \in Sym_n^+(\mathbb{R})$, then there exists a basis of eigenvectors u_i with $Bu_i = \mu u_i$ and $\mu_i > 0$. Writing $\mu_i = e_i^\lambda$ we define A by $Au_i = \lambda_i u_i$ and clearly $e^A = B$. This shows the map is onto. Since eigenvectors of A and e^A are the same, it also follows that the map is injective. Differentiability follows from the differentiability of the exponential map. $\square$

This in particular implies:

diffeoreal **Corollary 2.3** *$\mathrm{GL}(n,\mathbb{R})$ is diffeomorphic to $\mathrm{O}(n)\times\mathbb{R}^m$, $\mathrm{GL}^+(n,\mathbb{R})$ diffeomorphic to $\mathrm{SO}(n)\times\mathbb{R}^m$ and $\mathrm{SL}(n,\mathbb{R})$ is diffeomorphic to $\mathrm{SO}(n)\times\mathbb{R}^{m-1}$ with $m=n(n-1)/2$.*

Thus $\mathrm{GL}(n,\mathbb{R})$ has 2 components and $\mathrm{SL}(n,\mathbb{R})$ is connected.

There exists a vast generalization of Proposition 2.13 for any Lie group:

maxcompact **Proposition 2.4** *If G is a connected Lie group, then there exists a compact subgroup K, such that K is maximal in G among compact subgroups, and unique up to conjugacy.*

Exercises 2.5

(1) Show that in the polar decomposition of $A\in\mathrm{O}(p,q)$, $p,q\geq 1$, $R\in\mathrm{O}(p)\times\mathrm{O}(q)$. Thus these groups are non-compact, have 4 components, and $\mathrm{O}(p,q)$ and $\mathrm{O}(p',q')$ are isomorphic iff $(p,q)=(p',q')$ or $(p,q)=(q',p')$.

(2) Let $g=\mathrm{diag}(-1,1,\dots,1)$. Clearly g lies in $\mathrm{O}(n)$ but not in $\mathrm{SO}(n)$. Show that $\mathrm{Ad}(g)$ lies in $\mathrm{Aut}(\mathfrak{so}(n))$ but not in $\mathrm{Int}(\mathfrak{so}(n))$.

(3) Show that the Killing form of $\mathfrak{o}(n)$ is given by $B(X,X)=(n-2)\,\mathrm{tr}\,X^2$.

Unitary Groups

Let V be a complex vector space, with a positive definite Hermitian inner product $\langle\,,\,\rangle$, i.e. $\langle\lambda u,v\rangle=\bar\lambda\langle u,v\rangle$, $\langle u,\lambda v\rangle=\lambda\langle u,v\rangle$, $\langle u,v\rangle=\overline{\langle v,u\rangle}$ and $\langle u,u\rangle>0$ iff $u\neq 0$. The analogue of the orthogonal group is the **unitary group**

$$\mathrm{U}(V)=\{A\in\mathrm{GL}(V)\mid (Av,Aw)=(v,w)\ \text{ for all } v,w\in V\}.$$

with Lie algebra

$$\mathfrak{u}(V)=\{A\in\mathfrak{gl}(V)\mid \langle Au,v\rangle+\langle u,Av\rangle=0 \text{ for all } u,v\in V\}$$

and the **special unitary group**

$$\mathrm{SU}(V)=\{A\in\mathrm{U}(V)\mid \det A=1\} \text{ with } \mathfrak{su}(V)=\{A\in\mathfrak{u}(V)\mid \mathrm{tr}\,A=0\}$$

If $V=\mathbb{C}^n$, we write $\mathrm{U}(n)$ instead of $\mathrm{U}(V)$. With respect to an orthonormal basis u_i, we have $\langle u,v\rangle=\sum\bar a_i b_i=a^T\bar b$ which easily shows that $\mathrm{U}(V)$ is isomorphic to

$$\mathrm{U}(n)=\{A\in\mathrm{GL}(n,\mathbb{C})\mid A^*A=Id\},\ \text{ with }\ \mathfrak{u}(n)=\{X\in\mathfrak{gl}(n,\mathbb{C})\big|\ A+A^*=0\}$$

where $A^* = \bar{A}^T$ is the transpose conjugate. Similarly,

$$\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \mid \det A = 1\} \text{ with } \mathfrak{su}(n) = \{A \in \mathfrak{u}(n) \mid \operatorname{tr} A = 0\}.$$

Recall that for $A \in \mathrm{U}(n)$, we have $|\det A| = 1$ and for $A \in \mathfrak{u}(n)$, $\operatorname{tr} A$ is imaginary. Thus $\dim \mathrm{U}(n) = n^2$ and $\dim \mathrm{SU}(n) = n^2 - 1$

For every unitary matrix there exists an orthonormal basis of eigenvectors u_i with eigenvalues λ_i and $|\lambda_i| = 1$. Thus any matrix in $\mathrm{U}(n)$ can be deformed within $\mathrm{U}(n)$ to the Identity matrix by changing the eigenvalues. Hence $\mathrm{U}(n)$ is connected. Clearly, the same is true for $\mathrm{SU}(n)$ and we also have that $\mathrm{U}(n)$ and $\mathrm{SU}(n)$ are compact.

Note that, although the matrices in $\mathfrak{u}(n)$ are complex, $\mathfrak{u}(n)$ is not a complex subspace of $\mathfrak{gl}(n, \mathbb{C})$, i.e. it is not a complex Lie algebra. If we complexify we claim

$$\mathfrak{u}(n) \otimes \mathbb{C} \simeq \mathfrak{gl}(n, \mathbb{C}) \quad \text{and} \quad \mathfrak{su}(n) \otimes \mathbb{C} \simeq \mathfrak{sl}(n, \mathbb{C}).$$

In fact, a complex matrix A is the sum of a hermitian and skew hermitian matrix: $A = (A + A^*)/2 + (A - A^*)/2$. Furthermore, i times a hermitian matrix is skew hermitian. Thus for $A \in \mathfrak{gl}(n, \mathbb{C})$ we have $A = P + iQ$ with P, Q skew hermitian, i.e. $P, Q \in \mathfrak{u}(n)$.

For complex matrices we have the analogue of a polar decomposition.

polarreal

Proposition 2.6 *Given $A \in \mathrm{GL}(n, \mathbb{C})$, there exists a unique decomposition $A = R\, e^S$ with $R \in \mathrm{U}(n)$ and S hermitian, i.e. $S = S^*$.*

The proof is the same as before, and hence

diffeocomplex

Corollary 2.7 *$\mathrm{GL}(n, \mathbb{C})$ is diffeomorphic to $\mathrm{U}(n) \times \mathbb{R}^m$, and $\mathrm{SL}(n, \mathbb{C})$ to $\mathrm{SU}(n) \times \mathbb{R}^{m-2}$ with $m = n^2$.*

Thus $\mathrm{GL}(n, \mathbb{C})$ and $\mathrm{SL}(n, \mathbb{R})$ are connected and noncompact.

We finally discuss an embedding $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. For this we use the identification $\mathbb{C}^n \simeq \mathbb{R}^{2n}$:

$$\mathbb{R}^{2n} \simeq \mathbb{R}^n \oplus \mathbb{R}^n \to \mathbb{C}^n \;:\; (u, v) \to u + iv \tag{2.8}$$ Cn=R2n

which induces an embedding:

$$\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})\colon A + iB \to \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \tag{2.9}$$ Cn=R2n

since $(A + iB)(u + iv) = Au - Bv + i(Av + Bu)$. This is clearly an injective Lie group homomorphism. One gets further embeddings:

$$\mathrm{U}(n) \subset \mathrm{SO}(2n), \quad \text{in fact } \mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{SO}(2n) \cap \mathrm{GL}(n, \mathbb{C})$$

Indeed, the real part of the hermitian inner product on $\mathbb{C}^n$ is the euclidean inner product on $\mathbb{R}^{2n}$:

$$\langle u+iv, u'+iv'\rangle = (u+iv)\cdot(u'-iv') = u\cdot u' + v\cdot v' + i(v\cdot u' - u\cdot v').$$

Furthermore, A preserves the hermitian inner product iff it preserves its length, which, since real, is the same as preserving the euclidean length and hence the euclidean inner product.

Exercises 2.10

(1) Show that $\mathrm{U}(n)$ is diffeomorphic to $\mathrm{SU}(n)\times \mathrm{S}^1$, but not isomorphic (not even as groups). On the other hand, show that $\mathrm{SU}(n)\times\mathrm{S}^1$ is a n-fold cover of $\mathrm{U}(n)$.

(2) Show that $\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar b \\ b & \bar a \end{pmatrix} \mid a,b\in\mathbb{C} \text{ with } |a|^2+|b|^2=1 \right\}$. Hence $\mathrm{SU}(2)$ is diffeomorphic to $\mathbb{S}^3(1)\subset\mathbb{C}^2$.

(3) Develop the definition and properties of $\mathrm{U}(p,q)$ and $\mathrm{SU}(p,q)$.

(4) Show that the automorphism $A\to\bar A$ is outer for $\mathrm{U}(n), n\geq 2$ and $\mathrm{SU}(n), n\geq 3$, but inner for $\mathrm{SU}(2)$.

(4) Show that the Killing form of $\mathfrak{su}(n)$ is given by $B(X,X) = 2n\,\mathrm{tr}\,X^2$.

Quaternions and symplectic groups

Besides $\mathbb{R}$ and $\mathbb{C}$ there is another important division algebra, the quaternions, denoted by $\mathbb{H}$. It consists of the elements $\pm1, \pm i, \pm j \pm k$ which satisfy $i^2=j^2=k^2=-1$, i,j,k anti-commute with each other, and 1 commutes with i,j,k. An element $q\in\mathbb{H}$ has the form $q=a+bi+cj+dk$. We denote by $\bar q = a-bi-cj-dk$ the conjugate of q. Note that it satisfies $\overline{qr}=\bar r\bar q$. Under the identification $\mathbb{H}\simeq\mathbb{R}^4$, $q\to(a,b,c,d)$ the Euclidean inner product is given by $\langle q,r\rangle = \mathrm{Re}(\bar q r)$ with norm $|q|^2 = q\bar q = \bar q q = a^2+b^2+c^2+d^2$.

One easily checks that $|qr| = |q|\cdot|r|$ which implies that $\mathbb{S}^3(1) = \{q\in\mathbb{H} \mid |q|=1\}\subset\mathbb{R}^4$ is a Lie group. The same is of course also true for $\mathbb{S}^1(1)=\{q\in\mathbb{C}\mid |q|=1\}\subset\mathbb{R}^2$. These are the only spheres that can be Lie groups, see Excercise 6.

Linear Algebra over the Quaternions must be done carefully since quaternions do not commute. When defining a vector space V over $\mathbb{H}$, we let the scalars act from the right. Thus a linear map $L\colon V\to W$ is $\mathbb{H}$ linear if $L(vq)=L(v)q$ for $v\in V, q\in\mathbb{H}$. It has the following advantage. If we choose a basis u_i of V over $\mathbb{H}$, and associate as usual to L the matrix $A=(a_{ij})$

with $L(u_i) = \sum_{ji} u_j a_{ji}$, then L acts via matrix multiplication: If $u = \sum u_i b_i$ and $b = (b_1, \dots, b_n)^T$ then $L(u) = \sum_j (\sum_i a_{ji} b_i) u_j$, i.e. Lu is equal to the vector Ab in the basis u_i. Thus composition of two $\mathbb{H}$ linear maps L and M corresponds to the usual matrix multiplication of the matrices associated to L and M as above.

On the other hand, certain things that we are used to from Linear Algebra are not allowed, e.g. with the usual definition of the determinant, $\det(AB) \neq \det A \det B$. Also $\operatorname{tr} AB \neq \operatorname{tr} BA$ and $\overline{AB} = \bar{B}\bar{A}$ in general. Eigenvectors are not well behaved either: If $A(v) = vq$, then $A(vr) = v(qr) = vr(r^{-1}qr)$ and thus if q is an eigenvalue, so is $r^{-1}qr$ for any $r \in \mathbb{H}$. And the endomorphism $q \operatorname{Id}$ is different from scalar multiplication by q. Somewhat surprisingly, $\mathbb{H}$ 'holomorhic' maps are linear, i.e. if $F\colon \mathbb{H}^n \to \mathbb{H}^n$ is differentiable with dF_p $\mathbb{H}$-linear for all p, then F is $\mathbb{H}$-linear.

But it still makes sense to talk about $\mathrm{GL}(n, \mathbb{H})$ as a Lie group since multiplication of matrices corresponds to composition of $\mathbb{H}$ linear maps, and hence the product of invertible matrices are invertible. Its Lie algebra is $\mathfrak{gl}(n, \mathbb{H})$, the set of all $n \times n$ matrices whose entries are quaternions. One easily sees that exp and Ad satisfy the same properties as for $\mathrm{GL}(n, \mathbb{C})$. Thus the Lie bracket is still $[A, B] = AB - BA$. But notice that $\mathrm{SL}(n, \mathbb{H})$ and $\mathfrak{sl}(n, \mathbb{H})$ cannot be defined in the usual fashion, although we will find a different definition shortly.

A quaternionic inner product is a bilinear form with $\langle qu, v\rangle = \bar{q}\langle u, v\rangle$, $\langle u, qv\rangle = \langle u, v\rangle q$ and $\langle u, v\rangle = \overline{\langle v, u\rangle}$ as well as $\langle u, u\rangle > 0$ iff $u \neq 0$. We can thus define the **symplectic group** $\mathrm{Sp}(V)$ for a quaternionic vector space with a quaternionic inner product as the set of isometries: $\langle Av, Aw\rangle = \langle v, w\rangle$. After a choice of an orthonormal basis, we identify V with $\mathbb{H}^n$ and the inner product becomes $\langle v, w\rangle = \bar{v}^T \cdot w$. The euclidean inner product on $\mathbb{H}^n \simeq \mathbb{R}^{4n}$ is then given by $\operatorname{Re} \bar{v} \cdot w$. Notice that it follows as usual that $(AB)^* = B^* A^*$, where again $A^* = \bar{A}^T$. Thus $\mathrm{Sp}(V)$ is isomorphic to:

$$\mathrm{Sp}(n) = \{A \in \mathrm{GL}(n, \mathbb{H}) \mid A^* A = Id\}, \quad \mathfrak{sp}(n) = \{X \in \mathfrak{gl}(n, \mathbb{H}) \mid A + A^* = 0\}$$

In particular, $\dim \mathrm{Sp}(n) = 2n^2 + n$, and clearly, $\mathrm{Sp}(n)$ is again compact.

Next we discuss the embedding $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(2n, \mathbb{C})$. We identify $\mathbb{H}^n \simeq \mathbb{C}^{2n}$:

$$\mathbb{C}^{2n} \simeq \mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{H}^n \;:\; (u, v) \to u + jv \tag{2.11}$$

Cn=R2n

This gives rise to the Lie group embedding:

$$\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(2n, \mathbb{C})\colon A + jB \to \begin{pmatrix} A & -\bar{B} \\ B & \bar{A} \end{pmatrix} \tag{2.12}$$

Hn=C2n

since $(A+jB)(u+jv) = Au+jBjBv+Ajv+jBu = Au-\bar{B}v+j(\bar{A}v+Bu)$. Here we have used $jA = \bar{A}j$ for $A \in \mathfrak{gl}(n,\mathbb{C})$. The claim that the embedding is a Lie group homomorphism follows from the fact that matrix multiplication corresponds to a composition of linear maps.

As a consequence

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{GL}(n,\mathbb{H}) = \mathrm{SU}(2n) \cap \mathrm{GL}(n,\mathbb{H}).$$

Indeed, the 'complex' part of the quaternionic inner product on $\mathbb{H}^n$ is the hermitian inner product on $\mathbb{C}^{2n}$:

$$\langle u+jv, u'+jv'\rangle = (\bar{u}-jv)\cdot(u'+jv') = \bar{u}\cdot u' + \bar{v}\cdot v' + j(v\cdot u' - u\cdot v').$$

Furthermore, $\mathrm{Sp}(n) \subset \mathrm{SU}(2n)$, or equivalently $\mathfrak{sp}(n) \subset \mathfrak{su}(2n)$, follows from (2.12) since the image is skew hermitian with trace 0.

Under the identification $\mathbb{C}^n \oplus \mathbb{C}^n \to \mathbb{H}^n$ right multiplication by j corresponds to the complex antilinear endomorphism $\bar{J}(u,v) = (-\bar{v},\bar{u})$. Thus a complex linear endomorphism is $\mathbb{H}$ linear iff it commutes with $\bar{J}$. We could thus equivalently define:

$$\mathrm{GL}(n,\mathbb{H}) = \{A \in \mathrm{GL}(2n,\mathbb{C}) \mid A\bar{J} = \bar{J}A\}$$

and

$$\mathrm{Sp}(n) = \{A \in \mathrm{U}(2n) \mid A\bar{J} = \bar{J}A\}.$$

This can be useful if one is doubtful wether certain operations of matrices are allowed over $\mathbb{H}$. It also enables us to define:

$$\mathrm{SL}(n,\mathbb{H}) = \{A \in \mathrm{SL}(2n,\mathbb{C}) \mid A\bar{J} = \bar{J}A\}$$

with Lie algebra

$$\mathfrak{sl}(n,\mathbb{H}) = \{A \in \mathfrak{sl}(2n,\mathbb{C}) \mid A\bar{J} = \bar{J}A\}.$$

It is also the best way to prove the polar decomposition theorem:

polarreal **Proposition 2.13** *Given $A \in \mathrm{GL}(n,\mathbb{H})$, there exists a unique decomposition $A = R\,e^S$ with $R \in \mathrm{Sp}(n)$ and $S = S^*$. Thus $\mathrm{GL}(n,\mathbb{H})$ is diffeomorphic to $\mathrm{Sp}(n) \times \mathbb{R}^m$ with $m = 2n^2 - n$.*

Exercises 2.14

(1) Show that a quaternionic vector space has an orthonormal basis and hence $\mathrm{Sp}(n)$ is connected.
(2) Show that $\mathfrak{sl}(n,\mathbb{H}) \subset \mathfrak{sl}(2n,\mathbb{C})$ is a real form of $\mathfrak{sl}(2n,\mathbb{C})$.

(3) Show that the following properties do not hold for quaternionic matrices: $(AB)^T = B^T A^T$, $\overline{AB} = \bar{B}\bar{A}$, $\det(AB) = \det A \det B$, $\operatorname{tr}(AB) = \operatorname{tr}(BA)$. But it is true that $(AB)^* = B^* A^*$.

(4) Convince yourself that letting the scalar act on the left in a quaternionic vector space and changing the identification in (2.12) to $(u, v) \to u + vj$ does not work as well as above.

(5) Show that the Killing form of $\mathfrak{sp}(n)$ is given by $B(X, X) = 2n \operatorname{tr} X^2$.

(6) The geometrically inclined reader is encouraged to show that for a non-abelian compact Lie group G, the left invariant 3-form $\omega(X, Y, Z) = \langle [X, Y], Z\rangle$, $X, Y, Z \in \mathfrak{g}$, is parallel, where $\langle \cdot, \cdot \rangle$ is an inner product invariant under left and right translations. Show that this implies that ω is harmonic, i.e. closed and co-closed, and hence non-zero in the De Rham cohomology group $H^3_{DR}(G)$. Thus $\mathbb{S}^n$ is (homotopy equivalent) to a Lie group iff $n = 1, 3$.

Non-compact symplectic groups

Let V be a real or complex vector space with a skew-symmetric non-degenerate bilinear form

$$\omega : V \times V \to \mathbb{R} \text{ (or } \mathbb{C})$$

We then define the **symplectic** group:

$$\mathrm{Sp}(V, \omega) = \{A \in \mathrm{GL}(V) \mid \omega(Av, Aw) = \omega(v, w) \text{ for all } v, w \in V\}.$$

with Lie algebra

$$\mathfrak{sp}(V, \omega) = \{A \in \mathfrak{gl}(V) \mid \omega(Au, v) + \omega(u, Av) = 0 \text{ for all } u, v \in V\}$$

One easily sees that there exists a *symplectic* basis, i.e., a basis $x_1, \dots, x_n$, $y_1, \dots y_n$ of V such that $\omega(x_i, x_j) = \omega(y_i, y_j) = 0$, and $\omega(x_i, y_j) = \delta_{ij}$. In particular, $\dim V$ is even. If we identify the basis with its dual basis, we have $\omega = \sum_i dx_i \wedge y_i$.

The matrix associated to ω with respect to a symplectic basis is

$$J := \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

in other words $\omega(u, v) = u^T J v$. Thus $\mathrm{Sp}(\mathbb{R}^{2n}, \omega)$ can be identified with a matrix group:

$$\mathrm{Sp}(n, \mathbb{R}) = \{A \in \mathrm{GL}(2n, \mathbb{R}) \mid A^T J A = J\}.$$

Notice that we can embed $\mathrm{GL}(n,\mathbb{R})$ in $\mathrm{Sp}(n,\mathbb{R})$ by

$$B \to \begin{pmatrix} B & 0 \\ 0 & \left(B^T\right)^{-1} \end{pmatrix}.$$

since one easily checks that $A^T JA = J$. In particular $\mathrm{Sp}(n,\mathbb{R})$ is not compact. The Lie algebra of $\mathrm{Sp}(n,\mathbb{R})$ is

$$\begin{aligned}\mathfrak{sp}(n,\mathbb{R}) =& \{X \in \mathfrak{gl}(2n,\mathbb{R}) \mid XJ + JX^T = 0\} \\ =& \left\{ X = \left(\begin{array}{c|c} B & S_1 \\ \hline S_2 & -B^T \end{array}\right) \mid B \in \mathfrak{gl}(n,\mathbb{R})\ ,\ S_i \in Sym_n(\mathbb{R}) \right\}\end{aligned}$$

Thus dim $\mathrm{Sp}(n,\mathbb{R}) = 2n^2 + n$.

In a similar way we define the symplectic group over $\mathbb{C}$:

$$\mathrm{Sp}(n,\mathbb{C}) = \{A \in \mathrm{GL}(2n,\mathbb{C}) \mid A^T JA = J\}$$

which preserves the complex symplectic form $\omega(u,v) = u^T Jv,\ u,v \in \mathbb{C}^n$. Notice that ω is defined to be skew-symmetric and not skew-hermitian.

Next, we observe that $\mathrm{U}(n) \subset \mathrm{Sp}(n,\mathbb{R})$. Indeed, $\langle u + iv, u' + iv' \rangle = u \cdot u' + v \cdot v' + i(u \cdot v' - v \cdot u') = |(u,v)|^2 + i\omega(u,v)$. Thus $A \in \mathrm{U}(n)$ iff it preserves the euclidean norm as well as the symplectic form:

$$\mathrm{U}(n) = \mathrm{O}(2n) \cap \mathrm{Sp}(n,\mathbb{R}) = \mathrm{SO}(2n) \cap \mathrm{Sp}(n,\mathbb{R}).$$

Similarly,

$$\mathrm{Sp}(n) = \mathrm{U}(2n) \cap \mathrm{Sp}(n,\mathbb{C}) = \mathrm{SU}(2n) \cap \mathrm{Sp}(n,\mathbb{C}).$$

We finally discuss the polar decomposition theorem for symplectic groups.

polarsymp

Proposition 2.15

(a) *Given $A \in \mathrm{Sp}(n,\mathbb{R})$, there exists a unique decomposition $A = R\,e^S$ with $R \in \mathrm{U}(n)$ and $S \in \mathfrak{sp}(n,\mathbb{R}) \cap Sym_{2n}(\mathbb{R})$.*
(b) *Given $A \in \mathrm{Sp}(n,\mathbb{C})$, there exists a unique decomposition $A = R\,e^S$ with $R \in \mathrm{Sp}(n)$ and $S \in \mathfrak{sp}(n,\mathbb{C}) \cap Sym_{2n}(\mathbb{C})$.*

Proof Given a matrix A with $AJA^T = J$, we write it as $A = RL$ with $R \in \mathrm{O}(2n)$ and $L \in Sym^+_{2n}(\mathbb{R})$. Hence $RLJLR^T = J$ or equivalently $R^T JR = LJL$ or $(R^T JR)L^{-1} = LJ = J(-JLJ)$ since $J^2 = -\,\mathrm{Id}$. Now notice that in the equation $(R^T JR)L^{-1} = J(-JLJ)$ the first matrix on the left and right is orthogonal, and the second one symmetric. Hence by the uniqueness of polar decompositions, $R^T JR = J$ and $L^{-1} = -JLJ$, or $LJL = J$. This says that $R^T = R^{-1} \in \mathrm{Sp}(n, \mathbb{R})$, or equivalently $R \in \mathrm{Sp}(n, \mathbb{R})$, and $L \in \mathrm{Sp}(n, \mathbb{R})$. Thus $R \in \mathrm{O}(2n) \cap \mathrm{Sp}(n, \mathbb{R}) = \mathrm{U}(n)$ and $L = e^S$ with $S \in \mathfrak{sp}(n, \mathbb{R}) \cap Sym_{2n}(\mathbb{R})$. A similar proof works for complex matrices. □

In particular, $\mathrm{Sp}(n, \mathbb{R})$ and $\mathrm{Sp}(n, \mathbb{C})$ are connected and diffeomorphic to $\mathrm{U}(n) \times \mathbb{R}^m$ and $\mathrm{Sp}(n) \times \mathbb{R}^m$ respectively.

Exercises 2.16

(1) Show that $\mathrm{Sp}(1, \mathbb{R})$ is isomorphic to $\mathrm{SL}(2, \mathbb{R})$ and $\mathrm{Sp}(1, \mathbb{C})$ to $\mathrm{SL}(2, \mathbb{C})$..
(2) Show that $\mathfrak{sp}(n) \otimes \mathbb{C} \simeq \mathfrak{sp}(n, \mathbb{C})$.
(3) Show that $A \in \mathrm{Sp}(n, \mathbb{R})$ satisfies $\det A = 1$.

Coverings of classical Lie groups

There are interesting coverings among the examples in the previous sections in low dimensions, which we discuss now. These will also follow from the general theory developed in later sections, but we find it illuminating to describe them explicitly. Recall that if H, G are connected, then $\phi\colon H \to G$,, is a covering iff $d\phi$ is an isomorphism. Since the Lie algebra of $\ker \phi$ is $\ker d\phi$, it follows that a homomorphism ϕ with $\ker \phi$ discrete and $\dim H = \dim G$, must be a covering.

We start with the fundamental groups of the compact Lie groups in the previous sections.

fundgroup

Proposition 2.17

(a) $\pi_1(\mathrm{SO}(n) = \mathbb{Z}_2$ *for* $n \geq 3$ *and* $\pi_1(\mathrm{SO}(2) = \mathbb{Z}$.
(a) $\pi_1(\mathrm{U}(n) = \mathbb{Z}$ *and* $\pi_1(\mathrm{SU}(n) = 0$ *for* $n \geq 1$.
(b) $\pi_1(\mathrm{Sp}(n) = 0$ *for* $n \geq 1$.

Proof (a) Clearly $\pi_1(\mathrm{SO}(2)) = \pi_1(\mathrm{S}^1) = \mathbb{Z}$. For $n \geq 3$, the proof is by induction. By Proposition 2.20, $\pi_1(\mathrm{SO}(3)) = \mathbb{Z}_2$. Now let $\mathrm{SO}(n+1)$ act on $\mathbb{S}^n$ via $p \to Ap$. The isotropy at e_1 is $\{A \in \mathrm{SO}(n+1) \mid Ae_1 = e_1\} = \{\mathrm{diag}(1, B) \mid B \in \mathrm{SO}(n)\} \simeq \mathrm{SO}(n)$. Since the action is transitive, $\mathbb{S}^n = \mathrm{SO}(n+1)/\mathrm{SO}(n)$. One thus has a fibration $\mathrm{SO}(n) \to \mathrm{SO}(n+1) \to \mathbb{S}^n$ and we will use the long homotopy sequence $\pi_i(\mathrm{SO}(n)) \to \pi_i(\mathrm{SO}(n+1)) \to \pi_i(\mathbb{S}^n)$. Since $\pi_i(\mathbb{S}^n) = 0$

when $i = 1, \ldots, n-1$, it follows that the inclusion $\pi_1(\mathrm{SO}(n)) \to \pi_1(\mathrm{SO}(n+1))$ is onto for $n \geq 2$ and injective for $n \geq 3$. Thus $\mathbb{Z}_2 = \pi_1(\mathrm{SO}(3)) = \pi_1(\mathrm{SO}(n))$ for $n \geq 4$ which proves our claim.

(b) The proof is similar. $\mathrm{U}(n+1)$, as well as $\mathrm{SU}(n+1)$, acts transitively on $\mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$ with isotropy $\mathrm{U}(n)$ resp. $\mathrm{SU}(n)$. From the long homotopy sequence it follows that $\pi_1(\mathrm{U}(n)) \to \pi_1(\mathrm{U}(n+1))$ is an isomorphism for $n \geq 1$ and similarly for $\mathrm{SU}(n)$. Since $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$ and $\pi_1(\mathrm{SU}(1)) = 0$, the claim follows.

(c) Similarly, $\mathrm{Sp}(n+1)$ acts transitively on $\mathbb{S}^{4n+3} \subset \mathbb{H}^{n+1}$ with isotropy $\mathrm{Sp}(n)$ and $\pi_1(\mathrm{Sp}(1)) = \pi_1(\mathbb{S}^3)) = 0$. $\square$

The polar decomposition theorems imply

pi1class

Corollary 2.18

(a) $\pi_1(\mathrm{GL}^+(n,\mathbb{R})) = \pi_1(\mathrm{SL}(n,\mathbb{R})) = \mathbb{Z}_2$ *for* $n \geq 3$ *and* $\mathbb{Z}$ *for* $n = 2$.
(b) $\pi_1(\mathrm{GL}(n,\mathbb{C})) = \mathbb{Z}$ *and* $\pi_1(\mathrm{SL}(n,\mathbb{C})) = 0$ *for* $n \geq 2$.
(c) $\pi_1(\mathrm{GL}(n,\mathbb{H})) = \pi_1(\mathrm{SL}(n,\mathbb{H})) = 0$ *for* $n \geq 2$.
(d) $\pi_1(\mathrm{Sp}(n,\mathbb{R})) = \mathbb{Z}$ *for* $n \geq 1$.

The fact that $\pi_1(\mathrm{Sp}(n,\mathbb{R})) = \mathbb{Z}$ is particularly important in symplectic geometry since it gives rise to the Maslov index of a loop of symplectic matrices.

As we saw in Chapter 1, coverings are closely related to the center of a Lie group and we will now compute the center of the classical Lie groups. In any of the examples we saw so far, they all consist of diagonal matrices. This is in fact true in many cases. One easily shows:

$$\begin{gathered} Z(\mathrm{GL}(n,\mathbb{R})) = \mathbb{R}^*,\ Z(\mathrm{GL}(n,\mathbb{C})) = \mathbb{C}^*,\ Z(\mathrm{Sp}(n,\mathbb{R})) = Z_2 \\ Z(\mathrm{U}(n)) = \mathrm{S}^1,\ Z(\mathrm{SU}(n)) = Z(\mathrm{SL}(n,\mathbb{C})) == \mathbb{Z}_n,\ Z(\mathrm{Sp}(n)) = \mathbb{Z}_2 \\ Z(\mathrm{O}(n)) = Z_2,\ Z(\mathrm{SL}(n,\mathbb{R})) = Z(\mathrm{SO}(n)) = Z_2 \text{ if } n \text{ even, and } \{\mathrm{Id}\} \text{ if } n \text{ odd.} \end{gathered} \tag{2.19}$$

There are some explicit covers that are often useful.

2foldcovers1

Proposition 2.20

(a) $\mathrm{Sp}(1)$ *is a two-fold cover of* $\mathrm{SO}(3) \simeq \mathbb{RP}^3$.
(b) $\mathrm{Sp}(1) \times \mathrm{Sp}(1)$ *is a two-fold cover of* $\mathrm{SO}(4)$.

Proof (a) We can regard the adjoint representation of $\mathrm{Sp}(1) = \{q \in \mathbb{H} \mid |q| = 1\}$ as the two fold cover:

$$\phi\colon \mathrm{Sp}(1) \to \mathrm{SO}(3) \;:\; q \to \{v \to qv\bar{q}\} \in \mathrm{SO}(\mathrm{Im}\,\mathbb{H}) \simeq \mathrm{SO}(3).$$

Indeed, notice that $v \to qv\bar{q}$ is an isometry of $\mathbb{H}$ since $|qv\bar{q}| = |v|$. Furthermore $\phi(q)(1) = 1$ and hence $\phi(q)$ preserves $(\mathbb{R}\cdot 1)^\perp = \mathrm{Im}\,\mathbb{H}$ and lies in $\mathrm{SO}(3)$ since $\mathrm{Sp}(1)$ is connected. The center of $\mathrm{Sp}(1)$ is clearly $\{\pm 1\}$ and thus $\ker\phi = \{\pm 1\}$. Since both groups are 3 dimensional, ϕ a covering. This implies that $\mathrm{SO}(3) = \mathrm{Sp}(1)/\{\pm 1\}$ which is diffeomorphic to $\mathbb{S}^3/\{v \to -v\} = \mathbb{RP}^3$.

(b) Similar, we have a cover

$$\psi\colon \mathrm{Sp}(1)\times\mathrm{Sp}(1) \to \mathrm{SO}(4) \;:\; (q,r) \to \{v \to qv\bar{r}\} \in \mathrm{SO}(\mathbb{H}) \simeq \mathrm{SO}(4)$$

One easily sees that $\ker\psi = \{\pm(1,1)\}$ and hence ψ is a 2-fold cover as well. □

Somewhat more difficult are the following 2-fold covers:

2foldcovers2

Proposition 2.21

(a) $\mathrm{SL}(4,\mathbb{C})$ *is a two-fold cover of* $\mathrm{SO}(6,\mathbb{C})$.
(a) $\mathrm{Sp}(2)$ *is a two-fold cover of* $\mathrm{SO}(5)$.
(b) $\mathrm{SU}(4)$ *is a two-fold cover of* $\mathrm{SO}(6)$.

Proof (a) Consider $\mathbb{C}^4$ with the standard hermitian inner product $\langle\cdot,\cdot\rangle$. It induces an hermitian inner product on $\wedge^2\mathbb{C}^4 \simeq \mathbb{C}^6$ given by

$$\langle v_1\wedge v_2, w_1\wedge w_2\rangle := \det(\langle v_i, w_j\rangle_{i,j=1,2}).$$

If $A \in \mathrm{GL}(4,\mathbb{C})$, we define the linear map

$$\wedge^2 A : \wedge^2\mathbb{C}^4 \to \wedge^2\mathbb{C}^4 \;:\; \wedge^2 A(v\wedge w) := (Av)\wedge(Aw).$$

If $A \in \mathrm{U}(4)$, then $\wedge^2 A \in \mathrm{U}(6)$.

Next, we consider the bilinear form α on $\wedge^2\mathbb{C}^4$ given by

$$\alpha : \wedge^2\mathbb{C}^4 \times \wedge^2\mathbb{C}^4 \to \wedge^4\mathbb{C}^4 \simeq \mathbb{C}, :\; (u,v) \to u\wedge v.$$

α is symmetric since $v\wedge w = (-1)^{\deg v \deg w} w\wedge v$. One also easily sees that it is non-degenerate and thus the matrices that preserve α is $\mathrm{SO}(6,\mathbb{C})$. If $A \in \mathrm{SL}(4,\mathbb{C})$, then

$$\alpha((\wedge^2 A)u, (\wedge^2 A)v) \;=\; (\wedge^2 A)u\wedge(\wedge^2 A)v = (\wedge^4 A)(u\wedge v)$$

$$= \quad \det(A)(u \wedge v) = u \wedge v = \alpha(u, v)$$

so $\wedge^2 A$ preserves α. This defines a map

$$\psi : \mathrm{SL}(4, \mathbb{C}) \to \mathrm{SO}(6, \mathbb{C}) \ , \ A \to \wedge^2 A,$$

which is a homomorphism since $\wedge^2(AB)(v \wedge w) = ABv \wedge ABw = (\wedge^2 A)(Bv \wedge Bw) = (\wedge^2 A)(\wedge^2 B)(v \wedge w)$. If $A \in \ker \psi$, then $Au \wedge Av = u \wedge v$ for all u, v, which implies that A preserve planes and hence lines as well. Thus $Ae_i = \pm 1 e_i$ and one easily sees that this can only be if $A = \pm \mathrm{Id}$. Thus $\ker \psi = \{\pm \mathrm{Id}\}$ and hence ψ is a 2-fold cover since both have the same dimension.

(b) If $A \in \mathrm{SU}(4) \subset \mathrm{SL}(4, \mathbb{C})$, then $B := \psi(A)$ lies in $\mathrm{U}(6)$, and since it also preserves α, in $\mathrm{SO}(6, \mathbb{C})$ as well. Thus $\bar{B}B^T = \mathrm{Id}$ and $BB^T = \mathrm{Id}$ and hence $\bar{B} = B$ which means that $B \in \mathrm{SO}(6)$. Thus ψ also induces the desired 2-fold cover from $\mathrm{SU}(4)$ to $\mathrm{SO}(6)$.

(c) Now let $A \in \mathrm{Sp}(2)$. Recall that $\mathrm{Sp}(2) = \mathrm{SU}(4) \cap \mathrm{Sp}(2, \mathbb{C})$ and let ω be the symplectic 2-form on $\mathbb{C}^4$ that defines $\mathrm{Sp}(2, \mathbb{C})$. It can be regarded as a linear map $\tilde{\omega}\colon\ \wedge^2 \mathbb{C}^4 \to \mathbb{C}, :\ v \wedge w \to \omega(v, w)$. Since A preserves ω, we also have $\tilde{\omega}(\wedge^2 A(v \wedge w)) = \omega(Av, Aw) = \omega(v, w) = \tilde{\omega}(v \wedge w)$ and thus $\wedge^2 A$ preserves $\ker \tilde{\omega}$. Thus $\wedge^2 A \in \mathrm{SO}(6)$ has a fixed vector, which implies that $\wedge^2 A \in \mathrm{SO}(5)$. Hence ψ also induces the desired 2-fold cover from $\mathrm{Sp}(2)$ to $\mathrm{SO}(5)$. $\square$

One can also make the coverings in Proposition 2.21 more explicit by observing the following. If e_i is the standard orthonormal basis with respect to the hermitian inner product on $\mathbb{C}^4$, then $e_i \wedge e_j,\ i < j$ is an orthonormal basis of $\wedge^2 \mathbb{C}^4$. One easily see that the six 2-forms

$$\frac{1}{\sqrt{\pm 2}}(e_1 \wedge e_2 \pm e_3 \wedge e_4),\ \ \frac{1}{\sqrt{\pm 2}}(e_1 \wedge e_3 \pm e_2 \wedge e_4),\ \ \frac{1}{\sqrt{\pm 2}}(e_1 \wedge e_4 \pm e_2 \wedge e_3)$$

form an orthonormal basis with respect to α, and with respect to $\langle \cdot, \cdot \rangle$ as well. Let B be the matrix of $\wedge^2 A$ in this basis. If $A \in \mathrm{SL}(4, \mathbb{C})$, then B lies in $\mathrm{SO}(6, \mathbb{C})$. If $A \in \mathrm{SU}(4, \mathbb{C})$, B is a real matrix and is hence an element in $\mathrm{SO}(6)$. One easily shows that the kernel of $\tilde{\omega}$ is spanned by the above six vectors with $\frac{1}{\sqrt{-2}}(e_1 \wedge e_3 - e_2 \wedge e_4)$ removed. Hence $\wedge^2 A,\ A \in \mathrm{Sp}(2)$ preserves this 5 dimensional space and thus B lies in $\mathrm{SO}(5)$. This enables one, if desired, to write down explicitly the matrix $\wedge^2 A$.

Notice that in Proposition 2.20 and Proposition 2.21, the groups on the left are connected and simply connected and hence are the universal covers of the groups on the right.

Since $\pi_1(\mathrm{SO}(n)) = \mathbb{Z}_2$ for $n \geq 3$, there exists a 2-fold universal cover which is again a Lie group. These are the **spinor groups** $\mathrm{Spin}(n)$. By the above Proposition 2.20 and Proposition 2.21, we have

$$\mathrm{Spin}(3) = \mathrm{Sp}(1), \mathrm{Spin}(4) = \mathrm{Sp}(1) \times \mathrm{Sp}(1), \mathrm{Spin}(5) = \mathrm{Sp}(2), \mathrm{Spin}(6) = \mathrm{SU}(4)$$

The higher spin groups have no simple description as above. We will come back to them later, and will also see how to represent them as matrix groups.

Another important but non-trivial questions about the spin groups is what their center is. From $Z(\mathrm{SO}(2n+1)) = \{e\}$ and $Z(\mathrm{SO}(2n)) = \mathbb{Z}_2$ it follows that $Z(\mathrm{Spin}(2n+1)) = \mathbb{Z}_2$ and $|Z(\mathrm{Spin}(2n))| = 4$. But notice that $Z(\mathrm{Spin}(4)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and $Z(\mathrm{Spin}(6)) = \mathbb{Z}_4$. As we will see:

$$Z(\mathrm{Spin}(n)) \simeq \begin{cases} \mathbb{Z}_2 & n = 2k+1 \\ \mathbb{Z}_4 & n = 4k+2 \\ \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n = 4k \end{cases}$$

This will in particular imply that besides $\mathrm{SO}(4n)$ there is another Lie group, called $\mathrm{SO}'(4n)$, whose 2-fold universal cover is $\mathrm{Spin}(4n)$ as well.

matrix **Example 2.22** We now apply the above results to show that there are Lie groups which are not matrix groups, i.e. are not Lie subgroups of $\mathrm{GL}(n, \mathbb{R})$ or $\mathrm{GL}(n, \mathbb{C})$ for any n.

We saw above that $\mathrm{SL}(n, \mathbb{R})$ is not simply connected and we let G be the 2-fold universal cover of $\mathrm{SL}(n, \mathbb{R})$. We claim that G cannot be a matrix group. So assume that there exists an injective Lie group homomorphism $\phi \colon G \to \mathrm{GL}(n, \mathbb{C})$ for some n. Recall that $\mathfrak{g} \otimes \mathbb{C} = \mathfrak{sl}(n, \mathbb{C})$ and that $\mathrm{SL}(n, \mathbb{C})$ is simply connected. Thus the Lie algebra homomorphism $d\phi \otimes \mathbb{C} \colon \mathfrak{sl}(n, \mathbb{C}) \to \mathfrak{gl}(n, \mathbb{C})$ can be integrated to a homomorphism $\psi \colon \mathrm{SL}(n, \mathbb{C}) \to \mathrm{GL}(n, \mathbb{C})$ with $d\psi = d\phi \otimes \mathbb{C}$. In the following diagram

$$\begin{tikzcd} G = \widetilde{\mathrm{SL}(n, \mathbb{R})} \arrow[r, "\phi"] \arrow[d, "\pi"] & \mathrm{GL}(n, \mathbb{C}) \\ \mathrm{SL}(n, \mathbb{R}) \arrow[r, "i"] & \mathrm{SL}(n, \mathbb{C}) \arrow[u, "\psi"'] \end{tikzcd}$$

π denotes the 2-fold cover and i the inclusion. The above diagram is commutative since all groups are connected, and the diagram is by definition commutative on the level of Lie algebras. But this is a contradiction since ϕ is injective but $\pi \circ i \circ \psi$ is not since π is a 2-fold cover.

Exercises 2.23

(1) Determine all Lie groups with Lie algebra $\mathfrak{o}(n)$ for $n \leq 6$ up to isomorphism. In particular show that $\mathrm{SO}(4)$ and $\mathrm{S}^3 \times \mathrm{SO}(3)$ are not isomorphic (not even as groups).

(2) Determine all Lie groups with Lie algebra $\mathfrak{u}(n)$ up to isomorphism. How are they related to $\mathrm{U}(n)$?

(3) Find a 2-fold cover $\mathrm{SL}(2,\mathbb{R}) \to \mathrm{SO}^+(2,1)$ and $\mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(3,1)$. Thus the first 2 Lorenz groups are $\mathrm{SO}^+(2,1) \simeq \mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{SO}^+(3,1) \simeq \mathrm{PSL}(2,\mathbb{C})$. They can also be regarded as the isometry group of 2 and 3 dimensional hyperbolic space.

(3) Show that there are infinitely many covers of $\mathrm{Sp}(n,\mathbb{R})$ which are not matrix groups.

3
Basic Structure Theorems

Although it will not be the focus of what we will cover in the future, we will discuss here the basics of what one should know about nilpotent and solvable groups. We do not include the proofs of all results, but present the basic ideas.

Nilpotent and Solvable Lie algebras

Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{R}$ or $\mathbb{C}$. We define inductively:

$$\begin{aligned} \mathfrak{g}^0 &= \mathfrak{g}; \qquad & \mathfrak{g}^k &= [\mathfrak{g}, \mathfrak{g}^{k-1}] \\ \mathfrak{g}_0 &= \mathfrak{g}; \qquad & \mathfrak{g}_k &= [\mathfrak{g}_{k-1}, \mathfrak{g}_{k-1}] \end{aligned}$$

where, for any two linear subspace $a, b \subset \mathfrak{g}$, $[a, b]$ refers to the subalgebra *spanned* by the Lie brackets $[u, v]$, $u \in a, v \in b$. The first is usually called the lower cental series and the second the derived series. Clearly both $\{\mathfrak{g}^k\}$ and $\{\mathfrak{g}_k\}$ are decreasing sequences of subalgebras.

Definition 3.1 *A Lie algebra* $\mathfrak{g}$ *is called* k**-step nilpotent** *if* $\mathfrak{g}^k = 0$ *and* $\mathfrak{g}^{k-1} \neq 0$, *i.e., if the sequence* $\{\mathfrak{g}^k\}$ *terminates.* $\mathfrak{g}$ *is called* k **step solvable** *if* $\mathfrak{g}_k = 0$ *and* $\mathfrak{g}_{k-1} \neq 0$.

A connected Lie group G is called nilpotent (solvable) if its Lie algebra $\mathfrak{g}$ is nilpotent (solvable). The basic properties for both types of Lie algebras is given by the following Proposition.

solvbasic

Proposition 3.2 *Let $\mathfrak{g}$ be a Lie algebra which is k step nilpotent resp. k step solvable. The following are some basic facts:*

(a) $\mathfrak{g}_i \subset \mathfrak{g}^i$ for all i. In particular, $\mathfrak{g}$ is solvable if it is nilpotent.
(b) $\mathfrak{g}^i$ and $\mathfrak{g}_i$ are ideals in $\mathfrak{g}$.
(c) If $\mathfrak{g}$ is nilpotent, then $\left\{\mathfrak{g}^{k-1}\right\}$ lies in the center. If $\mathfrak{g}$ is solvable, $\{\mathfrak{g}_{k-1}\}$ is abelian.
(d) A subalgebra of a nilpotent (solvable) Lie algebra is nilpotent (solvable).
(e) If $\mathfrak{a} \subset \mathfrak{b}$ is an ideal of the Lie algebra $\mathfrak{b}$, we let $\mathfrak{a}/\mathfrak{b}$ be the quotient algebra. If $\mathfrak{a}$ is solvable (nilpotent), $\mathfrak{a}/\mathfrak{b}$ is solvable (nilpotent).

solvexact

(f) Let

$$0 \to \mathfrak{a} \to \mathfrak{b} \to \mathfrak{c} \to 0$$

be an exact sequence of Lie algebras. If $\mathfrak{a}$ and $\mathfrak{c}$ are both solvable, then $\mathfrak{b}$ is solvable. In general the corresponding statement is not true for for nilpotent Lie algebras.
(g) Let $\mathfrak{a}$, $\mathfrak{b}$ be solvable (nilpotent) ideals, then the vector sum $\mathfrak{a}+\mathfrak{b}$ is a solvable (nilpotent) ideal.

Proof We only present the proof of some of them, since most easily follow by using the Jacobi identity and induction on i.

(b) The Jacobi identity implies that $\mathfrak{g}^i$ is an ideal in $\mathfrak{g}$, and similarly $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-1}$. To see that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}$, one shows by induction on k that $\mathfrak{g}_i$ is an ideal in $\mathfrak{g}_{i-k}$.

(f) Let $\phi\colon \mathfrak{a} \to \mathfrak{b}$ and $\psi\colon \mathfrak{b} \to \mathfrak{c}$ be the Lie algebra homomorphisms in the exact sequence. Clearly, $\psi(\mathfrak{b}_k) \subset \mathfrak{c}_k$. Since $\mathfrak{c}_k = 0$ for some k, exactness implies that $\mathfrak{b}_k \subset \mathrm{Im}(\mathfrak{a}_k)$ and since $\mathfrak{a}_m = 0$ for some m, we also have $\mathfrak{b}_m = 0$.

(g) Consider the exact sequence of Lie algebras

$$0 \to \mathfrak{a} \to \mathfrak{a}+\mathfrak{b} \to (\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \to 0.$$

Since $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a} \simeq \mathfrak{b}/(\mathfrak{a}\cap\mathfrak{b})$, and since $\mathfrak{b}$, $\mathfrak{a}\cap\mathfrak{b}$ are solvable ideals, $(\mathfrak{a}+\mathfrak{b})/\mathfrak{a}$ is a solvable ideal as well. Thus (f) implies that $\mathfrak{a}+\mathfrak{b}$ is a solvable ideal.

The nilpotent case follows by showing that $(\mathfrak{a}+\mathfrak{b})^k \subset \sum_i \mathfrak{a}^i \cap \mathfrak{b}^{k-i}$ via induction. $\square$

Example 3.3 a) The set of $n \times n$ upper-triangular matrices is an n-step solvable Lie subalgebra of $\mathfrak{gl}(n,\mathbb{R})$, and the set of $n \times n$ upper-triangular matrices with zero entries on the diagonal is an $(n-1)\times(n-1)$-step nilpotent subalgebra of $\mathfrak{gl}(n,\mathbb{R})$. Thus any subalgebra of each is solvable resp. nilpotent. We will shortly see that any solvable (nilpotent) Lie algebra is a subalgebra of such upper triangular matrices.

b) Recall that an affine transformation of $\mathbb{R}$ is a map $f:\mathbb{R}\to\mathbb{R}$ so $f(x)=ax+b$ for $a\neq 0$. The group of affine transformations is isomorphic to the Lie group consisting of matrices

$$\left\{\begin{pmatrix} a & b \\ 0 & 1\end{pmatrix} \mid a\neq 0\right\}$$

The Lie algebra $\mathfrak{g}$ of this Lie group is the algebra generated by $X=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$ and $Y=\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$ Because these are upper-triangular matrices, $\mathfrak{g}$ is solvable.

However, $[X,Y]=Y$, so $\mathfrak{g}$ is not nilpotent. It also provides an example which shows that Proposition 3.2(f) is not true in the nilpotent case: Consider the exact sequence of Lie algebras

$$0\to\mathbb{R}\cdot X\to\mathfrak{g}\to\mathbb{R}\cdot Y\to 0$$

Both $\mathbb{R}\cdot X$ and $\mathbb{R}\cdot Y$ are nilpotent but $\mathfrak{g}$ is not.

Since the sum of solvable ideals is solvable, we can make the following definition.

Definition 3.4

(a) *Given a Lie algebra* $\mathfrak{g}$*, the* **radical** *of* $\mathfrak{g}$*, denoted by* $\mathrm{rad}(\mathfrak{g})$*, is the unique maximal solvable ideal.*

(b) $\mathfrak{g}$ *is called* **semisimple** *if* $\mathfrak{g}$ *has no solvable ideals, i.e.* $\mathrm{rad}(\mathfrak{g})=0$. *Equivalently,* $\mathfrak{g}$ *is semisimple if it has has no abelian ideal.*

(c) $\mathfrak{g}$ *is called* **simple** *if the only ideals are* $\{0\}$ *and* $\mathfrak{g}$*, and* $\dim\mathfrak{g}>1$.

The assumption that $\dim\mathfrak{g}>1$ guarantees that a simple Lie algebra has trivial center. We first observe that

Proposition 3.5 *For any Lie algebra* $\mathfrak{g}$*, we have that* $\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ *is semisimple.*

Proof Assume $\mathfrak{a}\subset\mathfrak{g}/\mathrm{rad}(\mathfrak{g})$ is a solvable ideal. Since the quotient map

$\pi : \mathfrak{g} \to \mathfrak{g}/\operatorname{rad}(\mathfrak{g})$ is a homomorphism, $\pi([\pi^{-1}(\mathfrak{a}), \mathfrak{g}]) = [\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$ and thus $\pi^{-1}(\mathfrak{a})$ is an ideal in $\mathfrak{g}$. From the exact sequence

$$0 \to rad(\mathfrak{g}) \to \pi^{-1}(\mathfrak{a}) \to \mathfrak{a} \to 0$$

it follows that $\pi^{-1}(\mathfrak{a})$ is solvable, and hence by definition $\pi^{-1}(\mathfrak{a}) \subset rad(\mathfrak{g})$. Thus $\mathfrak{a}$ must be trivial. □

We mention, without proof the following theorem, which sometimes reduces a proof to the semisimple and the solvable case.

[Levi-Malcev

Theorem 3.6 *[Levi-Malcev] Given a Lie algebra $\mathfrak{g}$, there exists a semisimple subalgebra $\mathfrak{h} \subset \mathfrak{g}$, unique up to an inner automorphism, such that $\mathfrak{h} + \operatorname{rad}(\mathfrak{g}) = \mathfrak{g}$, and $\mathfrak{h} \cap \operatorname{rad}(\mathfrak{g}) = 0$.*

We now discuss the main result about nilpotent Lie algebras.

Engel

Theorem 3.7 [Engel] *Let $\mathfrak{g} \in \mathfrak{gl}(V)$ be a Lie subalgebra whose element are nilpotent, i.e., for each $A \in \mathfrak{g}$ there exists a k such that $A^k = 0$. Then there exists a basis, such that $\mathfrak{g}$ lies in the subalgebra of upper triangular matrices with zeros along the diagonal entries, and hence is nilpotent.*

Proof This will easily follow from:

nilplem

Lemma 3.8 *Let $\mathfrak{g} \in \mathfrak{gl}(V)$ be a Lie subalgebra of nilpotent elements. Then there exists a vector v such that $Av = 0$ for every $A \in \mathfrak{g}$.*

Proof Recall that for $X, Y \in \mathfrak{g}$, we have $[X, Y] = XY - YX$. Thus $A^k = 0$ implies that $(\mathrm{ad}_X)^{2k} = 0$ since e.g. $(\mathrm{ad}_X)^2(Y) = X^2Y - 2XYX + Y^2$ and the powers of X or Y accumulate on the left or the right. The rest of the proof is by induction on the dimension of $\mathfrak{g}$, the case $\dim \mathfrak{g} = 1$ being trivial. Now suppose the lemma holds in dimension $< n$, and suppose $\dim \mathfrak{g} = n$. Let $\mathfrak{h} \in \mathfrak{g}$ be a subalgebra of maximal dimension. Since $\mathrm{ad}_X(\mathfrak{h}) \subset \mathfrak{h}$ for $X \in \mathfrak{h}$, we can define $\mathfrak{h}' = \{\mathrm{ad}_X : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h} \mid X \in \mathfrak{h}\} \subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ as a Lie subalgebra. By the above observation, every element of $\mathfrak{h}'$ is nilpotent and since $\dim \mathfrak{h}' < \dim \mathfrak{g}$, by induction, there exists an element $\bar{A} \in \mathfrak{g}/\mathfrak{h}$ such that $L(\bar{A}) = 0$ for every $L \in \mathfrak{h}'$ and thus an element $A \in \mathfrak{g}$ such that $\mathrm{ad}_X(A) \in \mathfrak{h}$ for all $X \in \mathfrak{h}$. In particular $\mathfrak{h} + A$ is a subalgebra of $\mathfrak{g}$ and by maximality $\mathfrak{h} + A = \mathfrak{g}$.
Now let $W := \{w \in V \mid Xw = 0 \text{ for all } X \in \mathfrak{h}\}$, which is non-zero by induction hypothesis. A takes W to W since $0 = [X, A]w = XAw - AXw =$

XAw. But $A_{|W}$ is also nilpotent, and hence there exists a $v \in W$ with $Av = 0$. This vector v is clearly the desired vector. $\square$

We now return to the proof of Engel's Theorem. We inductively choose a basis $e_1, \dots, e_n$ with $Ae_1 = 0$ and $A(V_i) \subset V_{i-1}$, $i = 2, \dots, n$, where $V_i := \operatorname{span}\{e_1, \dots, e_i\}$. The existence of e_1 follows from the Lemma. Now assume that we defined $V_1, \dots V_k$ such that $A(V_i) \subset V_{i-1}$, $i = 1, \dots, k$. Then A induces a linear map $\bar{A}\colon V/V_k \to V/V_k$ and $\bar{A}$ is again nilpotent. Thus there exists a vector $u \in V/V_k$ with $\bar{A}u = 0$ for all $A \in \mathfrak{g}$. If e_k is a preimage of u in V, we define $V_{k+1} = \operatorname{span}\{e_1, \dots, e_{k+1}\}$ and by definition, $A(V_{k+1}) \subset V_k$. This basis clearly has the desired properties. $\square$

We now derive several consequences.

uniquedph **Corollary 3.9** ad_X *is nilpotent for all* $X \in \mathfrak{g}$, *iff* $\mathfrak{g}$ *is nilpotent.*

Proof For one direction apply Engel's Theorem to $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$. For the other observe that if $\mathfrak{g}$ is nilpotent, then $\operatorname{ad}_X(\mathfrak{g}^k) \subset \mathfrak{g}^{k+1}$ and hence ad_X is nilpotent. $\square$

uniquedph **Corollary 3.10** *A nilpotent Lie algebra* $\mathfrak{g}$ *is isomorphic to a subalgebra of the nilpotent Lie algebra in* $\mathfrak{gl}(n, \mathbb{R})$ *of upper triangular matrices with zeros along the diagonals.*

Proof Apply Ado's Theorem..... $\square$

nilexp **Theorem 3.11**

(a) *If* G *is a connected Lie group with nilpotent Lie algebra* G, *then* $\exp : \mathfrak{g} \to G$ *is a covering. In particular, if* $\pi_1(G) = 0$, *then* G *is diffeomorphic to* $\mathbb{R}^n$;
(b) *If* $G \subset \mathrm{GL}(V)$ *is such that* $\mathfrak{g}$ *has only nilpotent elements, then* $\exp : \mathfrak{g} \to G$ *is a diffeomorphism.*

Proof Proof to be added... $\square$

Corollary 3.12 *If* G *is a Lie group whose Lie algebra has only nilpotent elements, and* $\pi_1(G) \neq 0$, *then* G *is NOT a matrix group.*

This gives rise to many examples which are not matrix groups since a nilpotent Lie algebra always has a nontrivial center and hence there are many discrete subgroups of $Z(G)$.

The basic structure theorem for solvable Lie algebras is Lie's Theorem. Its proof is again by induction, in fact very similar to Engel's Theorem, and we will thus omit it here, but derive several important consequences.

Lie

Proposition 3.13 [Lie] *Let $\mathfrak{g} \in \mathfrak{gl}(\mathbb{C}^n)$ be a solvable Lie subalgebra. Then there exists a $\lambda \in \mathfrak{g}^*$ and a vector v such that $Av = \lambda(A)v$ for every $A \in \mathfrak{g}$.*

Unlike Engel's Theorem, this holds only over the complex numbers. As before, this has the following consequence:

Corollary 3.14 *If $\mathfrak{g} \in \mathfrak{gl}(V)$ be a solvable Lie algebra, then there exists a basis such that $\mathfrak{g}$ lies in the subalgebra of upper triangular matrices*

Combining this with Ado's Theorem one has

Corollary 3.15 *Every solvable Lie algebra is a subalgebra of the Lie algebra of upper triangular matrices in $\mathfrak{gl}(n, \mathbb{C})$.*

Corollary 3.16 *$\mathfrak{g}$ is solvable iff $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof If $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent, clearly $\mathfrak{g}$ is solvable. If $\mathfrak{g}$ is solvable, we can embed it as a subalgebra of the upper triangular matrixs, whose commutator has $0's$ along the diagonal and hence is nilpotent. □

The following result we will also present without proof.

Theorem 3.17 [Cartan's Second Criterion] *$\mathfrak{g}$ is solvable iff $B = 0$ on $[\mathfrak{g}, \mathfrak{g}]$.*

In one direction this is clear since $\mathfrak{g}$ solvable implies $[\mathfrak{g}, \mathfrak{g}]$ nilpotent and the Killing form of a nilpotent Lie algebra is 0 since ad_X is nilpotent. Notice that Cartan's Second Criterion implies that a Lie algebra with $B = 0$ must be solvable.

Exercises 3.18

(a) Give an example of a Lie algebra which is not nilpotent, but whose Killing form vanishes.
(b)
(c)

Semisimple Lie algebras

Recall that $\mathfrak{g}$ is called semisimple if it has no solvable ideals, or equivalently no abelian ideals. In particular the center of $\mathfrak{g}$ is trivial. A more useful characterization is:

semisimplenondegB

Theorem 3.19 [Cartan's Second Criterion] *$\mathfrak{g}$ is semisimple iff B is non-degenerate.*

Proof Recall that $\ker B$ is an ideal in $\mathfrak{g}$ and that $B_{\mathfrak{a}} = B_{\mathfrak{g}|\mathfrak{a}}$ for an ideal $\mathfrak{a}$ in $\mathfrak{g}$. Thus the Killing form of $\ker B$ is trivial which, by Cartan's first criterium implies that $\ker B$ is solvable. Hence, if $\mathfrak{g}$ is semisimple, B is non-degenerate.

If, on the other hand, $\mathfrak{g}$ is not semisimple, let $\mathfrak{a} \subset \mathfrak{g}$ be an abelian ideal. Choose a basis $e_1, \ldots, e_k, f_1, \ldots f_l$ such that $e_1, \ldots, e_k$ is a basis of $\mathfrak{a}$. Then for any $X \in \mathfrak{a}$ and $Y \in \mathfrak{g}$ we have $\operatorname{ad}_Y \operatorname{ad}_X(e_i) = 0$ and $\operatorname{ad}_Y \operatorname{ad}_X(f_i) \in \operatorname{span}\{e_1, \ldots e_k\}$ and thus $B(X,Y) = \operatorname{tr}(\operatorname{ad}_Y \operatorname{ad}_X) = 0$. This implies that $\mathfrak{a} \subset \ker B$ and hence B is degenerate. $\square$

Notice that we actually proved two general statements: For any Lie algebra $\mathfrak{g}$, $\ker B$ is a solvable ideal in $\mathfrak{g}$ and $\mathfrak{a} \subset \ker B$ for any abelian ideal $\mathfrak{a}$ in $\mathfrak{g}$.

Example 3.20 The Lie algebras $\mathfrak{sl}(n,\mathbb{R}), \mathfrak{sl}(n,\mathbb{C}), \mathfrak{su}(n), \mathfrak{so}(n), \mathfrak{sp}(n,\mathbb{R}), \mathfrak{sp}(n)$ and $\mathfrak{o}(p,q)$ are all semisimple. To see this, we first observe that $B(X,Y) = \alpha \operatorname{tr}(XY)$, where α is a non-zero constant that depends on the Lie algebra. We saw this for some of the above Lie algebras in the exercises, and leave it for the reader to check it for the others. Furthermore, one easily sees that $X^T \in \mathfrak{g}$ if $X \in \mathfrak{g}$. We thus have

$$B(X, X^T) = \operatorname{tr}(XX^T) = \alpha \sum_{i,j} x_{ij}^2, \qquad X = (x_{ij}),$$

and thus B is non-degenerate.

The basic structure theorem for semisimple Lie algebras is:

semisimpleideals

Theorem 3.21 *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then:*

(a) $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_k$ *where $\mathfrak{g}_i$ are simple ideals.*
(b) if $\mathfrak{a} \in \mathfrak{g}$ is an ideal, then $\mathfrak{a} = \bigoplus_{j \in I} \mathfrak{g}_j$ for some $I \subset \{1, \ldots, k\}$. In particular, the decomposition in (a) is unique up to order.
(c) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.
(d) $\operatorname{Aut}(\mathfrak{g}) / \operatorname{Int}(\mathfrak{g})$ *is discrete, or equivalent every derivation is inner.*

Proof Part (a) follows from the following 3 claims.
1. If $\mathfrak{a} \subset \mathfrak{g}$ is an ideal, then $\mathfrak{a}^\perp = \{X \in \mathfrak{g} \mid B(X,Y) = 0 \text{ for all } Y \in \mathfrak{a}\}$ is an ideal.

Indeed, if $X \in \mathfrak{a}^\perp$, $Z \in \mathfrak{a}$, $Y \in \mathfrak{g}$, then $B([Y,X],Z)- = B([Y,Z],X) = 0$ where we have used that ad_Y is skew symmetric w.r.t. B.
2. If $\mathfrak{a}$ is an ideal, then $\mathfrak{a} \cap \mathfrak{a}^\perp = 0$.

First observe that for $X, Y \in \mathfrak{a} \cap \mathfrak{a}^\perp$, $Z \in \mathfrak{g}$, we have $B([X,Y],Z) = -B([X,Z],Y) = 0$ since $[X,Z] \in \mathfrak{a}$ and $Y \in \mathfrak{a}^\perp$. Thus $\mathfrak{a} \cap \mathfrak{a}^\perp$ is an abelian ideal which by semisimplicity must be 0.
3. $\dim \mathfrak{a} + \dim \mathfrak{a}^\perp = \dim \mathfrak{g}$.

This is a general fact about non-degenerate bilinear forms. To see this, let $e_1, \dots, e_k$ be a basis of $\mathfrak{a}$. The equations $B(e_i, X) = 0$, $i = 1, \dots, k$ are linearly independent since otherwise $0 = \sum \lambda_i B(e_i, X) = B(\sum e_i, X)$ for all $X \in \mathfrak{g}$ implies that B is degenerate. Thus the solution space has dimension $\dim \mathfrak{g} - \dim \mathfrak{a}$.

Putting all three together, we have $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$, a direct sum of 2 ideals. Continuing in this fashion, we write $\mathfrak{g}$ as a sum of simple ideals.

For (b) it is sufficient to show that if $\mathfrak{b}$ is a simple ideal in $\mathfrak{g}$, then $\mathfrak{b} = \mathfrak{a}_i$ for some i. But for a fixed j we have that $\mathfrak{b} \cap \mathfrak{a}_j$ is in ideal in $\mathfrak{a}_j$ and thus either $\mathfrak{b} \cap \mathfrak{a}_j = 0$ or $\mathfrak{b} \cap \mathfrak{a}_j = \mathfrak{a}_j$. Thus there must exists an i with $\mathfrak{b} = \mathfrak{a}_i$.

(c) For a simple Lie algebra $\mathfrak{g}$ we have $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ since it is an ideal and cannot be 0. The general claim now follows by combining with (a).

(d) Recall that the Lie algebra of inner derivations $\mathfrak{Int}(\mathfrak{g})$ is an ideal in all derivations $\mathfrak{Der}(\mathfrak{g})$. Also, since the center of a semisimple Lie algebra is 0, $\mathfrak{Int}(\mathfrak{g}) \simeq \mathrm{ad}(\mathfrak{g}) \simeq \mathfrak{g}$ and thus $\mathfrak{Int}(\mathfrak{g})$ is semisimple. Next we observe that part 1.-3. in the proof of (a) also hold if we only assume that $B_{|\mathfrak{a}}$ is non-degenerate (Notice that $B_{|\mathfrak{a}}$ is the Killing form of $\mathfrak{a}$ since $\mathfrak{a}$ is an ideal). Thus $\mathfrak{Der}(\mathfrak{g}) = \mathfrak{Int}(\mathfrak{g}) \oplus (\mathfrak{Int}(\mathfrak{g}))^\perp$. If $D \in (\mathfrak{Int}(\mathfrak{g}))^\perp$ and $\mathrm{ad}_X \in \mathfrak{Int}(\mathfrak{g})$ then $0 = [D, \mathrm{ad}_x] = \mathrm{ad}_{DX}$ and hence $DX \in \mathfrak{z}(\mathfrak{g}) = 0$. This implies that $D = 0$ and hence $\mathfrak{Der}(\mathfrak{g}) = \mathfrak{Int}(\mathfrak{g})$. □

Exercises 3.22

(a) Show that $\mathfrak{g}$ is semisimple (nilpotent, solvable) iff $\mathfrak{g}_{\mathbb{C}}$ is semisimple (nilpotent, solvable).
(b) Show that if $\mathfrak{g}_{\mathbb{C}}$ is simple, then $\mathfrak{g}$ is simple.
(c) If $\mathfrak{g}$ is a Lie algebra such that ad_X is skew symmetric with respect to some non-degenerate bilinear form, show that $\mathfrak{g} \simeq \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{b}$ with $\mathfrak{b}$ semisimple.

Compact Lie algebras

It is convenient to make the following definition.

Definition 3.23 *A Lie algebra is called compact if it is the Lie algebra of a compact Lie group.*

Notice that in this terminology an abelian Lie algebra is compact. The following is one of its basic properties.

biinvariant

Proposition 3.24

(a) *If $\mathfrak{g}$ is a compact Lie algebra, then there exists an inner product on $\mathfrak{g}$ such that ad_X is skew-symmetric for all $X \in \mathfrak{g}$..*

(b) *If G is compact Lie group, then there exists a biinvariant metric on G, i.e. a Riemannian metric such that L_g, R_g act by isometries for all $g \in G$.*

Proof This is a standard averaging procedure. Let G be a compact Lie group with Lie algebra $\mathfrak{g}$. Choose any inner product $\langle \cdot\ ,\ \cdot \rangle_0$ on $\mathfrak{g}$ and define a new inner product on $\mathfrak{g}$:

$$\langle X, Y \rangle = \int_G \langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y \rangle_0\ \omega$$

where ω is a biinvariant volume form, i.e. $L_g^*\omega = R_g^*\omega = \omega$. One can easily see the existence of such a volume form by first choosing a volume form ω_0 on $\mathfrak{g}$ and define it on G by $\omega_g = L_g^*\omega_0$. One now defines a function $f\colon G \to \mathbb{R}$ by $R_g^*\omega = f(g)\omega$. Notice that $f(g)$ is constant since both ω and $R_g^*\omega$ are left invariant. One easily sees that $f(gh) = f(g)f(h)$ and thus $f(G)$ is a compact subgroup of $\mathbb{R}^*$. But this implies that $f(g) = 1$ and hence ω is right invariant.

We now claim that $\mathrm{Ad}(h)$ is an isometry in the new inner product:

$$\begin{aligned}\langle \mathrm{Ad}(h)X, \mathrm{Ad}(h)Y \rangle &= \int_G \langle \mathrm{Ad}(gh)X, \mathrm{Ad}(gh)Y \rangle_0\ \omega \\ &= \int_G \langle \mathrm{Ad}(g)X, \mathrm{Ad}(g)Y \rangle_0\ R_{h^{-1}}^*(\omega) = \langle X, Y \rangle\end{aligned}$$

Since $\mathrm{Ad}(G)$ acts by isometries, and $d\,\mathrm{Ad} = \mathrm{ad}$, it follows that ad_X is skew symmetric. This proves part (a).

For part (b) we choose the inner product on $\mathfrak{g}$ as above and define a metric on G by

$$\langle X, Y\rangle_g = \langle d(L_{g^{-1}})_g X, d(L_{g^{-1}})_g Y\rangle.$$

Then L_g is an isometry by definition, and R_g is one as well, since $d(R_{gh})_e = d(R_h)_g d(R_g)_e$ shows it is sufficient to prove $d(R_g)_e$ is an isometry, which holds since

$$\begin{aligned}\langle d(R_g)_e X, d(R_g)_e Y\rangle_g &= \langle d(L_{g^{-1}})_g d(R_g)_e X, d(L_{g^{-1}})_g d(R_g)_e Y\rangle \\ &= \langle \mathrm{Ad}(g^{-1})X, \mathrm{Ad}(g^{-1})Y\rangle = \langle X, Y\rangle\end{aligned}$$

□

We now prove a basic structure theorem for compact Lie algebras.

compact

Proposition 3.25 *Let $\mathfrak{g}$ be a Lie real algebra.*

(a) $B < 0$ iff $\mathfrak{g}$ is compact with $\mathfrak{z}(\mathfrak{g}) = 0$.
(b) If $\mathfrak{g}$ is compact, then $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ with $[\mathfrak{g}, \mathfrak{g}]$ semisimple.

Proof (a) If $B < 0$, then $\mathfrak{g}$ is semisimple and hence has trivial center. Since elements of $\mathrm{Aut}(\mathfrak{g})$ are isometries of B, and $-B$ is an inner product, $\mathrm{Aut}(\mathfrak{g}) \subset \mathrm{O}(\mathfrak{g})$. Since $\mathrm{Aut}(\mathfrak{g})$ is also closed, it is compact. Since $\mathrm{Aut}(\mathfrak{g})/\operatorname{Int}(\mathfrak{g})$ is finite when $\mathfrak{g}$ is semisimple, $\operatorname{Int}(\mathfrak{g})$ is the identity component of $\mathrm{Aut}(\mathfrak{g})$ and hence compact as well. But since $\mathfrak{z}(\mathfrak{g}) = 0$, the Lie algebra of $\operatorname{Int}(\mathfrak{g})$ is isomorphic to $\mathfrak{g}$. Hence $\operatorname{Int}(\mathfrak{g})$ is the desired compact Lie group with Lie algebra $\mathfrak{g}$.

If $\mathfrak{g}$ is compact, let $\langle \cdot\ ,\ \cdot\rangle$ be an $\mathrm{Ad}(G)$ invariant inner product on $\mathfrak{g}$. Then $B(X, X) = \operatorname{tr}(\mathrm{ad}_X \mathrm{ad}_X) = -\operatorname{tr}(\mathrm{ad}_X (\mathrm{ad}_X)^T) \le 0$ and $B(X, X) = 0$ iff $\mathrm{ad}_X = 0$, i.e. $X \in \mathfrak{z}(\mathfrak{g})$. But since we assume that $\mathfrak{z}(\mathfrak{g}) = 0$, B is negative definite.

For part (b) we first observe that in the proof of Proposition 3.21 (a), we only use the skew symmetry of ad_X and non-degeneracy of B to show that, for any ideal $\mathfrak{a}$, $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{a}^\perp$ with $\mathfrak{a}^\perp$ an ideal. Thus, using an $\mathrm{Ad}(G)$ invariant inner product $\langle \cdot\,,\cdot\,\rangle$, the same is true here. Hence $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{b}$ with $\mathfrak{b} = (\mathfrak{z}(\mathfrak{g}))^\perp$ and $\mathfrak{b}$ an ideal. As we saw above, $B(X, X) \le 0$ and $B(X, X) = 0$ iff $X \in \mathfrak{z}(\mathfrak{g})$. This implies that $B_\mathfrak{b} = (B_\mathfrak{g})_{|\mathfrak{b}}$ is non-degenerate and hence $\mathfrak{b}$ is semisimple. Thus $[\mathfrak{g}, \mathfrak{g}] = [\mathfrak{b}, \mathfrak{b}] = \mathfrak{b}$ which proves our claim. □

Corollary 3.26 *A compact Lie group with finite center is semisimple.*

A more difficult and surprising result is compactpi1

Proposition 3.27 [Weyl] *If G is a compact Lie group with finite center, then $\pi_1(G)$ is finite and hence every Lie group with Lie algebra $\mathfrak{g}$ is compact.*

Proof The proof will be geometric and assume some knowledge from Riemannian geometry, since it will be a consequence of the Bonnet-Myer's theorem. As follows from Proposition 3.25 (a), $-B$ is an $\mathrm{Ad}(G)$ invariant inner product and as in the proof of Proposition 3.24 (b) extends to a Riemannian metric on G such that L_g and R_g are isometries. We will show that $\mathrm{Ric} \geq \frac{1}{4}$, and the claim follows from Bonnet Myers. We first claim that the formulas for the connection and the curvature tensor are given by:

$$\nabla_X Y = \frac{1}{2}[X,Y] \ , \ \text{and } R(X,Y)Z = -\frac{1}{4}[[X,Y],Z]$$

where $X, Y, Z \in \mathfrak{g}$. For convenience, set $\langle , \rangle = -B$. For the connection, we use the fact that $\langle X, Y \rangle$ is constant since X, Y and $\langle , \rangle$ are left invariant. We fist show that $\nabla_X X = 0$, which follows from

$$\begin{aligned}\langle \nabla_X X, Z\rangle &= X(\langle X, Z\rangle) - \langle X, \nabla_X Z\rangle = -\langle X, \nabla_X Z\rangle \\ &= -\langle X, \nabla_Z X\rangle + \langle X, [X,Z]\rangle = -\langle X, \nabla_Z X\rangle = -\frac{1}{2}Z(-\langle X, X\rangle) = 0\end{aligned}$$

for all $Z \in \mathfrak{g}$, where we also used $\langle X, [X,Z]\rangle = 0$ by skew symmetry of ad_X. For the reader with a more sophisticated back ground in geometry we observe that a quicker argument for $\langle \nabla_X X, Z\rangle = -\langle X, \nabla_Z X\rangle$ is the fact that the vector field X is Killing since its flow is $R_{\exp(tX)}$ and hence ∇X is skew symmetric.

For the curvature tensor we use the Jacobi identity:

$$\begin{aligned}\langle R(X,Y)Z\rangle &= \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \\ &= \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ &= -\frac{1}{4}[Z,[X,Y]] - \frac{1}{2}[[X,Y],Z] = -\frac{1}{4}[[X,Y],Z]]\end{aligned}$$

Finally, we compute the Ricci tensor:

$$\mathrm{Ric}(X,Y) = \mathrm{tr}\{Z \to R(Z,X)X\} = -\frac{1}{4}\,\mathrm{tr}\{Z \to \mathrm{ad}_X \,\mathrm{ad}_X(Z)\} = -\frac{1}{4}B(X,X)$$

which finishes the proof. □

Thus if $\mathfrak{g}$ is a compact Lie algebra with trivial center, every Lie group with Lie algebra $\mathfrak{g}$ is compact. As a consequence one can reduce the classification of compact Lie groups to that of simple Lie groups.

Corollary 3.28 *Every compact Lie group is isomorphic to*

$$(\mathrm{T}^n \times G_1 \times \cdots \times G_m)/\Gamma$$

where G_i are compact, simply connected, simple Lie groups and Γ is a finite subgroup of the center.

Proof Let G be a compact Lie group. By Proposition 3.25 (b) and Proposition 3.21 (a) $\mathfrak{g}$ is isomorphic to $\mathbb{R}^n \times \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_m$ with $\mathfrak{g}_i$ simple. Hence the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$, which is also the universal cover of G, is isomorphic to $\tilde{G} = \mathbb{R}^n \times G_1 \times \cdots \times G_m$ with G_i simply connected and simple. By Weyl's Theorem G_i is compact as well. From general covering space theory we know that $G = \tilde{G}/\tilde{\Gamma}$ with $\tilde{\Gamma}$ a discrete subgroup of the center. But the center of G_i is finite and hence the projection from $\tilde{\Gamma}$ to $G_1 \times \cdots \times G_m$, which is a homomorphism, has finite index and its kernel $\bar{\Gamma}$ is a discrete subgroup of $\mathbb{R}^n$. Since G is compact, $\mathbb{R}^n/\bar{\Gamma}$ must be compact as well and hence isomorphic to a torus T^n. Thus the finite group $\Gamma = \tilde{\Gamma}/\bar{\Gamma}$ acts on $\mathrm{T}^n \times G_1 \times \cdots \times G_m$ with quotient G. □

The proof of Proposition 3.27 also implies:

expcompact **Corollary 3.29** *If G is a compact Lie group, then $\exp\colon \mathfrak{g} \to G$ is onto.*

Proof As we saw before, for a biinvariant metric we have $\nabla_X X = 0$. This implies that the geodesics through $e \in G$ are the one parameter groups $c(t) = \exp(tX)$. Indeed, c is an integral curve of X and $\nabla_{\dot c}\dot c = \nabla_X X = 0$ means that c is a geodesic. Thus exp is the usual exponential map of the biinvariant metric: $\exp_e\colon T_eG \to G$. Since G is compact, the metric is complete, and by Hopf-Rinow $\exp_e = \exp$ is onto. □

Exercises 3.30

(a) If G is a compact Lie group, show that it is isomorphic to $\mathrm{T}^n \times G_1 \times \cdots \times G_m$ with G_i compact and simple.
(b) If $\mathfrak{g}$ is a compact Lie algebra, show that $\mathrm{Int}(\mathfrak{g}) = \{e^{\mathrm{ad}_X} \mid X \in \mathfrak{g}\}$.
(c) If $\mathfrak{g}$ is a Lie algebra such that ad_X is skew symmetric with respect to some inner product, show that $\mathfrak{g}$ is compact. Similarly, if G is a Lie group which admits a biinvariant metric, show that some subcover of G is compact.

Maximal Torus and Weyl group

In order to understand compact (simple) Lie groups in more detail, we will use the following theorem. A Lie algebra $\mathfrak{g}$ certainly has abelian subalgebras, e.g. one dimensional ones. We say that $\mathfrak{t} \subset \mathfrak{g}$ is a **maximal abelian subalgebra** if $\mathfrak{t}$ is abelian, and $\mathfrak{t} \subset \mathfrak{t}'$ with $\mathfrak{t}'$ abelian implies $\mathfrak{t}' = \mathfrak{t}$.

maximaltorus

Proposition 3.31 *Let G be a compact Lie group and $\mathfrak{t} \subset \mathfrak{g}$ a maximal abelian subalgebra.*

(a) For any $X \in \mathfrak{g}$, there exists a $g \in G$ such that $\mathrm{Ad}(g)(X) \in \mathfrak{t}$
(b) If $\mathfrak{t}_1, \mathfrak{t}_2$ are two maximal abelian subalgebras, then there exists a $g \in G$ such that $\mathrm{Ad}(g)\mathfrak{t}_1 = \mathfrak{t}_2$

Proof Let $T = \exp(\mathfrak{t})$ and suppose T is not closed. Then $T' = \overline{\exp(\mathfrak{t})}$ is a (connected) closed subgroup of G, and hence a Lie group. Since T is abelian, so is T'. Since T' is strictly bigger than T, $\mathfrak{t}'$ is also strictly bigger than $\mathfrak{t}$, a contradiction. Thus T is compact and hence a torus which means we can choose an $X \in \mathfrak{t}$ such that $\exp(sX)$ is dense in T.

We first claim, after fixing the above choice of X, that $\mathfrak{t} = \{Z \mid [X, Z] = 0\}$, i.e. $[X, Z] = 0$ implies $Z \in \mathfrak{t}$. Indeed, $\mathrm{Ad}(\exp(sX))(Z) = e^{s\,\mathrm{ad}X} Z = Z$, and by density of $\exp(sX)$ in T, $\mathrm{Ad}(T)(Z) = Z$ which implies $[Z, \mathfrak{t}] = 0$ since $d\,\mathrm{Ad} = \mathrm{ad}$. By maximality of $\mathfrak{t}$, $Z \in \mathfrak{t}$.

Let $\langle , \rangle$ be an $\mathrm{Ad}(G)$-invariant inner product. For a fixed $Y \in \mathfrak{t}$, we define the function

$$F \colon G \to \mathbb{R} \quad g \to \langle X, \mathrm{Ad}(g)Y \rangle$$

Since G is compact, there exists a critical point g_0 of F. At such a point we have

$$\begin{aligned} dF_{g_0}(Z_{g_0}) &= \frac{d}{dt}_{|t=0} \langle X, \mathrm{Ad}(\exp(tZ)g_0)Y \rangle = \frac{d}{dt}_{|t=0} \langle X, \mathrm{Ad}(\exp(tZ))\,\mathrm{Ad}(g_0)Y \rangle \\ &= \langle X, [Z, \mathrm{Ad}(g_0)Y] \rangle = -\langle [X, \mathrm{Ad}(g_0)Y], Z \rangle = 0 \text{ for all } Z \in \mathfrak{g} \end{aligned}$$

Thus for a critical point g_0 we have

$$[X, \mathrm{Ad}(g_0)Y] = 0 \text{ and hence } \mathrm{Ad}(g_0)Y \in \mathfrak{t}$$

which proves part (a).

For part (b) choose $X_i \in \mathfrak{t}_i$ such that $\exp(sX_i)$ is dense in $\exp(\mathfrak{t}_i)$. By (a), there exists a $g \in G$, such that $\mathrm{Ad}(g)X_1 \in \mathfrak{t}_2$. Since $\mathfrak{t}_2$ is abelian,

$[Y, \mathrm{Ad}(g)X_1] = 0$ or $[\mathrm{Ad}((g^{-1})Y, X_1] = 0$ for all $Y \in \mathfrak{t}_2$. Thus $\mathrm{Ad}(g^{-1})\mathfrak{t}_2 \subset \mathfrak{t}_1$, which implies that $\dim \mathfrak{t}_2 \le \dim \mathfrak{t}_1$. Reversing the role of $\mathfrak{t}_1$ and $\mathfrak{t}_2$, we see that $\dim \mathfrak{t}_1 \le \dim \mathfrak{t}_2$ and hence $\dim \mathfrak{t}_2 = \dim \mathfrak{t}_1$. Thus $\mathrm{Ad}(g^{-1})\mathfrak{t}_2 = \mathfrak{t}_1$ or $\mathfrak{t}_2 = \mathrm{Ad}(g)\mathfrak{t}_1$. □

In terms of the group G this can be reformulated as follows. If $\mathfrak{t}$ is maximal abelian, we call $T = \exp(\mathfrak{t})$ a **maximal torus**.

Corollary 3.32 *Let G be a compact Lie group with maximal torus T. Then every element of G is conjugate to an element of T. Furthermore, any two maximal tori are conjugate to each other.*

Proof Given an element $g \in G$, Corollary 3.29 implies that there exists an $X \in \mathfrak{t}$ such that $\exp(X) = g$. By Proposition 3.31 there exists an $h \in G$ such that $\mathrm{Ad}(h)X \in \mathfrak{t}$. Then $hgh^{-1} = h\exp(X)h^{-1} = \exp(\mathrm{Ad}(h)X) \in \exp(\mathfrak{t}) = T$. The second claim follows similarly. □

We can thus define:

Definition 3.33 *Let G be a compact Lie group. The the dimension of a maximal torus is called the* **rank** *of G, denoted by $\mathrm{rk}(G)$, or $\mathrm{rk}(\mathfrak{g})$ for its Lie algebra.*

Before proceeding, we show:

torusmaxabelian

Lemma 3.34 *Assume that G is compact, $S \subset G$ is a torus, and g commutes with S. Then there exists a maximal torus containing both S and g. In particular, a maximal torus is its own centralizer.*

Proof Let A be the closure of the subgroup generated by g and S. Then A is closed, abelian and compact, and hence its identity component A_0 also a torus. Since gS generates A/A_0, A is isomorphic to $\mathrm{T}^k \times \mathbb{Z}_m$ for some k and m. One easily sees that one can thus choose an element $a \in A$ such that A is the closure of $\{a^n \mid n \in \mathbb{Z}\}$. Since a lies in a maximal torus T, so does S and g. □

An orbit $\mathrm{Ad}(G)X,\ X \in \mathfrak{g}$ is called an **adjoint orbit**. Proposition 3.31 says that every adjoint orbit meets $\mathfrak{t}$. To see how many times it meets $\mathfrak{t}$, we define the **Weyl group** $W = N(T)/T$, where N denotes the normalizer.

adjointorbit

Proposition 3.35 *Let G be a compact Lie group with maximal torus T and Weyl group $W = N(T)/T$.*

(a) *W is a finite group which acts effectively on $\mathfrak{t}$ via $w \cdot X = \mathrm{Ad}(n)X$ with $w = n \cdot T$.*
(b) *The adjoint orbit $\mathrm{Ad}(G)X,\ X \in \mathfrak{t}$ meets $\mathfrak{t}$ in the Weyl group orbit $W \cdot X$.*
(c) *Whenever the adjoint orbit $\mathrm{Ad}(G)X,\ X \in \mathfrak{g}$ meets $\mathfrak{t}$, it does so orthogonally.*

Proof (a) $N(T)$ is clearly a closed subgroup of G and hence a compact Lie group. We claim that T is its identity component, and hence W is finite. To see this, recall that the Lie algebra of $N(T)$ is $\mathfrak{n}(\mathfrak{t}) = \{X \in \mathfrak{g} \mid [X, \mathfrak{t}] \subset \mathfrak{t}\}$ and choose $X_0 \in \mathfrak{t}$ such that $\exp(sX)$ is dense in T. If $X \in \mathfrak{n}(\mathfrak{t})$, then $\langle [X, X_0], Z\rangle = -\langle [Z, X_0], X\rangle = 0$ whenever $Z \in \mathfrak{t}$. Since $[X, X_0] \in \mathfrak{t}$ by assumption, this implies that $[X, X_0] = 0$ and hence $X \in \mathfrak{t}$, which proves our claim.

Since $\mathrm{Ad}(T)_{|\mathfrak{t}} = \mathrm{Id}$, the action of W on $\mathfrak{t}$ is well defined. If $g \in N(T)$ and $\mathrm{Ad}(g)_{|\mathfrak{t}} = \mathrm{Id}$, then $g \exp(X) g^{-1} = \exp(\mathrm{Ad}(g)X) = \exp(X)$, for $X \in \mathfrak{t}$, and hence $g \in Z(T)$, the centralizer of T. But Lemma 3.34 implies that $g \in T$, which shows the action of W is effective.

(b) We need to show that if $\mathrm{Ad}(g)X = Y$ for some $X, Y \in \mathfrak{t}$, then there exists an $n \in N(T)$ with $\mathrm{Ad}(n)X = Y$. Let $Z_X = \{g \in G \mid \mathrm{Ad}(g)X = X\}$ be the centralizer of X in G. Then T $\subset Z_X$ and $g^{-1}Tg$ as well since $\mathrm{Ad}(g^{-1})Tg)X = \mathrm{Ad}(g^{-1})\,\mathrm{Ad}(T)Y = \mathrm{Ad}(g^{-1})Y = X$. We can now apply Proposition 3.31 (b) to the identity component of Z_X to find an $h \in Z_X$ with $hTh^{-1} = g^{-1}Tg$. Thus $gh \in N(T)$ and $\mathrm{Ad}(gh)X = \mathrm{Ad}(g)\,\mathrm{Ad}(h)X = \mathrm{Ad}(g)X = Y$, i.e. $n = gh$ is the desired element.

(c) If $\mathrm{Ad}(G)X$ meets $\mathfrak{t}$ in Y, then the tangent space of the orbit $\mathrm{Ad}(G)X = \mathrm{Ad}(G)Y$ is $T_Y(\mathrm{Ad}(G)Y) = \{[Z, Y] \mid Z \in \mathfrak{g}\}$. If $U \in \mathfrak{t}$, then $\langle [Z, Y], U\rangle = -\langle [U, Y], Z\rangle = 0$, which says that the orbit mets $\mathfrak{t}$ orthogonally at Y. □

Example 3.36 (a) Let $G = \mathrm{SO}(3)$ with $\mathrm{SO}(2) \subset \mathrm{SO}(3)$ a maximal torus and $N(T) = \mathrm{O}(2)$ and hence $W = \mathbb{Z}_2$. We can identify $\mathfrak{o}(3) \simeq \mathbb{R}^3$ such that the adjoint action of $\mathrm{SO}(3)$ on $\mathfrak{o}(3)$ is via ordinary rotations and $\mathrm{SO}(2)$ are the rotations fixing the z-axis. Thus the maximal abelian subalgebra $\mathfrak{t}$ is the z-axis and W acts on it via reflection in $0 \in \mathfrak{t}$. An $\mathrm{SO}(3)$ orbit is a sphere of radius r which meets $\mathfrak{t}$ orthogonally in the W orbit $\pm r$.

We now discuss maximal tori and Weyl groups for all the classical Lie groups.

Classical Lie groups

(a) $\mathbf{G} = \mathrm{U}(\mathbf{n})$

We claim that the maximal torus consists of the diagonal matrices

$$\mathrm{T}^n = \mathrm{diag}(z_1, \dots, z_n),\ |z_i| = 1 \quad \text{with Weyl group} \quad W = S_n$$

and thus $\mathrm{rk}(\mathrm{U}(n)) = n$. Indeed, if $B \in \mathrm{U}(n)$ commutes with a matrix $A \in T$, then B preserves the eigenspaces of A. Choosing the diagonal entries of A distinct, we see that B must be diagonal as well. If B normalizes elements of T, by the same argument it can at most permute the diagonal elements of $A \in T$. Thus the Weyl group is a subgroup of the permutation group S_n. Since

$$\mathrm{Ad}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} b & 0 \\ 0 & a \end{pmatrix}$$

it follows that every permutation is contained in the Weyl group, i.e $W = S_n$.

The adjoint orbits are given by $\mathrm{Ad}(G)H,\ H \in \mathfrak{t}$ and contain a unique H' with $h_1 \leq h_2 \leq \cdots \leq h_n$. They are homogeneous with G acting transitively with isotropy those matrices which commute with H. The isotropy depends on how many components in H are equal to each other and is thus given by $\mathrm{U}(n_1)U(n_2)\cdots U(n_k)$ with $\sum n_i = n$, if $h_1 = \cdots = h_{n_1}$ ect. Thus the adjoint orbits are $U(n)/\,\mathrm{U}(n_1)U(n_2)\cdots U(n_k)$. Another interpretation is as a flag manifold

$$F(n_1, \dots, n_k) = \{V_1 \subset V_2 \cdots \subset V_k \subset \mathbb{C}^n \mid \dim V_i - \dim V_{i-1} = n_i,\ i = 1, \dots, n\}$$

where $V_0 = 0$.

For $G = \mathrm{SU}(n)$, we add the restriction $z_1 \cdots z_n = 1$ but the Weyl group is the same. Thus $\mathrm{rk}(\mathrm{SU}(n)) = n - 1$.

(b) $\mathbf{G} = \mathrm{Sp}(\mathbf{n})$

Recall that $\mathrm{U}(n) \subset \mathrm{Sp}(n)$ and we claim that $T^n \subset \mathrm{U}(n) \subset \mathrm{Sp}(n)$ is also a maximal torus in $\mathrm{Sp}(n)$. Indeed, if $B + jC \in \mathfrak{sp}(n)$ commutes with $A \in \mathfrak{t}^n$, we have $AB = BA$ and $\bar{A}C = CA$ which is only possible when $C = 0$. The Weyl group contains all permutations as before. But now we can also change the sign of each individual diagonal entry since $jzj^{-1} = \bar{z}$. Thus $W = S_n \rtimes (\mathbb{Z}_2)^n$ with S_n acting on $(\mathbb{Z}_2)^n$ via permutations.

(c) $\mathbf{G} = \mathrm{SO}(\mathbf{2n})$

With an argument similar to the above, one sees that the maximal torus is

$$\mathrm{T}^n = \mathrm{diag}(R(\theta_1), \ldots, R(\theta_n)) \quad \text{with} \quad R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

Let E_{ij} be the matrix whose only non-zero entries are a 1 in row i and column j. The Weyl group contains all permutations of θ_i since conjugation with $A = E_{(2i+1)(2k+1)} + E_{(2i+2)(2k+2)} + E_{(2k+1)(2i+1)} + E_{(2k+2)(2i+2)}$ interchanges θ_i with θ_k, and $A \in \mathrm{SO}(2n)$. In addition, conjugation with $B = E_{(2i+1)(2i+2)} + E_{(2i+2)(2i+1)}$ changes the sign of θ_i. But $\det B = -1$. On the other hand, conjugating with $C = E_{(2i+1)(2i+2)} + E_{(2i+2)(2i+1)} + E_{(2k+1)(2k+2)} + E_{(2k+2)(2k+1)}$ changes the sign of both θ_i and θ_k, and $\det C = 1$. Thus all even sign changes are contained in the Weyl group. Hence $W = S_n \rtimes (\mathbb{Z}_2)^{n-1}$.

(d) $\mathbf{G} = \mathrm{SO}(\mathbf{2n+1})$

In this case one easily sees that $\mathrm{T}^n \subset \mathrm{SO}(2n) \subset \mathrm{SO}(2n+1)$ is the maximal torus. The Weyl group contains all permutations as before, but also all sign changes since conjugation with $E_{(2i+1)(2i+2)} + E_{(2i+2)(2i+1)} - E_{(2n+1)(2n+1)}$ has determinant 1 and changes the sign of θ_i only. Thus $W = S_n \rtimes (\mathbb{Z}_2)^n$

Exercises 3.37

(1) Show that $\mathrm{diag}(\pm 1, \ldots, \pm 1) \cap \mathrm{SO}(n) \subset \mathrm{SO}(n)$ is maximal abelian, but not contained in any torus.
(2) Determine the adjoint orbits of $\mathrm{Sp}(n)$ and $\mathrm{SO}(n)$.
(3)

4

Complex Semisimple Lie algebras

In this Chapter we will discuss the classification of complex semisimple Lie algebras and their relationships to real compact Lie algebras. Throughout this Chapter $\mathfrak{g}$ will be a complex semisimple Lie algebra. We sometimes write it as the complexification of a compact real Lie algebra, which we will denote by $\mathfrak{k}$, i.e. $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{g}$.

4.1 Cartan subalgebra and roots

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Recall that $A \in \mathrm{End}(V)$ is called **semisimple** if it can be diagonalized.

Cartansubalg **Definition 4.1** *$\mathfrak{h} \subset \mathfrak{g}$ is a* **Cartan subalgebra** *of $\mathfrak{g}$ if the following hold:*

(a) $\mathfrak{h}$ is a maximal abelian subalgebra.
(b) If $X \in \mathfrak{h}$, ad_X is semisimple.

Unlike in the compact case, the existence is non-trivial.

Cartanex **Theorem 4.2** *Every complex semisimple Lie algebra has a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Moreover, $\mathfrak{h}$ is unique up to inner automorphisms and $N_{\mathfrak{g}}(\mathfrak{h}) = Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Proof We will add a proof later on..... □

Thus we again define $\mathrm{rk}(\mathfrak{g}) = \dim \mathfrak{h}$.

As we will see later on, every complex semisimple Lie algebra has a compact real form. It may thus be comforting to the reader that in this case the existence of a Cartan subalgebra easily follows from Proposition 3.31.

Theorem 4.3 *Let $\mathfrak{k}$ be a compact semisimple Lie algebra with $\mathfrak{k}_{\mathbb{C}} \simeq \mathfrak{g}$ and $\mathfrak{t} \subset \mathfrak{k}$ a maximal abelian subalgebra. Then $\mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ with $N_{\mathfrak{g}}(\mathfrak{h}) = Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$.*

Proof Since a basis of $\mathfrak{t}$ is a (complex) basis of $\mathfrak{h}$, and $\mathfrak{t}$ is abelian, $\mathfrak{h}$ is abelian. With respect to an $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{k}$, ad_X is skew-symmetric for all $X \in \mathfrak{k}$ and can hence be diagonalized over $\mathbb{C}$. Since $\mathrm{ad}_X, \mathrm{ad}_Y$ commute: $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]} = 0$, they can be diagonalized simultaneously. Hence $\mathrm{ad}_{X+iY} = \mathrm{ad}_X + i\,\mathrm{ad}_Y$, $X + iY \in \mathfrak{t}_{\mathbb{C}}$ can be diagonalized as well. Let $\mathfrak{h}'$ be an abelian subalgebra of $\mathfrak{g}$ strictly containing $\mathfrak{h}$. Then for $X + iY \in \mathfrak{h}'$ we have $[\mathfrak{t}, X + iY] = [\mathfrak{t}, X] + i[\mathfrak{t}, Y] = 0$ and hence $[\mathfrak{t}, X] = [\mathfrak{t}, Y] = 0$ which implies $X, Y \in \mathfrak{t}$ by maximality of $\mathfrak{t}$. Thus $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. In the proof of Proposition 3.35 we saw that $N_{\mathfrak{g}}(\mathfrak{t}) = \mathfrak{t}$, and hence the same follows for $\mathfrak{h}$.

$\square$

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Since ad_X, $X \in \mathfrak{g}$ are diagonalizable and $[\mathrm{ad}_X, \mathrm{ad}_Y] = \mathrm{ad}_{[X,Y]} = 0$ for $X, Y \in \mathfrak{g}$, they can be diagonalized simultaneously. Thus there exists a common basis of eigenvectors $\{e_1, \ldots, e_n\}$ with $\mathrm{ad}_X(e_i) = \lambda_i(X)e_i$ for all $X \in \mathfrak{h}$. One easily sees that λ_i is linear in X. This motivates the following definition:

Definition 4.4 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Given a linear map $\alpha \in \mathfrak{h}^* = \mathrm{Hom}(\mathfrak{h}, \mathbb{C})$, define the* **root space** *of α as $\mathfrak{g}_\alpha = \{X \in \mathfrak{g} \mid [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h}\}$. If $\mathfrak{g}_\alpha \neq 0$, α is called a* **root** *of $\mathfrak{g}$ with respect to $\mathfrak{h}$, or simply a root. Let $\Delta \subset \mathfrak{h}^*$ be the set of all non-zero roots of $\mathfrak{g}$.*

As we will see later on, the root spaces are one dimensional. The roots characterize the Lie algebra up to isomorphism, and we will encode a complete description in a diagram called the Dynkin diagram.

From now on $\mathfrak{g}$ will always be a complex semisimple Lie algebra with a fixed Cartan subalgebra $\mathfrak{h}$. We will also use, without always saying explicitly, the following convenient convention. Vectors in $\mathfrak{h}$ will be denoted by H and vectors in $\mathfrak{g}_\alpha$ by X_α. Although 0 is not considered a root, we will sometimes denote $\mathfrak{h} = \mathfrak{h}_0$ since $Z_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$ implies that these are the only vectors with weight 0.

We start with some simple properties of the root system.

rootsbasic

Theorem 4.5 *Let Δ be the set of roots with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. Then one has the following properties.*

(a) $\mathfrak{g} = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha$.
(b) If $\alpha, \beta \in \Delta$, *then* $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$.
(c) $\{\alpha \mid \alpha \in \Delta\}$ *spans* $\mathfrak{h}^*$.
(d) If $\alpha, \beta \in \Delta$, *and* $\alpha + \beta \neq 0$, *then* $B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$.
(e) If $\alpha \in \Delta$, *then* $-\alpha \in \Delta$.
(f) $B|_\mathfrak{h}$ *is non-degenerate.*

Proof (a) This follows from the fact that $\mathfrak{g}$ is the sum of all common eigenspaces of $\mathrm{ad}_X, X \in \mathfrak{h}$.

(b) follows from the Jacobi identity:

$$[H, [X_\alpha, X_\beta]] = [[H, X_\alpha], X_\beta] + [X_\alpha, [H, X_\beta]] = (\alpha(H) + \beta(H))[X_\alpha, X_\beta]$$

(c) If the roots do not span $\mathfrak{h}^*$, there exists a vector H with $\alpha(H) = 0$ for all $\alpha \in \Delta$. This implies that $H \in \mathfrak{z}(\mathfrak{g})$ which is 0 since $\mathfrak{g}$ is semisimple.

(d) If $\alpha + \beta \neq 0$ and $\gamma \in \Delta$, then $\mathrm{ad}_{X_\alpha} \mathrm{ad}_{X_\beta} : \mathfrak{g}_\gamma \to \mathfrak{g}_{\gamma+\alpha+\beta}$ is a map between disjoint root spaces. Thus $\mathrm{ad}_{X_\alpha} \mathrm{ad}_{X_\beta}$ is nilpotent which implies that its trace is 0.

(e) Part (d) implies that only $\mathfrak{g}_{-\alpha}$ can have non-zero inner products with $\mathfrak{g}_\alpha$. Since B is non-degenerate, the claim follows.

(f) This also follows from non-degeneracy of B since $B(\mathfrak{h}, \mathfrak{g}_\alpha) = 0$ for $\alpha \neq 0$.

□

Since $B_{|\mathfrak{h}}$ is non-degenerate, we can identify $\mathfrak{h}^*$ with $\mathfrak{h}$ and denote the image of α by H_α, i.e. $B(H_\alpha, H) = \alpha(H)$. For simplicity, we denote from now on the inner product B on $\mathfrak{h}$, as well as the inner product induced on $\mathfrak{h}^*$, by $\langle \cdot, \cdot \rangle$. Thus $\langle \alpha, \beta \rangle = \langle \mathrm{H}_\alpha, \mathrm{H}_\beta \rangle = \alpha(H_\beta) = \beta(H_\alpha)$. Later on it will be convenient to re-normalize the Killing form. The reader may confirm that in all of the proofs in this section, all that is needed is that ad_X is skew symmetric with respect to a non-degenerate bilinear form.

We now derive some less elementary properties of the root system.

rootsadvanced

Theorem 4.6 *Let Δ be the set of roots with respect to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and $\alpha \in \Delta$. Then one has the following properties.*

(a) $[X_\alpha, X_{-\alpha}] = \langle X_\alpha, X_{-\alpha}\rangle \cdot H_\alpha$ *and* $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] = \mathbb{C} \cdot H_\alpha$.
(b) $\langle \alpha, \alpha\rangle \neq 0$.
(c) $\dim \mathfrak{g}_\alpha = 1$.

Proof (a) By skew-symmetry of ad_X, we have

$$\begin{aligned}\langle [X_\alpha, X_{-\alpha}], H\rangle &= -\langle [H, X_{-\alpha}], X_\alpha\rangle = \alpha(H)\langle X_{-\alpha}, X_\alpha\rangle = \\ &\langle H, H_\alpha\rangle\langle X_\alpha, X_{-\alpha}\rangle = \langle\ \langle X_\alpha, X_{-\alpha}\rangle H_\alpha, H\rangle\end{aligned}$$

for all $H \in \mathfrak{h}$, which implies the first claim. By non-degeneracy of B, there exists an $X_{-\alpha}$ for each X_α with $\langle X_\alpha, X_{-\alpha}\rangle \neq 0$, and the second claim follows as well.

(b) By non-degeneracy of $B|_{\mathfrak{h}}$, there exists a $\beta \in \Delta$ such that $\langle \alpha, \beta\rangle \neq 0$. Consider the subspace $V \subset \mathfrak{g}$ defined by

$$V = \sum_{n\in\mathbb{Z}} \mathfrak{g}_{\beta+n\alpha}.$$

Observe that V is preserved by $\mathrm{ad}_{\mathrm{H}_\alpha}$ since it preserves root spaces, and also by ad_{X_α} and $\mathrm{ad}_{X_{-\alpha}}$ since they 'move up and down' the summands. Now choose $X_\alpha, X_{-\alpha}$ such that $[X_\alpha, X_{-\alpha}] = H_\alpha$. Then

$$\mathrm{tr}(\mathrm{ad}_{H_\alpha})_{|V} = \mathrm{tr}(\mathrm{ad}_{[X_\alpha, X_{-\alpha}]})_{|V} = \mathrm{tr}(\mathrm{ad}_{X_\alpha}\mathrm{ad}_{X_{-\alpha}} - \mathrm{ad}_{X_{-\alpha}}\mathrm{ad}_{X_\alpha})_{|V} = 0.$$

We can also compute the trace differently since $(\mathrm{ad}_H)_{|\mathfrak{g}_\gamma} = \gamma(H) \cdot \mathrm{Id}$

$$\mathrm{tr}(\mathrm{ad}_{H_\alpha})_{|V} = \sum_n \mathrm{tr}(\mathrm{ad}_{H_\alpha})_{|\mathfrak{g}_{\beta+n\alpha}} = \sum_n (\beta + n\alpha)(H_\alpha)\, d_{\beta+n\alpha}$$

where we have set $\dim(\mathfrak{g}_\gamma) = d_\gamma$. Hence

$$-\beta(H_\alpha) \sum_n d_{\beta+n\alpha} = \alpha(H_\alpha) \sum_n n \cdot d_{\beta+n\alpha}$$

which implies that $\langle \alpha, \alpha\rangle = \alpha(H_\alpha) \neq 0$ since $d_\beta > 0$.

(c) Using the same method, we define

$$V = \mathbb{C}X_{-\alpha} + \mathbb{C}H_\alpha + \sum_{n\geq 1} \mathfrak{g}_{n\alpha}.$$

One checks that it is again preserved by $\mathrm{ad}_{\mathrm{H}_\alpha}, \mathrm{ad}_{X_\alpha}$ and $\mathrm{ad}_{X_{-\alpha}}$ and taking the trace of ad_{H_α} we get

$$\langle \alpha, -\alpha\rangle + \sum_{n\geq 1} d_{n\alpha}\langle \alpha, n\alpha\rangle = \langle \alpha, \alpha\rangle(-1 + d_1 + \sum_{n\geq 1} n\, d_{n\alpha}) = 0$$

Thus, since α is a root, $d_\alpha = 1$, which proves our claim. Notice that it also implies that $d_{n\alpha} = 0$ for $n \geq 2$, i.e, if $n\alpha,\ n \in \mathbb{Z}$, is a root, then $n = 0, \pm 1$. We will be able to use this fact later on. □

As we already saw in the previous proof, the next concept is useful.

`string` **Definition 4.7** *Let $\alpha, \beta \in \Delta$ be roots. The α* **string containing** *β is the set of roots of the form*

$$\{\beta + n\alpha \mid n \in \mathbb{Z}\}.$$

Here is its main property and some applications.

`strings` **Theorem 4.8** *Let Δ be the set of roots. Then*

(a) There exists integers p, $q \geq 0$, such that the α string containing β consists of consecutive roots, i.e., $-p \leq n \leq q$ and

$$\frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle} = p - q \qquad (4.9)$$ `integer`

(b) If $\beta = c\alpha$ with $\alpha, \beta \in \Delta$ and $c \in \mathbb{C}$, then $c = 0$ or $c = \pm 1$.

(c) If $\alpha, \beta, \alpha + \beta \in \Delta$, then $[X_\alpha, X_\beta] \neq 0$ for any $X_\alpha \in \mathfrak{g}_\alpha$, $X_\beta \in \mathfrak{g}_\beta$.

Proof Let $[r, s]$ be a component of the set of integers $n \in \mathbb{Z}$ such that $\beta + n\alpha \in \Delta$. Setting $V = \sum_{n=r}^{s} \mathfrak{g}_{\beta+n\alpha}$, we compute $\mathrm{tr}(\mathrm{ad}_{H_\alpha})_{|V}$ as before:

$$\begin{aligned}\mathrm{tr}(\mathrm{ad}_{h_\alpha})_{|V} &= \sum_{n=r}^{s}(\beta + n\alpha)(H_\alpha) = (s - r + 1)\beta(H_\alpha) + \alpha(H_\alpha)\sum_{n=r}^{s} n \\ &= (s - r + 1)\beta(H_\alpha) + \alpha(H_\alpha)\frac{(s + r)(s - r + 1)}{2} = 0\end{aligned}$$

and thus $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = -(r+s)$. If $[r', s']$ is another disjoint interval of consecutive roots in the α string containing β, it follows that $s + r = s' + r'$, which is clearly impossible. Thus the string is connected, and since $n = 0$ belongs to it, our claim follows.

(b) If $\beta = c\alpha$, we can apply (4.9) to the α string containing β and the β string containing α and obtain $2c \in \mathbb{Z}$ and $\frac{2}{c} \in \mathbb{Z}$. Thus if $c \neq 0, \pm 1$, it follows that $c = \pm\frac{1}{2}$ or ± 2. But we already saw in the proof of Proposition 4.5 (c), that $\beta = 2\alpha$ and $\alpha = 2\beta$ is not possible.

(c) Assume that $[X_\alpha, X_\beta] = 0$. The α string containing β satisfies $-p \leq n \leq q$. This implies that $V = \sum_{n=-p}^{0} \mathfrak{g}_{\beta+n\alpha}$ is invariant under $\mathrm{ad}_{\mathrm{H}_\alpha}, \mathrm{ad}_{X_\alpha}$ and $\mathrm{ad}_{X_{-\alpha}}$ and taking the trace of ad_{H_α} we get

$$\mathrm{tr}(\mathrm{ad}_{H_\alpha})_{|V} = \sum_{n=-p}^{0} (\beta + n\alpha)(H_\alpha) = (p+1)\beta(H_\alpha) - \frac{p(p+1)}{2}\alpha(H_\alpha) = 0$$

Thus $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = p$. On the other hand, (4.9) implies that $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = p - q$ and hence $q = 0$. But $q \geq 1$ since $\alpha + \beta$ is a root, a contradiction. □

We next consider a real subspace of the Cartan subalgebra:

$$\mathfrak{h}_{\mathbb{R}} = \sum_{\alpha} \mathbb{R} \cdot H_\alpha \subset \mathfrak{h}.$$

Its importance is given by:

Proposition 4.10

(a) B is positive definite on $\mathfrak{h}_{\mathbb{R}}$.
(b) $\mathfrak{h}_{\mathbb{R}}$ is a real form of $\mathfrak{h}$, i.e. $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} + i\mathfrak{h}_{\mathbb{R}}$.

Proof (a) First notice that, given $H, H' \in \mathfrak{h}$,

$$\langle H, H\rangle = B(H, H') = \mathrm{tr}(\mathrm{ad}_H \, \mathrm{ad}_{H'}) = \sum_{\alpha\in\Delta} \alpha(H)\alpha(H').$$

If we let $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} = n_{\alpha,\beta} \in \mathbb{Z}$, we obtain

$$\langle\alpha, \alpha\rangle = \sum_{\gamma\in\Delta} \gamma(H_\alpha)^2 = \frac{1}{4}\langle\alpha, \alpha\rangle^2 \sum_{\gamma\in\Delta} n_{\alpha,\gamma}^2$$

and since $\langle\alpha, \alpha\rangle \neq 0$,

$$\langle\alpha, \alpha\rangle = \frac{2}{\sum_\gamma n_{\alpha,\gamma}^2} \in \mathbb{Q} \ , \quad \text{and} \quad \langle\alpha, \beta\rangle = \frac{n_{\alpha,\beta}}{\sum_\gamma n_{\alpha,\gamma}^2} \in \mathbb{Q}.$$

Thus $\gamma_{|\mathfrak{h}_{\mathbb{R}}}, \gamma \in \Delta$ and hence $B_{|\mathfrak{h}_{\mathbb{R}}}$ are real valued. Moreover, for each $0 \neq H \in \mathfrak{h}_{\mathbb{R}}$ there exists a root β such that $\beta(H) \neq 0$, and hence

$$B(H, H) = \sum_{\gamma\in\Delta} \gamma(H)^2 \geq \beta(H)^2 > 0,$$

(b) Clearly $\mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C} = \mathfrak{h}$ and thus $\mathfrak{h}$ is spanned by $\mathfrak{h}_{\mathbb{R}}$ and $i\mathfrak{h}_{\mathbb{R}}$. If $X \in$

$\mathfrak{h}_\mathbb{R} \cap i\mathfrak{h}_\mathbb{R}$, i.e. $X = iY$ with $Y \in \mathfrak{h}_\mathbb{R}$, then $\langle X, X\rangle \geq 0$ by (a) and $\langle X, X\rangle = \langle iY, iY\rangle = -\langle Y, Y\rangle \leq 0$, which implies $X = Y = 0$. $\square$

If $\mathfrak{k}$ is a compact semisimple Lie algebra with $\mathfrak{t}$ a maximal abelian subalgebra, then $\mathfrak{t}\otimes\mathbb{C} = \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g} := \mathfrak{k}_\mathbb{C}$. The roots of ad_X, $X \in \mathfrak{t}$ are purely imaginary since it is real and skew-symmetric. It has the following normal form: On $\mathfrak{t}$ ir is 0, and $\mathfrak{t}^\perp \subset \mathfrak{k}$ is the direct sum of orthogonal 2 dimensional 'root spaces' in which ad_X has the form

$$\begin{pmatrix} 0 & \alpha(X) \\ -\alpha(X) & 0 \end{pmatrix}$$

in an orthonormal basis $\{v, w\}$ and for some $\alpha \in \mathfrak{t}^*$. In the basis $\{v + iw, v - iw\}$ it becomes $\mathrm{diag}(i\alpha(X), -i\alpha(X))$ and hence $\mathfrak{g}_{i\alpha} = \mathbb{C}\cdot(v + iw)$ and $\mathfrak{g}_{-i\alpha} = \mathbb{C}\cdot(v - iw) = \bar{\mathfrak{g}}_\alpha$. If on the other hand β is a root of $\mathfrak{g}$ with respect to $\mathfrak{h}$, it takes on real values on $\mathfrak{h}_\mathbb{R}$ and thus $\mathfrak{h}_\mathbb{R} = i\mathfrak{t}$.

Conversely, if we start with a complex semisimple Lie algebra $\mathfrak{g}$ and $\mathfrak{k}$ is a compact real form with Cartan subalgebra $\mathfrak{h} = \mathfrak{t}\otimes\mathbb{C}$, then $\mathfrak{g}_{-\alpha} = \bar{\mathfrak{g}}_\alpha$ and $\mathfrak{g}_\alpha^\mathbb{R} := (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}) \cap \mathfrak{t}$ is a real 2 dimensional subspace of $\mathfrak{k}$ on which ad_X, $X \in i\mathfrak{h}_\mathbb{R}$ takes on the above form.

We now discuss the root systems of the classical Lie groups.

Classical Lie groups

Although convenient in the the proofs, it is in practice better to normalize the Killing form. We will do so for each classical Lie group separately, in order to make the standard basis of the Cartan subalgebra into an orthonormal basis. Notice that all statements in the above propositions are unaffected.

$$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$$

A Cartan subalgebra is given by

$$\mathfrak{h} = \{H = \mathrm{diag}(h_1, \ldots, h_n) \mid h_i \in \mathbb{C},\ \sum h_i = 0\}\ \subset \mathbb{C}^{n+1}$$

Let E_{ij} be the matrix whose only non-zero entry is a 1 in row i and column j. Then $[H, E_{ij}] = (h_i - h_j)E_{ij}$. If we let e_i be the standard basis of $\mathbb{C}^{n+1}$ with $H = \sum h_i e_i$ and ω_i the dual basis of $(\mathbb{C}^{n+1})^*$. Then we have the roots

$$\Delta = \{\pm(\omega_i - \omega_j) \mid i < j\} \text{ with root spaces } \mathfrak{g}_{\omega_i-\omega_j} = \mathbb{C}E_{ij} \ \text{ for } i \neq j$$

The Killing form is given by

$$B(H,H)=\sum_{\alpha\in\Delta}\alpha(H)\alpha(H)=2\sum_{i<j}(h_i-h_j)^2=2n\sum h_i^2$$

since $0=(\sum h_i)^2=\sum h_i^2+2\sum_{i<j}h_ih_j$. We normalize the inner product $\langle\cdot\,,\,\cdot\rangle=\frac{1}{2n}B$ so that e_i is an orthonormal basis of $\mathbb{C}^{n+1}$ and thus $H_{\omega_i-\omega_j}=e_i-e_j$. Clearly

$$\mathfrak{h}_{\mathbb{R}}=\{\mathrm{diag}(h_1,\dots,h_n)\mid h_i\in\mathbb{R}\}.$$

A word of caution: Notice that ω_i is *not* orthonormal basis of $\mathfrak{h}^*$.Hence the inner product should only be applied when $\sum h_i=0$ or $\sum\omega_i=0$. All roots have length $\sqrt{2}$.

$$\mathfrak{g}=\mathfrak{so}(2n,\mathbb{C})$$

Although there are other choices for a maximal torus, which are sometimes more convenient, we prefer, due to geometric applications later on, the complexification of the maximal torus of $\mathrm{SO}(2n)$ discussed in. Let $F_{ij}=E_{ij}-E_{ji}$. A simple way to remember the Lie brackets $[F_{ij},F_{kl}]$ in $\mathfrak{so}(n)$ is as follows. If all 4 indices are distinct, the Lie bracket is 0. Otherwise

$$[F_{ij},F_{jk}]=F_{ik}\text{ where }i,j,k\text{ are all distinct}$$

Since $F_{ji}=-F_{ij}$, this determines all Lie brackets.

The Cartan subalgebra is given by

$$\mathfrak{h}=\{H=h_1F_{12}+h_2F_{34}+\cdots+h_nF_{2n-1,2n}\mid h_i\in\mathbb{C}\}.$$

In order to describe the root vectors, define

$$X_{kl}=F_{2k-1,2l-1}+F_{2k,2l}+i(-F_{2k,2l-1}+F_{2k-1,2l})$$

$$Y_{kl}=F_{2k-1,2l-1}-F_{2k,2l}+i(F_{2k,2l-1}+F_{2k-1,2l})$$

A straightforward computation shows that

$$[H,X_{k,l}]=-i(h_k-h_l)X_{k,l}\ ,\ [H,Y_{k,l}]=i(h_k+h_l)Y_{k,l}.$$

The remaining eigenvalues and eigenvectors are obtained by conjugating the above ones. We set $e_i=iF_{2i-1,2i}\in\mathfrak{h}_{\mathbb{R}}$ and let ω_i be the basis dual to e_i. We therefore have the root system

$$\Delta=\{\pm(\omega_i+\omega_j),\ \pm(\omega_i-\omega_j)\mid i<j\}$$

One easily shows that the Killing form is $B=4(n-1)\sum\omega_i^2$ and we use the normalized inner product $\langle \cdot\ ,\ \cdot \rangle = \frac{1}{4(n-1)}B$. Then e_i and ω_i become an orthonormal basis of $\mathfrak{h}$ and $\mathfrak{h}^*$, and $H_{\pm\omega_i \pm \omega_j} = \pm e_i \pm e_j$. Notice that all roots again have length $\sqrt{2}$.

$$\mathfrak{g} = \mathfrak{so}(2n+1, \mathbb{C})$$

We choose the same maximal torus as in $\mathfrak{so}(2n, \mathbb{C}) \subset \mathfrak{so}(2n+1, \mathbb{C})$. The roots and root vectors in $\mathfrak{so}(2n, \mathbb{C})$ remain roots in $\mathfrak{so}(2n+1, \mathbb{C})$. But we have additional root vectors:

$$[H, F_{2k-1,2n+1} + iF_{2k,2n+1}] = ih_k(F_{2k-1,2n+1} + iF_{2k,2n+1})$$

and their conjugates. Thus the root system is

$$\Delta = \{\pm\omega_i,\ \pm(\omega_i + \omega_j),\ \pm(\omega_i - \omega_j) \mid i < j\}$$

with Killing form $B = 2(2n-1)\sum \omega_i^2$. In the inner product $\langle \cdot\ ,\ \cdot \rangle = \frac{1}{2(2n-1)}B$ e_i and ω_i are orthonormal basis and the root vectors are $\pm e_i, \pm e_i \pm e_j$. Notice there are short roots of length 1 and long roots of length $\sqrt{2}$.

$$\mathfrak{g} = \mathfrak{sp}(n, \mathbb{C})$$

Recall that

$$\mathfrak{sp}(n, \mathbb{C}) = \left\{ \left(\begin{array}{c|c} B & S_1 \\ \hline S_2 & -B^T \end{array} \right) \mid B \in \mathfrak{gl}(n, \mathbb{C})\ ,\ S_i \in Sym_n(\mathbb{C}) \right\}$$

For a Cartan subalgebra we choose

$$\mathfrak{h} = \{H = \operatorname{diag}(h_1, \dots, h_n, -h_1, \dots, h_n) \mid h_i \in \mathbb{C}\}.$$

If we let

$$X_{k,l} = E_{n+k,l} + E_{n+l,k} \quad , \quad \bar{X}_{k,l} = E_{k,n+l} + E_{l,n+k},\ k \le l$$

$$Y_{k,l} = E_{k,l} - E_{n+l,n+k},\ k < l$$

then

$$[H, X_{k,l}] = -(h_k + h_l)X_{k,l} \quad , \quad [H, \bar{X}_{k,l}] = (h_k + h_l)\bar{X}_{k,l}$$

$$[H, Y_{k,l}] = (h_k - h_l)Y_{k,l}$$

Notice that this includes $[H, X_{k,k}] = -2h_kX_{k,k}$, $[H, \bar{X}_{k,k}] = 2h_k\bar{X}_{k,k}$. We normalize the Killing form so that $e_i = E_{i,i} - E_{n+i,n+i} \in \mathfrak{h}_\mathbb{R}$, and its dual ω_i are orthonormal basis. The roots are

$$\Delta = \{\pm 2\omega_i,\ \pm(\omega_i + \omega_j),\ \pm(\omega_i - \omega_j) \mid i < j\}$$

The Killing form is $B = 4(n+1)\sum \omega_i^2$ and we use the normalized inner product $\langle \cdot\ ,\ \cdot \rangle = \frac{1}{4(n+1)}B$ in which e_i and ω_i become an orthonormal basis of $\mathfrak{h}$ and $\mathfrak{h}^*$. The root vectors are thus $\pm e_i \pm e_j$. Here there are roots of length 2 and of length $\sqrt{2}$. Notice the difference between the root system for $\mathfrak{so}(2n+1, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$. Both Lie algebras have the same dimension and rank, but the difference in root systems will enable us to prove they are not isomorphic.

Exercises 4.11

(1) Find a basis for the root spaces of $\mathfrak{sl}(n, \mathbb{C})$ and $\mathfrak{sp}(n, \mathbb{C})$ in their compact real form $\mathrm{SU}(n)$ and $\mathrm{Sp}(n)$ with respect to the maximal torus from previous sections.
(2) For each classical Lie algebra, start with a fixed root and find all other roots by using only the string property in Theorem 4.8.
(3) Show that the only complex semisimple Lie algebra of rank 1 is isomorphic $\mathfrak{sl}(2, \mathbb{C})$.

4.2 Dynkin diagram and classification

Recall that the α-string containing β is of the form $\{\beta + n\alpha \mid -p \le n \le q\}$ and

$$n_{\alpha\beta} := \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = p - q$$

We will call $n_{\alpha\beta}$ the **Cartan integers**. Note that $\langle \alpha, \beta \rangle < 0$ implies $\alpha + \beta \in \Delta$, and $\langle \alpha, \beta \rangle > 0$ implies $\alpha - \beta \in \Delta$. If, on the other hand $\langle \alpha, \beta \rangle = 0$, then all we can say is that one can add α as many times as one can subtract it.

But notice that we always have the following important property:

$$\text{If } \alpha, \beta \in \Delta \text{ then } \beta - \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle}\alpha \in \Delta \tag{4.12}$$ subtract

Moreover

$$n_{\alpha\beta} \cdot n_{\beta\alpha} = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \cdot \frac{2\langle \beta, \alpha \rangle}{\langle \beta, \beta \rangle} = 4\cos^2(\measuredangle(H_\alpha, H_\beta)) \in \{0, 1, 2, 3, 4\}$$

since the product above is an integer, and is ≤ 4. Note that $4\cos^2(\measuredangle(H_\alpha, H_\beta)) = 4$ iff $\alpha = \pm\beta$ and otherwise $n_{\alpha\beta} = 0, \pm 1, \pm 2, \pm 3$. Also, if α and β are neither parallel nor orthogonal, one of $n_{\alpha\beta}, n_{\beta\alpha}$ is ± 1, and the other is $\pm 1, \pm 2, \pm 3$. We summarize the possible options in the following table. Notice though that the α string containing β could have more roots in it if $\beta + \alpha$ (in the case $n_{\alpha\beta} > 0$) is a root as well.

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\measuredangle(H_\alpha, H_\beta)$	Relative Size	α String Containing β
0	0	$\frac{\pi}{2}$	N/A	N/A
1	1	$\frac{\pi}{3}$	$\lvert\beta\rvert^2 = \lvert\alpha\rvert^2$	$\beta, \beta - \alpha$
-1	-1	$\frac{2\pi}{3}$	$\lvert\beta\rvert^2 = \lvert\alpha\rvert^2$	$\beta, \beta + \alpha$
2	1	$\frac{\pi}{4}$	$\lvert\beta\rvert^2 = 2\lvert\alpha\rvert^2$	$\beta, \beta - \alpha, \beta - 2\alpha$
-2	-1	$\frac{3\pi}{4}$	$\lvert\beta\rvert^2 = 2\lvert\alpha\rvert^2$	$\beta, \beta + \alpha, \beta + 2\alpha$
3	1	$\frac{\pi}{6}$	$\lvert\beta\rvert^2 = 3\lvert\alpha\rvert^2$	$\beta, \beta - \alpha, \beta - 2\alpha, \beta - 3\alpha$
-3	- 1	$\frac{5\pi}{6}$	$\lvert\beta\rvert^2 = 3\lvert\alpha\rvert^2$	$\beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha$

rootstable

Table 4.13. *Relationship between α, β if $|\beta| \geq |\alpha|$.*

As we will see, the last possibility $|\alpha|^2 = 3|\beta|^2$ is only possible for one simple Lie group, namely for the exceptional Lie group G_2.

These relationships are clearly very restrictive. As an example, we use them to classify semisimple Lie algebras of rank 2. For simplicity, we normalize the Killing form so that short root vectors have length 1. One has the following 4 possibilities.

1) There are 2 orthogonal roots $\pm\alpha, \pm\beta$ and no others, i.e. all roots are orthogonal. This is the Lie algebra $\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$.

2) There are 2 roots α, β with angle $\frac{\pi}{3}$, and thus $|\alpha|^2 = |\beta|^2 = 1$ and $\langle \alpha, \beta \rangle = \frac{1}{2}$. Hence $\beta - \alpha$ is a root with $|\beta - \alpha|^2 = 1$. The 6 roots are arranged at the vertices of a regular hexagon. There is clearly no room for any further roots satisfying the angle conditions in Table 4.13. This is the Lie algebra $\mathfrak{sl}(3, \mathbb{C})$.

3) There are 2 roots α, β with angle $\frac{\pi}{4}$, and thus $|\alpha|^2 = 1$, $|\beta|^2 = 2$, $\langle \alpha, \beta \rangle = 1$ and $\beta - \alpha, \beta - 2\alpha$ are roots as well. This implies $|\beta - \alpha|^2 =$

1, $|\beta - 2\alpha|^2 = 2$ and $\langle \alpha, \beta - \alpha \rangle = \langle \beta, \beta - 2\alpha \rangle = 0$. Thus there are 8 roots, arranged on the corners and midpoints of a square of side length 1. There is no room for any further roots satisfying the angle conditions. This is the Lie algebra $\mathfrak{so}(5, \mathbb{C})$.

4) There are 2 roots α, β with angle $\frac{\pi}{6}$, and thus $|\alpha|^2 = 1$, $|\beta|^2 = 3$, $\langle \alpha, \beta \rangle = \frac{3}{2}$ and $\beta, \beta - \alpha, \beta - 2\alpha, \beta - 3\alpha$ are roots as well. Since $2\langle \beta, \beta - 3\alpha \rangle / \langle \beta, \beta \rangle = -1$, $(\beta - 3\alpha) + \beta = 2\beta - 3\alpha$ must be a root as well. The 6 roots $\pm\alpha, \pm(\beta - \alpha), \pm(\beta - 2\alpha)$ have length 1 and form a regular hexagon. The 6 roots $\pm\beta, \pm(\beta - 3\alpha), \pm(2\beta - 3\alpha)$ have length squared 3 and form another regular hexagon. Two adjacent root vectors have angle $\frac{\pi}{6}$. There is no room for further root vectors. The Lie algebra has dimension 14. At this point it is not clear that such a Lie algebra exists, but we will see later on that it is one of the exceptional Lie algebras $\mathfrak{g}_2$.

To study root systems of higher rank, we need to organize the roots more systematically. For this we introduce a partial ordering.

order **Definition 4.14**

(a) *We call $H_0 \in \mathfrak{h}$* **regular** *if $\alpha(X) \neq 0$ for all $\alpha \in \Delta$ and* **singular** *otherwise.*

(b) *We say that $\alpha \leq \beta$ for $\alpha, \beta \in \Delta$ if $\alpha(H_0) \leq \beta(H_0)$ and set*

$$\Delta^+ = \{\alpha \in \Delta \mid \alpha(H_0) > 0\}.$$

(c) *A root $\alpha \in \Delta^+$ is called* **simple** *or* **fundamental** *if $\alpha \neq \beta + \gamma$ for any $\beta, \gamma \in \Delta^+\}$.*

(d) *We call $F = \{\alpha_1, \ldots, \alpha_n\}$ the set of fundamental roots.*

From now on we fix a regular element H_0 and hence Δ^+ and F.

Proposition 4.15

(a) *If $\alpha_i, \alpha_j \in F$, then $\langle \alpha_i, \alpha_j \rangle \leq 0$ for $i \neq j$.*

(b) *The elements of F are linearly independent.*

(c) *If $\alpha \in \Delta^+$, then $\alpha = \sum n_i \alpha_i$ with $n_i \geq 0$. In particular, F is a basis of $\mathfrak{h}_{\mathbb{R}}^*$*

Proof (a) If $\langle \alpha_i, \alpha_j \rangle > 0$, then $\alpha_i - \alpha_j \in \Delta$, so either $\alpha_i - \alpha_j \in \Delta^+$ or $\alpha_j - \alpha_i \in \Delta^+$. Hence either $\alpha_i = \alpha_j + (\alpha_i - \alpha_j)$ or $\alpha_j = \alpha_i + (\alpha_j - \alpha_i)$ is not in F, a contradiction.

(b) Let $F = \{\alpha_1, ..., \alpha_k\}$. Assume $\sum n_i\alpha_i = 0$. Moving negative coefficients to the right hand side, we have $\sum p_i\alpha_i = \sum q_i\alpha_i$ with $p_i, q_i > 0$ and each α_i appears in at most one summand. Then

$$\langle \sum p_i\alpha_i, \sum p_j\alpha_j\rangle = \langle \sum p_i\alpha_i, \sum q_j\alpha_j\rangle = \sum p_i q_j \langle \alpha_i, \alpha_j\rangle \le 0,$$

as each product of *distinct* fundamental roots is non-positive and p_i, q_i are positive. Therefore $\sum p_i\alpha_i = \sum q_i\alpha_i = 0$, and hence all n_i are zero.

(c) If α is not simple, it is the sum of 2 positive roots and continuing in this fashion, we can write α as a sum of simple roots, possibly with repetitions. □

We encode the roots as follows, where we have set $n_{ij} = n_{\alpha_i\alpha_j}$:

$\boxed{\texttt{dynkin}}$ **Definition 4.16** *Let Δ be the root system with fundamental roots $F = \{\alpha_1, \ldots, \alpha_n\}$. We define its* **Dynkin diagram** *as follows:*

(a) Draw one circle for each $\alpha_i \in F$.

(b) Connect $\alpha_i, \alpha_j \in F$ by $n_{ij}\, n_{ji}$ many lines.

(c) If $n_{ij}\, n_{ji} > 1$, make the circle corresponding to the shorter root solid.

Another convention in (c) is to draw an arrow from the longer root to the shorter root. We often denote the Dynkin diagram of $\mathfrak{g}$ by $D(\mathfrak{g})$.

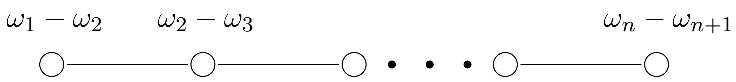
One can also encode the root system via its **Cartan Matrix** $A = (n_{ij})$. This matrix in fact plays a significant role in many areas of mathematics.

Classical Lie groups

We now exhibit the Dynkin diagrams of the classical Lie groups. It is conventional to use the following abbreviations for the Dynkin diagrams and their corresponding Lie algebras, where the index denotes their rank. A_n for $\mathfrak{sl}(n+1, \mathbb{C})$, B_n for $\mathfrak{so}(2n+1, \mathbb{C})$, C_n for $\mathfrak{sp}(n, \mathbb{C})$, and D_n for $\mathfrak{so}(2n, \mathbb{C})$. We choose the Cartan algebras as in the previous section and retain the notation.

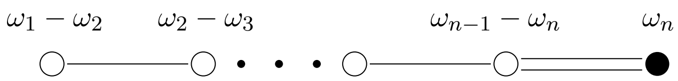
$\mathbf{A_n = \mathfrak{sl}(n+1, \mathbb{C})}$

The roots are $\Delta = \{\pm(\omega_i - \omega_j) \mid 1 \le i < j \le n+1\}$. We choose a regular element $H_0 = \mathrm{diag}(h_1, \ldots, h_{n+1})$ with $h_1 > h_2 > \cdots > h_{n+1}$ and $\sum h_i = 0$. Thus $\Delta^+ = \{\omega_i - \omega_j \mid i < j\}$. One easily sees that the simple roots are $F = \{\omega_1 - \omega_2, \omega_2 - \omega_3, \ldots, \omega_n - \omega_{n+1}\}$ and all Cartan integers are 1. Thus the Dynkin diagram is:



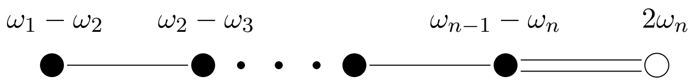
$\mathbf{B_n} = \mathfrak{so}(\mathbf{2n+1}, \mathbb{C})$

The roots are $\Delta = \{\pm\omega_i,\ \pm(\omega_i + \omega_j),\ \pm(\omega_i - \omega_j) \mid i < j\}$ and we choose a regular element $H_0 = \operatorname{diag}(h_1, \ldots, h_n)$ with $h_1 > h_2 > \cdots > h_n$. Then $\Delta^+ = \{\omega_i,\, \omega_i + \omega_j,\, \omega_i - \omega_j \mid i < j\}$ and $F = \{\omega_1 - \omega_2, \ldots, \omega_{n-1} - \omega_n,\, \omega_n\}$. Thus the Dynkin diagram is:



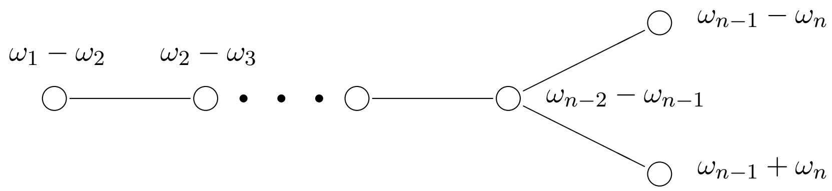
$\mathbf{C_n} = \mathfrak{sp}(\mathbf{n}, \mathbb{C})$

The roots are $\Delta = \{\pm 2\omega_i,\ \pm(\omega_i + \omega_j),\ \pm(\omega_i - \omega_j) \mid i < j\}$ and we choose a regular element $H_0 = \operatorname{diag}(h_1, \ldots, h_n)$ with $h_1 > h_2 > \cdots > h_n$. Then $\Delta^+ = \{2\omega_i,\, \omega_i + \omega_j,\, \omega_i - \omega_j \mid i < j\}$ and $F = \{\omega_1 - \omega_2, \ldots, \omega_{n-1} - \omega_n, 2\omega_n\}$. Thus the Dynkin diagram is:



$\mathbf{D_n} = \mathfrak{so}(\mathbf{2n}, \mathbb{C})$

The roots are $\Delta = \{\pm(\omega_i + \omega_j),\ \pm(\omega_i - \omega_j) \mid i < j\}$ and we choose a regular element $H_0 = \operatorname{diag}(h_1, \ldots, h_n)$ with $h_1 > h_2 > \cdots > h_n$. Then $\Delta^+ = \{\omega_i + \omega_j,\, \omega_i - \omega_j \mid i < j\}$ and $F = \{\omega_1 - \omega_2, \ldots,\, \omega_{n-1} - \omega_n,\, \omega_{n-1} + \omega_n\}$. Thus the Dynkin diagram is:



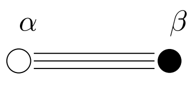
We will now show that one can recover the root system from the Dynkin diagram. Let $F = \{\alpha_1, \dots, \alpha_n\}$ be the fundamental roots of Δ. If $\alpha \in \Delta^+$, write $\alpha = \sum n_i \alpha_i$ with $n_i \in \mathbb{Z}$, $n_i \geq 0$ and define the **level** of α to be $\sum_i n_i$.

level **Lemma 4.17** *Given $\alpha \in \Delta^+$ of level n, there exists an α^* of level $n-1$ and a simple root $\alpha_i \in F$ such that $\alpha = \alpha^* + \alpha_i$.*

Proof Let $0 \neq \alpha = \sum n_i \alpha_i$, with $\sum n_i = n$. Then $\langle \alpha, \alpha \rangle = \sum n_i \langle \alpha, \alpha_i \rangle > 0$. Since $n_i \geq 0$, there exists at least one simple root, say α_k, with $\langle \alpha, \alpha_k \rangle > 0$ and $n_k \geq 1$. Thus $\alpha - \alpha_k$ is a root, and is positive since $n_k \geq 1$. But $\alpha - \alpha_k$ has level $n-1$ and $\alpha = (\alpha - \alpha_k) + \alpha_k$, which proves our claim. $\square$

From the Dynkin diagram we recover the integers $n_{ij}\, n_{ji}$. Since the values of n_{ij} are only $\pm 1, \pm 2, \pm 3$, and since the diagram tells us which root the shorter one is, Table 4.13 determines the values of n_{ij}. We thus recover the lengths and inner products between all simple roots. Next, we reconstruct all positive roots one level at a time. To go from one level to the next, we need to decide if a simple root can be added. But this is determined by the string property of roots since we already know how many times it can be subtracted.

g2 **Example 4.18** The exceptional Lie group G_2 has Dynkin diagram



According to Table 4.13, we have $\frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} = -1$ and $\frac{2\langle \alpha, \beta \rangle}{\langle \beta, \beta \rangle} = -3$. Thus we have only one root of level one, namely $\alpha + \beta$. According to the string property, we cannot add α, but we can add β to obtain $\alpha + 2\beta$, the only root of level 2. To this we cannot add α since twice a root cannot be a root, but we can add β again to obtain $\alpha + 3\beta$, the only root of level 4. We are not allowed to add β anymore, but we can add α since $\frac{2\langle\alpha+3\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = 2 + \frac{6\langle\beta,\alpha\rangle}{\langle\alpha,\alpha\rangle} = -1$. To the root $2\alpha + 3\beta$ of level 5 we clearly cannot add α or beta, and hence we are done. There are 6 positive roots, 6 negative ones, and hence the Lie algebra has dimension 14.

Next we will show that one can reduce the classification to connected Dynkin diagrams.

Proposition 4.19 *The Lie algebra $\mathfrak{g}$ is simple iff the Dynkin diagram $D(\mathfrak{g})$ is connected.*

Proof Assume that $\mathfrak{g} \simeq \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and $\mathfrak{h}_i \subset \mathfrak{g}_i$ Cartan subalgebras with root systems Δ_i. Then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ clearly is a Cartan subalgebra of $\mathfrak{g}$. If $\alpha_i \in \Delta_i$ with root spaces $\mathfrak{g}_{\alpha_i}$, then $\mathfrak{g}_{\alpha_1} \oplus 0$ and $0 \oplus \mathfrak{g}_{\alpha_2}$ are root spaces with roots $\beta_i(H_1, H_2) = \alpha_i(H_i)$. The ideals $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are clearly orthogonal with respect to the Killing form and hence $\langle H_{\beta_1}, H_{\beta_2}\rangle = 0$. If H_i are regular elements of $\mathfrak{h}_i$, then (H_1, H_2) is a regular element for $\mathfrak{h}$. Now one easily sees that the Dynkin diagram $D(\mathfrak{g})$ breaks up into two components, namely the Dynkin diagrams $D(\mathfrak{g}_1)$ and $D(\mathfrak{g}_2)$ with no arrows connecting them.

Conversely, given two connected components D_i of a Dynkin diagram, all simple roots in one are orthogonal to all simple roots in the other. Let $\mathfrak{h}_i$ be the subspace generated by the simple root vectors of each. Then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\langle \mathfrak{h}_1, \mathfrak{h}_2\rangle = 0$. Constructing all positive roots from the simple ones as above, one level at a time, we see that the set of roots brake up into two sets, $\Delta^+ = \Delta_1^+ \cup \Delta_2^+$ with the root vectors of Δ_i^+ lying in $\mathfrak{h}_i$. Furthermore, if $\alpha_i \in \Delta_i^+$, $\alpha_1 + \alpha_2$ is never a root. Of course $\Delta^- = -\Delta^+$. We can thus let $\mathfrak{g}_i$ be the span of $\mathfrak{h}_i$ and the root spaces corresponding to roots in Δ_i. This implies that $\mathfrak{g}_i$ are ideals and $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. □

In particular

Corollary 4.20 *The classical Lie algebras $\mathfrak{sl}(n, \mathbb{C})$, $\mathfrak{sp}(n, \mathbb{C})$, as well as $\mathfrak{so}(n, \mathbb{C})$ with $n \neq 2, 4$, are all simple.*

Notice that for $\mathfrak{so}(4, \mathbb{C})$ the roots are $\pm(\omega_1 + \omega_2)$, $\pm(\omega_1 - \omega_2)$. Hence the 2 positive roots are orthogonal and its Dynkin diagram is the disconnected union of 2 circles, corresponding to the fact that $\mathfrak{so}(4, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C}) \oplus \mathfrak{so}(3, \mathbb{C})$.

For the classification of the Dynkin diagrams, it is sufficient to make the following assumptions on the roots.

rootsystem

Definition 4.21 *An abstract* **root system** *is the data* $(V, \langle\cdot,\cdot\rangle, \Delta)$*, where* V *is a real vector space,* $\langle\cdot,\cdot\rangle$ *a positive definite inner product, and* D *a set of (root) vectors in* V *that span* V*, and such that:*

(a) If $\alpha, \beta \in \Delta$*, then* $\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$*;*
(b) If $\alpha, \beta \in \Delta$*, and* $\beta = c \cdot \alpha$*, then* $c = 0, \pm 1$*;*
(c) If $\alpha, \beta \in \Delta$*, then* $\beta - \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}\alpha \in \Delta$*.*

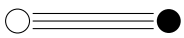
Notice that (c) is weaker than the string property. Given this definition of a root system, however, it is possible to re-obtain many of the results we proved. The Dynkin diagram is defined similarly: A vector $H_0 \in V$ is called regular if $\langle H_0, v\rangle \neq 0$ for all $v \in \Delta$. This determines the positive root vectors $\Delta^+ = \{v \in \Delta \mid \langle H_0, v\rangle > 0\}$ and one defines the Dynkin diagrams as before. A root system is called simple if Δ cannot be decomposed into two mutually orthogonal sets of root vectors. As above, one shows that the root system is simple iff its Dynkin diagram is connected. This gives rise to a classification of Dynkin diagrams.

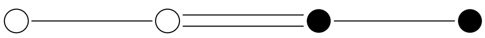
diagramclass

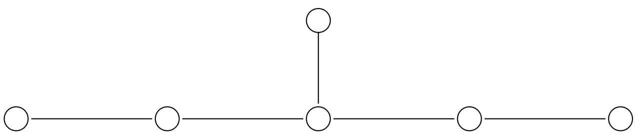
Theorem 4.22 *Given a simple root system* $(V, \langle\cdot,\cdot\rangle, \Delta)$*, its Dynkin diagram is one of the following:*

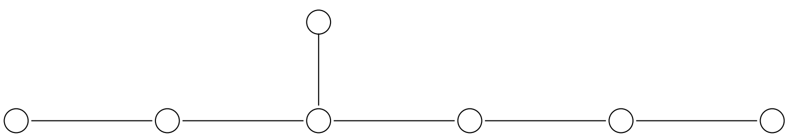
(a) The Dynkin diagram associated to one of the classical semisimple Lie algebras A_n*,* $n \geq 1$*;* B_n*,* $n \geq 3$*;* C_n*,* $n \geq 2$*;* D_n*,* $n \geq 4$*.*
(b) One of the exceptional Dynkin diagrams G_2, F_4, E_6, E_7, E_8*.*

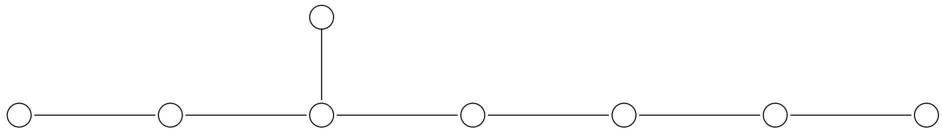
The exceptional diagrams are given by:

G_2 

F_4 

E_6 

E_7 

E_8 

The proof of this classification is purely combinatorial and pretty straight forward but tedious. We will thus omit the proof.

Notice that $\mathfrak{sl}(3,\mathbb{C})$, $\mathfrak{so}(3,\mathbb{C})$, $\mathfrak{sp}(1,\mathbb{C})$ all have the same Dynkin diagram and hence, by the above, are isomorphic. Furthermore, $B_2 = C_2$ and $A_3 = D_3$, corresponding to the fact that $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$, resp. $\mathfrak{su}(4)$ and $\mathfrak{so}(6)$ are isomorphic as real Lie algebras, and hence via complexification as complex ones as well, see Proposition 2.21. This explains the restriction on the indices in Theorem 4.22. Notice though that the equality of Dynkin diagrams does not (yet) imply that the corresponding compact Lie algebras are isomorphic since a complex Lie algebra can have many real forms.

To finish the classification of semisimple complex Lie algebras we still need to prove the following.

diagramclass2

Theorem 4.23 *Let $\mathfrak{g}$ be a complex simple Lie algebra.*

(a) The Dynkin diagram of $\mathfrak{g}$ is independent of the choice of a Cartan subalgebra and a choice of ordering.
(b) Two simple Lie algebras are isomorphic iff their Dynkin diagrams are the same.
(c) Each Dynkin diagram arises from a complex simple Lie algebra.

In the next two sections we will be able to prove part (a) and (b). For part (c) one can either exhibit for each Dynkin diagram a simple Lie group, as we have done for the classical ones, or give an abstract proof that such Lie algebras exist, see... We prefer the first approach and will construct the exceptional Lie algebras in a later chapter.

Exercises 4.24

(1) Show that a string has length at most 4.
(2) For some of the classical Lie algebras of low rank, start with the Dynkin diagram and find all roots by the process performed in Example 4.18
(3) Given a root system as defined in Definition 4.21, prove the string

property of roots and convince yourself that we recover all the properties we proved for the roots and Dynkin diagram of a semisimple Lie algebra in the case of a root system.

4.3 Weyl Chevally Normal Form

There exists a basis of a simple Lie algebra which is almost canonical and has important consequences. Before we describe the basis, we introduce another useful concept. Recall that associated to a root $\alpha \in \Delta$, we have the root vectors H_α defined by $\langle H_\alpha, H\rangle = \alpha(H)$. We call the renormalized vector

$$\tau_\alpha = \frac{2}{\langle \alpha, \alpha\rangle} H_\alpha$$

the **co-root** or **inverse root**. Notice in particular that $\alpha(\tau_\alpha) = 2$ and that, unlike the root vectors H_α, the coroots τ_α are independent of the scaling of the inner product. Another reason why they are important, although we will see further ones later on, is that

$$[\tau_\alpha, X_\beta] = \frac{2\langle \alpha, \beta\rangle}{\langle \alpha, \alpha\rangle} X_\beta = \beta(\tau_\alpha) X_\beta = n_{\alpha,\beta} X_\beta$$

and thus the coefficients are integers. From now on we will also choose $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}$ such that

$$[X_\alpha, X_{-\alpha}] = \tau_\alpha.$$

Much less trivial is that we can choose the whole basis of $\mathfrak{g}$ such that all Lie brackets have integer coefficients. This is the content of the Weyl Chevally normal form:

Weylbasis

Theorem 4.25 *Let $\mathfrak{g}$ be a complex simple Lie algebra with root system Δ, positive roots Δ^+ ordered by some H_0, and fundamental roots $F = \{\alpha_1, \ldots, \alpha_n\}$. Then there exists a basis for $\mathfrak{g}$ consisting of the coroots $\tau_{\alpha_i} \in \mathfrak{h}, 1 \le i \le n$ and $X_\alpha \in \mathfrak{g}_\alpha, X_{-\alpha} \in \mathfrak{g}_{-\alpha}$, for all $\alpha \in \Delta^+$, such that:*

(a) $[\tau_{\alpha_i}, X_\beta] = n_{\alpha_i,\beta} X_\beta$,
(b) $[X_\alpha, X_{-\alpha}] = \tau_\alpha$,
(c) If $\alpha, \beta, \alpha+\beta \in \Delta$, then $[X_\alpha, X_\beta] = \pm(t+1) X_{\alpha+\beta}$, where t is the largest integer such that $\beta - t\alpha \in \Delta$,

for an appropriate choice of signs.

Some comments are in order. We already saw that (a) holds, and (b) is clearly possible for appropriate choice of X_α. So the main content is part (c). If $\alpha, \beta, \alpha+\beta \in \Delta$, we already saw that $[X_\alpha, X_\beta] = N_{\alpha,\beta}X_{\alpha+\beta}$ for some non-zero $N_{\alpha,\beta} \in \mathbb{C}$. One first shows that $N_{\alpha,\beta}N_{-\alpha,-\beta} = -(t+1)^2$ for any choice of X_α satisfying (b). Then one shows that one can choose X_α inductively so that $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ by induction on the level of the roots (after fixing an ordering). Along the way one is forced to make certain sign changes. To explain the issue, say that γ has level k and $\gamma = \alpha+\beta$ where the level of α, β is less than k. If one chooses a sign for $N_{\alpha,\beta}$, then the signs for N_{α^*,β^*} with $\gamma = \alpha^* + \beta^*$ are determined. It turns out that one can make an arbitrary choice for one of these at each level.

The vectors X_α can of course be changed by c_α as long as $c_\alpha c_{-\alpha} = 1$. Up to these choices of c_α, and the above choice of signs at each level, the basis is unique.

We now derive several consequences.

integralbasis **Corollary 4.26** *There exists a basis such that all structure constants are integers.*

In particular, the classical and exceptional Lie algebras exist over any field. More importantly

iso **Proposition 4.27** *Let $\mathfrak{g}_i$ be two semisimple Lie algebras with Cartan subalgebras $\mathfrak{h}_i$ and root systems Δ_i. If $f\colon \Delta_1 \to \Delta_2$ is a bijection such that $f(-\alpha) = -f(\alpha)$ and $f(\alpha+\beta) = f(\alpha) + f(\beta)$ whenever $\alpha, \beta, \alpha+\beta \in \Delta_1$, then there exists an isomorphism $\phi\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ with $\phi(\mathfrak{h}_1) = \mathfrak{h}_2$ which induces f on the respective set of roots.*

Proof Since f preserves addition of roots, the chain properties of roots imply that f also preserves the length and inner products of all roots (after scaling both inner products appropriately). Hence we can define $f^*\colon \mathfrak{h}_1 \to \mathfrak{h}_2$ first by sending n linearly independent root vectors in $\mathfrak{h}_1$ to the corresponding ones in $\mathfrak{h}_2$. This defines an isometry, after appropriate scaling, since length and inner products are determined by the string property. By the same reasoning f^* carries all root vectors to corresponding root vectors. Next we define an order with respect to an arbitrary choice of a regular vector $H_0 \in \mathfrak{h}_1$ and $f(H_0) \in \mathfrak{h}_2$, which one easily sees is regular as well. We then choose the basis $\tau_{\alpha_i}, X_\alpha$ for both inductively as in the proof of Theorem 4.25, making the same sign changes along the way. By setting $f^*(X_\alpha) = X_{f(\alpha)}$ we get the desired isomorphism. $\square$

As we saw earlier, one can recover the whole root system from its Dynkin diagram. Thus

Dynkiniso **Corollary 4.28** *Two semisimple Lie algebras with the same Dynkin diagram are isomorphic.*

Exercises 4.29

(1) Prove the existence of the covers in Proposition 2.20 and Proposition 2.21 just from the Dynkin diagrams.
(2) If Δ is a root system as defined in Definition 4.21, let $\Delta^* = \{\tau_\alpha \mid \alpha \in \Delta\}$ be the set of coroots. Show that Δ^* is a root system as well. For the simple Lie algebras with root system Δ, identify the Lie algebra with root system Δ^*.

4.4 Weyl group

To obtain more information we will study the Weyl group of a complex semisimple Lie algebra. In the next section we will see that it is equal to the Weyl group of its compact real form.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and root system Δ. For $\alpha \in \Delta$ we define the reflections in the hyperplane $\{X \in \mathfrak{h}_\mathbb{R} \mid \alpha(X) = 0\}$ which are given by

$$s_\alpha\colon \mathfrak{h} \to \mathfrak{h} \;:\qquad s_\alpha(H) = H - \frac{2\langle H, \alpha\rangle}{\langle \alpha, \alpha\rangle} H_\alpha = H - \langle H, \alpha\rangle \tau_\alpha.$$

This is indeed the desired reflection since $\alpha(H) = 0$ implies $s_\alpha(H) = H$ and $s_\alpha(H_\alpha) = H_\alpha - 2H_\alpha = -H_\alpha$. It also induces a reflection in $\mathfrak{h}^*$, which we again denote by s_α, which acts on roots:

$$s_\alpha(\beta) = \beta - \frac{2\langle \beta, \alpha\rangle}{\langle \alpha, \alpha\rangle}\alpha = \beta - \beta(\tau_\alpha)\alpha.$$

Notice that the string property implies that $s_\alpha(\beta) \in \Delta$ if $\alpha, \beta \in \Delta$, i.e. s_α permutes roots (resp. root vectors). Thus s_α preserves $\mathfrak{h}_\mathbb{R}$. It also follows from Proposition 4.27 that s_α is the restriction of an automorphism of $\mathfrak{g}$ to $\mathfrak{h}$, although we will later see that it is even an inner automorphism. Notice that property (c) of a root system has the geometric interpretation that the root system is invariant under the Weyl group.

Definition 4.30

(a) The **Weyl group** $W(\mathfrak{g})$ *is the group generated by the reflections* s_α, $\alpha \in \Delta$.

(b) The **Weyl chambers** *are the components of the complement of* $\cup_{\alpha\in\Delta} \ker \alpha$.

Notice that the union of the Weyl chambers are precisely the regular elements of $\mathfrak{g}$. We will denote a generic Weyl Chamber by WC.

Weyl-ordering

Proposition 4.31

(a) If two regular elements H_1, H_2 lie in the same Weyl chamber, then they define the same ordering. Thus Weyl chambers are in one-to-one correspondence with orderings of the root system.

(b) $W(\mathfrak{g})$ takes Weyl chambers to Weyl chambers and acts transitively on the Weyl chambers.

Proof (a) The ordering is determined by the set of positive roots. But a Weyl chamber is convex and hence connected, so if a root is positive on one vector it must be positive on all.

(b) To see that Weyl chambers are taken to Weyl chambers we need to show that singular vectors are taken to singular ones. But by duality $s_\alpha(\beta)(H) = \beta(s_\alpha(H))$ or $s_\alpha(\beta)(s_\alpha(H)) = \beta(H)$. Since $s_\alpha(\beta) \in \Delta$ if $\alpha, \beta \in \Delta$, this implies that if H is singular, so is $s_\alpha(H)$.

If WC' is another Weyl chamber, we can choose a sequence of Weyl chambers $WC = WC_1, \ldots, WC_k = WC'$ such that WC_i shares a "wall" $\ker \alpha_i$ with WC_{i+1}. The reflection in this wall clearly takes WC_i to WC_{i+1} and hence the composition of such reflections takes WC to WC'. Thus w also takes Cartan integers to Cartan integers and these determine the Dynkin diagram. □

If we specify an ordering, we sometimes denote the Weyl chamber corresponding to this ordering as WC^+.

We are now ready to show that the Dynkin diagram is independent of any choices.

G-Dynkin

Proposition 4.32 *Let $\mathfrak{g}_1, \mathfrak{g}_2$ be two semisimple Lie algebras with Cartan algebras $\mathfrak{h}_i$ and roots Δ_i.*

(a) *If $A\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ is an isomorphism with $A(\mathfrak{h}_1) = \mathfrak{h}_2$, then $\alpha \circ A \in \Delta_1$ if $\alpha \in \Delta_2$.*
(b) *The Dynkin diagram of $\mathfrak{g}$ does not depend on the choice of Cartan subalgebra and ordering.*
(c) *Isomorphic Lie algebras have the same Dynkin diagram.*

Proof (a) Since $\alpha \in \Delta_2$, we have $[H, X_\alpha] = \alpha(H)X_\alpha$ and applying the isomorphism A^{-1} we get $[A^{-1}(H), A^{-1}(X_\alpha)] = \alpha(H)A^{-1}(X_\alpha)$ or $[H', A^{-1}(X_\alpha)] = \alpha(A(H'))A^{-1}(X_\alpha)$ which proves our claim.

(b) Let $\mathfrak{h}_i$ be two different Cartan subalgebras of $\mathfrak{g}$ and $WC_i \subset \mathfrak{h}_i$ two Weyl chambers defining an ordering and hence fundamental roots F_i. Then by Theorem 4.2 there exists $A \in \mathrm{Aut}(\mathfrak{g})$ such that $A(\mathfrak{h}_1) = \mathfrak{h}_2$ and by part (a) A takes roots to roots. This implies that A takes Weyl chambers to Weyl chambers and hence by Proposition 4.31 (b) there exists $w \in W(\mathfrak{g}_2)$ with $w(A(WC_1)) = WC_2$. w is also a restriction of an automorphism B and replacing A by BA we can assume that $A(WC_1) = WC_2$. Hence A takes positive roots to positive ones and simple ones to simple ones. Since an automorphism is an isometry with respect to the Killing form, inner products are also the same, which implies the Dynkin diagrams are the same.

(c) Let $A\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ be the isomorphism and $\mathfrak{h}_1$ a Cartan subalgebra of $\mathfrak{g}_1$ and H_0 a regular element. Clearly $A(\mathfrak{h}_1)$ is a Cartan subalgebra in $\mathfrak{g}_2$ and A takes roots with respect to $\mathfrak{h}_1$ to roots with respect to $A(\mathfrak{h}_1)$. Hence $A(H_0)$ is regular in $A(\mathfrak{h}_1)$. Now we proceed as in (b). □

We have now established the desired one-to-one correspondence between semisimple Lie algebras and Dynkin diagrams, and between simple Lie algebras and connected Dynkin diagrams.

Exercises 4.33

(1) Show that if α is a root, then there exists a $w \in W$ such that $w\alpha$ is a simple root.
(2) Show that there are at most two possible values for the length of roots. We can thus talk about long roots and short roots. Show that the set of long roots form a subalgebra and identify it for each of the classical simple Lie algebras.

4.5 Compact forms

In this section we relate our classification results to compact Lie algebras and show the two definitions of a Weyl group are the same.

compactrealform **Proposition 4.34** *Every complex semisimple Lie algebra $\mathfrak{g}$ has a compact real form.*

Proof Choose a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and an ordering Δ^+. Next, choose a basis $\tau_\alpha, X_\alpha, X_{-\alpha}$ as in Theorem 4.25 and let $\mathfrak{k}$ be the real span of $i\tau_\alpha$, $i(X_\alpha + X_{-\alpha})$, $X_\alpha - X_{-\alpha}$, $\alpha \in \Delta^+$. We claim that this is indeed a compact real form. First, we check that it is a subalgebra:

$$[i\tau_\alpha, i(X_\beta + X_{-\beta})] = [-\tau_\alpha, X_\beta] - [\tau_\alpha, X_{-\beta}] = -\frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle}(X_\beta - X_{-\beta})$$

$$[i\tau_\alpha, (X_\beta - X_{-\beta})] = \frac{2\langle\alpha,\beta\rangle}{\langle\alpha,\alpha\rangle} i(X_\beta + X_{-\beta})$$

For the remaining brackets, recall that in the proof of Theorem 4.25 one shows that $N_{\alpha,\beta} = -N_{-\alpha,-\beta}$ where $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$. Thus for example

$$[X_\alpha - X_{-\alpha}, i(X_\beta + X_{-\beta}] = N_{\alpha,\beta} i(X_{\alpha+\beta} + X_{-\alpha-\beta}) + N_{\alpha,-\beta} i(X_{\alpha-\beta} + X_{\beta-\alpha}).$$

and similarly for the others.

It is also clear that $\mathfrak{k}$ is a real form. To see that it is compact, we will show that $B_{|\mathfrak{k}} < 0$. Since the Killing form restricted to a real form is again the Killing form, i.e. $B_{\mathfrak{g}|\mathfrak{k}} = B_{\mathfrak{k}}$, Proposition 3.25 finishes our proof. Recall that $\langle X_\alpha, X_\beta\rangle = 0$ unless $\alpha + \beta = 0$ and hence the above root space vectors are orthogonal to each other, and to $\mathfrak{k}$ as well. Furthermore, from $[X_\alpha, X_{-\alpha}] = \langle X_\alpha, X_{-\alpha}\rangle H_\alpha$ it follows that $\langle X_\alpha, X_{-\alpha}\rangle = 2/\langle\alpha,\alpha\rangle$. Thus $\langle X_\alpha - X_{-\alpha}, X_\alpha - X_{-\alpha}\rangle = -2/\langle\alpha,\alpha\rangle < 0$ and $\langle i(X_\alpha + X_{-\alpha}), i(X_\alpha + X_{-\alpha})\rangle = -2/\langle\alpha,\alpha\rangle < 0$. Since $B_{|\mathfrak{h}_\mathbb{R}} > 0$, we have $B_{|i\mathfrak{h}_\mathbb{R}} < 0$, and the claim follows. □

As a special case, we observe the following. Since

$$[\tau_\alpha, X_\alpha] = 2X_\alpha,\ [\tau_\alpha, X_{-\alpha}] = -2X_{-\alpha},\ [X_\alpha, X_{-\alpha}] = \tau_\alpha$$

we obtain a subalgebra

$$\mathfrak{sl}_\alpha = \operatorname{span}_\mathbb{C}\{\tau_\alpha, X_\alpha, X_{-\alpha}\} \subset \mathfrak{g} \tag{4.35}$$ sla

which isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, the isomorphism given by

$$\tau_\alpha \to \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_\alpha \to \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad X_{-\alpha} \to \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Furthermore,

$$\mathfrak{k}_\alpha = \mathrm{span}_{\mathbb{R}}\{i\tau_\alpha,\, i(X_\alpha + X_{-\alpha}),\, X_\alpha - X_{-\alpha}\} \subset \mathfrak{k} \tag{4.36}$$ ka

is a compact subalgebra isomorphic to $\mathfrak{su}(2)$, the isomorphism given by

$$i\tau_\alpha \to \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad i(X_\alpha + X_{-\alpha}) \to \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad X_\alpha - X_{-\alpha} \to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Notice that $\mathfrak{k}_\alpha \otimes \mathbb{C} \subset \mathfrak{k} \otimes \mathbb{C} \simeq \mathfrak{g}$ agrees with sl_α. Furthermore, $\mathfrak{k}_\alpha$ generates a subgroup of K which is isomorphic to SU(2) or SO(3) depending on the choice of α.

These 3-dimensional subalgebras $\mathfrak{sl}_\alpha$ and $\mathfrak{k}_\alpha$, generated by each root $\alpha \in \Delta$, are crucial in understanding the structure of $\mathfrak{g}$.

The uniqueness of the compact real form is closely connected to the classification of all real forms.

realform

Theorem 4.37 *Let $\mathfrak{g}$ be a complex semisimple lie Algebra, and $\mathfrak{k}$ a compact real form. Given $A \in \mathrm{Aut}(\mathfrak{k})$ with $A^2 = Id$, we can decompose $\mathfrak{k}$ as*

$$\mathfrak{k} = \mathfrak{m} \oplus \mathfrak{n}, \quad A|_{\mathfrak{m}} = \mathrm{Id}, \quad A|_{\mathfrak{n}} = -\,\mathrm{Id}\,.$$

Then

(a) $\mathfrak{q} = \mathfrak{m} \oplus i\mathfrak{n}$ is a subalgebra of $\mathfrak{g}$ and is a real form of $\mathfrak{g}$.
(b) Every real form of $\mathfrak{g}$ arises in this fashion, unique up to inner automorphisms of $\mathfrak{g}$.

The proof is non-trivial and we will omit it here. But we make the following observations.

Since $A[X,Y] = [AX, AY]$, we have:

$$[\mathfrak{m},\mathfrak{m}] \subset \mathfrak{m}, \quad [\mathfrak{m},\mathfrak{p}] \subset \mathfrak{p}, \quad [\mathfrak{p},\mathfrak{p}] \subset \mathfrak{m},$$

This implies in particular that $\mathfrak{q}$ is a subalgebra of $\mathfrak{g}$. And $\mathfrak{k} \otimes \mathbb{C} = \mathfrak{g}$ clearly implies that $\mathfrak{q} \otimes \mathbb{C} = \mathfrak{g}$, i.e. $\mathfrak{q}$ is a real form. Since $\mathfrak{k}$ is compact, its Killing form is negative definite. This implies that it is negative definite on $\mathfrak{m}$ and positive definite on $i\mathfrak{n}$. Thus $\mathfrak{q}$ is a non-compact real form, unless $\mathfrak{n} = 0$. In particular:

uniquecompactform **Corollary 4.38** *Any two compact real forms of $\mathfrak{g}$ are conjugate in $\mathfrak{g}$.*

This implies a classification for compact semisimple Lie algebras:

compactclass **Corollary 4.39** *There is a one-to-one correspondence between complex semisimple Lie algebras and compact real forms. Hence there is a one-to-one correspondence between (connected) Dynkin diagrams and compact simple Lie algebras.*

As we saw, for the classical simple Lie algebras we have:

$$(\mathfrak{g}, \mathfrak{k}) = (\mathfrak{sl}(n, \mathbb{C}), \mathfrak{su}(n)), \ , \ (\mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n)), \ , \ (\mathfrak{sp}(n, \mathbb{C}), \mathfrak{sp}(n)).$$

Hence $\mathrm{SU}(n)$, $\mathrm{SO}(n)$, $\mathrm{Sp}(n)$ are, up to covers, the classical simple compact Lie groups, and the exceptional simply connected ones are, using the name name as the one for the complex Lie algebra, G_2, F_4, E_6, E_7, E_8. Any other compact Lie group is up to covers, a product of these simple ones, possibly with a torus as well.

Theorem 4.37 enables one to classify all non-compact real simple Lie algebras as well.

Example 4.40 The real forms of $\mathfrak{sl}(n, \mathbb{C})$ are $\mathfrak{sl}(n, \mathbb{R})$, $\mathfrak{su}(p, q)$ and $\mathfrak{sl}(n, \mathbb{H})$. Include proof....

We now show that the two definitions of a Weyl group agree, which enables us to prove a further property of the Weyl group.

WeylKG **Proposition 4.41** *Let K be a compact semisimple Lie group and $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ with $\mathfrak{t} \subset \mathfrak{k}$ maximal abelian and $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C} \subset \mathfrak{g}$ a Cartan subalgebra.*

(a) the action of the Weyl group $N(T)/T$ of K on $\mathfrak{t}$ agrees with the restriction of the action of $W(\mathfrak{g})$ to $i\mathfrak{h}_{\mathbb{R}}$.
(b) The Weyl group acts simply transitively on the set of Weyl chambers, i.e. if $w \in W(\mathfrak{g})$ takes a Weyl chamber to itself, $w = \mathrm{Id}$.

Proof For one direction we need to show that for all $w \in W(\mathfrak{g})$, there exists an $n \in N(T)$ such that $w = i \,\mathrm{Ad}(n)_{|\mathfrak{t}}$. It is clearly sufficient to do this for the reflections s_α, $\alpha \in \Delta$.

Recall that the 3 vectors

$$A_\alpha = i\tau_\alpha, \quad B_\alpha = X_\alpha + X_{-\alpha}, \quad C_\alpha = i(X_\alpha - X_{-\alpha})$$

form a subalgebra of $\mathfrak{k}$ isomorphic to $\mathfrak{su}(2)$ with

$$[A_\alpha, B_\alpha] = 2C_\alpha, \quad [B_\alpha, C_\alpha] = 2A_\alpha, \quad [A_\alpha, C_\alpha] = -2B_\alpha.$$

We now consider the one parameter group $g(s) = \exp(sB_\alpha)$ and claim that $n = g(\frac{\pi}{2})$ is the desired reflection s_α. For this we need to show that $\mathrm{Ad}(n)H = H$ if $\alpha(H) = 0$ and $\mathrm{Ad}(n)H_\alpha = -H_\alpha$. In the first case $\mathrm{Ad}(g(s))H = Ad(\exp(sB_\alpha))(H) = e^{\mathrm{ad}_{sB_\alpha}}H = H$ since $[B_\alpha, H] = [X_\alpha - X_{-\alpha}, H] = \alpha(H)(X_\alpha + X_{-\alpha}) = 0$.

For the second case observe that

$$\mathrm{ad}_{sB_\alpha}(A_\alpha) = -2sC_\alpha,\ \mathrm{ad}^2_{sB_\alpha}(A_\alpha) = -4s^2A_\alpha,\ \mathrm{ad}^3_{sB_\alpha}(A_\alpha) = 8s^3C_\alpha$$

and hence

$$\begin{aligned} Ad(g(s))(A_\alpha) &= Ad(\exp(sB_\alpha))(A_\alpha) = e^{\mathrm{ad}_{sB_\alpha}}(A_\alpha) = \sum_i \frac{\mathrm{ad}^i_{sB_\alpha}(A_\alpha)}{i!} \\ &= \sum_i \frac{(2s)^{2i}(-1)^iA_\alpha}{(2i)!} - \sum_i \frac{(2s)^{2i+1}(-1)^iC_\alpha}{(2i+1)!} \\ &= \cos(2s)A_\alpha - \sin(2s)C_\alpha \\ &= -A_\alpha \qquad \text{if } s = \frac{\pi}{2} \end{aligned}$$

This finishes the proof of one direction. For the other direction we need to show that if $n \in N(T)$, then $\mathrm{Ad}(n)_{|\mathfrak{t}} = w_{|i\mathfrak{t}}$ for some $w \in W(\mathfrak{g})$.

Recall that if $A \in \mathrm{Aut}(\mathfrak{g})$ with $A(\mathfrak{h}) \subset \mathfrak{h}$, then A permutes roots. Thus $\mathrm{Ad}(n)$ permutes roots and hence takes Weyl chambers to Weyl chambers. Fix an ordering and hence a positive Weyl chamber WC^+. Then $\mathrm{Ad}(n)WC^+$ is another Weyl chamber and hence there exists $w \in W(\mathfrak{g})$ with $w\,\mathrm{Ad}(n)(WC^+) = WC^+$. We already saw that $w = \mathrm{Ad}(n')$ for some $n' \in N(T)$ and hence $\mathrm{Ad}(n'n)(WC^+) = WC^+$. We will now show that this implies $n'n \in T$ and hence $\mathrm{Ad}(n)_\mathfrak{t} = \mathrm{Ad}(n')^{-1}_\mathfrak{t} = w^{-1} = w$.

So let $g \in N(T)$ with $\mathrm{Ad}(g)(WC^+) = WC^+$ and choose $H_0 \in WC^+$. Since $N(T)/T$ is finite, there exists a k such that $Ad(g^k)_{|\mathfrak{t}} = \mathrm{Id}$ and we can define $H_0^* = \frac{1}{k}\sum_{i=1}^k Ad(g)(H_0)$. Since WC^+ is convex, $H_0^* \in WC^+$ is non-zero and clearly $\mathrm{Ad}(g)H_0^* = H_0^*$. Let S be the closure of the one parameter group $\exp(isH_0^*) \subset T$. Notice that $g\exp(isH_0^*)g^{-1} = \exp(is\,\mathrm{Ad}(g)(H_0^*)) = \exp(isH_0^*)$, i.e. $g \in Z_K(S)$ the centralizer of S in K. From Lemma 3.34 it follows that $Z_K(S)$ is the union of all maximal tori that contain S and is hence connected. Its Lie algebra $Z_\mathfrak{g}(\mathfrak{s})$ clearly contains $\mathfrak{t}$ but is also contained in $Z_\mathfrak{g}(H_0^*)$. But H_0^* is regular and hence $Z_\mathfrak{g}(H_0^*) = \ker(\mathrm{ad}_{H_0^*}) = \mathfrak{t}$ since $\mathrm{ad}_{H_0^*}$

does not vanish on any of the root spaces. Thus $Z_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{t}$ as well and hence $Z_K(S) = T$. But this implies that $g \in Z_K(S) = T$, which finishes the proof of (a). Notice that we have proved part (b) at the sam time. $\square$

Combining both we obtain:

WeylInt

Corollary 4.42 *If $\mathfrak{g}$ is a complex semisimple Lie algebra, then $W(\mathfrak{g}) \subset \mathrm{Int}(\mathfrak{g})$.*

Proof Recall that $\mathrm{Int}(\mathfrak{g})$ is the Lie subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\{\mathrm{ad}_X \mid X \in \mathfrak{g}\}$. In particular, it is generated be linear maps of the form e^{ad_X}, $X \in \mathfrak{g}$. If $\mathfrak{k}$ is a compact and semisimple (real) Lie algebra, then $\mathrm{Int}(\mathfrak{k})$ is compact as well since its Lie algebra $\mathfrak{Int}(\mathfrak{k}) \simeq \mathfrak{k}/\mathfrak{z}(\mathfrak{k}) \simeq \mathfrak{k}$ is compact and semisimple. Since the exponential map of a compact Lie group is onto, we can actually say $\mathrm{Int}(\mathfrak{k}) = \{e^{\mathrm{ad}_X} \mid X \in \mathfrak{k}\}$. In particulat there is a natural embedding $\mathrm{Int}(\mathfrak{k}) \to \mathrm{Int}(\mathfrak{k}_{\mathbb{C}})$ via complex extension of ad_X. In the proof of Proposition 4.34, starting with a complex Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$, we constructed a compact real form $\mathfrak{k}$ with maximal abelian subalgebra $\mathfrak{t} \subset \mathfrak{k}$ such that $\mathfrak{h} = \mathfrak{t} \otimes \mathbb{C}$. In the proof of Proposition 4.41 we showed that each $w \in W(\mathfrak{g})$ is of the form $\mathrm{Ad}(n)_{|\mathfrak{t}} \otimes \mathbb{C}$ for some $n \in N(T) \subset K$. By the above, since $\mathrm{Ad}(n) \in \mathrm{Int}(\mathfrak{k})$, we have that $w \in \mathrm{Int}(\mathfrak{g})$. $\square$

Our final application is:

DynkinSym

Proposition 4.43 *Let K be a compact semisimple simply connected Lie group and $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$ with Dynkin diagram $D(\mathfrak{g})$. Then*

$$\mathrm{Aut}(K)/\mathrm{Int}(K) \simeq \mathrm{Aut}(\mathfrak{k})/\mathrm{Int}(\mathfrak{k}) \simeq \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g}) \simeq \mathrm{Sym}(D(\mathfrak{g}))$$

where $\mathrm{Sym}(D(\mathfrak{g}))$ is the group of symmetries of the Dynkin diagram.

Proof The first equality is clear. For the second one we have a homomorphism $\phi\colon \mathrm{Aut}(\mathfrak{k}) \to \mathrm{Aut}(\mathfrak{g})$ via complex extension and as we saw in the proof of the previous Corollary, it also induces $\phi\colon \mathrm{Int}(\mathfrak{k}) \to \mathrm{Int}(\mathfrak{g})$ via complex extension of ad_X. Thus we get a homomorphism $\bar{\phi}\colon \mathrm{Aut}(\mathfrak{k})/\mathrm{Int}(\mathfrak{k}) \to \mathrm{Aut}(\mathfrak{g})/\mathrm{Int}(\mathfrak{g})$. To see that $\bar{\phi}$ is onto, let $A \in \mathrm{Aut}(\mathfrak{g})$. Then $A(\mathfrak{k})$ is another real form of $\mathfrak{g}$ and by Corollary 4.38, there exists an $L \in \mathrm{Int}(\mathfrak{g})$ with $L(A(\mathfrak{k})) = \mathfrak{k}$. Thus $LA \in \mathrm{Aut}(\mathfrak{k})$. To see that $\bar{\phi}$ is one-to-one, we use that fact that $\mathrm{Int}(\mathfrak{g}) \cap \mathrm{Aut}(\mathfrak{k}) = \mathrm{Int}(\mathfrak{k})$.

For the last isomorphism, we start with $A \in \mathrm{Aut}(\mathfrak{g})$. As we saw before, we can assume, modulo $\mathrm{Int}(\mathfrak{g})$, that $A(\mathfrak{h}) = \mathfrak{h}$, $A(\Delta^+) = \Delta^+$ and hence

$A(F) = F$. A thus permutes the simple roots, but since it is also an isometry in the Killing form, inner products are preserved and hence the induced permutation of vertices of the Dynkin diagram also preserves the connections, i.e. it induces a symmetry of the Dynkin diagram.

Conversely, if we start with a symmetry of the Dynkin diagram, we can apply Corollary 4.28 to get an automorphism that induces this symmetry. We are left with showing that if A induces a trivial symmetry of the Dynkin diagram, then $A \in \mathrm{Int}(\mathfrak{g})$. But this condition is equivalent to saying that $A_{|\mathfrak{h}} = \mathrm{Id}$. Thus to finish the proof we need the following Lemma which is of independent interest.

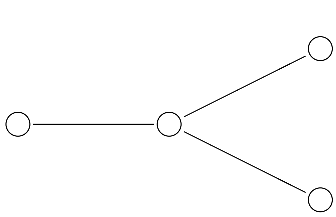
`trivial`

Lemma 4.44 *If $\mathfrak{h} \subset \mathfrak{g}$ is a Cartan subalgebra and $A \in \mathrm{Aut}(\mathfrak{g})$ with $A_{|\mathfrak{h}} = \mathrm{Id}$, then there exists an $X \in \mathfrak{h}$ such that $A = e^{\mathrm{ad}\, X}$.*

Proof An automorphism permutes roots and takes root spaces to corresponding root spaces. Since A fixes all roots by assumption, $A(\mathfrak{g}_\alpha) \subset \mathfrak{g}_\alpha$ and since $\dim \mathfrak{g}_\alpha = 1$, we have $A_{|\mathfrak{g}_\alpha} = c_\alpha \,\mathrm{Id}$ for some c_α. If α_i are the simple roots, then there exists a unique $X \in \mathfrak{h}$ with $c_\alpha = e^{\alpha_i(X)}$. Now let $B = e^{\mathrm{ad}\, X}$ and we want to show $A = B$. We can do this level by level, using the fact that both A and B are automorphisms and that the claim is true by choice at level one. $\square$

$\square$

Example 4.45 The Dynkin diagrams $B_n, C_n, G_2, F_4, E_7, E_8$ have no symmetries and hence for these Lie algebras and corresponding compact groups, every automorphism is inner. For $A_n = \mathfrak{sl}(n, C)$ there is one outer automorphism, up to inner ones, given by $\phi(A) = \bar{A}$, and the same for $\mathrm{SU}(n)$. For $\mathfrak{so}(2n, \mathbb{C})$ or $\mathrm{SO}(2n, \mathbb{R})$ we can choose conjugation with $\mathrm{diag}(-1, 1, \ldots, 1)$ to represent the outer automorphism. Most interesting is the diagram for $\mathfrak{so}(8, \mathbb{C})$:



which has the permutation group S_3 as its symmetry group. Rotation by 180 degrees gives rise to so called triality automorphisms, which we will discuss in a later section.

Exercises 4.46

(1) Show that the real forms of $\mathfrak{o}(n, \mathbb{C})$ are .... and those of $\mathfrak{sp}(n, \mathbb{C})$ are ....

(2)

4.6 Maximal root

A useful concept is that of a maximal root and the extended Dynkin diagram. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra, Δ^+ a system of positive roots and $F = \{\alpha_1, \dots, \alpha_n\}$ the set of fundamental roots.

Definition 4.47 $\alpha_m \in \Delta^+$ *is a* **maximal root** *if* $\alpha + \beta \notin \Delta^+$ *for all* $\beta \in \Delta^+$.

Its basic properties are given by

maxroot

Proposition 4.48 *Let $\mathfrak{g}$ be a complex simple Lie algebra.*

(a) There exists a unique maximal root α_m.
(b) If $\alpha_m = \sum n_i \alpha_i$, then $n_i > 0$.
(c) If $\alpha_m = \sum n_i \alpha_i$ and $\beta = \sum m_i \alpha_i \in \Delta^+$, then $m_i \le n_i$ for each $i = 1, \dots, n$. In particular $\beta < \alpha_m$ if $\beta \neq \alpha_m$.
(d) α_m is the unique maximal element in the ordering and the unique root of maximal level.

Proof (a) We first show by induction on the level of roots that if $\alpha_m + \alpha_i \notin \Delta^+$ for all $\alpha_i \in F$, then α_m is maximal. Indeed, let $\beta \in \Delta^+$ such that $\alpha_m + \beta \in \Delta^+$. If β has level one, we are done. If not, $\beta = \beta' + \alpha_i$ for some i and β' has smaller level. Then

$$0 \neq [X_{\alpha_m}, [X_{\alpha_i}, X_{\beta'}]] = [[X_{\alpha_m}, X_{\alpha_i}], X_{\beta'}]] + [[X_{\alpha_i}, [X_{\alpha_m}, X_{\beta'}]]$$

since $\alpha_m + \beta' + \alpha_i$ is a root. Since $\alpha_m + \alpha_i$ is not a root, we have $[X_{\alpha_m}, X_{\alpha_i}] = 0$, and hence $\alpha_m + \beta'$ must be a root.

Thus if α_m has maximal level, it must be a maximal root. This implies the existence of a maximal root. We will prove uniqueness after proving (b).

(b) First observe that $\langle \alpha_m, \alpha_i \rangle \geq 0$ since otherwise $\alpha_m + \alpha_i$ is a root. Also recall that $\langle \alpha_i, \alpha_j \rangle \leq 0$ for all $i \neq j$.

Let $\alpha_m = \sum n_i\alpha_i$ with $n_i \geq 0$ and assume there exists a k with $n_k = 0$. Then $\langle \alpha_m, \alpha_k\rangle = \sum n_i\langle\alpha_i, \alpha_k\rangle \leq 0$ which implies that $\langle\alpha_m, \alpha_k\rangle = 0$. This in turn implies that $\langle\alpha_i, \alpha_k\rangle = 0$ whenever $n_i \neq 0$. We can thus divide the simple roots $F = A \cup B$ with $A = \{\alpha_i \mid n_i = 0\}$ and $B = \{\alpha_i \mid n_i \neq 0\}$. We then have that $\langle\gamma, \delta\rangle = 0$ for all $\gamma \in A$ and $\delta \in B$. By induction on level, this implies that A and B generate 2 ideals, contradicting the assumption that $\mathfrak{g}$ is simple.

To prove uniqueness, let α_m, β_m be two roots at maximal level. Then $\langle\alpha_m, \beta_m\rangle \geq 0$ since otherwise $\alpha_m + \beta_m$ is a root. If $\langle\alpha_m, \beta_m\rangle = 0$, then $0 = \langle\alpha_m, \beta_m\rangle = \sum n_i\langle\alpha_i, \beta_m\rangle$. Since also $\langle\alpha_i, \beta_m\rangle \geq 0$ and $n_i > 0$, this implies $\langle\alpha_i, \beta_m\rangle = 0$ for all i. But α_i form a basis of $\mathfrak{h}_{\mathbb{R}}$, and hence $\langle\alpha_m, \beta_m\rangle > 0$. Thus $\alpha_m - \beta_m$ is a root and either $\alpha_m = \beta_m + (\alpha_m - \beta_m)$ or $\beta_m = \alpha_m + (\beta_m - \alpha_m)$ contradicts maximality.

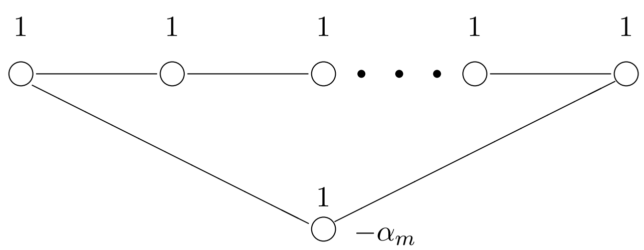
(c) Notice that if $\beta \neq \alpha_m$ has level i, then there exists an α_i such that $\beta + \alpha_i$ is a root of level $i+1$ since otherwise β is a maximal root by our first claim. This proves both (c) and (d). □

We can now define the **extended Dynkin diagram** as follows. To the simple roots $\alpha_1, \ldots \alpha_n$ add the root $-\alpha_m$ where α_m is maximal. Then draw circles and connect by lines according to the same rules as for the Dynkin diagram. Finally, put the integer n_i over the dot corresponding to α_i.
One easily sees that for the classical groups we have:

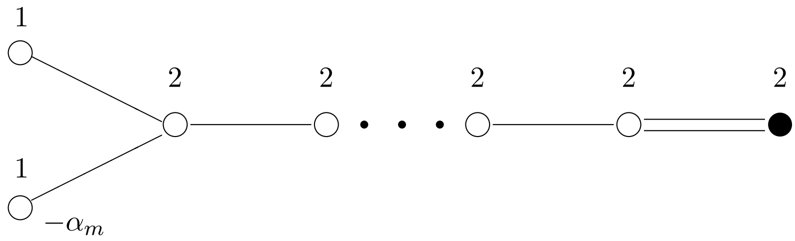
$$\begin{array}{lcll} A_n & : & \Delta^+ = \{\omega_i - \omega_j, i < j\} & \alpha_m = \omega_1 - \omega_{n+1} \\ B_n & : & \Delta^+ = \{\omega_i, \omega_i \pm \omega_j, i < j\} & \alpha_m = \omega_1 + \omega_2 \\ C_n & : & \Delta^+ = \{2\omega_i, \omega_i \pm \omega_j, i < j\} & \alpha_m = 2\omega_1 \\ D_n & : & \Delta^+ = \{\omega_i \pm \omega_j, i < j\} & \alpha_m = \omega_1 + \omega_2 \end{array}$$

and hence the extended Dynkin diagrams are

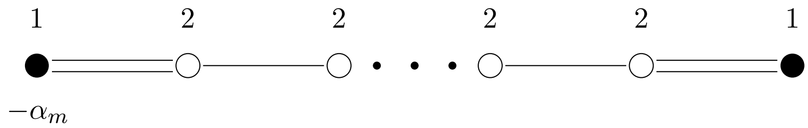
A_n :



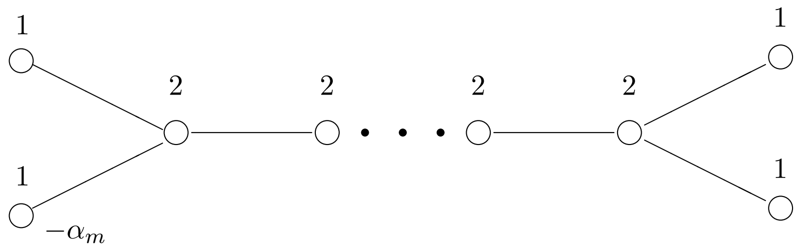
B_n :



C_n :



D_n :



`extendedDynkin` **Extended Dynkin diagrams**

(include exceptional ones as well)

We can use these extended Dynkin diagrams to give a classification of certain subalgebras of $\mathfrak{g}$.

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We say that $\mathfrak{h} \subset \mathfrak{g}$ is an equal rank subalgebra if $\operatorname{rk} \mathfrak{h} = \operatorname{rk} \mathfrak{g}$ and that $\mathfrak{h}$ is maximal if for every subalgebra $\mathfrak{k}$ with $\mathfrak{h} \subset \mathfrak{k} \subset \mathfrak{g}$ we have $\mathfrak{k} = \mathfrak{h}$ or $\mathfrak{k} = \mathfrak{g}$.

Notice that if we have a Dynkin diagram or an extended Dynkin diagram D and if we remove one of the circles, we obtain a Dynkin diagram $D' \subset D$ for a possibly non-simple Lie algebra. If we reconstruct the positive roots from the simple ones level by level, the diagram D' generates a subalgebra of $\mathfrak{g}$.

equalrank

Theorem 4.49 (Borel-Siebenthal) *Let $\mathfrak{g}$ be a complex simple Lie algebra with maximal root $\alpha_m = \sum n_i \alpha_i$ and extended Dynkin diagram D.*

(a) If D' is obtained from D by removing α_i with $n_i > 1$, then D' generates a maximal equal rank semisimple subalgebra $\mathfrak{h}$.
(b) If D' is obtained from D by removing $-\alpha_m$ and α_i with $n_i = 1$, then D' generates a subalgebra $\mathfrak{k}$ and $\mathfrak{h} = \mathfrak{k} \oplus \mathbb{R}$ is a maximal equal rank semisimple subalgebra.

The proof is non-trivial and we will omit it here. Since complex simple Lie algebras are in one-to-one correspondence with compact simple ones via their real form, this also gives a classification of the equal rank subalgebras for compact Lie groups.

Example 4.50 In the case of D_n we can delete a simple root with $n_i = 2$ to obtain the subalgebras $D_k \oplus D_{n-k}$ or a simple root with $n_i = 1$ to obtain $D_{n-1} \oplus \mathbb{R}$ or $A_{n-1} \oplus \mathbb{R}$. In terms of compact groups, this gives the block embeddings

$$\mathrm{SO}(2k) \times \mathrm{SO}(2n-2k) \subset \mathrm{SO}(2n) \quad \mathrm{U}(n) \subset \mathrm{SO}(2n)$$

Exercises 4.51

(1) Show that the maximal root has maximal length among all roots.
(2) Show that if $\mathfrak{h} \subset \mathfrak{g}_1 \oplus \mathfrak{g}_2$ is an equal rank subalgebra, then $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i \subset \mathfrak{g}_i$ equal rank subalgebras.
(3) Show that the equal rank subgroups (not necessarily just maximal ones) of the classical compact Lie groups are given by

$$\mathrm{S}(\mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_k)) \subset \mathrm{SU}(n) \text{ with } \sum n_i = n$$

$$U_1 \times \cdots \times U_k \subset \mathrm{Sp}(n), \text{ where } U_i = \mathrm{Sp}(n_i) \text{ or } \mathrm{U}(n_i) \text{ and } \sum n_i = n$$

$$U_1 \times \cdots \times U_k \subset \mathrm{SO}(2n) \text{ where } U_i = \mathrm{SO}(2n_i) \text{ or } \mathrm{U}(n_i) \text{ and } 2\sum n_i = n$$

$$U_0 \times \cdots \times U_k \subset \mathrm{SO}(2n+1) \text{ where } U_0 = \mathrm{SO}(2k+1) \text{ and}$$

$$U_i = \mathrm{SO}(2n_i) \text{ or } \mathrm{U}(n_i)\ , i \geq 1, \text{ and } 2\sum n_i = n-k$$

4.7 Lattices

We end this chapter by describing several lattices of $\mathfrak{h}_{\mathbb{R}}$ and $\mathfrak{h}_{\mathbb{R}}^*$. Recall that a lattice in a vector space V is a discrete subgroup which spans V. It follows that there exists a basis $v_1, \dots, v_n$ such that all lattice points are integral linear combinations of v_i. Conversely, given a basis v_i, the integral linear combinations of v_i form a lattice.

Let K be a compact Lie group with maximal torus T. If $\mathfrak{g} = \mathfrak{k}_{\mathbb{C}}$, then, as we saw, $\mathfrak{t} \otimes \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_{\mathbb{R}} = i\mathfrak{t}$. Denote by Δ the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$.

Definition 4.52 Let K be a compact real group, with $\mathfrak{k}_{\mathbb{C}} = \mathfrak{g}$. Then we have the following lattices in $\mathfrak{h}_{\mathbb{R}}$.

(a) The **central lattice** $\Gamma_Z = \{v \in \mathfrak{h}_{\mathbb{R}} \mid \alpha(v) \in \mathbb{Z} \quad \text{for all } \alpha \in \Delta\}$.
(b) The **integral lattice** $\Gamma_I = \{X \in \mathfrak{h}_{\mathbb{R}} \mid \exp_K(2\pi i X) = e\}$.
(c) The **coroot lattice** $\Gamma_C = \mathrm{span}_{\mathbb{Z}}\{\tau_{\alpha_i} = \frac{2H_{\alpha_i}}{\langle \alpha_i, \alpha_i \rangle} \mid \alpha_i \in F\}$.

These lattices can of course also be considered as lattices in $\mathfrak{t}$. Notice that the integral lattice depends on a particular choice of the Lie group K, while the root lattice and the central lattice only depend on the Lie algebra $\mathfrak{g}$. Also, Γ_I is indeed a lattice since $\exp\colon \mathfrak{t} \to T$ is a homomorphism and $d(\exp)_0$ an isomorphism. The basic properties of these lattices are:

lattices

Proposition 4.53 *Let K be a connected compact Lie group with universal cover $\tilde{K}$. Then we have*

(a) $\Gamma_C \subset \Gamma_I \subset \Gamma_Z$,
(b) $\Gamma_Z = \{X \in \mathfrak{h}_{\mathbb{R}} \mid \exp_K(2\pi i X) \in Z(K)\}$ and hence $\Gamma_Z/\Gamma_I = Z(K)$.
(c) $\Gamma_I/\Gamma_C = \pi_1(K)$, and hence $\Gamma_Z/\Gamma_C = Z(\tilde{K})$.

Proof We start with the proof of (b). Recall that $Z(K) \subset T$ and that $\exp\colon \mathfrak{t} \to T$ is onto. Thus, if $g \in Z(K)$, we can write $g = \exp(X)$ for some $X \in \mathfrak{t}$. Since $Z(K)$ is the kernel of Ad, it follows that $\mathrm{Ad}(\exp(X)) = e^{\mathrm{ad}_X} = \mathrm{Id}$, and since ad_X is skew symmetric, the eigenvalues of ad_X lie in $2\pi i\mathbb{Z}$. On the other hand, the eigenvalues of ad_X are 0 and $i\alpha(X)$, $\alpha \in \Delta$. This implies that $\alpha(X) \in 2\pi i\ \Gamma_Z$. The converse direction works similarly.

(a) By (4.9), $\beta(\tau_\alpha) \in \mathbb{Z}$ which implies that $\Gamma_C \subset \Gamma_Z$. To prove the stronger claim that $\Gamma_C \subset \Gamma_I$, let $\alpha \in \Delta$ and recall that $i\tau_\alpha$, $i(X_\alpha + X_{-\alpha})$, $X_\alpha - X_{-\alpha}$ form a subalgebra of $\mathfrak{k}$ isomorphic to $\mathfrak{su}(2)$. It is the image of a homomorphism $d\phi\colon \mathfrak{su}(2) \to \mathfrak{k}$ which integrates to a homomorphism $\phi\colon \mathrm{SU}(2) \to K$. Furthermore, $i\tau_\alpha = d\phi(X)$ with $X = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$. Since $\exp_{\mathrm{SU}(2)}(2\pi X) = e$, this implies that $\exp_K(2\pi i\tau_\alpha) = \exp_K(d\phi(2\pi X)) = \phi(\exp_{\mathrm{SU}(2)}(2\pi X)) = e$.

It remains to prove that $\Gamma_I/\Gamma_C = \pi_1(K)$. For this we construct a homomorphism $f\colon \Gamma_I \to \pi_1(K)$ as follows. If $X \in \Gamma_I$, then by definition $t \to \exp(2\pi i t X)$, $0 \le t \le 1$ is a loop in K and hence represents an element of $\pi_1(K)$. Recall that pointwise multiplication of loops in K is homotopic to the composition of loops. Since we also have $\exp(t(X+Y)) = \exp(tX)\exp(tY)$ for $X, Y \in \mathfrak{t}$, it follows that f is a homomorphism. As is well known, each element of $\pi_1(M)$ of a Riemannian manifold M can be represented by a geodesic loop (the shortest one in its homotopy class). We can apply this to K equipped with a biinvariant metric. The geodesics through e are of the form $t \to \exp(tX)$, $X \in \mathfrak{k}$. By the maximal torus theorem there exists a $g \in K$ such that $g\exp(tX)g^{-1} = \exp(t\,\mathrm{Ad}(X))$ with $\mathrm{Ad}(X) \in \mathfrak{t}$. This new loop is homotopic to the original one since K is connected. Thus each element $\pi_1(K)$ can be represented by a geodesic loop $t \to \exp(tX)$, $0 \le t \le 1$, $X \in \mathfrak{t}$ and hence $iX \in \Gamma_I$. Thus f is onto.

If $X \in \Gamma_C$, we can write the loop as the image of the corresponding loop in SU(2) under the above homomorphism ϕ. Since SU(2) is simply connected, Γ_C lies in the kernel of f. To see that $\ker f = \Gamma_C$ is more difficult and will be proved later on. □

Example 4.54 We now use Proposition 4.53 to compute the center of $\mathrm{Spin}(n)$, the universal cover of $\mathrm{SO}(n)$.

We start with $\mathrm{Spin}(2n+1)$. We use the basis e_i of $\mathfrak{t}$ as before, and hence the roots are $\pm\omega_i$, $\pm\omega_i \pm \omega_j$, $i < j$. Thus $\Gamma_I = \Gamma_Z$ is spanned by e_i and Γ_C by the coroots $\pm 2e_i$, $\pm e_i \pm e_j$, $i < j$. This implies that $\Gamma_Z/\Gamma_I = Z(\mathrm{SO}(2n+1)) = e$ and $\Gamma_I/\Gamma_C = \pi_1(\mathrm{SO}(2n+1)) = \mathbb{Z}_2$ spanned by e_1. Hence also $\Gamma_Z/\Gamma_C = Z(\mathrm{Spin}(2n+1)) = \mathbb{Z}_2$.

More interesting is the case of $\mathrm{SO}(2n)$. Here the roots are $\pm\omega_i \pm \omega_j, , i < j$.

Hence Γ_C is spanned by the coroots $\pm e_i \pm e_j$, $i < j$ and Γ_I by e_i. Furthermore, $\Gamma_Z = \{\sum a_i e_i \mid a_i \pm a_j \in \mathbb{Z}\}$ and hence spanned by $\frac{1}{2}\sum \pm e_i$, the sum being over an even number of indices. This implies that

$$\begin{aligned}
\pi_1(\mathrm{SO}(2n)) = \Gamma_I/\Gamma_C &= \mathbb{Z}_2 \qquad \text{generated by } e_1 \\
Z(\mathrm{SO}(2n)) = \Gamma_Z/\Gamma_I &= \mathbb{Z}_2 \qquad \text{generated by } \frac{1}{2}\sum e_i \\
Z(\mathrm{Spin}(2n)) = \Gamma_Z/\Gamma_C &= \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}_2 & n = 2k \\ \mathbb{Z}_4 & n = 2k+1 \end{cases}
\end{aligned}$$

Here the generators of Γ_Z/Γ_C are $e_1 + \frac{1}{2}\sum_{i=2}^k e_i$ and $-e_1 + \frac{1}{2}\sum_{i=2}^k e_i$ if n even, and $\frac{1}{2}\sum e_i$ if n odd.

Remark 4.55 It is worth pointing out that besides the Lie group $\mathrm{SO}(4n)$ there exists another Lie group, which we will denote by $\mathrm{SO}'(4n)$, which has center and fundamental group equal to $\mathbb{Z}_2$. For this, recall that for $n > 4$, there exists only one outer automorphism A of $\mathrm{Spin}(n)$, and it descends to an outer automorphism of $\mathrm{SO}(4n)$ as well. This automorphism acts on the center $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ non-trivially. Hence there exists a basis $(1,0),(0,1)$ with $A(1,0) = (0,1)$. Since A descends, $\mathrm{SO}(4n)$ is obtained by dividing by $\mathbb{Z}_2$ generated by $(1,1)$. Since there exists no automorphism taking $(1,0)$ to $(1,1)$, the Lie group obtained by dividing by $\mathbb{Z}_2$ generated by $(1,0)$ is not isomorphic to $\mathrm{SO}(4n)$. Notice though that $\mathrm{SO}(8) \simeq \mathrm{SO}'(8)$ due to the triality automorphism. Furthermore, $\mathrm{SO}'(4) \simeq \mathrm{SU}(2) \times \mathrm{SO}(3)$.

We also have the following dual lattices, which will be important in the next chapter.

Definition 4.56 Let $\mathfrak{g}$ be a semisimple Lie algebra. Then we have the following lattices in $\mathfrak{h}^*_{\mathbb{R}}$.

(a) The **weight lattice** $\Gamma_W = \{\lambda \in \mathfrak{h}^*_{\mathbb{R}} \mid \lambda(\tau_\alpha) \in \mathbb{Z} \quad \text{for all } \alpha \in \Delta\}$.
(b) The **root lattice** $\Gamma_R = \mathrm{span}_{\mathbb{Z}}\{\alpha_i \mid \alpha_i \in F\}$.

colattices **Proposition 4.57** *$\Gamma_R \subset \Gamma_W$ and $\Gamma_W/\Gamma_R \simeq Z(\tilde{K})$.*

Proof The inclusion is a basic property of roots. For the second claim, we first the following general comments. A lattice $\Gamma \subset V$ defines a dual lattice in V^* via $\Gamma^* = \{\lambda \in V^* \mid \lambda(v) \in \mathbb{Z} \text{ for all } v \in \Gamma\}$. If two lattices $\Gamma_i \subset V$ satisfy $\Gamma_1 \subset \Gamma_2$, then $\Gamma_2^* \subset \Gamma_1^*$ and $\Gamma_2/\Gamma_1 \simeq \Gamma_1^*/\Gamma_2^*$. Now it is clear that Γ_W is dual to Γ_C and Γ_R is dual to Γ_Z. The claim now follows from Proposition 4.53.

□

Exercises 4.58

(1) Compute the various lattices for $K = \mathrm{SU}(n)$ and $K = \mathrm{Sp}(n)$ and use them to determine center and fundamental group of K.

(2) Show that $\pi_1(\mathrm{Ad}(K) = \Gamma_Z/\Gamma_C$ if K is a compact semisimple Lie group.

(3) Show that all 6 lattices are invariant under the Weyl group.

5
Representation Theory

Our goal in this chapter is to study the representation theory of Lie algebras. We will see that every representation of a complex semisimple Lie algebra splits into a sum of irreducible representations, and that the irreducible representations can be described explicitly in terms of simple data.

5.1 General Definitions

Let G be a real or complex Lie group, with corresponding Lie algebra $\mathfrak{g}$.

A **real** (resp. **complex**) **representation** of G on a real (resp. complex) vector space V is a Lie group homomorphism

$$\pi : G \to GL(V);$$

A **real** (resp. **complex**) **representation** of $\mathfrak{g}$ on a real (resp. complex) vector space V is is a Lie algebra homomorphism

$$\pi : \mathfrak{g} \to \mathfrak{gl}(V);$$

Of course if $\mathfrak{g}$ is complex, V must be complex as well. When a representation is fixed, we often denote $\pi(X)(v)$ by $X \cdot v$ or just Xv and $\pi(g)(v)$ by $g \cdot v$ or gv.

A representation π of either a Lie group or a Lie algebra is **faithful** if π is injective. A representation π of a Lie group is **almost faithful** if $\ker(\pi)$ is discrete. Notice that since $\ker(\pi)$ is also a normal subgroup, it lies in the center of G. We can thus compute the kernel by just checking it on central elements.

If G is simply connected, there is a bijection between representations π of G and representations $d\pi$ of $\mathfrak{g}$. Notice that π is almost faithful iff $d\pi$ is faithful.

Because of this bijection, we will first study Lie algebras representations, and come back to Lie group representations at the end.

Let π and π' be two (real or complex) representations on a vector space V resp. V' by a Lie group G or Lie algebra $\mathfrak{g}$, distinguished by $\pi(g)$ and $\pi(X)$ for short.

We define the **direct sum** $\pi \oplus \pi'$ acting on $V \oplus V'$ as

$$\begin{aligned}(\pi \oplus \pi')(g) \cdot (v, v') &= (\pi(g) \cdot v, \pi'(g) \cdot v')) \\ (\pi \oplus \pi')(X) \cdot (v, v') &= (\pi(X) \cdot v, \pi'(X) \cdot v')\end{aligned}$$

The **tensor product** $\pi \otimes \pi'$ acting on $V \otimes V'$ as

$$\begin{aligned}(\pi \otimes \pi')(g) \cdot (v \otimes v') &= \pi(g) \cdot v \otimes \pi'(g) \cdot v' \\ (\pi \otimes \pi')(X) \cdot (v \otimes v') &= (\pi(X) \cdot v) \otimes v' + v \otimes (\pi'(X) \cdot v')\end{aligned}$$

The **k-th exterior power** $\Lambda^k \pi$ acting on $\Lambda^k V$ as

$$\begin{aligned}(\Lambda^k \pi)(g)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) &= gv_1 \wedge gv_2 \wedge \cdots \wedge gv_n \\ (\Lambda^k \pi)(X)(v_1 \wedge v_2 \wedge \cdots \wedge v_k) &= \sum_i v_1 \wedge \cdots \wedge Xv_i \wedge \cdots \wedge v_n\end{aligned}$$

and similarly the **k-th symmetric power** $S^k \pi$ acting on $S^k V$.

If π is a representation of G resp. $\mathfrak{g}$ and π' one of G' resp. $\mathfrak{g}'$, we define the **exterior tensor product** $\pi \widehat{\otimes} \pi'$, which is a representation of $G \times G'$ resp. $\mathfrak{g} \oplus \mathfrak{g}'$ acting on $V \otimes V'$ as

$$\begin{aligned}(\pi \widehat{\otimes} \pi')(g, g')(v \otimes v') &= \pi(g) \cdot v \otimes \pi'(g') \cdot v' \\ (\pi \widehat{\otimes} \pi')(X, X') \cdot (v \otimes v') &= \pi(X) \cdot v \otimes v' + v \otimes \pi'(X') \cdot v'\end{aligned}$$

Notice that the tensor product is a representation of the same Lie algebra, whereas the exterior tensor product is a representation of the sum of two Lie algebras.

Now assume that V has a real inner product in case of a real representation or an hermitian inner product in case of a complex representation. We can then associate to π the **contragredient** representation π^*. If we denote the adjoint of a linear map L by L^*, then on the group level it is defined by $\pi^*(g)v = (\pi(g^{-1}))^* v$ and on the Lie algebra level as $\pi^*(X)v = -(\pi(X))^* v$. We often think of a representation as acting on $\mathbb{R}^n$ endowed with its canonical inner product. In that case $\pi^*(g)v = \pi(g^{-1})^T$ and $\pi^*(X)v = -\pi(X)^T v$. If G is a Lie group or $\mathfrak{g}$ is a real Lie algebra we call a real representation π **orthogonal** if $\pi(G) \subset O(V)$ resp. $\pi(\mathfrak{g}) \subset \mathfrak{o}(V)$ and a complex representation **unitary** if $\pi(G) \subset U(V)$ resp. $\pi(\mathfrak{g}) \subset \mathfrak{u}(V)$. Notice that this definition does not make sense if $\mathfrak{g}$ is a complex Lie algebra since $\mathfrak{u}(n)$ is not a complex vector space.

Given a complex representation, we define the **complex conjugate** representation $\bar{\pi}$ by $\bar{\pi}(g) = \overline{\pi(g)}$ resp. $\bar{\pi}(X) = \overline{\pi(X)}$. Notice that if π is unitary, $\bar{\pi} = \pi^*$, and if π is orthogonal, $\pi = \pi^*$.

Let $\mathfrak{h}$ be a *real* Lie algebra, and $\pi : \mathfrak{h} \to \mathfrak{gl}(V)$ a *complex* representation. We define

$$\tilde{\pi} : \mathfrak{h} \otimes \mathbb{C} \to \mathfrak{gl}(V) \quad : \quad (X + iY) \cdot v = (X \cdot v) + i(Y \cdot v)$$

Given a complex Lie algebra $\mathfrak{g}$, a complex representation π, and a real form $\mathfrak{h} \subset \mathfrak{g}$, we obtain a complex representations of $\mathfrak{h}$ via restriction. One thus has a one-to-one correspondence between complex representations of $\mathfrak{h}$ and $\mathfrak{h} \otimes \mathbb{C}$.

If $\mathfrak{g}$ be a *real* Lie algebra, and $\pi : \mathfrak{g} \to \mathfrak{gl}(V)$ a *real* representation. We define

$$\pi_{\mathbb{C}} = \pi \otimes \mathbb{C} : \mathfrak{g} \to \mathfrak{gl}(V \otimes \mathbb{C}) \quad : \quad X \cdot (v + iw) = (X \cdot v) + i(X \cdot w)$$

as a complex representation of $\mathfrak{g}$. Unlike in the previous construction, the correspondence between π and $\pi_{\mathbb{C}}$ is not so clear since in general a complex representation is not the complexification of a real representation. We postpone a discussion of this relationship to a later section.

Finally, if π is a complex representation on V we denote by $\pi_{\mathbb{R}}$ the underlying real representation on $V_{\mathbb{R}}$ where we forget about the complex multiplication on V.

Let π, π' two representations of G resp. $\mathfrak{g}$. We say that π and π' are **equivalent**, denoted by $\pi \simeq \pi'$, if there is an isomorphism $L : V \to V'$ such that $\pi'(X)(Lv) = L(\pi(X)v)$ for every $X \in \mathfrak{g}$, $v \in V$. Such an L is also called an **intertwining map**.

A (real or complex) representation π is called **irreducible** if there are no non-trivial subspaces $W \subset V$ with $\pi(W) \subset W$.

A useful observation is the following.

schur **Lemma 5.1 (Schur's Lemma)**

(a) If π is an irreducible complex representation on V and L an intertwining map, then $L = a\,\mathrm{Id}$ for some $a \in \mathbb{C}$.
(b) If π is an irreducible real representation on V, then the set of intertwining maps is an associative division algebra and hence isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.
(c) If π is an irreducible real (resp. complex) representation of G on V and $\langle \cdot, \cdot \rangle_1$, $\langle \cdot, \cdot \rangle_2$ two inner products (resp. hermitian inner products) on V such that $\pi(g)$ is an isometry with respect to both, then $\langle \cdot, \cdot \rangle_1 = a \langle \cdot, \cdot \rangle_2$ for some $a \in \mathbb{R}$ (resp. $a \in \mathbb{C}$).

Proof (a) Since we are over the complex numbers, there exists an eigenvalue a and corresponding eigenspace $W \subset V$ of L. The representation π preserves W since $L(\pi(g)w) = \pi(g)(Lw) = a\pi(g)w$ for $w \in W$. By irreducibility, $W = V$, i.e., $L = a\,\mathrm{Id}$.

(b) First notice that if L is an intertwining map, the kernel and image are invariant subspaces, and hence L is an isomorphism. Sums and compositions of intertwining maps are clearly again intertwining maps, and so is the inverse. This shows it is a division algebra. A theorem of Frobenius states that an associative division algebra is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

(c) Define $L\colon V \to V$ by $\langle v, w \rangle_1 = \langle v, Lw \rangle_2$ for all v, w. Since $\langle \cdot, \cdot \rangle_1$ is symmetric, L is self adjoint with respect to $\langle \cdot, \cdot \rangle_2$. Since π acts by isometries,

$$\langle gv, gLw \rangle_2 = \langle v, Lw \rangle_2 = \langle v, w \rangle_1 = \langle gv, gw \rangle_1 = \langle gv, Lgw \rangle_2$$

and thus $\pi(g)L = L\pi(g)$. Hence $\pi(g)$ preserves eigenspaces of L and by irreducibility, and since L is self adjoint, $L = a\,\mathrm{Id}$, i.e. $\langle \cdot, \cdot \rangle_1 = a \langle \cdot, \cdot \rangle_2$. □

The same holds in (c) if π is a representation of a Lie algebra $\mathfrak{g}$ and $\pi(X)$ is skew symmetric (resp. skew hermitian).

Remark 5.2 According to (b), real irreducible representations fall into 3 categories according to wether the set of intertwining maps is isomorphic to $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. We say that the representation if of real, complex, resp. quaternionic type.

Real representations of complex type correspond to those whose image lies in $\mathrm{GL}(n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{R})$. In this case the intertwining maps are of the form $a\,\mathrm{Id} + bI$, $a, b \in \mathbb{R}$, where I is the complex structure on $\mathbb{R}^{2n}$.

Representations of quaternionic type correspond to those whose image lies in $\mathrm{GL}(n, \mathbb{H}) \subset \mathrm{GL}(4n, \mathbb{R})$. In this case the intertwining maps are of the form

$a\,\mathrm{Id} + bI + cJ + dK$, $a, b, c, d \in \mathbb{R}$, where I, J, K are the complex structure on $\mathbb{R}^{4n} \simeq \mathbb{H}^n$ given by multiplication with $i, j, k \in \mathbb{H}$ componentwise.

We will study this division of real representations into representation of real, complex, resp. quaternionic type in more detail later on.

We say that a representation π is **completely decomposable** if there exists a decomposition $V = W_1 \oplus \cdots \oplus W_k$ and representations π_i on W_k such that $\pi \simeq \pi_1 \oplus \cdots \oplus \pi_r$ with π_k irreducible. It is easy to see that a representation is completely decomposable iff every invariant subspace has a complementary invariant subspace.

What makes representations of semisimple Lie algebras special is:

compred

Proposition 5.3 *Let K be a compact Lie group and $\mathfrak{g}$ a real or complex semisimple Lie algebra.*

(a) Every real (complex) representation of K is orthogonal (unitary) and completely decomposable.
(b) Every real or complex representation of $\mathfrak{g}$ is completely decomposable.
(b) If $\mathfrak{g}$ is real, every real (complex) representation is orthogonal (unitary)

Proof (a) Let π be a complex rep of K. Starting with an arbitrary hermitian inner product $\langle \cdot, \cdot \rangle$ on V we can average as usual over K:

$$\langle X, Y \rangle' = \int_K \langle \psi(k)X, \psi(k)Y \rangle dk$$

such that $\pi(k)$ acts as an isometry in $\langle \cdot, \cdot \rangle'$. Thus π is unitary. Similarly for a real representation. In either case, if W is an invariant subspace, the orthogonal complement is also invariant. This proves complete decomposability.

(b) Let $\mathfrak{g}$ be a complex semisimple Lie algebra with a real or complex representation. Let $\mathfrak{k} \subset \mathfrak{g}$ be the compact real form and let K be the unique simply connected Lie group with Lie algebra $\mathfrak{k}$. Since $\mathfrak{k}$ is semisimple, K is compact. Restricting we obtain a representation $\pi_{|\mathfrak{k}} \colon \mathfrak{k} \to \mathfrak{gl}(V)$ which integrates to a representation $\psi \colon K \to \mathrm{GL}(V)$ with $d\psi = \pi$. Thus ψ is orthogonal (unitary) and hence also $d\psi = \pi_{|\mathfrak{k}}$ and by complexification also π.

(c) If $\mathfrak{g}$ is real semisimple, and π a rep of $\mathfrak{g}$, then $\pi_{\mathbb{C}}$ is a rep of $\mathfrak{g}_C$, and arguing as before is completely decomposable. Restricting, this implies that π is completely decomposable $\square$

Notice that a complex rep of a complex Lie algebra can not be unitary since $\mathfrak{su}(n) \subset \mathfrak{gl}(n, \mathbb{C})$ is not a complex subspace.

We end this section with the following general claim.

tensorprod

Proposition 5.4 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and π a complex irreducible representation.*

(a) *If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then there exist irreducible representations π_1 of $\mathfrak{g}_1$, π_2 of $\mathfrak{g}_2$, such that $\pi = \pi_1 \widehat{\otimes} \pi_2$;*
(b) *Conversely, if π_i are irreducible representations of $\mathfrak{g}_i$, then $\pi_1 \widehat{\otimes} \pi_2$, is an irreducible representation of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$.*
(c) *Every representation of an abelian Lie algebra is one dimensional. If $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{t}$ where $\mathfrak{g}$ is semisimple and $\mathfrak{t}$ abelian, then $\pi = \pi_1 \widehat{\otimes} \pi_2$ for some representation π_1 of $\mathfrak{g}_1$ and $\pi_2(x) = f(x), x \in \mathfrak{t}$ for some $f \in \mathfrak{t}^*$. π_2 is effective iff $\dim \mathfrak{t} = 1$ and f injective.*

Proof (a) Let π be an irrep of $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ on R. We will consider $\mathfrak{g}_i$ as embedded in the i-th coordinate. Let σ be the restriction of π to $\mathfrak{g}_1$. Then we can decompose R into σ irreducible subspaces: $X = V_1 \oplus V_2 \oplus \cdots \oplus V_k$. Let $pr_i \colon V \to V_i$ be the projection onto the i-th coordinate. This is clearly a $\mathfrak{g}_1$ equivariant map. Fixing $y \in \mathfrak{g}_2$, we have a linear map $L_{ij}(y) \colon V_i \to V_j, : v \to pr_j(y \cdot v)$. Since the action of $\mathfrak{g}_1$ and $\mathfrak{g}_2$ commute, $L_{ij}(y)$ is $\mathfrak{g}_1$ equivariant. Thus by Schur's Lemma it is either 0 or an isomorphism. We claim that $V_i \simeq V_j$ for all i, j. If not, fix i and let W be the direct sum of all irreps V_j not isomorphic to V_i. If W is not all of V, there exists a k with Then the above observation would imply that $L_{ij}(y) = 0$ for all $\mathfrak{g}_1 \oplus \mathfrak{g}_2$ invariant. Thus V_i must be isomorphic to V_j for all i, j. Fix $\mathfrak{g}_1$ equivariant isomorphisms $V := V_1,\ V \simeq V_i$ and define $\pi_1 \colon v \to x \cdot v$ to be the rep of $\mathfrak{g}_1$ on V. Hence $R = V \oplus V \oplus \cdots \oplus V$ with $\mathfrak{g}_1$ acting as $x \cdot (v_1, \dots, v_k) = (x \cdot v_1, \dots, x \cdot v_k)$. If ϕ_i is the embedding into the ith coordinate, we have linear maps $M_{ij}(y) \colon V \to V$ given by $M_{ij}(y)(v) = pr_j(y \cdot \phi_i(v))$. This is again $\mathfrak{g}_1$ equivariant and by Schur's Lemma, for each $y \in \mathfrak{g}_2$, $M_{ij}(y) = b_{ij}(y)\,\mathrm{Id}$ for some constant b_{ij}, or equivalently $y \cdot \phi_i(v) = \sum_j b_{ij}(y)\phi_j(v)$. This can be interpreted as a representation of $\mathfrak{g}_2$. To be explicit, let W be a k-dimensional vector space and fix a basis $w_1, \dots, w_k$. Then $y \cdot w_i := \sum_j b_{ij}(y) w_j$ for $y \in \mathfrak{g}_2$ defines a representation π_2 of $\mathfrak{g}_2$ on W. We now claim that $\pi \simeq \pi_1 \widehat{\otimes} \pi_2$. To see this, define an isomorphism $F \colon V \otimes W \to R$ by $F(v \otimes w_i) = \phi_i(v)$. F is $\mathfrak{g}_1$ equivariant since $F(x \cdot (v \otimes w_i)) = F((x \cdot v) \otimes w_i) = \phi_i(x \cdot v) = x \cdot \phi_i(v) = x \cdot F(v \otimes w_i)$ for $x \in \mathfrak{g}_1$. Furthermore, $F(y \cdot (v \otimes w_i)) = F(v \otimes (y \cdot w_i)) =$

$\sum b_{ij}(y)v\otimes w_j = \sum b_{ij}(y)\phi_j(v)$ and $y\cdot F(v\otimes w_i) = y\cdot\phi_i(v) = \sum_j b_{ij}(y)\phi_j(v)$ for all $y \in \mathfrak{g}_2$ implies that it is $\mathfrak{g}_2$ equivariant as well, and hence F is a $\mathfrak{g}$ equivariant isomorphism.

(b) Let $v_i \in V_i$ be maximal weight vectors of π_i and F_i the fundamental roots of $\mathfrak{g}_i$. Starting with $v_1\otimes v_2$ we can apply $X_{-\alpha}\cdot(v_1\otimes v_2) = (X_{-\alpha}\cdot v_1)\otimes v_2$, $\alpha \in F_1$, and separately $X_{-\beta}\cdot(v_1\otimes v_2) = v_1\otimes(X_{-\beta})v_2$, $\alpha \in F_2$. Applying these repeatedly, we can generate all of $V_1\otimes V_2$ starting with $v_1\otimes v_2$.

(c) The first part follows immediately from Schur's Lemma since all endomorphisms commute and can be diagonalized simultaneously. The rep is then clearly given by some $f \in \mathfrak{t}^*$. The second claim follows as in part (a) and the last claim is obvious. □

Thus understanding complex representations of complex semisimple Lie algebras reduces to classifying irreducible complex representations of simple Lie algebras. This will be the topic of the next section.

We added the case of an abelian Lie algebra as well since in a later section we will study representations of compact Lie algebras and compact Lie groups.

Exercises 5.5

(1) If $\mathfrak{g}$ is real and π a complex representation with $\tilde{\pi}$ is extension to $\mathfrak{g}\otimes\mathbb{C}$, show that π is irreducible iff $\tilde{\pi}$ is irreducible.
(2) Let π be the (irreducible) representation of $G = \mathrm{SO}(2)$ acting on $\mathbb{R}^2$ via rotations. Show explicitly how $\pi\otimes\mathbb{C}$ decomposes.
(3) If π is a complex representation and $\pi_{\mathbb{R}}$ the underlying real one, show that π is irreducible iff $\pi_{\mathbb{R}}$ is irreducible.
(5) If π is a rep of the Lie group G, show that π is irreducible iff $d\pi$ is irreducible.
(6) If π is a complex representation of $\mathfrak{g}$ (real or complex), show that $(\pi_{\mathbb{R}})_{\mathbb{C}}$ is isomorphic to $\pi\oplus\bar{\pi}$.
(6) Show that π is completely decomposable iff every invariant subspace has a complementary subspace. Give an example that a representation of a nilpotent or solvable Lie group is not completely decomposable.

5.2 Representations of sl(2,C)

From now on, $\mathfrak{g}$ is a complex semisimple Lie algebra, with a fixed Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and corresponding roots Δ. We choose an ordering defined by a regular vector $H_0 \in \mathfrak{h}_\mathbb{R}$ and thus positive roots Δ^+ and fundamental roots $F = \{\alpha_1, \ldots, \alpha_n\}, n = \operatorname{rk} \mathfrak{g}$. We will use the abbreviations $X_{\alpha_i} = X_i$ and $X_{-\alpha_i} = X_{-i}$ whenever convenient. We often write Xv or $X \cdot v$ instead of $\pi(X)v$ and will use the short form “rep” for representation and “irrep" for irreducible representation.

A functional $\mu \in \mathfrak{h}^*$ is called a **weight** of π if

$$V_\mu = \{v \in V \mid H \cdot v = \mu(H)v \quad \text{for all } H \in \mathfrak{h}\}$$

is nonempty. In that case, V_μ is the **weight space**, and $m_\mu = \dim V_\mu$ the **multiplicity** of μ. We denote by W_π the set of weights of π.

Note that, in particular, if π is the adjoint representation of $\mathfrak{g}$, then the weights of π are precisely the roots of $\mathfrak{g}$, as previously defined, with the weight spaces V_α being the root spaces $\mathfrak{g}_\alpha$. Moreover, $m_\alpha = 1$ for every root α. On the other hand, 0 is also a weight and has multiplicity $\dim \mathfrak{h}$.

rootweight

Lemma 5.6 *For all $\alpha \in \Delta$, $X_\alpha \in \mathfrak{g}_\alpha$ and weights μ we have*

$$X_\alpha \cdot V_\mu \subset V_{\mu+\alpha}$$

Proof Let $v \in V_\mu$, i.e. $H \cdot v = \mu(H)v$. Since π preserves Lie brackets, $\pi([X,Y] = \pi(X)\pi(Y) - \pi(Y)\pi(X)$ or in our short form $XY \cdot v = YX \cdot v + [X,Y] \cdot v$ for all $X, Y \in \mathfrak{g}$. Thus

$$\begin{aligned} H \cdot (X_\alpha \cdot v) &= HX_\alpha \cdot v = X_\alpha H \cdot v + [H, X_\alpha] \cdot v \\ &= \mu(H)X_\alpha \cdot v + \alpha(H)X_\alpha \cdot v = (\mu + \alpha)(H)X_\alpha \cdot v \end{aligned}$$

which means that $X_\alpha \cdot v \in V_{\mu+\alpha}$. $\square$

The basis for understanding irreps of $\mathfrak{g}$ is the classification of irreps of $\mathfrak{sl}(2,\mathbb{C})$ since, as we saw in (4.35), every root α spans a unique subalgebra $\mathfrak{sl}_\alpha$ generated by $\tau_\alpha, X_\alpha, X_{-\alpha}$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$.

Recall that we can represent $\mathfrak{sl}(2,\mathbb{C})$ as $\operatorname{span}\{H, X_+, X_-\}$, where

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad X_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad X_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{5.7}$$ sl2mat

and

$$[H, X_+] = 2X_+, \quad [H, X_-] = -2X_-, \quad [X_+, X_-] = H. \tag{5.8}$$ sl2br

A Cartan subalgebra is given by $\mathbb{C} \cdot H$ with respect to which the roots are $\pm\alpha$ with $\alpha(H) = 2$. Furthermore, $X_+ \in \mathfrak{g}_\alpha$, $X_- \in \mathfrak{g}_{-\alpha}$ are an appropriate choice for X_α and $X_{-\alpha}$, and $\tau_\alpha = H$ since $\alpha(\tau_\alpha) = 2$.

su2pol **Example 5.9** We fist give an explicit example of a representation of $\mathrm{SL}(2, \mathbb{C})$ and, via its derivative, of $\mathfrak{sl}(2, \mathbb{C})$. Define V_k be the space of homogeneous polynomials of degree k in two complex variables z, w. Thus

$$V_k = \mathrm{span}_{\mathbb{C}}\{z^k,\ z^{k-1}w,\ z^{k-2}w^2, \ldots,\ zw^{k-1},\ w^k\} \qquad \dim V_k = k+1$$

The Lie group $\mathrm{SL}(2, \mathbb{C})$ acts on V_k via $g \cdot p = p \circ g^{-1}$. If we let $g = \exp(tH) = \mathrm{diag}(e^t, e^{-t})$ then $g \cdot z^r w^{k-r} = (e^{-t}z)^r (e^t w)^{k-r} = e^{t(k-2r)} z^r w^{k-r}$. By differentiating, we obtain a representation of $\mathfrak{sl}(2, \mathbb{C})$ with

$$H \cdot z^r w^{k-r} = (k-2r) z^r w^{k-r}$$

Thus the weights are $k - 2r,\ r = 0, \ldots, k$ with weight space $\mathbb{C} \cdot z^r w^{k-r}$. Furthermore,

$$\begin{aligned} X_+ \cdot z^r w^{k-r} &= \frac{d}{dt}_{|t=0} e^{tX_+} \cdot z^r w^{k-r} = \frac{d}{dt}_{|t=0} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \cdot z^r w^{k-r} \\ &= \frac{d}{dt}_{|t=0} (z - tw)^r (w)^{k-r} = -r z^{r-1} w^{k-r+1}. \end{aligned}$$

Similarly $X_- \cdot z^r w^{k-r} = -(k-r) z^{r+1} w^{k-r-1}$, which easily implies that the representation is irreducible since X_- increases the degree of z and X_+ decreases it.

We will now show that these are in fact all of the irreducible representations of $\mathfrak{sl}(2, \mathbb{C})$.

sl2rep

Proposition 5.10 *For every integer $k \geq 1$, there exists an irrep π_k : $\mathfrak{sl}(2,\mathbb{C}) \to V_k$, with $\dim V_k = k+1$ and conversely every irrep of $\mathfrak{sl}(2,\mathbb{C})$ is isomorphic to π_k for some k. Moreover, there exists a basis $v_0, v_1, \ldots, v_k$ of V such that $X_+ \cdot v_0 = 0$, $X_- \cdot v_k = 0$ and*

$$H \cdot v_i = (k-2i)v_i, \quad X_- \cdot v_i = v_{i+1}, \quad X_+ \cdot v_i = \gamma_i v_{i-1}.$$

with $\gamma_i = i(k-i+1)$. In particular, the weights are given by

$$W_{\pi_k} = \{\mu \in \mathfrak{h}^* \mid \mu(H) = k, k-2, k-4, \ldots, -k\}$$

or equivalently

$$\begin{aligned} W_{\pi_k} &= \{\mu, \mu-\alpha, \mu-2\alpha, \ldots, \mu-k\alpha \mid \mu(\tau_\alpha) = k\} \\ &= \{\frac{k}{2}\alpha, (\frac{k}{2}-1)\alpha, \ldots, -\frac{k}{2}\alpha\} \end{aligned}$$

and all weights have multiplicity one.

Proof Let π be an irrep of $\mathfrak{sl}(2,|C)$. Since we work over $\mathbb{C}$, there exists an eigenvector v for $\pi(H)$, i.e. $H \cdot v = av$, $a \in \mathbb{C}$. The sequence of vectors v, $X_+ \cdot v$, $X_+^2 \cdot v = X_+ \cdot X_+ \cdot v, \ldots$ terminates, since the vectors, having eigenvalue (weights) $a, a+2, \ldots$ by Lemma 5.6, are linearly independent. Thus we set $v_0 = (X_+)^s \cdot v$ with $X_+ \cdot v_0 = 0$. We rename the weight of v_0 to be $b \in \mathbb{C}$, i.e. $H \cdot v_0 = bv_0$. We now inductively define the new sequence of vectors $v_{i+1} = X_- \cdot v_i = X_-^i \cdot v_0$. They have eigenvalues $b, b-2, \ldots$, i.e. $H \cdot v_i = (b-2i)v_i$ and there exists a largest k such that $v_k \neq 0$ but $v_{k+1} = X_- \cdot v_k = 0$. Next we claim that

$$X_+ \cdot v_i = \gamma_i v_{i-1}, \quad \text{with } \gamma_i = i(b-i+1).$$

The proof is by induction. It clearly holds for $i = 0$ since $X_+ v_0 = 0$. Furthermore,

$$X_+ v_{i+1} = X_+ X_- v_i = [X_+, X_-]v_i + X_- X_+ v_i = (b-2i)v_i + \gamma_i v_i,$$

i.e., $\gamma_{i+1} = \gamma_i + b - 2i$ which easily implies the claim.

It is now clear that $W = \mathrm{span}_{\mathbb{C}}\{v_0, v_1, \ldots, v_s\}$ is invariant under the action of $\mathfrak{sl}(2,\mathbb{C})$ since X_- moves down the chain and X_+ moves up the chain of vectors. Hence by irreducibility $W = V$. Furthermore, $0 = X_+ v_{k+1} = \mu_{k+1} v_k$ implies that $\mu_{k+1} = 0$, i.e., $b = k$. Putting all of these together, we see that the irrep must have the form as claimed in the Proposition.

Conversely, such a representation exists as we saw in Example 5.42. □

We can thus reperesent the representation also in matrix form:

$$\pi_k(H) = \begin{pmatrix} k & & & 0 \\ & k-2 & & \\ & & \ddots & \\ 0 & & & -k \end{pmatrix},$$

$$\pi_k(X_-) = \begin{pmatrix} 0 & & & 0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix},$$

$$\pi_k(X_+) = \begin{pmatrix} 0 & \gamma_1 & & 0 \\ & 0 & \ddots & \\ & & \ddots & \gamma_k \\ 0 & & & 0 \end{pmatrix}.$$

Notice that $\pi(H)$ is semisimple and $\pi(X_\pm)$ are nilpotent. This illustrates one of the differences with a complex rep of a compact Lie algebra since in that case every rep is unitary and hence all matrices $\pi(X)$ can be diagonalized.

so3pol

Example 5.11 (a) The rep π_k of $\mathfrak{sl}(2,\mathbb{C})$ integrates to a rep of $\mathrm{SL}(2,\mathbb{C})$ since the group is simply connected. But this representation does not have to be faithful. If it has a kernel, it must lie in the center and hence we only have to test the rep on central elements. In Example 5.42 we gave an explicit description of the rep of $\mathrm{SL}(2,\mathbb{C})$ which must be the integrated version of π_k by uniqueness. Thus the central element $-\mathrm{Id}$ acts trivially for k even and non-trivially for k odd which means that π_{2k+1} is a faithful irrep of $\mathrm{SL}(2,\mathbb{C})$ and π_{2k} a faithful irrep of $PSL(2,\mathbb{C})$.

The rep of $\mathrm{SL}(2,\mathbb{C})$ also induces a rep of the compact subgroup $\mathrm{SU}(2) \subset \mathrm{SL}(2,\mathbb{C})$. Since $\mathfrak{su}(2) \otimes \mathbb{C} \simeq \mathfrak{sl}(2,\mathbb{C})$, this rep is irreducible, and conversely every complex irrep of $\mathrm{SU}(2)$ is of this form. Again it follows that π_{2k+1} is a faithful irrep of $\mathrm{SU}(2)$ and π_{2k} a faithful irrep of $\mathrm{SO}(3)$. Thus $\mathrm{SO}(3)$ has complex irreps only in odd dimension, in fact one in each up to isomorphism.

(b) There also exists a natural real irrep for $\mathrm{SO}(3)$. Let V_k be the vector space of homogeneous polynomials in the real variables x, y, z. The Laplace operator is a linear map $\Delta\colon V_k \to V_{k-2}$ and one easily sees that it is onto.

The kernel $H_k = \{p \in V_k \mid \Delta p = 0\}$ is the set of harmonic polynomials. SO(3) acts on V_k as before, and since Δ is invariant under SO(3), it acts on H_k as well. One can show that this rep is irreducible, has dimension $2k+1$, and its complexification is irreducible as well and hence isomorphic to π_{2k} in Example (a). Thus all real irreps of SO(3) are odd dimensional and unique. The story for real irreps of SU(2) is more complicated. As we will see, there exists one in every odd dimension (only almost faithful) and a faithful one in every dimension $4k$, given by $(\pi_{2k-1})_{\mathbb{R}}$.

We will use this information about the representations of $\mathfrak{sl}(2,\mathbb{C})$ now to study general complex representations of complex semisimple Lie algebras. A key property of the reps of $\mathfrak{sl}(2,\mathbb{C})$ is that the eigenvalues of $\pi(\tau_\alpha)$ are integers, and are symmetric about 0. Furthermore, if μ is the weight of π_k with $\mu(\tau_\alpha) = k$, then the other weights are of the form $\mu, \mu-\alpha, \mu-2\alpha, \dots, \mu-k\alpha = s_\alpha(\mu)$.

5.3 Representations of semisimple Lie algebras

Recall that we have an ordering, after choosing a fixed regular element $H_0 \in \mathfrak{h}_{\mathbb{R}}$, defined on $\mathfrak{h}^*_{\mathbb{R}}$ by $\mu_1 \le \mu_2$ if $\mu_1(H_0) \le \mu_2(H_0)$ and $\mu_1 < \mu_2$ if $\mu_1(H_0) < \mu_2(H_0)$. Furthermore, $\Gamma_W = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid \mu(\tau_\alpha) \in \mathbb{Z} \text{ for all } \alpha \in \Delta\}$ is the weight lattice.

irrep1

Proposition 5.12 *Let $\mathfrak{g}$ be a complex semisimple Lie algebra and π a complex irreducible representation with weights W_π.*

(a) *V is the direct sum of its weight spaces.*
(b) *If $\mu \in W_\pi$, then $\mu(\tau_\alpha) = \langle \mu, \tau_\alpha\rangle = \frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}$ for all $\alpha \in \Delta$, i.e., $\mu \in \Gamma_W$.*
(c) *There exists a unique (strictly) maximal weight λ, i.e. $\mu < \lambda$ for all $\mu \in W_\pi$, $\mu \ne \lambda$. Furthermore, $\langle\lambda, \tau_\alpha\rangle \ge 0$ and $m_\lambda = 1$.*
(d) *Each weight is of the form $\lambda - m_1\alpha_1 - \cdots - m_n\alpha_n$ with $\alpha_i \in F$ and $m_i \in \mathbb{Z}$, $m_i \ge 0$.*
(e) *λ uniquely determines the representation π, i.e. if π, π' are two representations with equal highest weight λ, then $\pi \simeq \pi'$.*

Proof By definition $\pi(H) = \mu(H)\,\mathrm{Id}$ on V_μ for all $H \in \mathfrak{h}$. Recall that for each $\alpha \in \Delta$, we have the subalgebras $\mathfrak{sl}_\alpha = \mathrm{span}_{\mathbb{C}}\{\tau_\alpha, X_\alpha, X_{-\alpha}\}$ isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ with $\tau_\alpha = H,\ X_\alpha = H_+,\ X_{-\alpha} = H_-$ the basis in (5.7). By Proposition 5.10 the eigenvalues of $\pi(\tau_\alpha)$ (on each irreducible summand of $\pi_{|\mathfrak{sl}_\alpha}$) are integers, which implies (b) since $\pi(\tau_\alpha) = \mu(\tau_\alpha)\,\mathrm{Id}$ on V_μ. Furthermore, $\pi(\tau_\alpha)$ can be diagonalized for all τ_α and since τ_{α_i} is a basis of $\mathfrak{h}$, and $\mathfrak{h}$ is abelian, the commuting endomorphisms $\pi(H),\ H \in \mathfrak{h}$, have a common basis of eigenvectors. This implies (a). Notice that it can happen that $\pi_{|\mathfrak{sl}_\alpha}$ is trivial, since $\mu(\tau_\alpha) = 0$ for all $\mu \in W_\alpha$ is possible.

(c) It is clear that (weakly) maximal weights (in the ordering on $\mathfrak{h}^*_{\mathbb{R}}$) exist. Let λ be a maximal weight, and let $v \in V_\lambda$. Then by definition $X_\alpha \cdot v = 0$ for any $\alpha \in \Delta^+$. Now consider

$$V_0 = \mathrm{span}\{X_{-\beta_1} \cdots X_{-\beta_s} \cdot v\}, \qquad \beta_1, \dots \beta_s \in F.$$

where β_i are not necessarily distinct. We claim that V_0 is invariant under $\mathfrak{g}$. It is clearly invariant under $X_{-\alpha}$, $\alpha \in \Delta^+$, and under $\mathfrak{h}$. Thus we just need to prove that V_0 is invariant under the action of X_α, $\alpha \in F$, which we prove by induction on s. If $s = 0$, we know that $X_\alpha \cdot v = 0$.

Suppose now that $X_\alpha \cdot X_{-\beta_1} \cdots X_{-\beta_r} \cdot v \in V_0$ for any $r < s$. Then

$$X_\alpha X_{-\beta_1} \cdots X_{-\beta_s} \cdot v = [X_\alpha, X_{-\beta_1}] X_{-\beta_2} \cdots X_{-\beta_s} \cdot v + X_{-\beta_1} \cdot X_\alpha \cdots X_{-\beta_s} \cdot v$$

By induction, the second term on the right hand side belongs to V_0. Furthermore, $[X_\alpha, X_{-\beta_1}]$ is either a multiple of τ_α if $\beta_1 = \alpha$, or 0 otherwise since $\alpha - \beta_1$ cannot be a root be the definition of F. In either case, the first term belongs to V_0 as well, and hence V_0 is invariant under $\mathfrak{g}$. By irreducibility of π, $V_0 = V$. In particular, all weights are of the form

$$\mu = \lambda - m_1\alpha_1 - \cdots - m_n\alpha_n,\ m_i \geq 0.$$

which implies that if $\mu \in W_\pi$ is a weight different from λ, its order is strictly less than λ. Furthermore, V_λ is spanned by v, and hence $m_\lambda = 1$. If $\lambda(\tau_\alpha) < 0$, the fact that the eigenvalues of $\pi(\tau_\alpha)$ are centered around 0 implies that $\lambda + \alpha$ is an eigenvalue of $\pi(\tau_\alpha)$ and hence a weight of π as well. But this contradicts maximality of λ and hence $\lambda(\tau_\alpha) \geq 0$.

(d) To prove that λ uniquely determines the representation, suppose we have two representations π, π' acting on V, V' with the same maximal weight λ. Choose $v \in V_\lambda$, $v' \in V'_\lambda$. Then $(v, v') \in V \oplus V'$ is a weight vector of $\pi \oplus \pi'$ with weight λ. By the same argument as above, the space $W \subset V \oplus V'$ generated by $X_{-\alpha} \cdot (v, v')$, $\alpha \in \Delta^+$, induces an irreducible representation

$$\sigma = \pi \oplus \pi'|_W \colon \mathfrak{g} \to \mathfrak{gl}(W)$$

If we let $p_1 \colon V \oplus V' \to V$ and $p_2 \colon V \oplus V' \to V'$ be the projections onto the first and second factor, then $p = p_{1|W}$ is an intertwining map between σ and π and similarly $p = p_{2|W}$ is an intertwining map between σ and π'. By irreducibility, these intertwining maps are isomorphisms, and hence $\pi \simeq \sigma \simeq \pi'$, which proves (d). □

The uniquely determined weight λ in the above Proposition will be called the **highest weight** of π. Notice that it can also be characterized by the property that $\lambda + \alpha_i \notin W_\pi$ for all $\alpha_i \in F$.

In general, an element $\mu \in \Gamma_W$ (not necessarily associated to any representation) is called a **dominant weight** if $\mu(\tau_\alpha) \geq 0$ for all $\alpha \in \Delta^+$. Notice that in this terminology we do not specify a representation. In fact in an irreducible representation there can be other dominant weights besides the highest weight.

We denote by $\Gamma_W^d \subset \Gamma_W$ the set of all dominant weights. If π is a rep with highest weight λ, then Proposition 5.12 (b) implies that $\lambda \in \Gamma_W^d$. We also have the following existence theorem:

exrep **Theorem 5.13** *If $\lambda \in \Gamma_W^d$ is a dominant weight, then there exists an irreducible representation π with highest weight λ.*

Thus there is a one-to-one relationship between dominant weights and irreducible representations. We therefore denote by π_λ the unique irrep with highest weight $\lambda \in \Gamma_W^d$.

There are abstract constructions of the representations π_λ, see e.g., [Ha], 200-230. We will content ourselves with giving an explicit construction of the irreducible reps of the classical Lie groups.

Before doing so, we prove some further properties of weights. Recall that we denote by W the Weyl group of $\mathfrak{g}$, which is generated by the reflections $s_\alpha(\beta) = \beta - \langle \beta, \tau_\alpha \rangle \alpha$ acting on $\mathfrak{h}_\mathbb{R}$ and $\mathfrak{h}_\mathbb{R}^*$, for any $\alpha \in \Delta$. Furthermore, $WC^+ = \{v \in \mathfrak{h}_\mathbb{R} \mid \alpha(v) = \langle v, \alpha \rangle > 0 \text{ for all } \alpha \in \Delta^+\}$ is the positive Weyl chamber with respect to our chosen ordering. Dually, the positive Weyl chamber in $\mathfrak{h}_\mathbb{R}^*$ is defined by $WC^* = \{\mu \in \mathfrak{h}_\mathbb{R}^* \mid \mu(v) > 0 \text{ for all } v \in WC^+\}$. The closure of these Weyl chambers is denoted by $\overline{WC^+}$ and $\overline{WC^*}$. Finally, recall that for any $v \in \mathfrak{h}_\mathbb{R}$, there exists a unique $w \in W$ such that $wv \in \overline{WC^+}$ and similarly for $\mathfrak{h}_\mathbb{R}^*$.

irrep2

Proposition 5.14 *Let $\pi = \pi_\lambda$ be an irreducible representation of $\mathfrak{g}$.*

(a) If $\mu \in W_\pi$ and $w \in W$, then $w\mu \in W_\pi$ and $m_{w\mu} = m_\mu$.

(b) λ has maximal length and W acts transitively on weights of length $|\lambda|$ with λ being the only element among them in $\overline{WC^*}$.

(c) Let $\mu \in W_\pi$ and $\alpha \in \Delta^+$ with $\mu(\tau_\alpha) = r > 0$. Then in the string of weights $\mu, \mu - \alpha, \mu - 2\alpha, \ldots, \mu - r\alpha = s_\alpha(\mu)$ the multiplicities are are weakly increasing on the first half, and weakly decreasing on the second half. The same holds if $\mu(\tau_\alpha) < 0$ for the string $\mu, \mu + \alpha, \mu + 2\alpha, \ldots, s_\alpha(\mu)$.

Proof (a) It is sufficient to prove the claim for $w = s_\alpha$ and any $\mu \in W_\pi$. Set $r := \mu(\tau_\alpha)$ and hence $w\mu = \mu - r\alpha$. We can assume that $r > 0$ since the claim is obvious when $r = 0$ and we can replace α by $-\alpha$ otherwise. Every $v \in V_\mu$ generates an irreducible representation σ of $\mathfrak{sl}_\alpha$ by repeatedly applying X_α and $X_{-\alpha}$ to v. σ is clearly one of the irreducible summands in $\pi_{|\mathfrak{sl}_\alpha}$ and hence the weights of σ are restrictions of the weights of π. The weight of $X_{-\alpha}^s \cdot v$ is $\mu - s\alpha$. According to Proposition 5.10, the eigenvalues of $\sigma(\tau_\alpha)$ are symmetric around 0. Since $(\mu - s\alpha)(\tau_\alpha) = r - 2s$, it follows that $\mu - s\alpha$, is a weight of σ (and hence π) and $X_{-\alpha}^s \cdot v \neq 0$ for $s = 1, \ldots r$. In particular, $\mu - r\alpha = w\mu$ is a weight and $X_{-\alpha}^r \colon V_\mu \to V_{\mu - r\alpha}$ is injective, i.e. $m_\mu \leq m_{\mu - r\alpha}$. Since w is an involution, $m_\mu \geq m_{\mu - r\alpha}$ as well and hence they are equal. The same argument implies (c) since $(\mu - s\alpha)(\tau_\alpha)$ is positive on the first half of the string, and the second half are the Weyl group images of the first half.

(b) Let μ be a weight of maximal length and $w \in W$ such that $w\mu \in \overline{WC^*}$. Thus $\langle w\mu, \alpha\rangle \geq 0$ for all $\alpha \in \Delta^+$. Since w acts by isometries, $|w\mu| = |\mu|$ and hence $w\mu$ has maximal length as well. Thus $|w\mu + \alpha| > |w\mu|$ which implies that $w\mu + \alpha$ cannot be a weight. This means that $w\mu$ is a highest weight and by uniqueness $w\mu = \lambda$. □

A useful consequence is

irrepweyl

Corollary 5.15 *If π is a representation in which the Weyl group acts transitively on all weights, and one, and hence all weights have multiplicity one, then π is irreducible.*

Proof Let λ be some highest weight. Then $v_0 \in V_\lambda$ generates an irreducible subrep π_λ acting on $W \subset V$. By applying Proposition 5.14 (a) to π_λ, it follows that all weights of π are already weights of π_λ. Since the weight spaces are all one dimensional, they are contained in W as well, which implies that $V = W$. $\square$

We can determine all weights of a representation from the highest one in the following fashion. According to Proposition 5.12 (e), all weights $\mu \in W_{\pi_\lambda}$ are of the form $\mu = \lambda - m_1\alpha_1 - \cdots - m_n\alpha_n$ with $\alpha_i \in F$ and $m_i \in \mathbb{Z},\ m_i \geq 0$. We call $\sum m_i$ the level of the weight. Thus λ is the only weight of level 0. We can now apply Proposition 5.14 (c) to inductively determine all weights, level by level. It is convenient to apply Proposition 5.14 (c) immediately for all r although this goes down several levels. Notice that every weight of level k is indeed reached from level $k-1$ since otherwise it would be a highest weight. This process is similar to our construction of all positive roots from the simple ones. But notice that the level of the roots are defined in the opposite way to the level of the weights. In general, one problem though is that it is difficult to determine the multiplicity of the weights. The reason is that we can land in V_μ, where μ has level r, possibly in several ways starting with weight vectors at level $r-1$ by applying some X_{-i}. It is not clear when these vectors are linearly independent or not. Thus multiplicities, and hence the dimension of π_λ, are not so easy to determine just by knowing λ. We come back to this problem later on. We illustrate this process in two examples.

g2reps **Example 5.16** (a) Recall from Example 4.18 that $\mathfrak{g}_2$ has two simple roots α, β with $\alpha(\tau_\beta) = -3$ and $\beta(\tau_\alpha) = -1$.

Let us consider the representation π_λ with $\lambda(\tau_\alpha) = 0$ and $\lambda(\tau_\beta) = 1$. We now apply Proposition 5.12 and the chain property in the Excercise below. It is convenient to record the pair $(\mu(\tau_\alpha), \mu(\tau_\beta))$ next to every weight. Notice that $(\alpha(\tau_\alpha), \alpha(\tau_\beta)) = (2, -3)$ and $(\beta(\tau_\alpha), \beta(\tau_\beta)) = (-1, 2)$ and these numbers must be subtracted from the pair, whenever subtracting α resp. β from a weight. The weight of level 0 is λ $(0, 1)$ and hence $\lambda - \beta$ $(1, -1)$ is the only weight at level 1. We can only subtract α to obtain $\lambda - \alpha - \beta$ $(-1, 2)$ at level 2, and similarly $\lambda - \alpha - 2\beta$ $(0, 0)$ at level 3, $\lambda - \alpha - 3\beta$ $(1, -2)$ at level 4, $\lambda - 2\alpha - 3\beta$ $(-1, 1)$ at level 5 and $\lambda - 2\alpha - 4\beta$ $(0, -1)$ at level 6. No further simple roots can be subtracted. Since there is only one weight at each level, all weights have multiplicity one, and the dimension of the representation is 7. We will later see that this corresponds to the fact that the compact group $G_2 \subset \mathrm{SO}(7)$ as the automorphism group of the octonians.

(b) Recall that for $\mathfrak{sl}(4, \mathbb{C})$ we have 3 simple roots α, β, γ and all Cartan integers are -1. We study the rep with the following highest weight λ where we also record the values that need to be subtracted when subtracting a simple root.

$$\begin{aligned}(\lambda(\tau_\alpha),\lambda(\tau_\beta),\lambda(\tau_\gamma)) &= (0,1,0)\\ (\alpha(\tau_\alpha),\alpha(\tau_\beta),\alpha(\tau_\gamma)) &= (2,-1,0)\\ (\beta(\tau_\alpha),\beta(\tau_\beta),\beta(\tau_\gamma)) &= (-1,2,-1)\\ (\gamma(\tau_\alpha),\gamma(\tau_\beta),\gamma(\tau_\gamma)) &= (0,-1,2)\end{aligned}$$

One then obtains weights $\lambda-\beta$ $(1,-1,1)$ at level 1, $\lambda-\beta-\alpha$ $(-1,0,1)$ and $\lambda-\beta-\gamma$ $(1,0,-1)$ at level 2, $\lambda-\beta-\alpha-\gamma$ $(-1,1,-1)$ at level 3, and $\lambda-2\beta-\alpha-\gamma$ $(0,-1,-1)$ at level 4. Notice that since there were two ways to go from level 2 to level 3, it is not clear if $\lambda-\beta-\alpha-\gamma$ has multiplicity one or two. But Proposition 5.14 (c) implies that if it has multiplicity two, so does $\lambda-2\beta-\alpha-\gamma$. Thus the dimension is either 6 or 8. Using the isomorphism $\mathfrak{sl}(4,\mathbb{C})\simeq\mathfrak{so}(6,\mathbb{C})$, we will shortly see that this is the rep ρ_6 that defines $\mathfrak{so}(6,\mathbb{C})$. Thus all multiplicities are actually 1. But this example shows that this process, although it illustrates the geometry of the weights, is not efficient in general.

To get a better understanding of the geometry of the weights of an irrep, the following facts a are instructive (will include a proof later).

convexityweights

Proposition 5.17 *Let $\pi=\pi_\lambda$ be an irreducible representation of $\mathfrak{g}$.*

(a) All weights of π_λ occur in the convex hull of the Weyl orbit of λ.
(b) An element of the weight lattice in the convex hull of $W\cdot\lambda$ is a weight of π_λ iff $\mu=\lambda-m_1\alpha_1-\cdots-m_n\alpha_n$ with $m_i\in\mathbb{Z}$ and $m_i\geq 0$..

Notice that by Proposition 5.14 (c) the full weight diagram exhibits a certain Weyl symmetry since strings are Weyl group symmetric as well. In fact this easily implies Proposition 5.17 (a).

We also add the action of the Weyl group has a few more properties we have not discussed yet:

Weyl geom

Proposition 5.18 *Let W be the Weyl group of $\mathfrak{g}$ and WC^+ the positive Weyl chamber.*

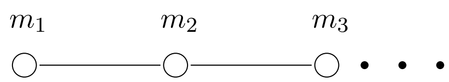
(a) If $x,y\in\overline{WC^+}$ with $w\cdot x=y$ for some $w\in W$, then $w=\mathrm{Id}$.
(b) If $x\in\overline{WC^+}$, then $w\cdot x<x$ for all $w\in W$.

We will also include a proof later on. [Now include more pictures of weight diagrams].

Since λ uniquely characterizes the representation π_λ, we can describe the representation by the n integers

$$m_i = \lambda(\tau_{\alpha_i}) = \frac{2\langle \lambda, \alpha_i \rangle}{\langle \alpha_i, \alpha_i \rangle} \qquad \alpha_i \in F.$$

We thus obtain the diagram of a representation by placing these integers above the corresponding simple root. For simplicity, a 0 integer is not displayed. Every diagram with integers over its dots



thus corresponds to an irrep of the corresponding Lie algebra. For example, the irrep π_k of $\mathfrak{sl}(2,\mathbb{C})$ of dimension $k+1$ is denoted by

π_k : 

We will always use the following notation for the defining representation of the classical Lie groups, which we also call tautological representations.

$$\mu_n\colon\ \mathfrak{su}(n) \text{ on } \mathbb{C}^n, \quad \rho_n\colon\ \mathfrak{so}(n,\mathbb{C}) \text{ on } \mathbb{C}^n, \quad \nu_n\colon\ \mathfrak{sp}(n,\mathbb{C}) \text{ on } \mathbb{C}^{2n} \qquad (5.19)$$ defreps

We often use the same letter for the action of $\mathrm{SU}(n)$or $\mathfrak{su}(n)$ on $\mathbb{C}^n$, the action of $\mathrm{SO}(n)$ or $\mathfrak{so}(n)$ on $\mathbb{R}^n$, and $\mathrm{Sp}(n)$ or $\mathfrak{sp}(n)$ on $\mathbb{C}^{2n}$. We now determine their weights and diagrams, using the previously established notation for the roots and weights.

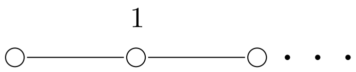
For μ_n the weights are $\omega_1, \dots \omega_n$ with weight vectors e_i. The highest weight is clearly ω_1.

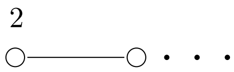
For ρ_n (both n even and odd), the weights are $\pm\omega_1, \dots, \pm\omega_n$ with weight vectors $e_{2i+1} \pm ie_{2i+2}$. ω_1 is again the highest weight.

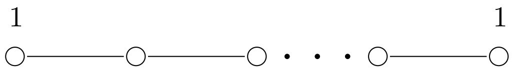
For ν_n the weights are $\omega_1, \dots \omega_n$ with weight vectors $e_1, e_2, \dots, e_n$ and $-\omega_1, \dots - \omega_n$ with weight vectors $e_{n+1}, \dots, e_{2n}$. The highest weight is again ω_1.

Thus in all 3 cases $m_1 = 1$ and $m_2 = m_3 =, \dots = 0$ and hence the diagram has only one integer over it, a 1 over the first dot.

It is also easy to determine the diagram of the adjoint representation of the classical Lie groups. The highest weight is the maximal root α_m, which we determined earlier: $\omega_1 - \omega_{n+1}$ for A_n, $\omega_1 + \omega_2$ for B_n and D_n and $2\omega_1$ for C_n. Thus their diagrams (where we need to assume $n \geq 6$ for $\mathfrak{so}(n)$) are given by:

$\mathrm{ad}_{\mathfrak{so}(n)}$: 

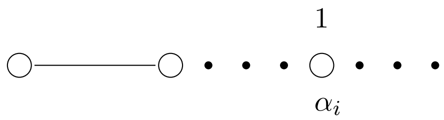
$\mathrm{ad}_{\mathfrak{sp}(n)}$: 

$\mathrm{ad}_{\mathfrak{su}(n)}$: 

It is natural to define as a basis the **fundamental weights**

$$\lambda_i, i = 1, \dots, n, \text{ where } \lambda_i(\tau_{\alpha_j}) = \delta_{ij}$$

We call the corresponding representations π_{λ_i} the **fundamental representations**. Thus their diagram is

π_{λ_i}: 

Clearly the dominant weights λ_i form a basis of the weight lattice Γ_W, and every highest weight λ of an irrep is of the form $\lambda = \sum m_i \lambda_i$ with $m_i = \lambda(\tau_{\alpha_i}) \geq 0$.

If we want to write λ as a linear combination of roots, we have:

inverse

Lemma 5.20 *Let $C = (c_{ij})$, $c_{ij} = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle} = \alpha_i(\tau_{\alpha_j})$ be the Cartan matrix of a simple Lie algebra $\mathfrak{g}$ with inverse $C^{-1} = (b_{ij})$, an λ_i the fundamental dominant weights.*

(a) $\alpha_i = \sum c_{ij} \lambda_j$ and $\lambda_i = \sum b_{ij} \alpha_j$ with b_{ij} positive rational numbers with denominators dividing $\det C$.
(b) $\det C = |Z(\tilde{K})|$ where K is the compact simply connected Lie group with $\mathfrak{k}$ a compact real form of $\mathfrak{g}$.

Proof If $\alpha_i = \sum_k a_{ik} \lambda_k$, then $c_{ij} = \langle \alpha_i, \tau_{\alpha_j} \rangle = \sum_k a_{ik} \langle \lambda_k, \tau_{\alpha_j} \rangle = \sum_k a_{ik} \delta_{kj} = a_{ij}$. This explains the first half of (a) and that b_{ij} are rational numbers with denominators dividing $\det C$.

Next we claim that $b_{ij} \geq 0$, which follows from the following easy geometric fact about a dual basis: τ_{α_i} is a basis of $\mathfrak{h}_{\mathbb{R}}$ with $\langle \tau_{\alpha_i}, \tau_{\alpha_j} \rangle \leq 0$ if $i \neq j$, i.e. all angles are obtuse. This implies that the dual basis λ_i has only obese angles, i.e. $\langle \lambda_i, \lambda_j \rangle \geq 0$ for $i \neq j$. But we also have $\langle \lambda_i, \lambda_j \rangle = \langle \lambda_i, \sum_k b_{jk}\alpha_k \rangle = \sum_k b_{jk} \langle \lambda_i, \alpha_k \rangle = \sum_k b_{jk} \frac{|\alpha_k|^2}{2} \langle \lambda_i, \tau_{\alpha_k} \rangle = b_{ji} \frac{|\alpha_i|^2}{2}$ and hence $b_{ji} \geq 0$.

To see that they are positive, assume that $b_{ij} = 0$ for some i, j. Then $\langle \lambda_i, \alpha_j \rangle = \sum_{k \neq j} b_{ik} \langle \alpha_k, \alpha_j \rangle \leq 0$ since $\langle \alpha_k, \alpha_j \rangle \leq 0$ if $k \neq j$ and $b_{ik} \geq 0$. Since also $\langle \lambda_i, \alpha_j \rangle \geq 0$, this implies that each term must vanish. Thus if $\langle \alpha_k, \alpha_j \rangle < 0$, i.e. α_k and α_j are connected in the Dynkin graph, then $b_{ik} = 0$. But $\mathfrak{g}$ is simple and hence the Dynkin diagram is connected. Connecting the simple root α_i to all other simple roots step by step, it follows that $b_{ik} = 0$ for all k. This cannot be since C is invertible.

(b) In Proposition 4.57 we proved that $Z(\tilde{K}) = \Gamma_W / \Gamma_R$ where Γ_R is the root lattice. Since the matrix that expresse a basis of Γ_R in terms of a basis of Γ_W is given by the Cartan matrix C, the claim follows. $\square$

If we apply Proposition 5.12 and Proposition 5.14 and Lemma 5.20 to the adjoint representation π_{ad} we obtain another proof of Proposition 4.48 since the highest weight of π_{ad} is the maximal root α_m.

From our previous study of the root system for the classical Lie groups and their connections to the compact group K, we obtain the following values for $\det C_{\mathfrak{g}}$:

$$\det C_{\mathfrak{sl}(n,\mathbb{C})} = n, \ \det C_{\mathfrak{so}(2n+1,\mathbb{C})} = \det C_{\mathfrak{sp}(n,\mathbb{C})} = 2, \ \det C_{\mathfrak{so}(2n,\mathbb{C})} = 4$$

A2B2reps **Example 5.21** We illustrate the concepts with two simple examples.

(a) The root system for $\mathfrak{so}(5, \mathbb{C})$ is given by $\Delta^+ = \{\omega_1 \pm \omega_2, \omega_1, \omega_2\}$ and the Weyl group acts by permutations and arbitrary sign changes on ω_i. The roots are the vertices and the midpoints of the sides in a unit square and $\alpha = \omega_1 - \omega_2$, $\beta = \omega_1$ the fundamental roots. The Cartan integers are $\alpha(\tau_\beta) = -2$ and $\beta(\tau_\alpha) = -1$. Thus

$$C = \begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix} \qquad C^{-1} = \frac{1}{2} \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$$

and hence the fundamental weights are

$$\lambda_1 = \alpha + \beta, \qquad \lambda_2 = \frac{1}{2}\alpha + \beta.$$

The root lattice is generated by α, β and the weight lattice by λ_1, λ_2. Clearly one has index two in the other.

For the fundamental representation π_{λ_2} it is clear from the picture that is has dimension 4: The Weyl group image of λ_2 are weights of multiplicity one and are the 4 points in the weight lattice closest to the origin. Furthermore, since λ_2 is not a sum of roots, 0 cannot be a weight. By using the isomorphism $\mathfrak{so}(5,\mathbb{C}) \simeq \mathfrak{sp}(2)$ we see that this rep is just ν_2.

We already saw that $\lambda_1 = \rho_5$. So besides the 4 weights which are Weyl group images of λ_2, 0 is a weight as well. [Need a picture here]

(b) The root system for $\mathfrak{sl}(3,\mathbb{C})$ is $\Delta^+ = \{\omega_i - \omega_j,\ i < j\}$ with simple roots $\alpha = \omega_1 - \omega_2,\ \beta = \omega_2 - \omega_3$. The roots lie at the vertices of a regular hexagon. They have the same length and hence

$$C = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \qquad C^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

and hence

$$\lambda_1 = \frac{2\alpha+\beta}{3}, \qquad \lambda_2 = \frac{\alpha+2\beta}{3}.$$

[Need a picture here and the weight diagram of a more complicated rep].

We will shortly give simple descriptions of the fundamental reps of all classical groups. But we first make some general remarks.

If $\pi_\lambda,\ \pi_{\lambda'}$ are two irreps of $\mathfrak{g}$ acting on V resp. V', then we can construct from them several new representations whose weights we now discuss. Let $W_\pi = \{\lambda_i\}$ and $W_{\pi'} = \{\lambda'_j\}$ be the weights of π resp. π' and v_0 resp. v'_0 a highest weight vector.

We can take the tensor product $\pi_\lambda \otimes \pi_{\lambda'}$ acting on $V \otimes V'$. If v_i resp. v'_j are weight vectors with weights λ_i resp. λ'_j, then clearly $v_i \otimes v'_j$ is a weight vector for $\pi_\lambda \otimes \pi_{\lambda'}$ with weight $\lambda_i + \lambda'_j$. Hence all weights are of the form $\lambda + \lambda' - m_1\alpha_1 - \cdots - m_n\alpha_n$ with $m_i \geq 0$ which implies that $\lambda + \lambda'$ is a highest weight with $m_{\lambda+\lambda'} = 1$. This says that $\pi_{\lambda+\lambda'}$ is an irreducible subrepresentation of $\pi_\lambda \otimes \pi_{\lambda'}$, which we simply write as $\pi_{\lambda+\lambda'} \subset \pi_\lambda \otimes \pi_{\lambda'}$. Furthermore, $\pi_{\lambda+\lambda'}$ has multiplicity one in $\pi_\lambda \otimes \pi_{\lambda'}$, i,e, it can occur only once as a subrep. In general there will be other highest weights generating further irreducible summands in $\pi_\lambda \otimes \pi_{\lambda'}$. The problem of decomposing $\pi_\lambda \otimes \pi_{\lambda'}$ into its irreducible summands can be quite difficult. As an example we derive the **Clebsch-Gordon** formula for the irreps π_k of $\mathfrak{sl}(2,\mathbb{C})$:

$$\pi_k \otimes \pi_\ell = \pi_{k+\ell} + \pi_{k+\ell-2} + \cdots + \begin{cases} 0, & k+\ell \text{ even;} \\ \pi_1, & k+\ell \text{ odd.} \end{cases} \tag{5.22}$$ clebsch

To see this, let $v_i,\ v'_i$ be the basis of weight vectors for the reps π_k and π_ℓ

constructed in Proposition 5.10. Then $v_0 \otimes v_0'$ is a maximal weight vector with weight $k+\ell$ giving rise to the irreducible summand in $\pi_{k+\ell}$. Furthermore,

$$X_- \cdot v_0 \otimes v_0' = v_1 \otimes v_0' + v_0 \otimes v_1', \; X_+ \cdot (\ell v_1 \otimes v_0' - k v_0 \otimes v_1') = k\ell v_0 \otimes v_0' - \ell k v_0 \otimes v_0' = 0$$

which means that $m_{k+\ell-2} = 2$ and $\ell v_1 \otimes v_0' - k v_0 \otimes v_1'$ is a maximal weight vector. Thus $\pi_{k+\ell-2}$ is also an irreducible summand in $\pi_{k+\ell}$. Continuing in this fashion, we obtain (5.22).

Given a rep π_λ acting on V, we can define a new rep $\Lambda^k \pi$ acting on $\Lambda^k V$. If $v_1, \dots, v_k$ are linearly independent weight vectors with weights $\lambda_1, \dots, \lambda_k$ then clearly $v_1 \wedge v_2 \wedge \dots \wedge v_k$ is a weight vector with weight $\lambda_1 + \lambda_2 + \dots + \lambda_k$. But the decomposition of $\Lambda^k \pi$ into irreducible subreps can be quite difficult.

Let us the discuss the simplest case $k = 2$ a little further. For each simple root $\alpha \in F$ with $\lambda(\tau_\alpha) > 0$, we obtain irreducible subrep $\pi_{2\lambda-\alpha}$ of $\Lambda^2 \pi_\lambda$ with multiplicity one. To see this, let v_0 be a highest weight vector. Then $v_0 \wedge X_{-\alpha} \cdot v_0$ is a weight vector with weight $2\lambda - \alpha$. Furthermore, $X_\alpha(v_0 \wedge X_{-\alpha} v_0) = v_0 \wedge X_\alpha X_{-\alpha} v_0 = v_0 \wedge [X_\alpha, X_{-\alpha}] v_0 = v_0 \wedge \lambda(\tau_\alpha) v_0 = 0$, and if β is a simple root distinct from α, then $X_\beta(v_0 \wedge X_{-\alpha} v_0) = v_0 \wedge X_\beta X_{-\alpha} v_0 = v_0 \wedge [X_\beta, X_{-\alpha}] v_0 = 0$ since $\beta - \alpha$ is not a root. Thus $v_0 \wedge X_{-\alpha} \cdot v_0$ is a maximal weight vector and $2\lambda - \alpha$ is a maximal weight. Clearly, $v_0 \wedge X_{-\alpha} \cdot v_0$ is the only weight vector with weight $2\lambda - \alpha$ and hence $\pi_{2\lambda-\alpha}$ has multiplicity one in $\Lambda^2 \pi_\lambda$.

Similarly, if $\lambda(\tau_\alpha) > 0$, then $\pi_{2\lambda} \subset S^2 \pi_\lambda$ with multiplicity one. The proof clearly shows more generally:

productweight

Proposition 5.23 *If $\pi_{\lambda_1}, \dots, \pi_{\lambda_k}$ resp π_λ are irreducible representations of $\mathfrak{g}$, then $\pi_{\lambda_1 + \dots + \lambda_k} \subset \pi_{\lambda_1} \otimes \dots \otimes \pi_{\lambda_k}$, $\pi_{k\lambda} \subset S^k(\pi_{\lambda_1})$ and $\pi_{2\lambda-\alpha} \subset \Lambda^2 \pi_\lambda$ whenever $\langle \lambda, \alpha \rangle > 0$, all with multiplicity one.*

Exercises 5.24

(1) Show that a simple Lie algebra has at most two different lengths among its roots and that the Weyl group acts transitively on roots of equal length.
(2) For each of the classical Lie algebras, and for each root α, determine the 3-dimensional subalgebras $\mathfrak{sl}_\alpha \simeq \mathfrak{sl}(2, \mathbb{C})$ up to inner automorphisms. For each of the compact Lie groups $K = \mathrm{SU}(n)$, $\mathrm{SO}(n)$ $\mathrm{Sp}(n)$, and for each root α, the subalgebra $\mathfrak{k}_\alpha$ gives rise to a compact subgroup $K_\alpha \subset K$. Classify K_α up to conjugacy, and in each case determine wether it is isomorphic to $\mathrm{SU}(2)$ or $\mathrm{SO}(3)$. Show that for the maximal root we always obtain an $\mathrm{SU}(2)$.

(3) Let π_λ be an irrep with $\lambda(\tau_\alpha) = k > 0$. Show that for all $1 \leq \ell \leq k$, $\pi_{2\lambda-\ell\alpha} \subset S^2\pi_\lambda$ if ℓ even and $\pi_{2\lambda-\ell\alpha} \subset \Lambda^2\pi_\lambda$ if ℓ odd, both with multiplicity one. You should be able to prove this easily for $k = 1, 2, 3$.

(4) A somewhat more difficult exercise is the following. We say that $\alpha_1, \ldots \alpha_k$ is a *chain of simple roots* if $\langle \alpha_i, \alpha_{i+1} \rangle \neq 0$ and $\langle \alpha_i, \alpha_j \rangle = 0$ whenever $j \geq i+2$. Given such a chain of roots, let π_λ be an irrep with $\langle \lambda, \alpha_i \rangle > 0$ for $i = 1, k$ and $\langle \lambda, \alpha_i \rangle = 0$ for $2 \leq i \leq k-1$. Show that $\pi_{2\lambda-\alpha_1-\cdots-\alpha_k}$ is an irreducible subrep of both $S^2\pi_\lambda$ and $\Lambda^2\pi_\lambda$.

(5) Show that $\pi \otimes \pi = \Lambda^2\pi \oplus S^2\pi$ and hence rule (4) and (5) apply to the tensor product as well.

5.4 Representations of classical Lie algebras

We will now discuss weight lattices, dominant weights and fundamental representations of the classical Lie algebras.

It follows from Proposition 5.23 that, starting with the fundamental representations π_{λ_i}, one can recover all other irreps as sub representations of tensor products and symmetric powers of π_{λ_i}. In order to prove Theorem 5.13 for the classical Lie groups, we thus only need to construct their fundamental representations. This will be done mostly with exterior powers of the tautological representations.

$$\mathbf{A_n} = \mathfrak{sl}(\mathbf{n+1}, \mathbb{C})$$

Recall that $\Delta^+ = \{\omega_i - \omega_j \mid i < j\}$and

$$F = \{\alpha_1 = \omega_1 - \omega_2, \;\; \alpha_2 = \omega_2 - \omega_3, \;\; \ldots, \alpha_n = \omega_n - \omega_{n+1}\}.$$

The Weyl group $W \simeq S_n$ acts as permutation on w_i and the inner product makes ω_i into an orthonormal basis (of $\mathbb{C}^{n+1}$).

If we define $\lambda_i = \omega_1 + \omega_2 + \cdots + \omega_i$ one clearly has $\langle \lambda_i, \tau_{\alpha_j} \rangle = \delta_{ij}$. Thus π_{λ_i} is the ith fundamental representation.

The weight lattice is

$$\Gamma_W = \left\{ \sum c_i\lambda_i \mid c_i \in \mathbb{Z} \right\} = \left\{ \sum k_i\omega_i \mid k_i \in \mathbb{Z} \right\}$$

Furthermore

$$\Gamma_W^d = \left\{ \sum n_i\lambda_i \middle| \, n_i \geq 0 \right\} = \left\{ \sum k_i\omega_i \middle| \, k_1 \geq k_2 \ldots k_{n+1} \geq 0 \right\}$$

Let μ_n be the tautological representation of $\mathfrak{sl}(n+1), \mathbb{C})$ on $\mathbb{C}^{n+1}$. It has weights $\omega_1, \dots \omega_n$ and thus $\mu_n = \pi_{\lambda_1}$. We now claim that $\pi_{\lambda_k} \simeq \Lambda^k \mu_n$. Indeed, we have a basis of $\Lambda^k \mathbb{C}^{n+1}$ given by the weight vectors

$$e_{i_1} \wedge \cdots \wedge e_{i_k}, \text{ with weight } \omega_{i_1} + \cdots + \omega_{i_n} \text{ for all } \quad i_1 < \cdots < i_k.$$

The weight vectors are linearly independent and all have distinct weights. Furthermore the Weyl group permutes all weights and thus Corollary 5.15 implies that $\Lambda^k \mu_n$ is irreducible. Clearly, $\omega_1 + \omega_2 + \cdots + \omega_k$ is the highest weight vector, which implies our claim.

Summarizing, we have

- Fundamental representations $\pi_{\lambda_k} = \wedge^k \mu_n$ with $\lambda_k = \omega_1 + \ldots + \omega_k$ and $\dim \pi_{\lambda_k} = \binom{n}{k}$.
- $\Gamma_W = \{\sum c_i \lambda_i \mid c_i \in \mathbb{Z}\} = \{\sum k_i \omega_i \mid k_i \in \mathbb{Z}\}$,
- $\Gamma_W^d = \left\{\sum n_i \lambda_i \middle| \, n_i \geq 0\right\} = \left\{\sum k_i \omega_i \middle| \, k_1 \geq k_2 \dots k_{n-1} \geq 0\right\}$.

$$\mathbf{B_n} = \mathfrak{so}(\mathbf{2n+1}, \mathbb{C})$$

Recall that $\Delta^+ = \{\omega_i \pm \omega_j, \, \omega_i, \mid 1 \leq i < j \leq n\}$ with coroots $\{\omega_i \pm \omega_j, \, 2\omega_i\}$ and simple roots

$$F = \{\alpha_1 = \omega_1 - \omega_2, \ \dots, \ \alpha_{n-1} = \omega_{n-1} - \omega_n, \ \alpha_n = \omega_n\}.$$

The Weyl group $W \simeq S_n \rtimes \mathbb{Z}_2^n$ acts as permutations and arbitrary sign changes on ω_i. The inner product makes ω_i into an orthonormal basis of $\mathfrak{h}^*$.

One easily sees that the fundamental weights are

$$\begin{cases} \lambda_i = \omega_1 + \ldots + \omega_i & i < n \\ \lambda_n = \frac{1}{2}(\omega_1 + \ldots \omega_n) \end{cases}$$

The $1/2$ is due to the fact that the coroot of α_n is $2\alpha_n$.

This implies that the weight lattice is

$$\Gamma_W = \left\{ \frac{1}{2} \sum k_i \omega_i \mid k_i \in \mathbb{Z}, \ k_i \text{ all even or all odd} \right\}$$

and

$$\Gamma_W^d = \left\{ \frac{1}{2} \sum k_i \omega_i \in \Gamma_W \mid k_1 \geq k_2 \geq \cdots \geq k_n \geq 0 \right\}$$

The tautological representation ρ_{2n+1} has weights $\pm\omega_1, \, \dots, \pm\omega_n, 0$ and highest weight ω_1. Thus $\pi_{\lambda_1} = \rho_{2n+1}$. One easily sees that $\Lambda^k \rho_{2n+1}$ has weights $\pm\omega_{i_1} \pm \cdots \pm \omega_{i_\ell}$ for all $i_1 < \cdots < i_\ell$ and any $\ell \leq k$, and we call ℓ the length of the weight (if we include an $e_i \wedge e_{n+i}$ term, the length of the weight decreases by two and decreases by one if e_{2n+1} is included). We now claim $\Lambda^k \rho_{2n+1}$ is irreducible for all $1 \le k \le n$ with highest weight $\omega_1 + \ldots + \omega_k$. (It is in fact irreducible for $n < k \le 2n$ as well, but $\Lambda^{2n+1-k}\rho_{2n+1} \simeq \Lambda^k \rho_{2n+1} \simeq$ for $1 \le k \le n$). First observe that $\lambda := \omega_1 + \ldots + \omega_k$ is certainly a highest weight (has largest value on our standard choice of regular vector defining the order) and thus defines an irreducible summand in $\pi_\lambda \subset \Lambda^k \rho_{2n+1}$. By applying the Weyl group to λ, we obtain all weights of length $\ell = k$. Since $\langle \lambda, \omega_k \rangle > 0$, $\lambda - \omega_k = \omega_1 + \ldots + \omega_{k-1}$ is a weight as well. Repeating and applying the Weyl group, we see that all weights belong to π_λ. Furthermore, since the weights for the usual wedge product basis are all distinct, each weight has multiplicity one. Thus Proposition 5.15 implies that $\Lambda^k \rho_{2n+1}$ is irreducible.

Comparing the highest weights, we see that $\pi_{\lambda_k} \simeq \Lambda^k \rho_{2n+1}$ for $k = 1, \ldots, n-1$ whereas $\Lambda^n \rho_{2n+1} \simeq \pi_{2\lambda_n}$. Indeed, π_{λ_n} is a new representation called the **spin representation**, and denoted by $\boldsymbol{\Delta}_n$, that cannot be described in an elementary fashion. A construction of this representation will be done in a later section. But for now we can say that $\frac{1}{2}(\pm\omega_1 \pm \cdots \pm \omega_n)$ are all weights of π_{λ_n} with multiplicity one since they are the Weyl orbit of λ_n. By going through the inductive procedure of constructing weights from the highest one, one easily sees that there are no further weights. Thus $\dim \pi_{\lambda_n} = 2^n$.

Notice that in the case of $n = 1$, where there is only one positive root ω, we can interpret Δ_1 as what we called π_1 in Proposition 5.10 since the highest weight is $\frac{1}{2}\omega$. A more interesting case is $n = 2$. Here we can use the isomorphism $\mathfrak{so}(5, \mathbb{C}) \simeq \mathfrak{sp}(2, \mathbb{C})$ to see that $\Delta_2 = \mu_2$. For $n = 3$ we will see that this representation gives rise to interesting subgroups of SO(8) isomorphic to Spin(7). Notice that the two half spin reps of $\mathfrak{so}(8, \mathbb{C})$ have dimension 8.

Summarizing, we have

- Fundamental representations $\pi_{\lambda_k} = \wedge^k \rho_{2n+1}$ with $\lambda_k = \omega_1 + \ldots + \omega_k$ for $k = 1, \ldots, n-1$ and $\dim \pi_{\lambda_k} = \binom{2n+1}{k}$, and the spin representation $\pi_{\lambda_n} = \Delta_n$ with $\dim \Delta_n = 2^n$.
- $\Gamma_W = \left\{ \frac{1}{2} \sum k_i \omega_i \mid k_i \in \mathbb{Z},\ k_i \text{ all even or all odd} \right\}$,
- $\Gamma_W^d = \left\{ \frac{1}{2} \sum k_i \omega_i \in \Gamma_W \mid k_1 \ge k_2 \ge \cdots \ge k_n \ge 0 \right\}$.

$\mathbf{D_n} = \mathfrak{so}(\mathbf{2n}, \mathbb{C})$

Recall that $\Delta^+ = \{\omega_i \pm \omega_j \mid 1 \le i < j \le n\}$ with simple roots

$$F = \{\alpha_1 = \omega_1 - \omega_2,\ \ldots,\ \alpha_{n-1} = \omega_{n-1} - \omega_n,\ \alpha_n = \omega_{n-1} + \omega_n\}.$$

Since all roots have the same length, roots and coroots agree. The Weyl group $W \simeq S_n \rtimes \mathbb{Z}_2^{n-1}$ acts as permutations and an even number of sign changes on ω_i. The inner product makes ω_i into an orthonormal basis of $\mathfrak{h}^*$.

One easily sees that the fundamental weights are

$$\begin{cases} \lambda_i = \omega_1 + \ldots + \omega_i & i \le n-2 \\ \lambda_{n-1} = \frac{1}{2}(\omega_1 + \cdots + \omega_{n-1} - \omega_n) & \\ \lambda_n = \frac{1}{2}(\omega_1 + \ldots + \omega_{n-1} + \omega_n) & \end{cases}$$

The weight lattice is the same as in the previous case, but the condition for $\frac{1}{2}\sum k_i\omega_i$ being dominant is now $k_1 \ge k_2 \ge \cdots \ge |k_n|$. The first string of inequalities is due to the requirement that $\omega_i - \omega_j, i < j$ must be non-negative, and the last since $\omega_{n-1} + \omega_n$ must be non-negative.

The tautological representation ρ_{2n} has weights $\pm\omega_1, \ldots, \pm\omega_n$ and highest weight ω_1. Thus $\pi_{\lambda_1} = \rho_{2n}$. One difference with the previous case is that $\Lambda^k\rho_{2n}$ has weights $\pm\omega_{i_1} \pm \cdots \pm \omega_{i_\ell}$ only for all $i_1 < \cdots < i_\ell$ with ℓ even (there is no 0 weight in this case). We claim that $\Lambda^k\rho_{2n}$ is irreducible with highest weight $\omega_1 + \ldots + \omega_k$ for all $k = 1, \ldots, n-1$. The argument is similar to the previous case. But since ω_i are not roots, we can lower the degree only by an even number: Since $\langle \lambda, \omega_{k-1} + \omega_k \rangle > 0$, $\lambda - (\omega_{k-1} + \omega_k) = \omega_1 + \ldots + \omega_{k-2}$ is a weight. The other difference is that the Weyl group allows only an even number of sign changes. But this is no problem for $k < n$ and the proof is finished as before.

Comparing highest weights, we see that $\pi_{\lambda_k} = \wedge^k\rho_{2n}$ for $k = 1, \ldots, n-2$. For the remaining ones we claim:

$$\Lambda^{n-1}\rho_{2n} = \pi_{\lambda_{n-1}+\lambda_n}, \qquad \Lambda^n\rho_{2n} = \pi_{2\lambda_{n-1}} \oplus \pi_{2\lambda_n}. \tag{5.25}$$ decompextson

For the first one we showed the rep is irreducible, and we just take inner products with coroots. For the second one we claim that both $\omega_1 + \cdots + \omega_{n-1} + \omega_n$ and $\omega_1 + \cdots + \omega_{n-1} - \omega_n$ are highest weights since adding any of the simple roots does not give a weight. Applying the Weyl group action one gets two irreps and since there are no other weights, the claim follows. This is particularly important for $n = 2$ (the discussion still works even though $\mathfrak{so}(4)$ is not simple) where it gives rise to self duality.

The representations $\pi_{\lambda_{n-1}}$ and π_{λ_n} are again new representations, called **half spin representations** and denoted by $\mathbf{\Delta}_n^-$ and $\mathbf{\Delta}_n^+$. Applying the Weyl group, we see that the weights of these reps are $\frac{1}{2}(\pm\omega_1 \pm \cdots \pm \omega_n)$ where the number of -1 is even for Δ_n^+, and odd for Δ_n^-. One again easily shows that they have multiplicity 1 and exhaust all weights of $\Delta_n^\pm$. Thus $\dim \Delta_n^\pm = 2^{n-1}$.

It is interesting to interpret the above discussion and spin reps for $n = 2, 3$ using low dimensional isomorphism (see the exercises below). In the case of $n = 4$ we get 3 fundamental irreps of SO(8) in dimension 8: Δ^+, Δ^- and ρ_8. Our theory implies that they are inequivalent, but we will see that they are outer equivalent.

Summarizing, we have

- Fundamental representations $\pi_{\lambda_k} = \wedge^k \rho_{2n+1}$ with $\lambda_k = \omega_1 + \ldots + \omega_k$ and $\dim \pi_{\lambda_k} = \binom{2n+1}{k}$ for $k = 1, \ldots, n-2$, and the half spin representation $\pi_{\lambda_{n-1}} = \Delta_n^-$ with $\lambda_{n-1} = \frac{1}{2}(\omega_1 + \cdots + \omega_{n-1} - \omega_n)$ and $\pi_{\lambda_n} = \Delta_n^+$ with $\lambda_n = \frac{1}{2}(\omega_1 + \ldots + \omega_{n-1} + \omega_n)$ and $\dim \Delta_n^\pm = 2^{n-1}$.
- $\Gamma_W = \left\{ \frac{1}{2} \sum k_i \omega_i \mid k_i \in \mathbb{Z},\ k_i \text{ all even or all odd} \right\}$,
- $\Gamma_W^d = \left\{ \frac{1}{2} \sum k_i \omega_i \in \Gamma_W \mid k_1 \geq k_2 \geq \cdots \geq |k_n| \right\}$.

$\mathbf{C_n} = \mathfrak{sp}(\mathbf{n}, \mathbb{C})$

Here we have the roots $\Delta^+ = \{\omega_i \pm \omega_j,\ 2\omega_i,\ \mid 1 \leq i < j \leq n\}$ with coroots $\{\omega_i \pm \omega_j,\ \omega_i\}$ and simple roots

$$F = \{\alpha_1 = \omega_1 - \omega_2,\ \ldots,\ \alpha_{n-1} = \omega_{n-1} - \omega_n,\ \alpha_n = 2\omega_n\}.$$

The Weyl group $W \simeq S_n \rtimes \mathbb{Z}_2^n$ acts as permutations and arbitrary sign changes on ω_i. The inner product makes ω_i into an orthonormal basis of $\mathfrak{h}^*$.

One easily checks that the fundamental weights are $\lambda_i = \omega_1 + \cdots + \omega_i$. The weights of μ_n acting on $\mathbb{C}^{2n}$ are $\pm\omega_1, \ldots, \pm\omega_n$ with highest weight ω_1. Thus $\pi_{\lambda_1} = \mu_n$. The rep $\Lambda^k \mu_n$ has weights $\pm\omega_{i_1} \pm \cdots \pm \omega_{i_\ell}$ for all $i_1 < \cdots < i_\ell$ with ℓ even. But now $\Lambda^k \mu_n$ is not irreducible anymore. To understand why, recall that the rep μ_n respects a symplectic form β on $\mathbb{C}^{2n}$ by definition. We can regard $\beta \in (\Lambda^2 \mathbb{C}^{2n})^*$, which enables one to define a contraction

$$\varphi_k : \Lambda^k \mathbb{C}^{2n} \to \Lambda^{k-2} \mathbb{C}^{2n} : \alpha \to \beta \lrcorner \alpha$$

where $\beta \lrcorner \alpha$ is defined by

$$\varphi(\beta \lrcorner v) = (\varphi \wedge \beta)(v), \qquad \forall \varphi \in (\Lambda^{k-2} \mathbb{C}^{2n})^*,\ v \in \Lambda^k \mathbb{C}^{2n}.$$

This can also be expressed as

$$\varphi_k(v_1 \wedge \cdots \wedge v_k) = \sum_{i<j} \beta(v_i, v_j) v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_k.$$

Now we claim that φ_k is equivariant with respect to the action induced by the action of $\mathfrak{sp}(n, \mathbb{C})$ on $\mathbb{C}^{2n}$. This is clear for the action of the Lie group $\mathrm{Sp}(n, \mathbb{C})$ since it respects β, and hence, via differentiating, for the action of $\mathfrak{sp}(n, \mathbb{C})$ as well. Hence $\ker I$ is an invariant subspace. We claim that this subspace is irreducible and is precisely the irrep π_{λ_k}. To see this, recall that in the standard basis of $\mathbb{C}^{2n}$ we have $\beta(e_i, e_{n+i}) = -\beta(e_{n+i}, e_i) = 1$ and all others are 0. Furthermore, e_i has weight ω_i and , e_{n+i} as weight $-\omega_i$. Thus all weight vectors with weight $\pm\omega_{i_1} \pm \cdots \pm \omega_{i_k}$ lie in $\ker I$. Since the Weyl group acts transitively on these weights, $\ker I$ is an irrep. Notice that the difference with $\mathfrak{so}(2n+1, \mathbb{C})$ is that here $\langle \lambda, \tau_{\omega_k} \rangle = 1$ and hence $\lambda - 2\omega_k = \omega_1 + \ldots + \omega_{k-1} - \omega_k$ is a weight, i.e. the length cannot be reduced.

By taking inner products with coroots, we see that $\ker I \simeq \Lambda^k \mu_n$. Inductively one shows:

$$\Lambda^k \mu_n = \pi_{\lambda_k} \oplus \pi_{\lambda_{k-2}} \oplus \cdots \oplus \begin{cases} \nu_n & k \text{ odd} \\ 1 & k \text{ even} \end{cases}$$

where 1 is the trivial one-dimensional representation.

Notice for example that $\Lambda^2 \mu_n = \pi_{\lambda_2} \oplus 1$. The existence of the trivial rep in $\Lambda^2 \mu_n$ is clear, since it represents the symplectic form $\beta \in \Lambda^2 \mathbb{C}^{2n}$.

Notice also that $S^2 \mu_n$ has weights $\pm\omega_i \pm \omega_j$ (which include $2\omega_i$ with multiplicity 1, and 0 with multiplicity n. These are precisely the roots of $\mathfrak{sp}(n, \mathbb{C})$ and hence $S^2 \mu_n = \pi_{\mathrm{ad}}$, a fact we saw earlier.

Summarizing, we have

- Fundamental representations π_{λ_k} with $\lambda_k = \omega_1 + \ldots + \omega_k$ and $\pi_{\lambda_k} \subset \wedge^k \mu_n$.
- $\Gamma_W = \{\sum k_i \omega_i \mid k_i \in \mathbb{Z}\}$,
- $\Gamma_W^d = \{\sum k_i \omega_i \in \Gamma_W \mid k_1 \geq k_2 \geq \cdots \geq k_n \geq 0\}$.

Exercises 5.26

(1) Check that the above discussion still holds for $\mathfrak{so}(4, \mathbb{C})$ and identify ρ_4 and the spin reps as exterior tensor products. Discuss (5.25) in this context and relate it to self duality in dimension 4 as discussed in Chapter 2.

(2) Classify the irreps of $\mathfrak{so}(4, \mathbb{C})$ and determine their weights, multiplicities and dimensions.

(3) The spin rep of $\mathfrak{so}(6, \mathbb{C})$ becomes a irrep of $\mathfrak{su}(4)$ under the isomorphism $\mathfrak{so}(6, \mathbb{C}) \simeq \mathfrak{su}(4)$. Relate the spin reps to reps of $\mathfrak{su}(4)$ and explain (5.25) in this case.

(4) Use the explicit description of the cover $\mathrm{Sp}(2) \to \mathrm{SO}(5)$ to determine the induced homomorphism of maximal tori and hence the induced map on weight lattice, root lattice and integral lattice. Discuss how the weights of their fundamental representations match.
(5) Repeat the discussion in exercise (4) for the two fold cover $\mathrm{SU}(4) \to \mathrm{SO}(6)$.
(6) Determine the dimension of π_{λ_k} for $\mathfrak{sp}(n, \mathbb{C})$.

5.5 Real Representations of Real Lie Groups

In the previous section we classified complex irreducible representations of complex semisimple Lie algebras. In this section we want to study how to derive from this knowledge real representations of real Lie algebras, and apply this e.g. to classify all subgroups of classical Lie groups up to conjugacy.

We start with the following definitions. Recall that a complex bilinear form b is called invariant under π if $b(X \cdot v, w) + b(v, X \cdot w) = 0$.

orthsympdef

Definition 5.27 *Let π be a complex representation of the complex Lie algebra $\mathfrak{g}$.*

(a) π is called **orthogonal** *if there exists a non-degenerate symmetric bilinear form invariant under π.*
(b) π is called **symplectic** *if there exists a non-degenerate skew-symmetric bilinear form invariant under π.*
(c) π is called of **complex** *if it neither orthogonal nor quaternionic.*
(d) π is called of **self dual** *if $\pi \simeq \pi^*$.*

In other words, π is orthogonal (resp. symplectic) if there exists a basis of V such that $\pi(\mathfrak{g}) \subset \mathfrak{o}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$ (resp. $\pi(\mathfrak{g}) \subset \mathfrak{sp}(n, C) \subset \mathfrak{gl}(2n, \mathbb{C})$).

If π acts on V, recall that we have the dual representation π^* acting on V^* via $\pi^*(X)(f)(v) = -f(\pi(X)(v))$. If we choose a basis of V, and the dual basis of V^*, we have $\pi^*(X) = -\pi(X)^T$ on the level of matrices. The choice of sign is necessitated due to the fact that on the group level we need to define $\pi^*(g) = \pi(g^{-1})^T$ in order for π^* to be a Lie group representation. In particular, if we have diagonalized $\pi(\mathfrak{h})$ with respect to some basis of eigenvectors in V, i.e. $X \cdot (e_i) = \mu_i(X) e_i$, then the dual basis f_i are eigenvectors of $\pi^*(\mathfrak{h})$ with weights $-\mu_i$ since $(X \cdot f_i)(e_j) = -f_i(X \cdot e_j) = -\mu_j(X) f_i(e_j) = -\mu_j(X)\delta_{ij}$, i.e. $X \cdot f_i = -\mu_i(X) f_i$. Thus $W_{\pi^*} = -W_\pi$. Notice also that μ_i and $-\mu_i$ have the same multiplicity.

orthsymp1

Proposition 5.28 *Let π be a complex representation of $\mathfrak{g}$.*

(a) π is orthogonal or symplectic iff π is self dual.
(b) If π is irreducible, then the property of being orthogonal, symplectic or complex are mutually exclusive.
(c) $\pi \oplus \pi^*$ is both orthogonal and symplectic. In particular, $\pi \oplus \pi$ is both orthogonal and symplectic if π is either orthogonal or symplectic.
(d) π is orthogonal iff $\pi \simeq \pi_1 \oplus \cdots \oplus \pi_k \oplus (\sigma_1 \oplus \sigma_1^*) \oplus \cdots \oplus (\sigma_\ell \oplus \sigma_\ell^*)$ where π_i are orthogonal irreducible representations and σ_i are complex or symplectic irreducible representations.
(e) π is symplectic iff $\pi \simeq \pi_1 \oplus \cdots \oplus \pi_k \oplus (\sigma_1 \oplus \sigma_1^*) \oplus \cdots \oplus (\sigma_\ell \oplus \sigma_\ell^*)$ where π_i are symplectic irreducible representations and σ_i are complex or orthogonal irreducible representations.

Proof (a) An invariant non-degenerate bilinear form b can equivalently be regarded as an equivariant isomorphism $V \simeq V^*$ via $v \to f_v \in V^*$ with $f_v(w) = b(v, w)\}$. Indeed, $X \cdot v \to f_{X \cdot v}(w) = b(X \cdot v, w) = -b(v, X \cdot w) = -f_v(X \cdot w)$.

Part (b) follows since by Schur's Lemma there can be only one equivariant linear map $V \to V^*$ up to complex multiples.

(c) Define the bilinear form b on $V \oplus V^*$ by $b((v, f), (w, g)) = f(w) + \epsilon g(v)$ for some $\epsilon = \pm 1$. Then b is clearly bilinear. Since $b((w, g), (v, f)) = g(v) + \epsilon f(w)$, b is symmetric if $\epsilon = 1$ and skew-symmetric if $\epsilon = -1$. b is non-degenerate since $B((v, f), (w, g)) = 0$ for all (w, g) implies that $v = 0$ by first setting $w = 0$ and $f = 0$ by setting $g = 0$. Finally, b is invariant since $b((X \cdot v, X \cdot f), (w, g)) = (X \cdot f)(w) + \epsilon g(X \cdot v) = -f(X \cdot w) - \epsilon(X \cdot g)(v) = -b((v, f), (X \cdot w, X \cdot g))$.

(d) and (e) Let b be the non-degenerate symmetric (skew-symmetric) bilinear form on V and $V = V_1 \oplus \cdots \oplus V_r$ a decomposition into irreducible sub-representations π_i acting on V_i. Then $b\colon V_i \times V_j \to \mathbb{C}$ induces an equivariant linear map $V_i \to V_j^*$ for all i, j. By irreducibility, this is either an isomorphism or 0. If $b_{|V_i \times V_i} \neq 0$, then π_i is orthogonal (resp. symplectic). If it is 0, there exists a $j \neq i$ with $b_{|V_i \times V_j} \neq 0$ and hence $\pi_i^* \simeq \pi_j$, which proves our claim.

$\square$

The following are also elementary but important properties.

orthsymp2

Proposition 5.29 *Let orth stand for an orthogonal representation and symp for a symplectic one. Then*

(a) $orth \otimes orth = orth$, $symp \otimes symp = orth$ and $orth \otimes symp = symp$.
(b) $\Lambda^k(orth) = orth$ and $S^k orth = orth$.
(c) $S^k symp$ and $\Lambda^k symp$ are orthogonal if k even and symplectic if k odd.
(d) If π is complex, then $\pi \otimes \pi^*$ is orthogonal.

Proof (a) If b_i are non-degenerate bilinear forms on V_i then $b(v_1 \otimes w_1, v_2 \otimes w_2) = b_1(v_1, w_1)b_2(v_2, w_2)$ is a bilinear form on $V_1 \otimes V_2$ which one easily sees is non-degenerate.

(b) and (c) If b is a non-degenerate bilinear form on V, then

$$b(v_1 \wedge \cdots \wedge v_k, w_1 \wedge \cdots \wedge w_k) = \det(b(v_i, w_j)_{1 \le i,j \le k})$$

is a non-degenerate bilinear form on $\Lambda^k V$.

Similarly,

$$b(v_1 \otimes \cdots \otimes v_k, w_1 \otimes \cdots \otimes w_k) = \prod_{\sigma \in S_k} b(v_1, w_{\sigma(1)})b(v_2, w_{\sigma(2)}) \dots b(v_k, w_{\sigma(k)})$$

is a non-degenerate bilinear form on $S^k V$.

(d) On $V \otimes V^*$ we have the symmetric bilinear form $b(v_1 \otimes f_1, v_2 \otimes f_2) = f_1(v_2)f_2(v_1)$ which one easily shows is non-degenerate. □

For the classical Lie groups we now want to determine which representations belong to which category. For this we need to first decide which representations are self dual. For this we define the **opposition element** $op \in W$ to be the unique Weyl group element which sends the positive Weyl chamber WC^+ to its negative, see Proposition 4.41 (b). Clearly if $-\operatorname{Id} \in W$, then $op = -\operatorname{Id}$.

selfdual1

Proposition 5.30 *Let π_λ be an irreducible representation.*

(a) π_λ is self dual iff $-W_\pi = W_\pi$.
(b) The highest weight of π_{λ^*} is $op(-\lambda)$ and hence π_λ is self dual iff $op(\lambda) = -\lambda$.
(c) $op = -\operatorname{Id}$ if $\mathfrak{g} \simeq B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$.

Proof (b) Since the weights of π^* are the negatives of the weights of π, $-\lambda$ is the minimal weight in π^*. Thus it is maximal in the reverse ordering. The element $op \in W$ reverses the order, and hence $op(-\lambda)$ is the highest weight of π^* in our given ordering.

(a) If $-\lambda \in W_\lambda$, then Proposition 5.14 (b) implies that there exists an element w in the Weyl group with $w(-\lambda) = \lambda$. But since the Weyl group acts simply transitively on the Weyl chambers, $w = op$ and thus (b) applies.

(c) It follows from Proposition 4.27 that there exists an automorphism $A \in \mathrm{Aut}(\mathfrak{g})$ such that $A_{|\mathfrak{h}} = -\mathrm{Id}$ since it simply takes all roots to their negative. For the Lie algebras listed, except for D_{2n}, every automorphism is inner since the Dynkin diagram has no automorphisms, see Proposition 4.43. Thus there exists an element k in the compact real form of $\mathfrak{g}$ such that $\mathrm{Ad}(k)_{|\mathfrak{t}} = -\mathrm{Id}$ which means that $k \in N(T)$ and hence $-\mathrm{Id} \in W$.

For D_{2n} recall that the Weyl group contains an even number of sign changes in $\omega_1, \dots, \omega_{2n}$ and thus $-\mathrm{Id} \in W$. □

selfdual2

Corollary 5.31 *Every representation of $B_n, C_n, D_{2n}, G_2, F_4, E_7, E_8$ is self dual. A representation of A_n, D_{2n+1} or E_6 is self dual iff its diagram is invariant under the (unique) non-trivial diagram automorphism.*

Proof The first part follows from Proposition 5.30 (b) and (c), except in the case of. For the Lie algebras A_n, D_{2n+1} the element op can be described as follows.

For A_n the Weyl group consists of all permutations in $\omega_1, \dots, \omega_{n+1}$ and $\alpha_1 = \omega_1 - \omega_2,\ \dots, \alpha_2 = \omega_n - \omega_{n+1}$ are the simple roots. Hence the permutation $\omega_i \to \omega_{n+2-i}$ takes $\alpha_i \to -\alpha_{n+1-i}$ and hence takes WC^+ to $-WC^*$. This also implies that if $\lambda = \sum a_i\lambda_i$, then the dominant weight of π^*_λ is $\sum a_{n+1-i}\lambda_i$.

For D_n with n odd, op is the Weyl group element that sends $\omega_i \to -\omega_i$ for $i = 1, \dots, n-1$ and fixes ω_n. Indeed, it sends the first $n-2$ roots to their negative and sends $\alpha_{n-1} = \omega_{n-1} - \omega_n \to \omega_{n-1} + \omega_n = -\alpha_n$ and $\alpha_n \to -\alpha_{n-1}$. Thus if $\lambda = \sum a_i\lambda_i$, the dominant weight of π^*_λ interchanges a_{n-1} and a_n.

A similar discussion for E_6 will be carried out in a later section. □

The last piece of information we need is the following.

sumtype

Proposition 5.32 *Let π_λ and $\pi_{\lambda'}$ be irreducible representations.*

(a) If π_λ and $\pi_{\lambda'}$ are self dual, then $\pi_{\lambda+\lambda'} \subset \pi_\lambda \otimes \pi_{\lambda'}$, as well as $\pi_{k\lambda} \subset S^k\pi_\lambda$ are of the same type as $\pi_\lambda \otimes \pi_{\lambda'}$ resp. $S^k\pi_\lambda$, as determined by Proposition 5.29 (a)-(c).
(b) If π_λ is complex, then $\pi_{\lambda+op(-\lambda)} \subset \pi_\lambda \otimes \pi^*_\lambda$ is orthogonal.

Proof (a) Recall that $\pi_{\lambda+\lambda'} \subset \pi_\lambda \otimes \pi_{\lambda'}$ and that it has multiplicity one. Since π_λ is self dual, $op(\lambda) = -\lambda$ and similarly for λ' and thus $op(\lambda+\lambda') = -\lambda-\lambda'$, i.e. $\pi_{\lambda+\lambda'}$ is self dual. Let $V = U \oplus W_1 \oplus \cdots \oplus W_k$ be the decomposition of $\pi_\lambda \otimes \pi_{\lambda'}$ into irreducible subspaces with U corresponding to $\pi_{\lambda+\lambda'}$. If b is the non-degenerate bilinear form on V, then either $b_{U\otimes U} \neq 0$, and U has the same type as V, or there exists an i with $b_{U\otimes W_i} \neq 0$. But then $U \simeq W_i^*$ and since U is self dual, $W_i \simeq U$, contradicting the fact that U has multiplicity one in V. A similar argument for for $S^k\pi$.

(b) Since $op(\lambda + op(-\lambda)) = -(\lambda + op(-\lambda))$, the representation $\pi_{\lambda+op(-\lambda)}$ is self dual. Furthermore, $(\pi_\lambda)^* = \pi_{op(-\lambda)}$, and hence the proof proceeds as in (a). □

Remark 5.33 The proof more generally shows that if π is a self dual rep and $\sigma \subset \pi$ occurs with multiplicity one and is self dual also, then σ has the same type as π.

We can now use all these rules to decide when an irrep of a classical Lie group is complex, orthogonal or symplectic.

slntype

Proposition 5.34 *For the representations of $\mathfrak{sl}(n+1,\mathbb{C})$ with dominant weight $\lambda = \sum a_i\lambda_i$ we have:*

(a) π_λ is self dual iff $a_i = a_{n+1-i}$.
(b) If $n = 2k$ or $n = 4k-1$, all self dual representations are orthogonal.
(c) If $n = 4k+1$, then a self dual representation is orthogonal if a_{2k+1} is even and symplectic if a_{2k+1} is odd.

Proof Part (a) follows from Corollary 5.31. Since $\pi^*_{\lambda_i} = \pi_{\lambda_{n+1-i}}$, Proposition 5.32 (b) implies that $\pi_{\lambda_i+\lambda_{n+1-i}}$ is orthogonal. Together with Proposition 5.32 (a), this implies (b) for $n = 2k$ since there is no "middle" root.

If $n+1 = 2k$, π_{λ_k} is self dual and we need to decide its type. But in this case we have the bilinear form $\Lambda^k\mathbb{C}^{2k} \times \Lambda^k\mathbb{C}^{2k} \to \Lambda^{2k}\mathbb{C}^{2k} \simeq \mathbb{C}$ given by

$(v, w) \to v \wedge w$ which is symmetric if k is even and skew symmetric when k is odd. Now Proposition 5.32 again finishes the proof. □

For the remaining classical Lie algebras we only determine the type of the fundamental representation π_{λ_k} since the type of all other irreps is determined by it.

ontype

Proposition 5.35

(a) The representations π_{λ_k}, $k = 1, \ldots, n-1$ *for* $\mathfrak{so}(2n+1, \mathbb{C})$ *and* π_{λ_k}, $k = 1, \ldots, n-2$ *for* $\mathfrak{so}(2n, \mathbb{C})$ *are orthogonal.*

(b) The representations π_{λ_k}, $k = 1, \ldots, n$ *for* $\mathfrak{sp}(n, \mathbb{C})$ *are symplectic if* k *is odd and orthogonal if* k *even.*

Proof Part (a) follows since ρ_n is orthogonal and $\pi_{\lambda_k} = \Lambda^k \rho_n$. For part (b) recall that $\Lambda^k \mu_n$ is not irreducible and that its irreducible summand with highest weight $\omega_1 + \cdots + \omega_k$ is π_{λ_k}. This irreducible summand is self dual since all irreps of $\mathfrak{sp}(n, \mathbb{C})$ are self dual. As in the proof of Proposition 5.32, one sees that π_{λ_k} has the same type as $\Lambda^k \mu_n$, which together with Proposition 5.29 finishes the proof. □

It remains to determine the type of the spin representations. We will supply the proof of the following claims in a later section.

spintype

Proposition 5.36

(a) The spin representation Δ_n *of* $\mathfrak{so}(2n+1, \mathbb{C})$ *is orthogonal if* $n = 4k$ *or* $4k+3$ *and symplectic otherwise.*

(b) For the spin representation $\Delta_n^{\pm}$ *of* $\mathfrak{so}(2n, \mathbb{C})$ *we have:*
$\Delta_{4k}^{\pm}$ *is orthogonal,* $\Delta_{4k+2}^{\pm}$ *symplectic, and* $\Delta_{2k+1}^{\pm}$ *is complex with* $(\Delta_{2k+1}^{+})^* = \Delta_{2k+1}^{-}$.

We now relate these results to studying representations of real Lie algebras. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{g}_0 \subset \mathfrak{g}$ a real form. As we saw, complex representations of $\mathfrak{g}_0$ are in one-to-one correspondence to complex representations of $\mathfrak{g}$ via restriction, i.e. if $\pi \colon \mathfrak{g} \to \mathfrak{gl}(V)$ is a rep on the complex vector space V, then $\pi_0 := \pi_{\mathfrak{g}_0}$ is a complex rep of $\mathfrak{g}_0$. Conversely, via extension, π_0 determines π since $\pi(X + iY) \cdot v = \pi_0(X) \cdot v + i\pi_0(Y) \cdot v$. Thus $R \subset V$ is an invariant subspace under π iff it is invariant under π_0 and hence irreps π are in one-to-one correspondence to irreps π_0. On the other hand, if $\sigma\colon \mathfrak{g}_0 \to \mathfrak{gl}(W)$ is a real irrep, then $\sigma_{\mathbb{C}} = \sigma \otimes \mathbb{C}$ is a complex rep on $W_{\mathbb{C}} = W \otimes \mathbb{C}$, but does not need to be irreducible.

For simplicity we will restrict ourselves to compact real forms. Thus let $\mathfrak{k}$ be a compact semisimple Lie algebra with $\mathfrak{k} \otimes \mathbb{C} \simeq \mathfrak{g}$. For clarity, in the following we will usually denote complex reps of $\mathfrak{k}$ by π and real reps by σ.

For complex reps of $\mathfrak{k}$ we can interpret our division of reps into 3 types in a different way, which is sometimes easier to use. Recall that $\tau\colon V \to V$ is called conjugate linear if $\tau(\lambda v) = \bar{\lambda}\tau(v)$ and is an intertwining map if $X \cdot \tau(v) = \tau(X \cdot v)$. Notice that conjugate linear intertwining map only makes sense for a real Lie algebra.

realconj

Proposition 5.37 *Let π be a complex irreducible representation of a compact semisimple Lie $\mathfrak{k}$ on V. Then there exists a symmetric (skew-symmetric) bilinear form on V invariant under π iff there exists a conjugate linear intertwining map τ with $\tau^2 = \mathrm{Id}$ (resp. $\tau^2 = -\mathrm{Id}$).*

Proof [Proof still incomplete.....] Recall that there exists a hermitian inner product $\langle \cdot, \cdot \rangle_h$ on V such $\mathfrak{k}$ acts by skew-hermitian linear maps (e.g., average an arbitrary inner product over a compact Lie group K with Lie algebra $\mathfrak{k}$).

Given an intertwining map τ with $\tau^2 = \mathrm{Id}$, we can define the eigenspaces $V_{\pm} = \{v \mid \tau(v) = \pm v\}$ and since τ is conjugate linear $iV_+ = V_-$. Clearly $V = V_+ \oplus V_-$ as well and π preserves each eigenspace since it commutes with τ. Thus $\sigma = \pi_{|V_+}$ is a real representation with $\sigma \otimes \mathbb{C} = \pi$. There exists an inner product on V_+ preserved by σ and the complex bilinear extension is an invariant symmetric bilinear form.

Conversely, given a bilinear form b we define L by $b(v,w) = \langle L(v), w \rangle_h$. Then L is conjugate linear since

$$\langle L(\lambda v), w \rangle_h = b(\lambda v, w) = \lambda b(v,w) = \lambda \langle L(v), w \rangle_h = \langle \bar{\lambda} L(v), w \rangle_h$$

Furthermore, L commutes with π since

$$\langle L(X \cdot v), w \rangle_h = b(X \cdot v, w) = -b(v, X \cdot w) = -\langle L(v), X \cdot w \rangle_h = \langle X \cdot L(v), w \rangle_h.$$

Although L is not hermitian, it is self adjoint with respect to the real inner product $Re\langle \cdot, \cdot \rangle_h$ and hence has only real eigenvalues. Since $L(iv) = -iL(v)$ there are as many positive ones as negative ones. Let W_1, resp. W_2 be the real span of the eigenvectors with positive eigenvalues resp. negative eigenvalues. π preserves W_i since it commutes with L. Thus if we define τ by $\tau_{|W_1} = \mathrm{Id}$ and $\tau_{|W_2} = -\mathrm{Id}$ it also commutes with π.

we claim that there exists an orthonormal basis $v_1, \dots, v_n$ such that $L(v_i) = \lambda_i v_i$ by mimiking the usual argument: Since V is a complex vector space, there exists an eigenvector v_1 with $L(v_1) = \lambda_1 v_1$. If $\langle w, v_1\rangle_h = 0$, then $\langle L(w), v_1\rangle_h = b(w, v_1) = b(v_1, w) = \langle L(v_1), w\rangle_h = \bar\lambda_1 \langle v_1, w\rangle_h = 0$ and hence $v_1^\perp$ is preserved and we can repeat the argument. Since L is conjugate linear, this implies $L(\sum x_i v_i) = \sum \lambda_i \bar x_i v_i$. By collecting equal eigenvalues, we write $V = V_1 \oplus, \cdots \oplus V_k$ and since π commutes with L, π preserves these eigenspaces. Hence we can replace λ_i by 1, i.e. define $\tau(\sum x_i v_i) = \sum \bar x_i v_i$, and τ still commutes with π. This is thus our desired conjugate linear involution.

The second case does not seem to work so easily.... Also, in the above, where do we use that the eigenvalues are real? Why does the proof not work in the second case the same? Still confused.... □

Next a simple Lemma. Recall that for a complex representation π acting on V we denote by $\pi_{\mathbb{R}}$ the real representation on $V_{\mathbb{R}}$ where we forget the complex structure. Furthermore, $\bar\pi$ denotes the rep on $\bar V$ which is V endowed with the complex structure $-J$, if J is the original complex structure on V.

complexify

Lemma 5.38

(a) If π is a complex representation of a real Lie algebra $\mathfrak{g}_0$ then $(\pi_{\mathbb{R}}) \otimes \mathbb{C} \simeq \pi \oplus \bar\pi$.

(b) if π is a complex representation of a compact Lie algebra $\mathfrak{k}$, then $\bar\pi \simeq \pi^*$.

Proof (a) On the level of vector spaces the claim is that there exists a natural isomorphism of complex vector spaces $(V_{\mathbb{R}}) \otimes \mathbb{C} \simeq V \oplus \bar V$. Let $J \colon V_{\mathbb{R}} \to V_{\mathbb{R}}$ be the complex structure that defines the complex vector space V. J extends complex linearly to $(V_{\mathbb{R}}) \otimes \mathbb{C}$ via $J(v \otimes z) = J(v) \otimes z$. Let $V_\pm := \{v \otimes 1 \mp Jv \otimes i \mid v \in V\}$ be the $\pm i$ eigenspaces of J. Clearly $V = V_+ \oplus V_-$. Furthermore, if we define $F_\pm \colon V \to V_\pm$, $F_\pm(v) = v \otimes 1 \mp Jv \otimes i$, then F_+ is complex linear since $F_+(Jv) = Jv \otimes 1 + v \otimes i = i(-Jv \otimes i + v \otimes 1) = iF_+(v)$ and F_- is conjugate linear since $F_-(Jv) = Jv \otimes 1 - v \otimes i = -i(Jv \otimes i + v \otimes 1) = -iF_-(v)$. Thus $V_+ \simeq V$ and $V_- \simeq \bar V$ as complex vector spaces. The representation π commutes with J by definition and hence preserves the $\pm i$ eigenspaces. Since $\mathfrak{g}$ acts via $X \cdot (v \otimes z) = (X \cdot v) \otimes z$ the linear maps $F_\pm$ are equivariant.

(b) If $\mathfrak{k}$ is a compact Lie algebra, there exists a hermitian inner product

$\langle \cdot, \cdot \rangle_h$ on V such that π acts via skew-hermitian endomorphisms. Clearly $G\colon V \to V^*$ defined by $G(v)(w) = \langle v, w\rangle_h$ is conjugate linear: $G(\lambda v)(w) = \langle \lambda v, w\rangle_h = \bar\lambda \langle v, w\rangle_h = \bar\lambda G(v)(w)$ and thus defines a complex isomorphism $\bar V \simeq V^*$. Furthermore, G commutes with π since $G(X\cdot v)(w) = \langle X\cdot v, w\rangle_h = -\langle v, X\cdot w\rangle_h = (X\cdot G(v))(w)$. $\square$

realcomp

Proposition 5.39 *Let σ be a real irreducible representation of $\mathfrak{k}$. Then one and only one of the following holds:*

(a) $\sigma\otimes\mathbb{C} \simeq \pi$ with π an orthogonal irreducible representation.
(b) $\sigma\otimes\mathbb{C} \simeq \pi\oplus\pi^*$ with π an irreducible complex representation.
(c) $\sigma\otimes\mathbb{C} \simeq \pi\oplus\pi$ with π an irreducible symplectic representation.

In case (b) and (c) $\sigma \simeq \pi_{\mathbb{R}}$.

Proof Let σ act on V and $V\subset V\otimes\mathbb{C}$ as $v\to v\otimes 1$. Furthermore, $\sigma\otimes\mathbb{C}$ acts as $X\cdot(v\otimes z) = (X\cdot v)\otimes z$. There exists a real inner product on V such that σ acts by skew-symmetric endomorphisms. Extending the inner product to a complex symmetric bilinear form, shows that $\sigma\otimes\mathbb{C}$ is orthogonal in the sense of Definition 5.27. If $\sigma\otimes\mathbb{C}$ is irreducible, we are in case (a). If not, let $W\subset V\otimes\mathbb{C}$ be a (non-trivial) irreducible invariant subspace. Then $\sigma\otimes\mathbb{C}$ also preserves $\bar W := \{w\otimes\bar z \mid w\otimes z\in W\}$. Since it thus also preserves $W\cap\bar W$, either $W=\bar W$ or $W\cap\bar W = 0$ by irreducibility of W. In the first case, $X = W\cap V$ is a real σ invariant subspace of dimension smaller than V, contradicting irreducibility of σ. Thus $W\cap\bar W = 0$ and hence $V\otimes\mathbb{C} = W\oplus\bar W$ since otherwise $(W\oplus\bar W)\cap V$ is a σ invariant subspace of dimension smaller than V. If we denote by π the representation induced on W, this implies that $\sigma\otimes\mathbb{C}\simeq\pi\oplus\bar\pi\simeq\pi\oplus\pi^*$. If $\pi\not\simeq\pi^*$, the representation π is complex and we are in case (b). Finally, we need to show if $\pi\simeq\pi^*$, then π is symplectic. If not, it must be orthogonal and hence there exists a conjugate linear map $\tau\colon W\to W$ with $\tau^2 = \mathrm{Id}$. But then $\{w\in W \mid \tau(w) = w\}$ is a real form of W and invariant under π, and the same for $\bar W$, contradicting irreducibility of σ.

To see the last claim, observe that any $w\in W$ we can write uniquely as $v_1\otimes 1 + v_2\otimes i$ and that both $v_i\neq 0$ since $W\cap\bar W = 0$. Thus the map $w\to v_1$ gives an isomorphism of real representations. $\square$

We say that the real representation σ is of **real type** in case (a), of **complex type** in case (b) and **quaternionic** type in case (c). To justify this terminology we show:

realcomp

Proposition 5.40 *Let σ be a real irreducible representation of $\mathfrak{k}$. Then with respect to some basis we have*

(a) If σ is of real type, $\sigma(\mathfrak{k}) \subset \mathfrak{o}(n) \subset \mathfrak{o}(n, \mathbb{C}) \subset \mathfrak{gl}(n, \mathbb{C})$.
(b) If σ is of quaternionic type, $\sigma(\mathfrak{k}) \subset \mathfrak{sp}(n) \subset \mathfrak{sp}(n, \mathbb{C}) \subset \mathfrak{gl}(2n, \mathbb{C})$.
(c) If σ is of complex type, $\sigma(\mathfrak{k}) \subset \mathfrak{u}(n) \subset \mathfrak{u}(n) \otimes \mathbb{C} \simeq \mathfrak{sl}(n, \mathbb{C})$.

Proof To be filled in.... □

We now come to our last interpretation. Recall that for the set of intertwining operators of a real irrep the only possibilities are $\mathbb{R}, \mathbb{C}$ or $\mathbb{H}$, whereas for a complex irrep it is always $\mathbb{C}$.

realcomp

Proposition 5.41 *Let σ be a real irreducible representation of $\mathfrak{k}$ and I_σ the algebra of intertwining operators.*

(a) If σ is of real type, then $I_\sigma \simeq \mathbb{R}$.
(b) If σ is of complex type, then $I_\sigma \simeq \mathbb{C}$.
(c) If σ is of quaternionic type, then $I_\sigma \simeq \mathbb{H}$.

Proof To be filled in.... □

Notice that this gives us now a recipe for finding all real irreps. Start with complex irreps π which are either complex or symplectic and take $\sigma = \pi_{\mathbb{R}}$. Notice that $\dim_{\mathbb{R}} \sigma = 2 \dim_{\mathbb{C}} \pi$. If π is complex, σ commutes with the complex structure on $V_{\mathbb{R}}$. If π is symplectic, σ commutes with the 3 complex structure on $V_{\mathbb{R}}$ given by right multiplication with i, j, k on $\mathbb{H}^n \simeq \mathbb{R}^{4n}$. For an orthogonal complex rep π there exists a conjugate linear intertwining operator τ with $\tau^2 = \mathrm{Id}$ and $V := \{v \mid \tau(v) = v\}$ is an invariant subspace which defines the real rep $\sigma = \pi_{|V}$. In this case $\dim_{\mathbb{R}} \sigma = \dim_{\mathbb{C}} \pi$. It is customary to use the notation $\sigma = [\pi]_{\mathbb{R}}$ in all 3 cases.

su2pol

Example 5.42 Let us try to find all real irreps of SU(2) and SO(3). We know that SU(2) has one complex irrep in every dimensions of the form $S^k \mu_2$ where μ_2 is the tautological rep on $\mathbb{C}^2$. They are all self dual, symplectic for odd k and orthogonal for even k. Hence SU(2) has one irrep in every odd dimension and one in every dimension $4k$. Only the ones in odd dimension descend to SO(3) and these are the irreps on the set of homogeneous harmonic polynomials in 3 real variables. The effective reps of SU(2) have dimension $4k$.

We will still fill in how to compute the effective representation of a compact Lie group. For this we need to solve the following problem. Given an irrep σ of $\mathfrak{k}$, let ψ be the unique representation of the compact simply connected Lie group K with Lie algebra $\mathfrak{k}$ such that $d\psi = \sigma$. We then need to determine the (finite) group $\ker\psi$ and would like to do this in terms of the dominant weight λ with $\sigma \simeq (\pi_\lambda)_\mathbb{R}$ resp. $\sigma \otimes \mathbb{C} \simeq \pi_\lambda$. To be added....

Exercises 5.43

(1) Verify that the bilinear forms in Proposition 5.29 are non-degenerate.
(2) Determine the real irreps of SO(4).
(3) For the orthogonal representation π_{2k} of SU(2), find the conjugate linear map τ with $\tau^2 = \mathrm{Id}$ and determine an orthonormal basis of the real subspace invariant under π_{2k}. Do this in terms of homogeneous polynomials of degree $2k$.
(4)
(5)

6

Symmetric Spaces

Our goal in this chapter is to give a geometric introduction to the theory of symmetric spaces. In many books, see e.g. [He], this is done, or quickly reduced to, an algebraic level. We will always try to relate to the geometry of the underlying Riemannian manifold and often use it in proofs. Some familiarity with Riemannian geometry (Levi-Cevita connection, geodesics, exponential maps, Jacobi fields, isometries and curvature) are assumed. We will denote by $B_r(p) = \{q \in M \mid d(p,q) < r\}$ a ball of radius r. It is called a normal ball if it is the diffeomorphic image of a ball in the tangent space. We denote by $\mathrm{I}(M)$ the isometry group and by $\mathrm{I}_0(M)$ the identity component. The following is a non-trivial fact, see e.g. [KN] for a proof.

isometrygroup

Theorem 6.1 *Assume that M is complete. Then*

(a) The isometry group $\mathrm{I}(M)$ is a Lie group and the stabalizer $\mathrm{I}(M)_p$ is compact for all $p \in M$.
(b) If M is compact, then $\mathrm{I}(M)$ is compact.

Recall that if G acts on M, then $G_p = \{g \in G \mid gp = p\}$ is the stabalizer, or the isotropy group, of $p \in M$. If G acts transitively on M with $H = G_p$, then $G/H \to M$, $gH \to gp$, is a diffeomorphism. On G/H we have the left translations $L_g \colon G/H \to G/H$, where $L_g(aH) = g(aH) = (ga)H$ which under the above identification become $L_g(aH) = gap = gp$, i.e. simply the action of G on M.

We also have the isotropy representation of H on T_pM given by $h \to d(L_h)_p$. If G acts by isometries, the isotropy action is effective, i.e. $d(L_h)_p = \mathrm{Id}$ implies $L_h = \mathrm{Id}$ since isometries are determined by their derivative at one point. This is not true for a general homogeneous space.

6.1 Basic geometric properties

We start with a local and global definition

symmetricspace

Definition 6.2 *Let $(M, \langle\cdot,\cdot\rangle\ )$ be a Riemannian manifold.*

(a) M is called (global) symmetric if for all $p \in M$ there exists an isometry $s_p \colon M \to M$ with $s_p(p) = p$ and $d(s_p)_p = -\operatorname{Id}$.
(b) M is called locally symmetric if for all $p \in M$ there exists a radius r and an isometry $s_p \colon B_r(p) \to B_r(p)$ with $s_p(p) = p$ and $d(s_p)_p = -\operatorname{Id}$.

In case (a) we will usually not include the word "global". We will often call s_p the symmetry around p.

Some simple consequences:

symmsimple

Proposition 6.3 *Let $(M, \langle\cdot,\cdot\rangle\ )$ be a symmetric space.*

(a) If γ is a geodesic with $\gamma(0) = p$, then $\sigma_p(\gamma(t)) = \gamma(-t)$.
(b) M is complete.
(c) M is homogeneous.
(d) If M is homogeneous and there exists a symmetry at one point, M is symmetric.

Proof (a) Since an isometry takes geodesics to geodesics, and since the geodesic $c(t) = s_p(\gamma(t))$ satisfies $c'(0) = d(s_p)_p(\gamma'(0)) = -\gamma'(0)$ the uniqueness property of geodesics implies that $c(t) = \gamma(-t)$.

(b) By definition, this means that geodesics are defined for all t. If M is not complete, let $\gamma \colon [0, t_0) \to M$ be the maximal domain of definition of the geodesic γ. Then applying $s_{\gamma(t_0-\epsilon)}$ to γ, and using part (a), enables one to extend the domain of definition to $[0, 2t_0 - 2\epsilon)$ and $2t_0 - 2\epsilon > t_0$ when ϵ is small.

(c) Let $p, q \in M$ be two points. We need to show that there exists an isometry f with $f(p) = q$. By completes, Hopf-Rinow implies that there exists a geodesic $\gamma \colon [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Thus by part (a) $s_{\gamma(\frac{1}{2})}(\gamma(0)) = \gamma(1)$.

(d) If s_p is the given symmetry, then one easily checks that $L_g \circ s_p \circ L_{g^{-1}}$ is a symmetry at gp. $\square$

A geometric way of interpreting the Definition is thus that s_p "flips" geodesics starting at p. This of course also holds in the case of locally symmetric spaces on a normal ball. Notice that for any Riemannian manifold the candidate s_p is hence always defined on a normal ball as $s_p(\exp(tv)) = \exp(-tv)$ and it is then a strong condition that s_p is an isometry. To be globally symmetric it is also a strong condition that if two geodesics go from p to q the ones in opposite direction need to end at the same point, clearly very unlikely in general.

Before continuing the general theory, a few examples.

symmexamples

Example 6.4 (a) Manifolds of constant curvature are locally symmetric, and simply connected ones are globally symmetric. For $\mathbb{R}^n$ with a flat metric the reflection around p given by $s_p(p+v) = p-v$ is clearly the desired symmetry.

For a sphere of radius 1, the reflection in the line $\mathbb{R}\cdot p$: $s_p(v) = -v + 2\langle v,p\rangle p$, where $||p|| = ||v|| = 1$ is an isometry. Ir fixes p, and on the tangent space $\{v \in \mathbb{R}^n \mid \langle v,p\rangle = 0\}$ the derivative, which is $ds_p = s_p$ by linearity, is equal to $-\operatorname{Id}$.

For hyperbolic space we use the Lorentz space model

$$\{v \in \mathbb{R}^{n+1} \mid \langle v,v\rangle = -1,\ x_{n+1} > 0\} \text{ with inner product } \langle x,y\rangle = \sum_{n=1}^{k=n} x_k y_k - x_{n+1}y_{n+1}.$$

Then the reflection $s_p(v) = -v - 2\langle v,p\rangle p$ does the job as well.

(b) A compact Lie group G with a bi-invariant inner product is a symmetric space. For this we first claim that $s_e(g) = g^{-1}$ is the symmetry at $e \in G$. Clearly $s_e(e) = e$ and since $s_p(\exp(tX) = \exp(-tX)$ for all $X \in \mathfrak{g}$, we also have $d(s_e)_e = -\operatorname{Id}$. Here we have used the fact that the exponential map of a biinvariant metric is the same as the exponential map of the Lie group, see .

We now show that it is an isometry. This is clearly true for $d(s_e)_e$. Since we have $s_e \circ L_g = R_{g^{-1}} \circ s_e$, it follows that $d(s_e)_g \circ d(L_g)_e = d(R_{g^{-1}})_e \circ d(s_e)_e$. Since left and right translations are isometries, $d(s_e)_g$ is an isometry as well. Using Proposition 6.40, we see that G is symmetric.

(c) The Grassmannians of k-planes: $G_k(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$, $G_k(\mathbb{H}^n)$ have a natural metric in which they are symmetric spaces. We carry out the argument for the real one $G_k(\mathbb{R}^n)$, the others being similar.

For this we use an embedding into the Euclidean vector space

$$V = \{P \in M(n,n,\mathbb{R}) \mid P = P^T\} \text{ with } \langle P,Q\rangle = \operatorname{tr}(PQ).$$

It sends $E \in G_k(\mathbb{R}^n)$ to the orthogonal projection $P = P_E \in V$ onto E, i.e. $P^2 = P$ with $\operatorname{Im}(P) = E$. Note that conversely, any $P \in V$ with $P^2 = P$, is an orthogonal projection onto $\operatorname{Im}(P)$ since $\mathbb{R}^n$ is the orthogonal sum of its 0 and 1 eigenspaces. In order for E to be k-dimensional, we require in addition that $\operatorname{tr} P = k$. Thus we can alternatively define

$$G_k(\mathbb{R}^n) = \{P \in V \mid P^2 = P,\ \operatorname{tr} P = k\}.$$

$A \in \mathrm{O}(n)$ acts on V via $P \to APA^T = APA^{-1}$ and hence takes the $k-$plane $\operatorname{Im} P$ to $A(\operatorname{Im} P)$. It thus acts transitively on $G_k(\mathbb{R}^n)$ with isotropy at $E_0 = \operatorname{span}\{e_1, \ldots, e_k\}$ equal to $\mathrm{O}(k)\,\mathrm{O}(n-k)$. Thus $G_k(\mathbb{R}^n) = \mathrm{O}(n)/\mathrm{O}(k)\,\mathrm{O}(n-k) = \mathrm{SO}(n)/\mathrm{S}(\mathrm{O}(k)\,\mathrm{O}(n-k))$ is a manifold. Here $\mathrm{O}(k)\,\mathrm{O}(n-k) = \{\operatorname{diag}(A, B) \mid A \in \mathrm{O}(k),\ B \in \mathrm{O}(n-k)\}$ is the block embedding, and $\mathrm{S}(\mathrm{O}(k)\,\mathrm{O}(n-k))$ satisfies $\det(AB) = 1$. In particular, $\dim G_k(\mathbb{R}^n) = n(n-k)$. Notice that $G_k(\mathbb{R}^n)$ is also an embedded submanifold of V since it is an orbit of the action of $\mathrm{O}(n)$.

The inner product on V induces a Riemannian metric on $G_k(\mathbb{R}^n)$. Now let $r_E \colon \mathbb{R}^n \to \mathbb{R}^n$ be the reflection in E, i.e. $(r_E)_{|E} = \mathrm{Id}$ and $(r_E)_{|E^\perp} = -\mathrm{Id}$. We claim that $s_E(Q) = r_E Q r_E$ is the symmetry at E. Before proving this, note that since $r_E^T = r_E = r_E^{-1}$, we can regard s_E either as conjugation with the isometry r_E, or as a basis change given by r_E. The latter implies that $\operatorname{Im}(s_E(Q)) = r_E(\operatorname{Im}(Q))$, i.e. s_E reflects k-planes.

To see that s_E is the desired symmetry, first observe that s_E take V to V and $d(s_E) = s_E$ preserves the inner product: $\langle d(s_E)(P), d(s_E)(Q)\rangle = \operatorname{tr}(r_E P r_E r_E Q r_E) = \operatorname{tr}(PQ) = \langle P, Q\rangle$. Furthermore, one easily checks that s_E takes projections to projections and preserves the trace, and hence induces an isometry on $G_k(\mathbb{R}^n)$. Clearly $s_E(P_E) = P_E$ since it takes E to E and is 0 on $E^\perp$. By differentiating a curve $P_0 + tQ + \cdots \in G_k(\mathbb{R}^n)$, i.e. $(P_E + tQ + \cdots)^2 = P_E + tQ + \cdots$, we see that $T_E(G_k(\mathbb{R}^n)) = \{Q \in V \mid P_E Q + Q P_E = Q$, and $\operatorname{tr} Q = 0\}$. If Q is a tangent vector and $v \in E$, then $P_E Q(v) + Q P_E(v) = Q(v)$ implies that $P_E Q(v) = 0$, or equivalently $Q(v) \in E^\perp$ and thus $r_E Q r_E(v) = r_E Q(v) = -v$. Similarly, if $v \in E^\perp$, then $P_E Q(v) + Q P_E(v) = Q(v)$ implies that $P_E Q(v) = Q(v)$, i.e. $Q(v) \in E$ and thus $r_E Q r_E(v) = r_E Q(-v) = -v$. Thus $d(s_E)_E = -\mathrm{Id}$.

Finally, notice that since $\langle P, P\rangle = \operatorname{tr}(P^2) = \operatorname{tr} P = k$, the image lies in a sphere of radius $\sqrt{k}$. Furthermore, since $\operatorname{tr}(P) = k$, it lies in an affine subspace of codimension 1, and hence in a round sphere of dimension $\frac{n(n+1)}{2} - 2$. This embedding is also called the Veronese embedding, and turns out to be a minimal submanifold.

We can add one more condition, namely prescribing the orientation on E. This gives rise to the oriented Grassmannian $G_k^0(\mathbb{R}^n) = \mathrm{SO}(n)/\mathrm{SO}(k)\,\mathrm{SO}(n-k)$, and is clearly a symmetric space as well with the same symmetry s_E. There is a 2-fold cover $G_k^0(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ which forgets the orientation. Notice that this is not possible for the Grassmanians $G_k(\mathbb{C}^n) = \mathrm{U}(n)/\mathrm{U}(k)\,\mathrm{U}(n-$

$k) = \mathrm{SU}(n)/\,\mathrm{S}(\mathrm{U}(k)\,\mathrm{U}(n-k))$ and $G_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\,\mathrm{Sp}(k)\,\mathrm{Sp}(n-k)$ since a complex or quaternionic subspace has a natural orientation given by the complex structure (which preserves E by definition).

Especially important is the case $k = 1$. These are the symmetric spaces of rank 1, i.e., $\mathbb{RP}^n$, resp. $\mathbb{S}^n$, with their constant curvature metric, and $\mathbb{CP}^n$, $\mathbb{HP}^n$ with their Fubuni-Study metric. Notice that the lowest dimensional Veronese surface is a (minimal) embedding of $\mathbb{RP}^2$ in $\mathbb{S}^4$. We will study these spaces in more detail later on and will see that they all have positive sectional curvature. There is one more rank one symmetric space, the Caley plane $\mathrm{Ca}\mathbb{P}^2$ which can be described as $F_4/\,\mathrm{Spin}(9)$.

Now now discuss the important concept of **transvections**

$$T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$$

defined for every geodesic γ in M. Its main properties are:

tranvections

Proposition 6.5 *Let M be a symmetric space and γ a geodesic.*

(a) T_t translates the geodesics, i.e., $T_t(\gamma(s)) = \gamma(t+s)$
(b) $d(T_t)_{\gamma(s)}$ is given by parallel translation from $\gamma(s)$ to $\gamma(t+s)$ along γ.
(c) T_t is a one-parameter group of isometries, i.e. $T_{t+s} = T_t \circ T_s$.

Proof (a) Notice that $s_{\gamma(r)}$ takes $\gamma(t)$ to $\gamma(2r-t)$. Thus

$$T_t(\gamma(s)) = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}(\gamma(s)) = s_{\gamma(\frac{t}{2})}(\gamma(-s)) = \gamma(t+s)$$

(b) Since symmetries are isometries, they takes parallel vector fields to parallel vector fields. Let X be a parallel vector field along the geodesic γ. Then $(s_{\gamma(0)})_*(X)$ is parallel and since $d(s_{\gamma(0)})_{\gamma(0)}(X) = -X$ we have $(s_{\gamma(0)})_*(X) = -X$ for all t. Applying a symmetry twice changes the sign again and hence $d(T_t)_{\gamma(s)}(X(\gamma(s)) = X(\gamma(t+s))$. This implies in particular that $d(T_t)_{\gamma(s)}$ is given by parallel translation.

(c) A basic property of isometries is that they are determined by their value and derivative at one point. Clearly $T_{t+s}(\gamma(0)) = \gamma(t+s) = T_t \circ T_s(\gamma(0))$ and by part (b) $d(T_{t+s})_{\gamma(0)}$ is given by parallel translation from $\gamma(0)$ to $\gamma(t+s)$. On the other hand, $d(T_t \circ T_s)_{\gamma(0)} = d(T_t)_{\gamma(s)} \circ d(T_s)_{\gamma(0)}$ is given by first parallel translating from $\gamma(0)$ to $\gamma(s)$ and then from $\gamma(s)$ to $\gamma(t+s)$. These are clearly the same, and hence T_{t+s} and $T_t \circ T_s$ agree. $\square$

We thus have:

geodesics

Corollary 6.6 *Let M be a symmetric space.*

(a) Geodesics in M are images of one parameter groups of isometries.
(b) $I_0(M)$ acts transitively on M.

Part (b) follows since $T_t \in I_0(M)$ and since any two points in M can be connected by a geodesic. It is also more generally true that if a Lie group G acts transitively, so does G_0. Part (a) on the other hand is very special, and is not satisfied, even for most homogeneous spaces.

Recall that for a Riemannian manifold and a fixed point p one defines the holonomy group $\mathrm{Hol}_p = \{P_\gamma \mid \gamma(0) = \gamma(1) = p\}$ given by parallel translation along piecewise smooth curves, and we let Hol_p^0 be its identity component. Notice that if q is another base point, and γ a path from p to q, then $\mathrm{Hol}_q = P_\gamma(Hol_p)P_\gamma^{-1}$ and thus they are isomorphic (though not naturally). We thus often denote it by Hol.

Its basic properties are:

holonomyLieGroup

Theorem 6.7 *Assume that M is complete. Then*

(a) Hol *is a Lie group and its identity component* Hol^0 *is compact.*
(b) Hol^0 *is given by parallel translation along null homotopic curves.*
(c) If M is simply connected, Hol_p *is connected*

,

There exists a natural surjective homomorphism $\pi_1(M) \to \mathrm{Hol}_p / \mathrm{Hol}_p^0$ given by $[\gamma] \to P_\gamma$ which, by part (b), is well defined. This clearly implies part (c), and that Hol_p has at most countably many components since this is true for the fundamental group of a manifold. To prove that Hol is a Lie group, it is thus sufficient to prove that Hol^0 is a Lie group. This follows from the (non-trivial) theorem that an arcwise connected subgroup of a Lie group is a Lie group. It was a long standing conjecture that $\mathrm{Hol}_p \subset \mathrm{O}(T_pM)$ is closed and hence compact. This turned out to be false, see [**?**].

Since $G = \mathrm{I}(M)$ acts transitively on M, we can write $M = G/K$ where $K = G_p$ is the isotropy group at p. Notice that Hol_p is a subgroup of $\mathrm{O}(T_pM)$ by definition, as is K via the isotropy representation.

holonomy

Corollary 6.8 *If $M = G/K$ is a symmetric space with $G = \mathrm{I}(M)$, then $\mathrm{Hol}_p \subset K$.*

Proof Every closed curve γ can be written as a limit of geodesic polygons γ_i. For example, cover γ with finitely many totally normal balls, and connect nearby points by minimal geodesics. By refining the subdivision, we can make the sequence converge in C^1. This implies that also $P_{\gamma_i} \to P_\gamma$ since parallel vector fields locally satisfy a differential equation, and its solutions depend continuously on the coefficients and initial conditions. Along a geodesic polygon parallel translation is given by a composition of isometries, namely tranvections along each side. This composition fixes the point p and hence lies in K. Since K is compact, $P_\gamma \in K$ as well. $\square$

Notice that the proof even works for a locally symmetric space.

This is an important property of symmetric spaces since it gives rise to many examples with small holonomy group. Generically one would expect that $\mathrm{Hol}_p = \mathrm{O}(n)$. As we will see, for a symmetric space $\mathrm{Hol}_p^0 = K_0$ in most cases. On the other hand, if $M = \mathbb{R}^n$ with its Euclidean inner product, we have $\{e\} = \mathrm{Hol}_p \subset K_0 = \mathrm{SO}(n)$. As it turns out, this is essentially the only exception.

We can now combine this information with one of the most important applications of holonomy groups, the DeRham decomposition theorem. Recall that M is called decomposable if M is a product $M = N_1 \times N_2$ and the Riemannian metric is a product metric. If this is not possible, M is called indecomposable.

deRham

Theorem 6.9 *Let M be a simply connected Riemannian manifold, $p \in M$ and Hol_p the holonomy group. Let $T_pM = V_0 \oplus V_1 \oplus \cdots \oplus V_k$ be a decomposition into Hol_p irreducible subspaces with $V_0 = \{v \in T_pM \mid hv = v \text{ for all } h \in \mathrm{Hol}_p\}$. Then M is a Riemannian product $M = M_0 \times \cdots \times M_k$, where M_0 is isometric to flat $\mathbb{R}^n$. If $p = (p_0, p_1, \ldots, p_k)$, then $T_{p_i}M_i \simeq V_i$ and M_i is indecomposable if $i \geq 1$. Furthermore, the decomposition is unique up to order and $\mathrm{Hol}_p \simeq \mathrm{Hol}_{p_1} \times \cdots \times \mathrm{Hol}_{p_k}$ with Hol_{p_i} the holonomy of M_i at p_i. Finally, $\mathrm{I}_0(M) = \mathrm{I}_0(M_0) \times \cdots \times \mathrm{I}_0(M_k)$.*

Since for a symmetric space $\mathrm{Hol}_p \subset K$, this implies

Kirred

Corollary 6.10 *If $M = G/K$ is a simply connected symmetric space, and M is indecomposable, then K acts irreducible on the tangent space.*

This motivates the definition:

symmirred

Definition 6.11 *A symmetric space G/K, where $G = \mathrm{I}(M)$ and $K = G_p$, is called irreducible if K_0 acts irreducibly on T_pM, and reducible otherwise.*

Notice that we do not assume that K acts irreducibly. One of the reason is that otherwise $\mathbb{S}^2(1) \times \mathbb{S}^2(1)$ would be an irreducible symmetric space, since switching the two factors is an isometry that generates another component. Notice also that the definition does not change if we replace G by $G = \mathrm{I}_0(M)$ since $G/K = G_0/(K \cap G_0)$ and $K \cap G_0$, although it may not be connected, at least has the same Lie algebra as K.

By the above, if a simply connected symmetric space is indecomposable as a Riemannian manifold, it is irreducible as a symmetric space. On the other hand, irreducible does not imply indecomposable, even in the simply connected case, since for flat $\mathbb{R}^n$ we have $K = \mathrm{O}(n)$ which acts irreducibly. On the other hand, this is essentially the only exception, as we will see later on: If $M = G/K$ is irreducible, then $M = \mathbb{R}^n \times M'$ with a product metric of a flat metric on $\mathbb{R}^n$ and a symmetric metric on M' which is indecomposable.

The DeRham decomposition theorem implies:

symmdecomp

Corollary 6.12 *If $M = G/K$ is a simply connected symmetric space, then M is isometric to $M_1 \times \cdots \times M_k$ with M_i irreducible symmetric spaces.*

Proof We can decompose T_pM into irreducible subspaces V_i under the isotropy representation of K_0. Since $\mathrm{Hol}_p = \mathrm{Hol}_p^0 \subset K_0$, these can be further decomposed into irreducible subspaces under Hol_p. Applying Theorem 6.9, M has a corresponding decomposition as a Riemannian product. Collecting factors whose tangent spaces lie in V_i, we get a decomposition $M_1 \times \cdots \times M_k$ with $M_1 \simeq \mathbb{R}^n$ flat (if a flat factor exists) and $T_{p_i}M_i \simeq V_i$. If s_p is the symmetry at $p = (p_1, \ldots, p_k)$, then the uniqueness of the decomposition also implies that $s_p = (s_{p_1}, \ldots, s_{p_k})$ since, due to $d(s_p)_p = -\mathrm{Id}$, s_p cannot permute factors in the decomposition. Thus each factor M_i is a symmetric space which is irreducible by construction. □

Thus a symmetric space which is reducible is locally an isometric product of symmetric spaces, which follows by going to the universal cover.

Another important consequence:

symmirredsimple

Corollary 6.13 *A simply connected symmetric space $M = G/K$ with G simple is irreducible.*

Indeed, if $M = M_1 \times \cdots \times M_k$, then $\mathrm{I}_0(M) = \mathrm{I}_0(M_1) \times \cdots \times \mathrm{I}_0(M_k)$ which implies that G is not simple. We will see that with one exception, the converse is true as well.

One easily sees:

symmEinstein **Corollary 6.14** *An irreducible symmetric space is Einstein, i.e. Ric =* $\lambda\langle\cdot\,,\cdot\,\rangle$ *for some constant* λ*. Furthermore, the metric is uniquely determined up to a multiple.*

Proof This follows from the following general useful Lemma:

SchurMetric **Lemma 6.15** *Let* B_1*,* B_2 *be two symmetric bilinear forms on a vector space* V *such that* B_1 *is positive definite. If a compact Lie group* K *acts irreducibly on* V *such that* B_1 *and* B_2 *are invariant under* K*, then* $B_2 = \lambda B_1$ *for some constant* λ*.*

Proof Since B_1 is non-degenerate, there exists an endomorphism $L\colon V \to V$ such that $B_2(u,v) = B_1(Lu,v)$. Since K acts by isometries, $B_1(kLu,v) = B_1(Lu,k^{-1}v) = B_2(u,k^{-1}v) = B_2(ku,v) = B_1(Lku,kv)$ and hence $Lk = kL$ for all $k \in K$. In addition, the symmetry of B_i implies that $B_1(Lu,v) = B_2(u,v) = B_2(v,u) = B_1(Lv,u) = B_1(u,Lv)$, i.e. L is symmetric with respect to B_1 and hence the eigenvalues of L are real. If $E \subset V$ is an eigenspace with eigenvalue λ, then $kL = Lk$ implies that E is invariant under K. Since K acts irreducibly, $E = \{0\}$ or $E = V$. Thus $L = \lambda\,\mathrm{Id}$ for some constant λ and hence $B_2 = \lambda B_1$. Notice that$\lambda \neq 0$ since otherwise $B_2 = 0$. □

This clearly implies that the metric is unique up to a multiple. Since isometries preserve the curvature, Ric is also a symmetric bilinear form invariant under K, which implies the first claim. □

Another reason why holonomy groups are important, is the *holonomy principle.* If S_p is a tensor on T_pM invariant under Hol_p, we can define a tensor S on all of M by parallel translating along any path. This is independent of the path since parallel translating along a closed path preserves S_p. It is an easy exercise to show that S is then smooth. Furthermore, S is parallel, i.e. $\nabla S = 0$, since $\nabla_X S = \frac{d}{dt}_{|t=0} P_\gamma^*(S_\gamma(t))$, where γ is a path with $\gamma'(0) = X$. For example, if the representation of Hol_p is a complex representation, then the complex structure on T_pM extends to a parallel complex structure on M, and such structures are integrable, and the metric is in fact Kähler. We will come back to applications of this principle to symmetric spaces later on.

We now discuss some properties of locally symmetric spaces. If R is the curvature tensor, then ∇R is the tensor defined by

$$(\nabla_X R)(Y,Z)W = \nabla_X(R(Y,Z)W) - R(\nabla_X Y,Z)W - R(Y,\nabla_X Z)W$$

$$- R(Y,Z)\nabla_X W.$$

This easily implies that $\nabla R = 0$ iff for every parallel vector fields Y, Z, W along a geodesic γ, $R(Y,Z)W$ is parallel along γ as well.

parallelR

Proposition 6.16 *Let M be a Riemannian manifold.*

(a) M is locally symmetric iff $\nabla R = 0$.
(b) If M is locally symmetric and simply connected, then M is globally symmetric.
(c) Let M_1 and M_2 be two symmetric spaces with M_1 simply connected and $p_i \in M_i$ fixed. Given an isometry $A\colon T_{p_1}M_1 \to T_{p_2}M_2$ with $A^*(R_2) = R_1$, there exists an isometric covering $f\colon M_1 \to M_2$ with $df_{p_1} = A$.

Proof (a) If M is locally symmetric with local symmetry s_p, then $\nabla R = 0$ since if we set $L = d(s_p)_p$ we have

$$\begin{aligned} -(\nabla_X R)(Y,Z)W &= L((\nabla_X R)(Y,Z)W) \\ &= (\nabla_{LX} R)(LY, LZ)LW = (\nabla_X R)(Y,Z)W \end{aligned}$$

since an isometry respects curvature. Notice that the same argument implies that any tensor of odd order invariant under s_p must vanish.

For the converse, we need to show that $s(\exp_p(tv)) = \exp_p(-tv)$ is an isometry on a small normal ball. For this we compute the derivative of $\exp_p$ via Jacobi fields. Recall that a Jacobi field along a geodesic γ is defined as $J(t) = \frac{\partial}{\partial s}_{|s=0}\gamma_s(t)$ where γ_s are geodesics with $\gamma_0 = \gamma$. Equivalently, Jacobi fields are solutions of the Jacobi equation $J'' + R(J,\gamma')\gamma' = 0$. Thus, by differentiating $\gamma_s(t) = \exp_p(t(v + sw))$, we see that $d(exp_p)_v(w) = J(1)$ where $J(t)$ is a Jacobi field along the geodesic $\gamma(t) = \exp_p(tv)$ with $J(0) = 0$ and $\nabla_v J = w$. Since the curvature is invariant under parallel translation, the Jacobi equation in an orthonormal parallel frame has the form $J'' + R \circ J = 0$ where R is a constant matrix, namely the curvature endomorphism $v \to R(v,\gamma')\gamma'$ at any point $\gamma(t)$. Since the coefficients of the second order linear differential equation are constant, it follows that if $J(t)$ is a solutions, so is $\bar{J}(t) = J(-t)$ along the geodesic $\exp_p(-tv)$ with initial conditions $\bar{J}(0) = 0$ and $\bar{J}'(0) = -\nabla_v J = -w$. Thus $|d(exp_p)_v(w)| = |J(1)| = |\bar{J}(1)| = |d(exp_p)_{-v}(-w)|$ which means that s_p is an isometry.

(c) Here we need the Cartan-Ambrose-Hicks Theorem, which we first recall. The setup is as follows. Let M_i be two complete Riemannian manifolds with M_1 simply connected and $p_i \in M_i$. Let $A\colon T_{p_1}M_1 \to T_{p_2}M_2$ be an isometry with the following property. If γ is a geodesic in M_1 with $\gamma(0) = p_1$, we denote by $\bar\gamma$ the geodesic in M_2 with $\bar\gamma(t) = \exp_{p_2}(tA(\gamma'(0))$. If γ is a piecewise geodesic starting at p_1 we also have a corresponding piecewise geodesic $\bar\gamma$ where the "break" vectors in $\bar\gamma$ are obtained by parallel translation the ones of γ to p_1, mapping them to M_2 with A, and parallel translating in M_2 by the corresponding distance. If P_γ denotes parallel translation, we require that $P^*_\gamma((R_1)_{\gamma(1)}) = A^*(P^*_{\bar\gamma}((R_2)_{\bar\gamma(1)}))$. Then Cartan-Ambrose-Hicks says that there exists a local isometry $f\colon M_1 \to M_2$ with $df_{p_1} = A$. It is an easy exercise that a local isometry is a covering. Notice that it is clear how f should be defined since it needs to take a broken geodesic γ to $\bar\gamma$. See [CE], Theorem 1.36 for details of the proof.

We can now apply this to the case where M_i are symmetric spaces. In that case $P^*_\gamma((R_1)_{\gamma(1)}) = (R_1)_{\gamma(0)}$ since (a) implies that $\nabla R_1 = 0$ and hence the curvature tensor is invariant under parallel translation. Similarly for R_2 and hence we only need $A^*(R_2) = R_1$ to obtain the existence of f. Since both M_i are assumed to be simply connected, the covering f is an isometry.

Part (b) is now an easy consequence. Since M is locally symmetric, and since the local symmetry preserves curvature, it follows that $s_p^*(R_p) = R_p$. We can now apply the Cartan-Ambrose-Hicks Theorem to $A := d(s_p)_p = -\operatorname{Id}$ to obtain a global isometry f with $f(p) = p$ and $df_p = -\operatorname{Id}$. $\square$

Part (c) says in particular that a globally symmetric space is determined, up to coverings, by the curvature tensor at one point. This is an analogue of the fact that a Lie algebra $\mathfrak{g}$ determines the Lie group G up to coverings.

It is instructive to use the Cartan-Ambrose-Hicks Theorem to give a proof of Theorem **??**, an important property we use frequently. This is in fact also the most important step in proving that the isometry group is a Lie group.

Kcompact

Proposition 6.17 *Let M be a Riemannian manifold with K the set of all isometries fixing a point $p \in M$. Then K is compact.*

Proof Recall that the topology in $\mathrm{I}(M)$, and hence K, is given by uniform convergence on compact subsets. Via the isotropy representation $K \subset \mathrm{O}(T_pM)$ and hence the claim is equivalent to K being closed in $\mathrm{O}(T_pM)$. So let $f_i \in K$ and choose a subsequence, again denoted by f_i, such that $d(f_i)_p$ converges to an isometry $A \in \mathrm{O}(T_pM)$. We need to show that A is the derivative of an isometry f and that $f_i \to f$ uniformly on compact subsets. Let γ_v be the geodesic starting at p with $\gamma_v'(0) = v$. Then $\gamma_i = f_i(\gamma_v)$ is a geodesic with $\gamma_i'(0) = \exp(d(f_i)_p(v)) \to Av$. Thus $\gamma_i(t) \to \gamma_{Av}(t)$ uniformly on compact subsets and hence $P_{\gamma_i} \to P_{\gamma_{Av}}$ and $R_{\gamma_i(t)} \to R_{\gamma_{Av}(t)}$ since parallel translation and curvature depends continuously on the parameter. The same holds for broken geodesics. Since f_i are isometries, they satisfy the requirements of the Cartan-Ambrose-Hicks Theorem, and by taking limits, so does the isometry A. Thus there exists an isometric covering $f\colon M \to M$ with $df_p = A$. f must be an isometry by either going over to the universal cover or by observing that f preserves the volume. Finally, we need to show that $f_i \to f$. First observe that $f_i(\gamma_v(t)) \to \gamma_{Av}(t) = \exp(tAv) = \exp(df_p(tv)) = f(\exp(tv)) = f(\gamma_v(t))$ and similarly for broken geodesics. Choose a point q, $B_r(q)$ a normal ball around q, and fix a geodesic γ from p to q. Now compose γ with the unique geodesic from q to a point $q' \in B_r(q)$. By applying Cartan-Ambrose-Hicks to this broken geodesic, we see that $(f_i)_{|B_r(q)} \to f_{|B_r(q)}$ uniformly.

□

Remark 6.18 If G is a transitive isometric action on a Riemannian manifold M with isotropy $G_p = K$, it may not always be true that K is compact. For example, recall that for $\mathbb{R}^n$ the the full isometry group is $G = \mathrm{O}(n) \rtimes \mathbb{R}^n$, a semidirect product of rotations and translations. Of course in this case $G_0 = \mathrm{O}(n)$ is compact. But now we can take any subgroup $L \subset \mathrm{O}(n)$ and write $\mathbb{R}^n = L \rtimes \mathbb{R}^n/L$ and the isotropy is compact iff L is compact. The general issue can be formulated as follows. Let $G \subset \mathrm{I}(M)$ be a subgroup, and $K = \mathrm{I}(M)_p$. Then $G_p = G \cap K$ and G_p is compact iff G is closed in $\mathrm{I}(M)$. On the other hand, any metric invariant under G is also invariant under the closure $\bar{G}$, and conversely. Thus it is natural to assume that G is closed in $\mathrm{I}(M)$ and hence G_p is compact. We will always make this assumption from now on.

Exercises 6.19

(1) Show that the only quotient of $\mathbb{S}^n(1)$ which is symmetric is $\mathbb{RP}^n$.
(i) Show that the Grassmannians $G_k^0(\mathbb{R}^n)$, $G_k(\mathbb{C}^n)$, $G_k(\mathbb{H}^n)$ are simply connected.
(2) If G acts by isometries, show that G_p is compact iff G is closed in the isometry group $\mathrm{I}(M)$, which holds iff the action of G on M is proper, i.e. $G \times M \to M \times M$, $(g, p) \to (p, gp)$ is proper.
(3) Show that if G acts transitively on M, then so does G_0.
(4) Show that a symmetric space is irreducible iff the universal cover is

irreducible. You first need to prove the following general claim. If a connected Lie group G acts on M, and $\pi\colon \tilde{M} \to M$ is a cover, then there exists a cover $\sigma\colon \tilde{G} \to G$ with an action of $\tilde{G}$ on $\tilde{M}$ such that π is equivariant, i.e. $\pi(g \cdot p) = \sigma(g) \cdot \pi(p)$.

(5) Show that the fundamental group of a symmetric space is abelian.

(6) If (M, ∇) is a manifold with connection, then M is called locally affine symmetric if the the local geodesic symmetry preserves ∇, and affine symmetric if this holds for a globally defined geodesic symmetry. Show that M is locally affine symmetric iff $T = \nabla R = 0$, where T is the torsion of ∇. Show that a simply connected locally affine symmetric space is affine symmetric.

6.2 Cartan involutions

Since Proposition 6.16 (c) shows that a simply connected symmetric space is determined by the curvature tensor at one point, it suggests that there should be an equivalent algebraic definition of a symmetric space, which we develop in this Section.

First some notation. If $M = G/H,\ H = G_{p_0}$ is a homogeneous space, we obtain an action of G on G/H, which we write as $p \to gp$ when thinking of M, or on the level of cosets $kH \to g(kH) = gkH$ which we also denote by L_g. If $h \in H$, L_h takes p_0 to p_0 and hence $d(L_h)_{p_0}\colon T_{p_0}M \to T_{p_0}M$. This defines a representation $H \to \mathrm{GL}(T_{p_0}M)$ called the **isotropy representation**, which we sometimes denote by $\chi = \chi_{G/H}$. This representation may be highly ineffective. But if G acts effectively and by isometries on M, then χ is effective since isometries are determined by their derivative. On the other hand, we will often write a homogeneous space in an ineffective presentation. Recall that if G acts on M then the ineffective kernel $N = \{g \in G \mid gp = p \text{ for all } p \in M\}$ is a normal subgroup of G and G/N induces an effective action on M. In the case of homogeneous spaces $N \subset H$ and hence N is a subgroup normal in G and H. Conversely, the ineffective kernel is the largest normal subgroup that G and H have in common. Indeed, if n lies in such a normal subgroup, $L_n(gH) = ngH = gn'H = gh$ since $n' \in H$. Notice in particular that $Z(G) \cap Z(H) \subset N$. In the examples we usually let G act almost effectively on G/H. The only exception is in the case of $\mathrm{U}(n)$ where we allow $Z(\mathrm{U}(n))$ to lie in the ineffective kernel as well. This makes explicit computations often simpler. As an example, consider $\mathbb{CP}^n = \mathrm{U}(n+1)/\,\mathrm{U}(n)\,\mathrm{U}(1) = \mathrm{SU}(n+1)/\,\mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1) =$

$(\mathrm{SU}(n+1)/\mathbb{Z}_{n+1})/(\mathrm{S}(\mathrm{U}(n)\,\mathrm{U}(1))/\mathbb{Z}_{n+1})$. The last presentation is effective, the second one almost effective, but the first one is the most convenient one.

We will often use the following observation. The long homotopy sequence $K \to G \to G/K$ implies that K is connected if M is simply connected and G is connected. Conversely, if G is simply connected and K connected, then M is simply connected.

We use the following convention when possible. We will denote by G/H a general homogeneous space and reserve the notation G/K for symmetric spaces. Recall that a symmetric space M can be written as $M = G/K$, where $G = \mathrm{I}_0(M)$ and $K = G_{p_0}$. It is important that from now on we let G be the identity component of $\mathrm{I}(M)$ and not the full isometry group. Notice that this in particular means that s_p does not necessarily lie in G. Nevertheless, conjugation by s_p preserves G.

symmauto1

Proposition 6.20 *Let $M = G/K$ with $G = \mathrm{I}_0(M)$ and $K = G_p$ be a symmetric space.*

(a) The symmetry s_p gives rise to an involutive automorphism

$$\sigma = \sigma_p \colon G \to G \quad , \quad g \to s_p g s_p.$$

(b) If $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ is the fixed point set of σ, then

$$G_0^\sigma \subset K \subset G^\sigma.$$

Proof (a) Since $s_p \in \mathrm{I}(M)$ and $s_p^{-1} = s_p$, σ, σ is a conjugation and hence an automorphism that preserves $\mathrm{I}_0(M)$. Since s_p is involutive, so is σ.

(b) To see that $K \subset G^\sigma$, let $h \in K$. Then $\sigma(h) \cdot p = s_p h s_p \cdot p = p = h \cdot p$. Furthermore, $d\sigma(h)_p = (ds_p)_p dh_p (ds_p)_p = dh_p$. Since isometries are determined by their derivatives, $\sigma(h) = h$ and hence $K \subset G^\sigma$.

To see that $G_0^\sigma \subset K$ let $\exp tX \subset G_0^\sigma$ be a 1-parameter subgroup. Since $\sigma(\exp tX) = \exp tX$, it follows that $s_p \exp tX s_p = \exp tX$ and hence $s_p(\exp tX \cdot p) = \exp tX \cdot p$. But s_p fixes only p in a normal ball about p since $d(s_p)_p = -\mathrm{Id}$ and hence $\exp tX \cdot p = p$ for all t. Thus $\exp tX \in K$ and since G_0^σ is generated by a neighborhood of e, the claim follows. □

The involution σ is called the **Cartan Involution** of the symmetric space.

Before we prove a converse, we need to discuss some general facts about Riemannian homogeneous spaces.

If G acts by isometries on a manifold M, we can associate to each $X \in \mathfrak{g}$ a vector field X^* on M called an **action field** which is defined by

$$X^*(p) = \frac{d}{dt}_{|t=0}(\exp(tX) \cdot p).$$

These action fields are Killing vector fields since their flow acts by isometries. A word of caution: $[X^*, Y^*] = -([X,Y])^*$ since the flow of $X^*$is given by left translation, but the flow of $X \in \mathfrak{g}$ is given by right translation.

We say that G/H, $H = G_{p_0}$, is a Riemannian homogeneous space if L_g is an isometry for all $g \in G$. We say that the homogeneous space is **reductive** if there exists a subspace $\mathfrak{p} \subset \mathfrak{g}$, such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ and $\mathrm{Ad}_H(\mathfrak{p}) \subset \mathfrak{p}$. We can then identify

$$\mathfrak{p} \simeq T_{p_0}M \text{ via } X \to X^*(p_0). \tag{6.21}$$ tangentident

This is an isomorphism since $X^*(p_0) = 0$ iff $X \in \mathfrak{h}$.

isorep **Lemma 6.22** *Let G/H be a homogeneous space and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a reductive decomposition.*

(a) If $\sigma \in \mathrm{Aut}(G)$ with $\sigma(H) = H$, then under the identification (6.21), we have $d\sigma = d\bar{\sigma}_{p_0}$, where $\bar{\sigma}\colon G/H \to G/H$ is defined by $\bar{\sigma}(gH) = \sigma(g)H$.

(b) Under the identification (6.21), the isotropy representation of G/H is given by $d(L_h)_{p_0} = \mathrm{Ad}(h)_{|\mathfrak{p}}$.

(c) A homogeneous metric on G/H, restricted to $T_{p_0}M$, induces an inner product on $\mathfrak{p}$ invariant under Ad_H.

(d) An inner product on $\mathfrak{p}$, invariant under Ad_H, can be uniquely extended to a homogenous metric on G/H.

Proof (a) Let $H = G_{p_0}$. Then

$$\begin{aligned}(d\sigma(X))^*(p_0) &= \frac{d}{dt}_{|t=0}(\exp(td\sigma(X)) \cdot p_0) = \frac{d}{dt}_{|t=0}(\sigma(\exp(tX)) \cdot p_0) \\ &= \frac{d}{dt}_{|t=0}\bar{\sigma}(\exp(tX)H) = d\bar{\sigma}_{p_0}(X^*(p_0))\end{aligned}$$

Part (b) follows from (a) by letting $\sigma = C_h$ be conjugation by $h \in H$ and observing that $\bar{\sigma}(gH) = hgh^{-1}H = hgH = L_h(gH)$. Clearly (b) implies (c). For part (d), the inner product on $\mathfrak{p}$ induces one on $T_{p_0}M$ which is preserved by $d(L_h)_{p_0}$. We then define the metric at gp_0 by using $(L_g)_{p_0}\colon T_{p_0}M \to$

$T_{gp_0}M$. This definition is independent of the choice of g since the metric at p_0 is L_H invariant. □

Remark 6.23 If G/H does not have a reductive decomposition, one can still prove an analogue of (a) and (b) by replacing (6.21) with $T_{p_0}M \simeq \mathfrak{g}/\mathfrak{h}$. But the isotropy representation is in general not effective, even if the action of G on M is.

A Riemannian homogeneous space G/H is always reductive since $\chi(H)$ is compact (resp. has compact closure in $O(T_{p_0}M)$). We simply choose an inner product on $\mathfrak{g}$ invariant under Ad_H and let $\mathfrak{p}$ be the orthogonal complement of $\mathfrak{h}$. Reductive decompositions are not necessarily unique. Notice that this is simply a representation theory problem since, given one reductive decomposition, it can only be unique if the representation of Ad_H on $\mathfrak{h}$ and $\mathfrak{p}$ do not have any equivalent sub-representations.

For a symmetric space we have a natural reductive decomposition, called the **Cartan decomposition**.

symmred

Proposition 6.24 *Let $M = G/K$ with $G_0^\sigma \subset K \subset G^\sigma$ for an involutive automorphism σ of G. Furthermore, let $\mathfrak{k}$ and $\mathfrak{p}$ be the $+1$ and -1 eigenspaces of $d\sigma$. Then $\mathfrak{k}$ is indeed the Lie algebra of K and*

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}, \quad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

Furthermore, $\mathrm{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$, in particular $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.

Proof $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$ simply says that $\mathfrak{k}$ is a subalgebra and $G_0^\sigma \subset K \subset G^\sigma$ implies that K and G^σ have the same Lie algebra, which is clearly the $+1$ eigenspace of $d\sigma$. Since the automorphism σ respects Lie brackets, $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ follows as well. It also implies that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, but $\mathrm{Ad}_K(\mathfrak{p}) \subset \mathfrak{p}$ is stronger if K is not connected. To prove this, observe that for any automorphism $\alpha\colon G \to G$ one has $d\alpha \circ \mathrm{Ad}(g) = \mathrm{Ad}(\alpha(g)) \circ d\alpha$. If $h \in K$ and $X \in \mathfrak{p}$, i.e., $\sigma(h) = h$ and $d\sigma(X) = -X$, this implies that $d\sigma(\mathrm{Ad}(h)X) = \mathrm{Ad}(h)d\sigma(X) = -\mathrm{Ad}(h)X$, i.e., $\mathrm{Ad}(h)X \in \mathfrak{p}$. □

We will use this reductive decomposition from now on. Notice that in this language the symmetric space is irreducible iff the Lie algebra representation of $\mathfrak{k}$ on $\mathfrak{p}$ given by Lie brackets is irreducible

We are now ready to prove a converse of Proposition 6.20.

symmauto2 **Proposition 6.25** *Let G be a connected Lie group and $\sigma\colon G \to G$ an involutive automorphism such that G_0^σ is compact. Then for any compact subgroup K with $G_0^\sigma \subset K \subset G^\sigma$, the homogeneous space G/K, equipped with any G-invariant metric, is a symmetric space, and such metrics exist.*

Proof Let $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ be the reductive decomposition in Proposition 6.24. Homogeneous metrics on G/K correspond to Ad_K invariant inner products on $\mathfrak{p}$. Such inner products exists since K is compact.

We now claim that any such metric is symmetric. Since it is homogeneous, it is sufficient to find a symmetry at one point. Since $\sigma(K) = K$, we get an induced diffeomorphism $\bar{\sigma}\colon G/K \to G/K$, $\bar{\sigma}(gK) = \sigma(g)K$, and we claim that this is the symmetry at the base point coset (K). Clearly σ fixes the base point and since $d\sigma_{|\mathfrak{p}} = -\mathrm{Id}$, Lemma 6.22 (a) implies that $d(\bar{\sigma})_{(K)} = -\mathrm{Id}$ as well. □

Remark 6.26 It may seem that we have proved that there is a one to one correspondence between symmetric spaces and Cartan involutions. There is one minor glitch: If we start with a Cartan involution as in Proposition 6.25, it may not be true that $G = \mathrm{I}(M)_0$, as required in Proposition 6.20. This is illustrated by the example $\mathbb{R}^n = L \rtimes \mathbb{R}^n/L$ for any $L \subset \mathrm{O}(n)$. Notice that in this case the Cartan involution is $\sigma(A, T_v) = (A, T_{-v})$. Notice also that this example shows that one symmetric space can have a presentation as in Proposition 6.25 in several different ways, clearly not desirable. We will see that $\mathbb{R}^n$ is essentially the only exception.

We can now reduce a symmetric space to an infinitesimal object.

symminfinitesimal **Proposition 6.27** *Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a decomposition (as vector spaces) with*

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \quad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p} \quad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

If G is the simply connected Lie group with Lie algebra $\mathfrak{g}$ and $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k}$, then

(a) There exists an involutive automorphism $\sigma\colon G \to G$ such that $K = G_0^\sigma$.
(b) If K is compact, then every G-invariant metric on G/K is symmetric.
(c) G/K is almost effective iff $\mathfrak{g}$ and $\mathfrak{k}$ have no ideal in common.

Proof (a) Let $L\colon \mathfrak{g} \to \mathfrak{g}$ be the linear map with $L_{|\mathfrak{k}} = \mathrm{Id}$, $L_{|\mathfrak{p}} = -\mathrm{Id}$.

Then one easily checks that the Lie bracket condition is equivalent to L being an automorphism of $\mathfrak{g}$. Since G is simply connected, there exists an automorphism σ with $d\sigma = L$, and since $d\sigma^2 = L^2 = \mathrm{Id}$, σ is involutive. The Lie algebra of G^σ is the fixed point set of $d\sigma = L$, i.e it is equal to $\mathfrak{k}$. This proves that $K = G_0^\sigma$. (b) now clearly follows from Proposition ??.

(c) Recall that G/K is almost effective if the kernel of the left action has finite ineffective kernel, and that this is equivalent to saying that the largest subgroup of K normal in G (and hence of course in K as well), is discrete, i.e. has trivial Lie algebra. Since normal subgroups correspond to ideals, the claim follows. □

A decomposition of a Lie algebra $\mathfrak{g}$ as above is again called a Cartan decomposition of $\mathfrak{g}$. It is called orthogonal if it satisfies the condition in (b), and effective, if it satisfies the condition in (c). Thus a symmetric space gives rise to an effective orthogonal Cartan decomposition, and conversely such a Cartan decomposition defines a symmetric space. Again, the correspondence is not quite one to one.

We point out an elementary result that will be useful when discussing the symmetric spaces involving classical Lie groups.

totgeod

Proposition 6.28

(a) If M is a symmetric space and $N \subset M$ is a submanifold such that for all $p \in M$, $s_p(N) = N$, then N is totally geodesic and symmetric.

(b) Let $\sigma\colon G \to G$ be an involutive automorphism and G/K the corresponding symmetric space. If $L \subset G$ with $\sigma(L) \subset L$, then $L\big/(L \cap K)$ is a symmetric space such that $L\big/(L \cap K) \subset G/K$ is totally geodesic.

Proof (a) Recall that $N \subset M$ is totally geodesic if every geodesic in N is also a geodesic in M, or equivalently the second fundamental form $B\colon T_pN \times T_pN \to (T_pN)^\perp$ vanishes. But the isometries s_p preserves T_pN and hence $(T_pN)^\perp$, and thus B as well. Since the tensor B has odd order, it vanishes, see the proof of Proposition 6.16 (a).

(b) This follows from (a) since the symmetry at eK is given by $\bar\sigma(gK) = \sigma(g)K$ and if $N = L\big/(L \cap K) \subset G/K$, then $\sigma(L) \subset L$ implies that $\bar\sigma(N) \subset N$. □

Exercises 6.29

(1) Show that when G acts on a manifold M (not necessarily transitive or Riemannian) then $(\mathrm{Ad}(g)X)^* = (L_g)_*(X^*)$.

(2) Let $\mathrm{GL}(n, \mathbb{R})$ act on $\mathbb{R}^n - \{0\}$ via matrix multiplication. Compute the isotropy at a point and the isotropy representation on $\mathfrak{g}/\mathfrak{h}$ and show that this homogeneous space has no reductive decomposition.

(3) Show that up to scaling, there exists a unique metric on $\mathbb{S}^n$ invariant under $\mathrm{SO}(n+1)$, a one parameter family invariant under $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ on $\mathbb{S}^{2n-1}$, and a 7 parameter family invariant under $\mathrm{Sp}(n) \subset \mathrm{SO}(4n)$ on $\mathbb{S}^{4n-1}$.

(4)

(5)

6.3 A Potpourri of Examples

In this section we will describe all of the symmetric space which are quotients of classical Lie groups in a geometric fashion. We also compute the isotropy representation and fundamental group and discuss some low dimensional isomorphisms.

Grassmann manifolds

We first revisit some examples we already studied in Section 6.1. We denote by $G_k(R^n)$ the set of unoriented k-planes in $\mathbb{R}^n$ and by $G_k^0(R^n)$ the set of oriented k-planes. The Lie group $\mathrm{O}(n)$, and also $G = \mathrm{SO}(n)$, clearly acts transitively on k planes. If p_0 is the k-plane spanned by the first k-basis vectors $e_1, \cdots, e_k$, then the isotropy is embedded diagonally:

$$G_{p_0} = S(\mathrm{O}(k)\,\mathrm{O}(n-k)) = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in \mathrm{O}(k),\ B \in \mathrm{O}(n-k), \right\}$$

with $\det A \det B = 1$. In the case of the oriented planes we clearly have $G_{p_0} = \mathrm{SO}(k)\,\mathrm{SO}(n-k)$ embedded diagonally.

We denote from now on by $I_{p,q}$ the $(p+q) \times (p+q)$ diagonal matrix with p entries of -1 on the diagonal and q entries of $+1$. Then $\sigma(A) = I_{k,n-k} A I_{k,n-k}$ is an automorphism of G which is $+\,\mathrm{Id}$ on the upper $k \times k$ and lower $(n-k) \times (n-k)$ block and $-\,\mathrm{Id}$ in the two off blocks. Thus $G^\sigma = S(\mathrm{O}(k)\,\mathrm{O}(n-k))$. It has two components, and according to Proposition 6.25, gives rise to two symmetric spaces, $G_k(\mathbb{R}^n)$ and $G_k^0(\mathbb{R}^n)$. The -1 eigenspace $\mathfrak{p}$ of $d\sigma$ and a computation shows that the isotropy representation is given by:

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \mid X \in M(p,q,\mathbb{R}) \right\} \text{ and } \operatorname{Ad}((A,B))X = AXB^T,$$

where $(A,B) \in S(\mathrm{O}(k)\,\mathrm{O}(n-k))$. Thus $\chi_{G/H} = \rho_p \hat{\otimes} \rho_q$ and this rep is irreducible, as long as $(p,q) \neq (2,2)$. Thus the Grassmannians, except for $G_2(\mathbb{R}^4)$ and $G_2^0(\mathbb{R}^4)$, are irreducible. (Notice that an exterior tensor product over $\mathbb{R}$ of real irreducible reps may not be irreducible, as it was over $\mathbb{C}$).

In order to obtain a geometric interpretation of what the symmetry does to a plane, let r_E be the reflection in the plane E. We claim that the symmetry s_E is simply reflection in E, i.e. if we let $v_1, \dots, v_k$ be basis of F, then $r_E(v_1), \dots, r_E(v_k)$ is a basis of $s_E(F)$. To see this, we can assume that $E = \operatorname{span}\{e_1, \dots, e_k\}$ and $F = \operatorname{span}\{g(e_1), \dots g(e_k)\}$ for some $g \in \mathrm{SO}(n)$. Then, as we saw in the proof of Proposition 6.25, the symmetry at E is given by $gH \to \sigma(g)H$. Thus in the first k columns of g, i.e. $g(e_1), \dots g(e_k)$, the first k components are fixed, and the last $n-k$ are changed by a sign. But this is precisely what the reflection in E does.

Recall that $\pi_1(\mathrm{SO}(m)) \to \pi_1(\mathrm{SO}(n))$ is onto for all $n > m \geq 2$. This easily implies that $G_k^0(\mathbb{R}^n)$ is simply connected, and thus $\pi_1(G_k(\mathbb{R}^n)) = \mathbb{Z}_2$. The 2-fold cover $G_k^0(\mathbb{R}^n) \to G_k(\mathbb{R}^n)$ of course simply forgets the orientation.

Similarly, for the complex and quaternionic Grassmannian $G_k(\mathbb{C}^n) = \mathrm{U}(n)/\,\mathrm{U}(k)\,\mathrm{U}(n-k)$ and $G_k(\mathbb{H}^n) = \mathrm{Sp}(n)/\,\mathrm{Sp}(k)\,\mathrm{Sp}(n-k)$ with Cartan involution again given by conjugation with $I_{k,n-k}$. Both are simply connected and no sub-covers are symmetric.

As mentioned before, the special cases with $k = 1$, i.e., $G_1(\mathbb{R}^{n+1}) = \mathbb{RP}^n$, $G_1(\mathbb{C}^{n+1}) = \mathbb{CP}^n$, $G_1(\mathbb{H}^{n+1}) = \mathbb{HP}^n$ are especially important. They are also called rank 1 symmetric spaces. There is one more rank 1 symmetric space, the Cayley plane $\mathrm{Ca}\mathbb{P}^2 = F_4/\,\mathrm{Spin}(9)$.

Compact Lie groups

If K is a compact Lie group, we have an action of $K \times K$ on it given by $(a,b) \cdot h = ahb^{-1}$ with isotropy $\Delta K = \{(a,a) \mid a \in K\}$. Thus we can also write $K = K \times K/\Delta K$. Notice that $K \times K$ acts by isometries in the bi-invariant metric on K. We have the involutive automorphism $\sigma(a,b) = (b,a)$ with $G^\sigma = \Delta K$ which makes $K \times K/\Delta K$ into a symmetric space. Furthermore, $\mathfrak{p} = \{(X,-X) \mid X \in \mathfrak{g}\}$ with isotropy representation the adjoint representation $\operatorname{Ad}(k)(X,-X) = (\operatorname{Ad}(k)X, -\operatorname{Ad}(k)X)$. Thus the symmetric space K is irreducible iff K is simple. Notice that the natural isomorphism $\mathfrak{p} \simeq T_e K \simeq \mathfrak{k}$ is given by $(X, -X) \to 2X$. One should keep in mind this multiplication by 2 when relating formulas for the symmetric space to formulas for K.

$\mathrm{SO}(2n)/\,\mathrm{U}(n)$: Orthogonal Complex structures

If $(V, \langle \cdot, \cdot \rangle)$ is an inner product space, we will study the set of complex structures which are isometries, i.e., $M = \{J \in \mathrm{O}(V) \mid J^2 = -\mathrm{Id}\}$. V must be even dimensional and we set $\dim V = 2n$. If $J \in M$, we can find a normal form as follows. Choose a unit vector $v_1 \in V$ arbitrarily and let $v_{n+1} = J(v_1)$. Then $J(v_{n+1}) = -v_1$ and hence $\operatorname{span} v_1, v_{n+1}$ is J invariant. Since J is also orthogonal, it preserves its orthogonal complement, and repeating we obtain an orthonormal basis $v_1, \ldots, v_n, v_{n+1}, \ldots v_{2n}$ in which J is the matrix

$$J = \begin{pmatrix} 0 & -\mathrm{Id} \\ \mathrm{Id} & 0 \end{pmatrix}.$$

This implies that the action of $\mathrm{O}(V)$ on M, given by $A \cdot J = AJA^{-1}$, is transitive on M. Indeed, $A \cdot J \in M$ if $J \in M$ and if J and J' are orthogonal complex structures, then the isometry A which takes one orthonormal basis of each normal form to the other, satisfies $A \cdot J = J'$. Let us fix one such orthogonal complex structures J_0 and let v_i be a corresponding choice of orthonormal basis. The isotropy at J_0 is the set of $A \in \mathrm{O}(V)$ with $A \circ J_0 = J_0 \circ A$, i.e. the set of J_0 complex linear maps w.r.t. J_0. Thus $M = \mathrm{O}(V)/\,\mathrm{U}(V)$. Notice that this has 2 components and we call the set of complex structures J with $\det J = \det J_0$ the oriented complex structures (w.r.t. the orientation induced by J_0). Let us call this component again M.

We can use J_0 to identify V with $\mathbb{R}^n \oplus \mathbb{R}^n = \mathbb{C}^n$ with its canonical complex structure $J_0(u, v) = (-v, u)$ and then $M = \{J \in \mathrm{SO}(2n) \mid J^2 = -\mathrm{Id}\} = \mathrm{SO}(2n)/\,\mathrm{U}(n)$ where $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ is the canonical embedding $A + iB \to \begin{pmatrix} A & B \\ -B & A \end{pmatrix}$.

We now take the involutive automorphism of $\mathrm{SO}(2n)$ given by $\sigma(A) = J_0 A J_0$. Then clearly $G^\sigma = \mathrm{U}(n)$ and thus $\mathrm{SO}(2n)/\,\mathrm{U}(n)$ is a symmetric space, and since G^σ is connected, no subcover of M is symmetric. Using the usual embedding $\mathrm{U}(n) \subset \mathrm{SO}(2n)$, we get

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix} \mid X, Y \in \mathfrak{gl}(n, \mathbb{R}),\ X = -X^T,\ Y = Y^T \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} X & Y \\ Y & -X \end{pmatrix} \mid X, Y \in \mathfrak{gl}(n,\mathbb{R}),\ X = -X^T,\ Y = -Y^T \right\}$$

The isotropy representation is more difficult to compute, but one can show that $\chi = \Lambda^2\mu_n$, which is irreducible. Thus M is an irreducible symmetric space. It is not hard to see that $\pi_1(\mathrm{U}(n)) \to \pi_1(\mathrm{SO}(2n))$ is onto (choose canonical representatives) and hence M is simply connected.

Notice that if J is a complex structure, then J is orthogonal iff J is skew symmetric. Thus there is a natural embedding $M \subset \mathfrak{o}(2n)$. The metric obtained by restricting the inner product $\langle A, B\rangle = \frac{1}{2}\operatorname{tr} AB$ on $\mathfrak{o}(2n)$ to M is the above symmetric metric since it is invariant under the adjoint action of $\mathrm{SO}(2n)$ on $\mathfrak{o}(2n)$. It leaves M invariant, in fact M is an orbit of the action, and since M is isotropy irreducible the metric is unique up to scaling, and hence must be symmetric.

If we look at low dimensional cases, we have $\mathrm{SO}(4)/\mathrm{U}(2) \simeq \mathbb{S}^2$. Indeed, $\mathrm{SU}(2)$ is a normal subgroup of $\mathrm{SO}(4)$ and $\mathrm{SO}(4)/\mathrm{SU}(2) \simeq \mathrm{SO}(3)$ and thus $\mathrm{SO}(4)/\mathrm{U}(2) = \mathrm{SO}(3)/\mathrm{SO}(2) = \mathbb{S}^2$. If $n = 3$ one easily sees that that $\mathrm{SO}(6)/\mathrm{U}(3) \simeq \mathrm{SU}(4)/\mathrm{S}(\mathrm{U}(3)\,\mathrm{U}(1)) = \mathbb{CP}^3$ and if $n = 4$ that $\mathrm{SO}(8)/\mathrm{U}(4) \simeq \mathrm{SO}(8)/\mathrm{SO}(2)\,\mathrm{SO}(6)$. Notice that the last claim seems at first sight somewhat peculiar since $\pi_1(U(3)) = \mathbb{Z}$ and $\pi_1(\mathrm{SO}(2)\,\mathrm{SO}(6)) = \mathbb{Z} \oplus \mathbb{Z}_2$.

$\mathrm{U}(n)/\mathrm{O}(n)$: Lagrangian subspaces of $\mathbb{R}^{2n}$

Let (V, ω) be a symplectic vector space and set $\dim V = 2n$. A subspace $L \subset V$ with $\omega_{|L} = 0$ and $\dim L = n$ is called Lagrangian. We will show that M, the set of all Lagrangian subspaces of V, is a symmetric space. As we saw in Chapter 3, there exists a symplectic basis with which we can identify $V \simeq \mathbb{R}^n \oplus \mathbb{R}^n$ and ω with the canonical symplectic form $\omega_0((u,v),(u',v')) = u\cdot v' - u'\cdot v = \langle J_0(u,v),(u'v')\rangle$, or equivalently $\omega_0 = \sum dx_i \wedge dy_i$. Thus there exists an inner product $\langle\cdot,\cdot\rangle$ and an orthogonal complex structure J on V such that $\omega(u,v) = \langle Ju, v\rangle$. Another way to say that L is Lagrangian is thus that $J(L) \perp L$, i.e., M is also the set of all totally real subspaces w.r.t. J.

A third interpretation is that M is the set of conjugate linear intertwining maps τ of the complex vector space (V, J) with $\tau^2 = \mathrm{Id}$, which are orthogonal. Indeed, as we saw in the proof of Proposition 5.37, if $V_\pm$ are the eigenspaces of τ with eigenvalues ± 1, then $JV_- = V+$ and $\langle V_+, V_-\rangle = 0$, i.e. $J(V_-) \perp V_-$ and hence V_- is Lagrangian. Conversely, if L is Lagrangian, we define τ as above. τ is sometimes also called a real structures since $V_- \otimes \mathbb{C} = V$ and conversely, a subspace $L \subset V$ with $L \otimes \mathbb{C} = V$, defines a conjugate linear intertwining maps τ.

For simplicity identify from now on $(V, \omega, J) \simeq (\mathbb{R}^{2n}, J_0, \omega_0)$. The symplectic group $\mathrm{Sp}(n, \mathbb{R})$ clearly takes Lagrangian subspaces to Lagrangian subspaces. Recall that $\mathrm{U}(n) \subset \mathrm{Sp}(n, \mathbb{R})$ and we claim that $\mathrm{U}(n)$ acts transitively on M. For this, let L be Lagrangian, choose an orthonormal basis $v_1, \ldots, v_n$ of L and let $v_{n+i} = J_0(v_i)$. Since L is Lagrangian, and hence $J_0(L) \perp L$, $v_1, \ldots, v_{2n}$ is an orthonormal basis and $\omega(v_i, v_j) = \omega(v_{n+i}, v_{n+j}) = 0$. Furthermore, $\omega(v_i, v_{n+j}) = \delta_{ij}$ and hence $\omega = \sum dv_i \wedge dv_{n+i}$. Thus the linear map A that takes v_i to the standard basis $e_1, \ldots, e_{2n}$ lies in $\mathrm{Sp}(n, \mathbb{R})$, but also in $\mathrm{O}(2n)$ and hence in $\mathrm{Sp}(n, \mathbb{R}) \cap \mathrm{O}(2n) = \mathrm{U}(n)$. It takes L into the Lagrangian subspace $L_0 = \{e_1, \ldots e_n\}$. This shows that $\mathrm{U}(n)$ indeed acts transitively on M. The isotropy at L_0 is $\mathrm{O}(n) \subset \mathrm{U}(n)$ since $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{U}(n) \subset \mathrm{SO}(2n)$ fixes L_0 iff $B = 0$. Hence $M = \mathrm{U}(n)/\mathrm{O}(n)$. We also have $M^o = \mathrm{U}(n)/\mathrm{SO}(n)$ which can be interpreted as the set of oriented Lagrangian subspaces. and is a 2-fold cover of M.

If we choose the automorphism of $\mathrm{U}(n)$ defined by $\sigma(A) = \bar{A}$, then $G^\sigma = \mathrm{O}(n)$ and thus M, as well as M^o is a symmetric space. By embedding $\mathrm{U}(n) \subset \mathrm{O}(2n)$ we clearly have

$$\mathfrak{h} = \left\{ \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix} \mid X \in \mathfrak{gl}(n, \mathbb{R}),\ X = -X^T \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & Y \\ -Y & 0 \end{pmatrix} \mid Y \in \mathfrak{gl}(n, \mathbb{R}),\ Y = Y^T \right\}$$

Identifying a matrix in $\mathfrak{p}$ with Y, the isotropy representation is given by $\mathrm{Ad}(\mathrm{diag}(A, A))(Y) = AYA^T$, i.e. $\chi = S^2\rho_n$. Notice that this rep is not irreducible since the inner product is an element of $S^2(\mathbb{R}^n)$ which is fixed by ρ_n. This corresponds to the fact that $Y = \mathrm{Id} \in \mathfrak{p}$ lies in the center of $\mathfrak{u}(n)$. Thus M is not an irreducible symmetric space. Notice that we have a submanifold $\mathrm{SU}(n)/\mathrm{SO}(n) \subset \mathrm{U}(n)/\mathrm{O}(n)$ and since σ preserves $\mathrm{SU}(n)$, Proposition 6.28 implies that the embedding is totally geodesic. $\mathrm{SU}(n)/\mathrm{SO}(n)$ is sometimes called the set of *special Lagrangian* subspaces. The isotropy representation of $\mathrm{SU}(n)/\mathrm{SO}(n)$ is irreducible, i.e., it is an irreducible symmetric space.

There is a natural tautological embedding $\mathrm{U}(n)/\mathrm{O}(n) \subset G_n(\mathbb{R}^{2n})$ and we claim it is totally geodesic. For this we just observe that conjugation with

$I_{n,n}$ takes $\begin{pmatrix} A & B \\ -B & A \end{pmatrix} \in \mathrm{U}(n) \subset \mathrm{O}(2n)$ to $\begin{pmatrix} A & -B \\ B & A \end{pmatrix}$ and thus $A+iB \to A-iB = \sigma(A+iB)$, i.e. the Cartan involution for the Grassmannian restricts to the Cartan involution for the set of Lagrangian subspaces.

Finally, we consider some low dimensional isomorphisms. Clearly, we have that $\mathrm{SU}(2)/\mathrm{SO}(2) = \mathbb{S}^2 = \mathbb{CP}^1$ is the set of Lagrangian subspaces of $\mathbb{R}^4$. The 5-dimensional manifold $\mathrm{SU}(3)/\mathrm{SO}(3)$ is sometimes called the Wu manifold. The long homotopy sequence of the homogeneous space implies that it is simply connected with $\pi_2 = \mathbb{Z}_2$, i.e. as close to a homology $\mathbb{S}^5$ as one can get. Finally, one easily sees that $\mathrm{SU}(4)/\mathrm{SO}(4) = \mathrm{SO}(6)/\mathrm{SO}(3)\,\mathrm{SO}(3)$, which seems natural since $\mathrm{SU}(4)$ is a 2-fold cover of $\mathrm{SO}(6)$ and $\mathrm{SO}(4)$ is a 2-fold cover of $\mathrm{SO}(3)\,\mathrm{SO}(3)$.

$\mathrm{U}(2n)/\mathrm{Sp}(n)$: Quaternionic structures on $\mathbb{C}^{2n}$

Recall that if $(V, J, \langle \cdot\,, \cdot\,\rangle)$ is a hermitian vector space, a conjugate linear intertwining maps τ with $\tau^2 = -\mathrm{Id}$ is called a quaternionic structure of V. Since τ is conjugate linear $J\tau = -\tau J$ and thus $J, \tau, J\circ\tau$ are 3 anti-commuting complex structures which make V into a vector space over $\mathbb{H}$. We can then define the compact symplectic group as $\mathrm{Sp}(V) = \{A \in \mathrm{U}(n) \mid A\tau = \tau A\}$. We denote by M the set of quaternionic structures which are unitary. In coordinates, $\mathbb{C}^n \oplus \mathbb{C}^n \simeq \mathbb{H}^n$, with $(u, v) \to u + jv$ and $\tau_0(u, v) = (-\bar{v}, \bar{u}) = (u, v)j$.

We claim that $\mathrm{U}(2n)$ acts transitively on M. Let $V_\pm$ be the eigenspaces of τ with eigenvalues $\pm i$. Then we have again that $JV_- = V_+$ and $\langle V_-, V_+\rangle = 0$. If we let $u_1, \ldots, u_n$ be an orthonormal basis of V_-, then $u_1, \ldots, u_n, J(u_1), \ldots, J(u_n)$ is an orthonormal basis of V. If we have two such structures τ, τ' the unitary map that takes the orthonormal basis for τ into that for τ' takes τ to τ' as well. The isotropy at τ_0 is equal to $\mathrm{Sp}(n)$ and thus $M = \mathrm{U}(2n)/\mathrm{Sp}(n)$.

The automorphism $\sigma(A) = \tau_0 A \tau_0^{-1}$ makes M into a symmetric space since $\mathrm{U}(2n)^\sigma = \mathrm{Sp}(n)$. It is not irreducible, but the totally geodesic submanifold $\mathrm{SU}(2n)/\mathrm{Sp}(n) \subset \mathrm{U}(2n)/\mathrm{Sp}(n)$ is an irreducible symmetric space.

$\tau \in M$ is a skew-hermitian matrix and thus M is embedded as an adjoint orbit in $\{A \in M(2n, 2n, \mathbb{C}) \mid A = -\bar{A}^T\}$ under the action of $\mathrm{U}(2n)$ by conjugation, with metric induced by the trace form.

In the first non-trivial dimension we have $\mathrm{SU}(4)/\mathrm{Sp}(2) = \mathrm{SO}(6)/\mathrm{SO}(5) = \mathbb{S}^5$.

$\mathrm{Sp}(n)/\mathrm{U}(n)$: Complex Lagrangian subspaces of $\mathbb{C}^{2n}$

As in the case of real Lagrangian subspaces, we let M be the set of subspaces of $\mathbb{C}^{2n}$ which are Lagrangian w.r.t. the complex skew symmetric bilinear form ω. One easily sees that $\mathrm{Sp}(n,\mathbb{C}) \cap \mathrm{U}(n) = \mathrm{Sp}(n)$ acts transitively on M with isotropy $\mathrm{U}(n)$. Thus $M = \mathrm{Sp}(n)/\mathrm{U}(n)$ and it is a symmetric space with respect to the involution $\sigma(A + jB) = A - jB$ with $\mathrm{Sp}(n)^\sigma = \mathrm{U}(n)$. It is a totally geodesic submanifold of $G_n(\mathbb{C}^{2n})$. M is irreducible symmetric and $\mathrm{Sp}(2)/\mathrm{U}(2) = \mathrm{SO}(5)/\mathrm{SO}(2)\,\mathrm{SO}(3) = G_2(\mathbb{R}^5)$.

$\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$: Inner products on $\mathbb{R}^n$

The set of inner products on $\mathbb{R}^n$ is a non-compact symmetric space. If $\langle \cdot\, , \cdot \rangle_0$ is the standard inner product on $\mathbb{R}^n$, then any other inner product can be written as $\langle u, v\, \rangle = \langle Lu, v\rangle$ for some self adjoint linear map L. Thus the set of inner products can be identified with the set $M = \{A \in \mathrm{GL}(n,\mathbb{R}) \mid A = A^T,\ A > 0\}$ of positive definite symmetric matrices. The inner product $\langle X, Y\rangle = \operatorname{tr} XY$ on the set of symmetric matrices translates via left translations to any other $A \in M$, i.e. $\langle X, Y\rangle_A = \operatorname{tr}(A^{-1}XA^{-1}Y)$. The Lie group $\mathrm{GL}(n,\mathbb{R})$ acts on M via $g{\cdot}A = gAg^T$ and one easily sees that it acts by isometries. The action is transitive since the linear map that takes an orthonormal basis of one inner product to an orthonormal basis of another clearly takes the inner products into each other (the action is by basis change). The isotropy at Id is clearly $\mathrm{O}(n)$ and hence $M = \mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n) = \mathrm{GL}^+(n,\mathbb{R})/\mathrm{SO}(n)$. The involutive automorphism $\sigma(A) = (A^T)^{-1}$ has fixed point set $\mathrm{O}(n)$ and hence M is a symmetric space. In the Cartan decomposition, $\mathfrak{h}$ is the set of skew symmetric matrices, and $\mathfrak{p}$ the set of symmetric matrices. The isotropy representation is given by conjugation, i.e. $\chi(A)X = AXA^{-1}$ for $A \in \mathrm{O}(n)$ and $X \in \mathfrak{p}$. In other words, $\chi = S^2\rho_n$. It has a fixed vector Id but is irreducible on its orthogonal complement. $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is a totally geodesic submanifold of M and an irreducible symmetric space.

Finally, we claim that the symmetry s_{Id} is given by $s_{\mathrm{Id}}(A) = A^{-1}$. On the level of cosets it takes $g\,\mathrm{O}(n)$ to $\sigma(g)\,\mathrm{O}(n) = (g^T)^{-1}\,\mathrm{O}(n)$. Since $A > 0$ we can find g with $A = gg^T$ and hence $s_{\mathrm{Id}}(A) = s_{\mathrm{Id}}(gg^T \cdot \mathrm{O}(n)) = ((gg^T)^T)^{-1} = (gg^T)^{-1} = A^{-1}$.

$\mathrm{SO}(p,q)/\mathrm{S}(\mathrm{O}(p)\,\mathrm{O}(q))$: Positive p-planes in $\mathbb{R}^{p,q}$

In order to discuss another non-compact symmetric space, let $\mathbb{R}^{p,q}$ be $\mathbb{R}^{p+q}$ with an inner product $\langle \cdot\, , \cdot \rangle$ of signature (p,q). Let $M = \{L \in G_p(\mathbb{R}^{p+q}) \mid \langle u,u \rangle > 0 \text{ for all } u \in L\}$. The group of isometries of $\langle \cdot\, , \cdot \rangle$ is $\mathrm{O}(p,q)$ and it clearly acts transitively on M with isotropy at $L = \{e_1, \dots, e_n\}$ is $\mathrm{O}(p)\,\mathrm{O}(q)$. Thus $M = \mathrm{O}(p,q)/\,\mathrm{O}(p)\,\mathrm{O}(q) = \mathrm{SO}^+(p,q)/\,\mathrm{S}(\mathrm{O}(p)\,\mathrm{O}(q))$. The involutive automorphism is conjugation by $I_{p,q}$ which makes M into a symmetric space. The set of oriented positive p-planes $M^o = \mathrm{SO}(p,q)/\,\mathrm{SO}(p)\,\mathrm{SO}(q)$ is a symmetric space as well. Particularly important is the hyperbolic space $\mathbb{H}^{n+1} = \mathrm{SO}(p,1)/\,\mathrm{O}(p)$.

Similarly, for $\mathrm{SO}(p,q,\mathbb{C})/\,\mathrm{S}(\mathrm{U}(p)\,\mathrm{U}(q))$ and $\mathrm{SO}(p,q,\mathbb{H})/\,\mathrm{Sp}(p)\,\mathrm{Sp}(q)$ with the complex and quaternionic hyperbolic spaces $\mathbb{CH}^{n+1} = \mathrm{SU}(p,1)/\,\mathrm{U}(p)$ and $\mathbb{HH}^{n+1} = \mathrm{Sp}(p,1)/\,\mathrm{Sp}(p)\,\mathrm{Sp}(1)$. These 3 hyperbolic spaces, together with $F_{4,-20}/\,\mathrm{Spin}(9)$, are the non-compact rank 1 symmetric spaces with sectional curvature between -4 and -1. Here $F_{4,-20}$ is the Lie group corresponding to a particular real form of $\mathfrak{f}_4 \otimes \mathbb{C}$.

There are several more non-compact symmetric space, but we will see shortly that they are obtained by a duality from the compact ones.

Exercises 6.30

(1) Show that the Grassmannian $G_2^0(\mathbb{R}^4)$ is isometric to $\mathbb{S}^2(1) \times \mathbb{S}^2(1)$, up to some scaling on each factor. Furthermore, $G_2(\mathbb{R}^4) = \mathbb{S}^2(1) \times \mathbb{S}^2(1)/\{(a,b) \sim (-a,-b)\}$. Decompose $\rho_2 \hat{\otimes} \rho_2$ into irreducible subrepresentations and discuss the relationship.
(2) Compute the fundamental groups of $\mathrm{U}(n)/\,\mathrm{O}(n)$, $\mathrm{U}(n)/\,\mathrm{SO}(n)$ and $\mathrm{SU}(n)/\,\mathrm{SO}(n)$.
(3) Show that $\mathrm{U}(n)/\,\mathrm{SO}(n)$ is diffeomorphic to $\mathrm{S}^1 \times \mathrm{SU}(n)/\,\mathrm{SO}(n)$.
(4) Show that $\mathrm{SO}(6)/\,\mathrm{U}(3) = \mathbb{CP}^3$ and $\mathrm{SU}(4)/\,\mathrm{SO}(4) = \mathrm{SO}(6)/\,\mathrm{SO}(3)\,\mathrm{SO}(3)$.
(5) What is the set of unitary complex structures on $\mathbb{C}^{2n}$.

6.4 Geodesics and Curvature

Motivated by Proposition 6.20 and Proposition 6.25, we define:

symmpair **Definition 6.31** (G,K,σ) *is called a* **symmetric pair** *if K is compact, σ is an involution of G with $G_0^\sigma \subset K \subset G^\sigma$, and G acts almost effectively on G/K.*

Remark 6.32 As we saw, a symmetric space gives rise to a symmetric pair with $G = \mathrm{I}_0(M)$, and a symmetric pair gives rise to a symmetric space, although at this point the correspondence in not yet one-to-one, and a symmetric space can give rise to many symmetric pairs in infinitely many ways. Notice that in terms of the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, the condition that G acts almost effectively is equivalent to saying that $\mathfrak{g}$ and $\mathfrak{p}$ do not have any ideal in common.

A word of caution: If a Riemannian homogeneous space G/H is a symmetric space, it may not be true that (G, H) is a symmetric pair unless $G = \mathrm{I}_0(M)$. For example, $\mathbb{S}^n(1) = \mathrm{SO}(n+1)/\mathrm{SO}(n)$ and $(\mathrm{SO}(n+1), \mathrm{SO}(n))$ is a symmetric pair. But $\mathrm{SU}(n) \subset \mathrm{SO}(2n)$ also acts transitively on $\mathbb{S}^{2n-1}(1)$ with isotropy $\mathrm{SU}(n-1)$, i.e. $\mathbb{S}^{2n-1}(1) = \mathrm{SU}(n)/\mathrm{SU}(n-1)$. But $(\mathrm{SU}(n), \mathrm{SU}(n-1))$ is not a symmetric pair since one easily shows that there exists no automorphism σ of $\mathrm{SU}(n)$ with $G^\sigma = \mathrm{SU}(n-1)$. On the other hand, if $\mathbb{R}^n = L \rtimes \mathbb{R}^n/L$ with $L \subset \mathrm{O}(n)$ as above, it is still true that $(L \rtimes \mathbb{R}^n, L)$ is a symmetric pair since $\sigma(A, T_v) = (A, T_{-v})$ preserves $L \rtimes \mathbb{R}^n$.

We start with a description of the geodesics of a symmetric space. Recall that we identify $\mathfrak{p} \simeq T_{p_0}M$ via $X \to X^*(p_0)$.

symmgeod

Proposition 6.33 *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $X \in \mathfrak{p}$, then $\gamma(t) = \exp(tX) \cdot p_0$ is the geodesic in M with $\gamma(0) = p_0$ and $\gamma'(0) = X \in \mathfrak{p} \simeq T_{p_0}M$.*

Proof : Recall that the automorphism σ induces the symmetry s at the point p_0 given by $s(gH) = \sigma(g)H$. With respect to the symmetric metric on G/H, let (G', K') be the symmetric pair with $G' = \mathrm{I}_0(M)$ and with Cartan involution $\sigma'(g) = sgs$ and corresponding Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$. We first prove the claim for the symmetric pair (G', K').

Let γ be the geodesic in M with $\gamma(0) = p_0$ and $\gamma'(0) \in T_{p_0}M$. Then the transvection $T_t = s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)}$ is the flow of a Killing vector field $X \in \mathfrak{g}'$. Since $\gamma(t) = T_t \cdot p_0$, it follows that $\gamma'(0) = X^*(p_0)$. Furthermore, $\sigma'(T_t) = s_{\gamma(0)} s_{\gamma(\frac{t}{2})} s_{\gamma(0)} s_{\gamma(0)} = s_{\gamma(0)} s_{\gamma(\frac{t}{2})} = (s_{\gamma(\frac{t}{2})} \circ s_{\gamma(0)})^{-1} = (T_t)^{-1} = T_{-t}$. Differentiating we obtain $d\sigma'(X) = -X$ and thus $X \in \mathfrak{p}'$.

Next we look at the symmetric pair (G, H). We can assume that G/H is effective, since we can otherwise divide by the finite ineffective kernel without changing the Lie algebras. We first show that $\sigma'_{|G} = \sigma$. Indeed, if $g \in G$ then $L_{sgs} = L_{\sigma(g)}$ since $L_{sgs}(hK) = L_{sg}(\sigma(h)K) = L_s(g\sigma(h)K) = \sigma(g)hK = L_{\sigma(g)}(hK)$. Thus effectiveness implies that $\sigma(g) = sgs = \sigma'(g)$.

Next we prove that $\mathfrak{p} = \mathfrak{p}'$ which finishes the claim. Since they have the same dimension, it is sufficient to show that $\mathfrak{p} \subset \mathfrak{p}'$. But if $u \in \mathfrak{p}$, i.e. $d\sigma(u) = -u$, then $\sigma'_{|G} = \sigma$ implies that $d\sigma'(u) = -u$, i.e. $u \in \mathfrak{p}'$. $\square$

There is a simple formula for the connection and curvature of a symmetric space. Recall that we identify $\mathfrak{p} \simeq T_{p_0}M$ via $X \to X^*(p_0)$.

symmcurv

Proposition 6.34 *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$.*

(a) For any vector field Y on G/K and $X \in \mathfrak{p}$, we have $(\nabla_{X^*}Y)(p_0) = [X^*, Y](p_0)$.
(b) If $X, Y, Z \in \mathfrak{p}$, then $(R(X^*, Y^*)Z^*)(p_0) = -[[X, Y], Z]^*(p_0)$.

Proof: (a) For $X \in \mathfrak{p}$, consider the geodesic $\gamma(t) = \exp tX \cdot p_0$ in M. We have the corresponding transvection $T_t = s_{\gamma(\frac{t}{2})} s_{\gamma(0)} = L_{\exp tX}$ which is the flow of X^*. Also $(dT_t)_{\gamma(s)}$ is parallel translation along γ. Thus if Y is any vector field on M, we have $\nabla_{X^*}Y = \frac{d}{dt}_{|t=0}(P_t^{-1}Y(\gamma(t))) = \frac{d}{dt}_{|t=0}(dT_t^{-1})_{\gamma(t)}Y(\gamma(t)) = [X^*, Y]$.

(b) We first compute (everything at p_0)

$$\nabla_{X^*}\nabla_{Y^*}Z^* = [X^*, \nabla_{Y^*}Z^*] = \nabla_{[X^*,Y^*]}Z^* + \nabla_{Y^*}[X^*, Z^*]$$

since isometries preserve the connection and the flow of X^* consists of isometries. Since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$ we have $[X, Y]^*(p_0) = 0$ and hence

$$\nabla_{X^*}\nabla_{Y^*}Z^* = \nabla_{Y^*}[X^*, Z^*] = -\nabla_{Y^*}[X, Z]^* = -[Y^*, [X, Z]^*] = [Y, [X, Z]]^*.$$

Thus

$$\begin{aligned} R(X^*, Y^*)Z^* &= \nabla_{X^*}\nabla_{Y^*}Z^* - \nabla_{Y^*}\nabla_{X^*}Z^* - \nabla_{[X^*,Y^*]}Z^* \\ &= [Y, [X, Z]]^* - [X, [Y, Z]]^* = -[[X, Y], Z]^* \end{aligned}$$

by the Jacobi identity. $\square$

We usually simply state

$$\nabla_X Y = [X, Y], \quad R(X, Y)Z) = -[[X, Y], Z]$$

with the understanding that this only holds at p_0.

Remark 6.35 Part (a) gives rise to a geometric interpretation of the Cartan decomposition in terms of Killing vector fields, assuming that $G = \mathrm{I}_0(M)$:

$$\mathfrak{k} = \{X \in \mathfrak{g} \mid X^*(p_0) = 0\} \quad \mathfrak{p} = \{X \in \mathfrak{g} \mid \nabla_v X^*(p_0) = 0 \text{ for all } v \in T_{p_0}M\}$$

The first equality is obvious and for the second one, we observe that (a) implies that $(\nabla_{X^*}Y^*)(p_0) = [X,Y]^*(p_0) = 0$ for $X, Y \in \mathfrak{p}$ since $[\mathfrak{p},\mathfrak{p}] \subset \mathfrak{k}$. Equality then follows by dimension reason. One also easily sees that $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition. Elements of $\mathfrak{p}$ are often called *infinitesimal transvections.*

We finish with a simple characterization of totally geodesic submanifolds of symmetric spaces:

totgeodclass

Proposition 6.36 *Let G/H be a symmetric space corresponding to the Cartan involution σ, and $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ a Cartan decomposition. If $\mathfrak{a} \subset \mathfrak{p}$ is a linear subspace with $[[\mathfrak{a},\mathfrak{a}],\mathfrak{a}] \subset \mathfrak{a}$, called a Lie triple system, then* $\exp(\mathfrak{a})$ *is a totally geodesic submanifold.*

Proof First observe that $\mathfrak{h}' = [\mathfrak{a},\mathfrak{a}] \subset \mathfrak{h}$, the subspace spanned by all $[u,v]$, $u, v \in \mathfrak{a}$, is a subalgebra: Using the Jacobi identity, $[\mathfrak{h}',\mathfrak{h}'] = [[\mathfrak{a},\mathfrak{a}],[\mathfrak{a},\mathfrak{a}]] = [[[\mathfrak{a},\mathfrak{a}],\mathfrak{a}],\mathfrak{a}] = [\mathfrak{a},\mathfrak{a}]$ since $\mathfrak{a}$ is a Lie triple system. Furthermore, $[\mathfrak{h}',\mathfrak{a}] = [[\mathfrak{a},\mathfrak{a}],\mathfrak{a}] \subset \mathfrak{a}$ and clearly $[\mathfrak{a},\mathfrak{a}] \subset \mathfrak{h}'$ Thus $\mathfrak{g}' = [\mathfrak{a},\mathfrak{a}] \oplus \mathfrak{a}$ is a subalgebra of $\mathfrak{g}$. Let $G' \subset G$ be the connected subgroup with Lie algebra $\mathfrak{g}$ and $\mathrm{H}' \subset H$ the one with Lie algebra $\mathfrak{h}'$. Since $d\sigma$ clearly preserves $\mathfrak{g}'$, it follows that σ preserves G and the claim follows from Proposition 6.28. $\square$

It may seem that the Proposition would enable one to easily classify totally geodesic submanifolds of symmetric spaces. Unfortunately, this is not the case. Even totally geodesic submanifolds of Grassmannians have not been classified. On the other had, we will see that it is of theoretical use.

6.5 Type and Duality

We now continue the general theory of symmetric spaces. We start with an important definition.

symmtype

Definition 6.37 *Let (G,K) be a symmetric pair with B the Killing form of $\mathfrak{g}$. The symmetric pair is called of* **compact type** *if $B_{|\mathfrak{p}} < 0$, of* **non-compact type** *if $B_{|\mathfrak{p}} > 0$ and of* **euclidean type** *if $B_{|\mathfrak{p}} = 0$.*

We first observe

Cartandecomp2

Proposition 6.38 *Let (G, K) be a symmetric pair.*

(a) If (G, K) is irreducible, it is either of compact type, non-compact type or euclidean type.
(b) If $M = G/K$ is simply connected, then M is isometric to a Riemannian product $M = M_0 \times M_1 \times M_2$ with M_0 of euclidean type, M_1 of compact type and M_2 of non-compact type.
(c) If (G, K) is of compact type, then G is semisimple and G and M are compact.
(d) If (G, K) is of non-compact type, then G is semisimple and G and M are non-compact.
(e) (G, K) is of euclidean type iff $[\mathfrak{p}, \mathfrak{p}] = 0$. Furthermore, if G/K is simply connected, it is isometric to $\mathbb{R}^n$.

Proof : (a) If (G, K) is irreducible, then Schur's Lemma implies that $B_{|\mathfrak{p}} = \lambda\langle\cdot,\cdot\rangle$, where $\langle\cdot,\cdot\rangle$ is the metric on $\mathfrak{p}$. Thus M is of compact type if $\lambda < 0$, of non-compact type if $\lambda > 0$ and of euclidean type if $\lambda = 0$.

(b) From Corollary 6.12 it follows that M is isometric to $M_1 \times \cdots \times M_k$ with M_i irreducible symmetric spaces. The claim thus follows from (a).

(c) and (d) If σ is the automorphism of the pair, then $d\sigma$ preserves B and hence $B(\mathfrak{k}, \mathfrak{p}) = 0$ since $d\sigma_{|\mathfrak{k}} = \mathrm{Id}$ and $d\sigma_{|\mathfrak{p}} = -\mathrm{Id}$. Next we claim that $B_{|\mathfrak{k}} < 0$. Indeed, since K is compact, there exists an inner product on $\mathfrak{g}$ such that $\mathrm{ad}_X : \mathfrak{g} \to \mathfrak{g}$ is skew symmetric if $X \in \mathfrak{k}$. Thus $B(X, X) = \mathrm{tr}\,\mathrm{ad}_X^2 \le 0$ and $B(X, X) = 0$ iff $X \in \mathfrak{z}(\mathfrak{g})$. But $\mathfrak{z}(\mathfrak{g}) \cap \mathfrak{k} = 0$ since $\mathfrak{g}$ and $\mathfrak{k}$ have no ideals in common. Hence, if $B(X, X) = 0$, we have $X = 0$, i.e. $B_{|\mathfrak{k}} < 0$.

Thus G is semisimple if (G, K) is of compact or non-compact type. If it is of compact type, then $B < 0$ and hence G, and thus also M, is compact. Similarly, for the non-compact type.

(e) If $B_{|\mathfrak{p}} = 0$, then $B(\mathfrak{k}, \mathfrak{p}) = 0$ and $B_{|\mathfrak{k}} < 0$ implies that $\mathfrak{p} = \ker B$. But $\ker B$ is an ideal in $\mathfrak{g}$ and, together with $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$, this implies that $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{p} \cap \mathfrak{k} = 0$. Conversely, if $[\mathfrak{p}, \mathfrak{p}] = 0$, together with $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$, one easily sees that $B_{|\mathfrak{p}} = 0$.

If $[\mathfrak{p}, \mathfrak{p}] = 0$, Proposition 6.34 implies that the sectional curvature is 0. If M is simply connected, it follows that M is isometric to $\mathbb{R}^n$. □

The symmetric spaces of euclidean type are thus not so interesting, and we will say that $M = G/K$ **has no (local) euclidean factor** if in the splitting of the universal cover, none of the irreducible factors are of euclidean type. This clearly holds iff each point has a neighborhood which does not split of a euclidean factor. For simplicity we often leavs out the word "local".

These spaces will from now be our main interest. We start with the special curvature properties of each type

symmsec **Proposition 6.39** *Let (G, K) be a symmetric pair with .*

(a) If (G, K) is of compact type, then $\sec \geq 0$.
(b) If (G, K) is of non-compact type, then $\sec \leq 0$.
(c) In both cases, a 2-plane spanned by $u, v \in \mathfrak{p}$ has zero curvature iff $[u, v] = 0$.
(d) If (G, K) is irreducible, and $\langle \cdot, \cdot \rangle = \pm B_{|\mathfrak{p}}$, then $\sec(X, Y) = \pm ||[X, Y]||^2$.

Proof: It is clearly sufficient to prove this for an irreducible symmetric space. In that case $B_{|\mathfrak{p}} = \lambda\langle \cdot, \cdot \rangle$ for some $\lambda \neq 0$, where $\langle \cdot, \cdot \rangle$ is the metric on $\mathfrak{p}$. If $u, v \in \mathfrak{p} \simeq T_pM$ is an orthonormal basis of a 2-plane, then

$$\begin{aligned} \lambda \sec(u, v) = \lambda\langle R(u, v)v, u\rangle &= -\lambda\langle [[u, v], v], u\rangle \\ &= -B([[u, v], v], u) = B([u, v], [u, v]) \end{aligned}$$

where we have used the fact that ad_u is skew symmetric for B. Since $[u, v] \in \mathfrak{k}$ and since $B_{|\mathfrak{k}} < 0$ we have $B([u, v], [u, v]) < 0$ and thus sec is determined by the sign of λ. □

This implies in particular that if (G, K) has non-compact type, then G is simple. We can reduce the classification of symmetric pairs easily to the case where G is simple

symmsimple **Proposition 6.40** *Let (G, K) be an irreducible symmetric pair which is not of eucildean type. Then either G is simple, or $(G, K) = (K \times K, \Delta K)$ and G/K is isometric to a compact simple Lie group with bi-invariant metric.*

Proof : Proposition 6.38 implies that $\mathfrak{g}$ is semisimple, and thus $\mathfrak{g} = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_r$ with $\mathfrak{g}_i$ simple. This decomposition into simple ideals is unique up to order, and hence the Cartan involution σ permutes $\mathfrak{g}_i$. We can thus write $\mathfrak{g} = \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_s$ as a sum of ideals such that $\mathfrak{h}_i$ is either $\mathfrak{g}_k$ for some k with $\sigma(\mathfrak{g}_k) = \mathfrak{g}_k$, or $\mathfrak{h}_i = \mathfrak{g}_k \oplus \mathfrak{g}_l$ for some k, l and $\sigma(\mathfrak{g}_k) = \mathfrak{g}_l$. $\sigma_{|\mathfrak{h}_i}$ induces a further decomposition $\mathfrak{h}_i = \mathfrak{k}_i \oplus \mathfrak{p}_i$ into ± 1 eigenspaces and hence $\mathfrak{k} = \mathfrak{k}_1 \oplus \cdots \oplus \mathfrak{k}_s$ and $\mathfrak{p} = \mathfrak{p}_1 \oplus \cdots \oplus \mathfrak{p}_s$. Notice that $\mathfrak{p}_i \neq 0$ for all i since otherwise $\sigma_{|\mathfrak{l}_i} = \mathrm{Id}$ which means that $\mathfrak{l}_i$ is an ideal that $\mathfrak{g}$ and $\mathfrak{k}$ have in common, contradicting effectiveness. Since $[\mathfrak{k}_i, \mathfrak{p}_j] = 0$ for $i \neq j$, we have

$[\mathfrak{k}, \mathfrak{p}_i] \subset \mathfrak{p}_i$ and irreducibility implies that $s = 1$. If G is not simple, this implies that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{h}$ with $\sigma(a, b) = (b, a)$ and hence $\mathfrak{g}^\sigma = \Delta\mathfrak{h}$. This is the symmetric pair $K \times K/\Delta K$, where K is any compact simple Lie group with Lie algebra $\mathfrak{k}$. □

We can now determine the isometry group.

symmisom

Proposition 6.41 *Let (G, K) be a symmetric pair with no euclidean factor and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Then*

(a) $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$.
(b) $\mathrm{Hol}_p^0 = K_0$*, where* Hol_p *is the holonomy group.*
(c) If G/K is effective, then $G = \mathrm{I}_0(M)$.

Proof: (a) By Proposition 6.12 we can assume that G/K is irreducible. Thus $B_{|\mathfrak{p}}$ is non-degenerate. If $a \in \mathfrak{k}$ with $B(a, [u, v]) = 0$ for all $u, v \in \mathfrak{p}$, then $0 = B(a, [u, v]) = -B(u, [a, v])$ for all u and hence $[a, v] = 0$ for all $v \in \mathfrak{p}$. Since $\mathrm{Ad}(\exp(ta)) = e^{t\,\mathrm{ad}_a}$ this implies that $\mathrm{Ad}(\exp(ta))_{|\mathfrak{p}} = \mathrm{Id}$. But G/K is almost effective and hence the isotropy representation has finite kernel. Thus $\exp(ta) = e$ for all t and hence $a = 0$. This implies that $\mathfrak{k} = \{[u, v] \mid u, v \in \mathfrak{p}\}$.

(b) Recall that we already saw that $\mathrm{Hol}_p^0 \subset K_0$. One geometric interpretation of the curvature is in terms of parallel translation: If $x, y, z \in T_pM$, consider small rectangles in M with one vertex at p, whose sides have length s and at p are tangent to x, y. Parallel translating z around these rectangles gives a curve $z(t) \in T_pM$ and $R(x, y)z = z''(0)$. Since $z(t) \in \mathrm{Hol}_p^0$, the skew symmetric endomorphism $R(x, y)$ lies in the Lie algebra of Hol_p^0. But Proposition 6.34 implies that $R(x, y) = -\mathrm{ad}_{[x,y]}$ restricted to $\mathfrak{p} \simeq T_pM$. Since $[\mathfrak{p}, \mathfrak{p}] = \mathfrak{k}$, this implies that Hol_p and K have the same Lie algebra.

(c) Recall that the involutive automorphism σ induces the symmetry s at the identity coset $p_0 = (K)$ by $s(gK) = \sigma(g)K$. Let $G' = \mathrm{I}_0(M)$ be the full isometry group and K' its isotropy group at p_0. Then $K \subset K'$ by effectiveness and hence $\mathfrak{k} \subset \mathfrak{k}'$. The symmetry s at p_0 induces the automorphism σ' of G' defined by $\sigma'(g) = sgs$ which makes (G', K') into a symmetric pair. This symmetric pair is also irreducible since M is locally irreducible. Thus we have another Cartan decomposition $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ into the ± 1 eigenspaces of $d\sigma'$. and part (a) implies that $[\mathfrak{p}', \mathfrak{p}'] = \mathfrak{k}'$. In the proof of Proposition 6.33 we showed that $\mathfrak{p} = \mathfrak{p}'$. Thus $\mathfrak{k} = [\mathfrak{p}, \mathfrak{p}] = [\mathfrak{p}', \mathfrak{p}'] = \mathfrak{k}'$ and hence $\mathfrak{g} = \mathfrak{g}'$. Effectiveness now implies that $G = \mathrm{I}_0(M)$ since they have the same Lie algebra. □

We next discuss the important concept of **duality**. Let (G,K) be a symmetric pair with $\pi_1(G/K)=0$. Since G is connected, K is connected as well. Let $\mathfrak{g}=\mathfrak{k}\oplus\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$. We can consider $\mathfrak{g}$ as a real subalgebra of $\mathfrak{g}\otimes\mathbb{C}$ and define a new real Lie algebra $\mathfrak{g}^*\subset\mathfrak{g}\otimes\mathbb{C}$ by $\mathfrak{g}=\mathfrak{k}\oplus i\mathfrak{p}$. This is indeed a subalgebra since $[\mathfrak{k},\mathfrak{k}]\subset\mathfrak{k}$, $[\mathfrak{k},\mathfrak{p}]\subset\mathfrak{p}$ and $[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$ and hence $[\mathfrak{k},i\mathfrak{p}]\subset i\mathfrak{p}$ and $[i\mathfrak{p},i\mathfrak{p}]=-[\mathfrak{p},\mathfrak{p}]\subset\mathfrak{k}$. Now let G^* be the simply connected Lie group with Lie algebra $\mathfrak{g}^*$ and K^* the connected subgroup with Lie algebra $\mathfrak{k}\subset\mathfrak{g}^*$. Then G^*/K^* is simply connected and almost effective since $\mathfrak{g}$ and $\mathfrak{k}$, and hence also $\mathfrak{g}^*$ and $\mathfrak{k}^*$, have no ideals in common. We call (G^*,K^*) the dual of (G,K). Notice that K and K^* have the same Lie algebra, but may not be isomorphic.

Thus, if (G,K) is a simply connected symmetric pair, we have a dual (G^*,K^*) which is another simply connected symmetric pair. This relationship has the following properties.

symmduality

Proposition 6.42 *Let (G,K) be a symmetric pair with dual symmetric pair (G^*,K^*) .*

(a) If (G,K) is of compact type, then (G^*,K^*) is of non-compact type and vice versa.
(b) If (G,K) is of Euclidean type, then (G^*,K^*) is of Euclidean type as well.
(c) The pairs (G,K) and (G^*,K^*) have the same (infinitesimal) isotropy representation and hence (G,K) is irreducible iff (G^*,K^*) is irreducible.
(d) If (G,K) and (G^*,K^*) are effective and simply connected without euclidean factors, then $K=K^*$.

Proof : (a) Recall that if $\mathfrak{g}$ is a real Lie algebra, then $B_{\mathfrak{g}}=B_{\mathfrak{g}_{\mathbb{C}|\mathfrak{g}}}$. By construction $\mathfrak{g}_{\mathbb{C}}\simeq\mathfrak{g}^*_{\mathbb{C}}$. If (G,K) is of compact type, i.e. $B_{\mathfrak{g}}(u,u)<0$ for $u\in\mathfrak{p}$, then $B_{\mathfrak{g}^*}(iu,iu)=-B(u,u)>0$, i.e. (G^*,K^*) is of non-compact type and vice versa. Part (b) clearly follows from Proposition 6.38 (e). Part (c) is clear as well since the action of $\mathfrak{k}$ on $\mathfrak{p}$ resp. $i\mathfrak{p}$ is the same.

(d) Since (G,K) has no euclidean factors, Proposition 6.41 (c) implies that $G=\mathrm{I}_0(M)$. Also recall that K is connected. Now observe that if R is the curvature tensor of G/K, then Proposition 6.16 (c) implies that $K=K_0=\{A\in\mathrm{GL}(T_{p_0})M\mid A^*(R)=R\}_0$ since G/K is effective and simply connected. Similarly, $K_0^*=K^*=\{A\in\mathrm{GL}(T_{p_0})M\mid A^*(R^*)=R^*\}_0$ where R^* is the curvature tensor of G^*/K^*. But Proposition 6.34 implies that

$R^*(iX,iY)iZ = -[[iX,iY],iZ] = [[X,Y],Z] = -R(X,Y)Z$ for $X,Y,Z \in \mathfrak{p}$ and hence $K = K^*$. $\square$

We thus have a one-to-one correspondence between simply connected effective symmetric pairs of compact type and simply connected effective symmetric pais of non-compact type, which also take irreducible ones to irreducible ones.

symmexamples

Example 6.43 (a) We will give several examples of duality. The most basic one is the duality between $\mathbb{S}^n$ and $\mathbb{H}^n$. Multiplication by i on the tangent space in the definition of duality illustrates why $\sin(x)$, $\cos(x)$ in spherical geometry is replaced by $\sinh(x) = \sin(ix)$ and $\cosh(x) = i\cos(ix)$. It is just as easy to discuss the duality between $G/K = \mathrm{SO}(p+q)/\mathrm{SO}(p)\,\mathrm{SO}(q)$ and $G^*/K = \mathrm{SO}(p,q)/\mathrm{SO}(p)\,\mathrm{SO}(q)$. Recall that in both cases the Cartan involution is given by $\sigma(A) = \mathrm{Ad}(I_{p,q})$. We write the matrices in block form, the upper left a $p \times p$ block, and the lower one a $q \times q$ block. Also recall that $\mathfrak{o}(p+q) = \{A \in M(p+q) \mid A + A^T = 0\}$ and $\mathfrak{o}(p,q) = \{A \in M(p+q) \mid AI_{p,q} + I_{p,q}A^T = 0\}$. Thus one easily sees that in the Cartan decomposition

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & X \\ -X^T & 0 \end{pmatrix} \mid X \in M(p,q) \right\}, \quad \mathfrak{p}^* = \left\{ \begin{pmatrix} 0 & X \\ X^T & 0 \end{pmatrix} \mid X \in M(p,q) \right\}.$$

Here it is of course not true that $\mathfrak{p}^* = i\mathfrak{p}$ but a computation shows that the inner automorphism $\mathrm{Ad}(\mathrm{diag}(i,\dots,i,-1,\dots-1))$ (the first p entries are i) of $\mathfrak{so}(n,\mathbb{C})$ takes $i\mathfrak{p}$ to $\mathfrak{p}^*$ and preserves $\mathfrak{h}$ and thus conjugates $\mathfrak{g}^*$ into a new Lie algebra $\mathfrak{g}'$ that satisfies the above setup of duality. The inner automorphism gives rise to an isomorphism $(G^*,K) \simeq (G',K)$ of symmetric pairs.

The same relationship holds if we replace $\mathbb{R}$ by $\mathbb{C}$ or $\mathbb{H}$. Thus $\mathbb{CP}^n$ is dual to $\mathbb{CH}^n$ and $\mathbb{HP}^n$ is dual to $\mathbb{HH}^n$.

(b) Maybe the simplest example of duality is between $\mathrm{SU}(n)/\mathrm{SO}(n)$ and $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$. Since the involutions are given by $d\sigma(A) = \bar{A}$ and $d\sigma(A) = -A^T$ we have $\mathfrak{k} = \{A \in M(n,n,\mathbb{R}) \mid A = -A^T\}$ in both cases, and $\mathfrak{p} = \{A \in \mathfrak{su}(n) \mid \bar{A} = -A\}$ as well as $\mathfrak{p}^* = \{A \in M(n,n,\mathbb{R}) \mid A = A^T\}$. But $\mathfrak{p}$ can also be written us $\mathfrak{p} = \{iA \mid A \in M(n,n,\mathbb{R})$ and $A = A^T\}$ and thus $\mathfrak{p}^* = i\mathfrak{p} \subset \mathfrak{sl}(n,\mathbb{C})$.

(c) Somewhat more subtle is the dual of the symmetric pair $(G,K) = (L \times L, \Delta L)$ corresponding to a compact Lie group $K \simeq L$. We claim that on the Lie algebra level it is the pair $((\mathfrak{l}_{\mathbb{C}})_{\mathbb{R}}, \mathfrak{l})$ with Cartan involution $\sigma(A) = \bar{A}$, i.e. $\mathfrak{g}^*$ is $\mathfrak{l}_{\mathbb{C}}$ regarded as a real Lie algebra.

To see this, recall that $\mathfrak{g} = \mathfrak{l} \oplus \mathfrak{l}$ with Cartan involution $\sigma(X,Y) = (Y,X)$ and thus $\mathfrak{k} = \{(X,X) \mid X \in \mathfrak{l}\}$ and $\mathfrak{p} = \{(X,-X) \mid X \in \mathfrak{l}\}$. We now want to describe $\mathfrak{g}^* = \mathfrak{k} + i\mathfrak{p} \subset \mathfrak{g}_{\mathbb{C}} = \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$ in a different fashion. For this let $\mathfrak{p}' = \{(iX, iX) \mid X \in \mathfrak{l}\} \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{g}' = \mathfrak{k} \oplus \mathfrak{p}'$. The linear isomorphism of $\mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$ defined by $(X_1 + iY_1, X_2 + iY_2) \to (X_1 + iY_1, X_2 - iY_2)$ is an isomorphism of real Lie algebras and takes $\mathfrak{k}$ to $\mathfrak{k}$ and $i\mathfrak{p}$ to $\mathfrak{p}'$. Thus the dual pair $(\mathfrak{g}^*, \mathfrak{k})$ is isomorphic to $(\mathfrak{g}', \mathfrak{k})$. But notice that $\mathfrak{g}' = \{(X + iY, X + iY) \mid X, Y \in \mathfrak{l}\} = \Delta\mathfrak{l}_{\mathbb{C}} \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$ and $\mathfrak{k} = \Delta\mathfrak{l} \subset \mathfrak{l}_{\mathbb{C}} \oplus \mathfrak{l}_{\mathbb{C}}$. Thus $(\mathfrak{g}', \mathfrak{k})$ is isomorphic to $(\mathfrak{l}_{\mathbb{C}}, \mathfrak{l})$. The Cartan involution is then clearly given by conjugation.

6.6 Symmetric Spaces of non-compact type

Although the classification of symmetric spaces is easier for the one's of compact type, and by duality implies the classification of symmetric spaces of non-compact type, the geometry of the one's of non-compact type have many special properties. We will study these in this Section.

We first remark the following. If (G, K) has compact type, we have a natural positive definite inner product $\mathfrak{g}$ given by $-B$. In the case of non-compact type, we also have such a natural inner product.

innprodnoncomptype

Lemma 6.44 *If (G, K) is a symmetric space of non-compact type, then the inner product $B^*(X, Y) = -B(\sigma(X), Y)$ on $\mathfrak{g}$ has the following properties:*

(a) B^* is positive definite,
(b) If $X \in \mathfrak{k}$, then $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ is skew symmetric,
(c) If $X \in \mathfrak{p}$, then $\mathrm{ad}_X \colon \mathfrak{g} \to \mathfrak{g}$ is symmetric.

Proof Part (a) follows from $B_{|\mathfrak{k}} < 0$, $B_{|\mathfrak{p}} > 0$, $B(\mathfrak{k}, \mathfrak{p}) = 0$ and $\sigma_{|\mathfrak{k}} = \mathrm{Id}$, $\sigma_{|\mathfrak{p}} = -\mathrm{Id}$. For (b) and (c), notice that ad_X and σ commute if $X \in \mathfrak{k}$ or $X \in \mathfrak{p}$. The claim now easily follows from the fact that ad_X is always skew symmetric for B. □

We can now state the main properties,

symmnoncompact

Proposition 6.45 *Let (G, K) be a symmetric pair of non-compact type with Cartan involution σ. Then*

(a) G is non-compact and semisimple and G^σ and K are connected.
(b) K is a maximal compact subgroup of G.
(c) $Z(G) \subset K$, or equivalently, if G/K is effective, then $Z(G) = \{e\}$.
(d) G is diffeomorphic to $K \times \mathbb{R}^n$ and G/K is diffeomorphic to $\mathbb{R}^n$ and simply connected.

Proof : Recall that by Proposition 6.39 the symmetric metric on G/K has non-positive curvature and by Proposition 6.33 the exponential map of $M = G/K$ is given by $\exp_M : \mathfrak{p} \to G/K : X \to \exp(X)K$. By Hadamard, $\exp_M$ is a local diffeomorphism and onto by Hopf-Rinow. We now show that it is injective and hence a diffeomorphism. So assume that $\exp(X)h = \exp(X')h'$ for some $h, h' \in K$. Then $\mathrm{Ad}(\exp(X))\,\mathrm{Ad}(h) = \mathrm{Ad}(\exp(X'))\,\mathrm{Ad}(h')$ and thus $e^{\mathrm{ad}_X}\,\mathrm{Ad}(h) = e^{\mathrm{ad}_{X'}}\,\mathrm{Ad}(h')$. But by 6.44 e^{ad_X} is symmetric and $\mathrm{Ad}(h)$ orthogonal. By uniqueness of the polar decomposition, $\mathrm{Ad}(h) = \mathrm{Ad}(h')$ and $\mathrm{ad}_X = \mathrm{ad}_{X'}$, and hence $X - X' \in \mathfrak{z}(\mathfrak{g}) = 0$ since G is semisimple. Thus $X = X'$ (and $h = h'$ as well). In particular, G/K is simply connected and hence K connected.

Next, we show that $f : \mathfrak{p} \times K \to G \; : \; (X, h) \to \exp(X)h$ is a diffeomorphism. Indeed, f is clearly differential and by the same argument as above, f is one-to-one. Given $g \in G$, there exists a unique $X \in \mathfrak{p}$ such that $\exp(X)K = gK$ and hence a unique $h \in K$ such that $\exp(X)h = g$. X, and hence h, clearly depends differentiably on g and hence f is a diffeomorphism.

The argument can be repeated for $f' : \mathfrak{p} \times Z(G) \cdot K \to G \; : \; (X, h) \to \exp(X)h$ and shows that f' is a diffeomorphism as well, and hence $Z(G) \cdot K$ is connected. Since G is semisimple, $Z(G)$ is finite and hence $Z(G) \subset K$.

We finally show that K is maximal compact in G. Let L be a compact group with $K \subset L \subset G$ and hence $\mathfrak{k} \subset \mathfrak{l} \subset \mathfrak{g}$. Since L is compact, there exists an inner product on $\mathfrak{g}$ invariant under $\mathrm{Ad}(L)$ and hence $B_{|\mathfrak{l}} \le 0$, and since $\mathfrak{z}(\mathfrak{g}) = 0$, in fact $B_{|\mathfrak{l}} < 0$. But since $B_{|\mathfrak{k}} < 0$ and $B_{|\mathfrak{p}} > 0$, it follows that $\mathfrak{k} = \mathfrak{l}$. Thus $K = L_0$ and hence K is normal in L. Since L is compact, L/K is a finite group. Thus there exists an $g \in L$, which we can write as $g = \exp(X)h$ with $h \in K$ and hence $g^n = \exp(nX)h' = \bar{h} = \exp(0)\bar{h} \in K$ where $h, \bar{h} \in K$. But this contradicts the fact that f is a diffeomorphism. $\square$

The next important property of non-compact semisimple groups is the following, whose proof we will supply later on.

Cartandecompnoncompact

Proposition 6.46 *Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra. Then there exists a Cartan involution $\sigma \in \mathrm{Aut}(\mathfrak{g})$ with corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ into ± 1 eigenspaces and σ is unique up to inner automorphisms.*

As a consequence we show

maxcompact

Proposition 6.47 *Let G be a non-compact semisimple Lie group with finite center. Then there exists a maximal compact subgroup K, unique up to conjugacy, such that G is diffeomorphic to $K \times \mathbb{R}^n$. Furthermore, if G/K is effectively G^*/K^*, then (G^*, K^*) is a symmetric pair of non-compact type. If $Z(G) = \{e\}$, then (G, K) is a symmetric pair.*

Proof Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k}$. We have a finite cover $\pi \colon G \to G/Z(G)$ and define $K^* = \pi(K)$. Since G^* has no center, $G^* \simeq \mathrm{Ad}(G^*) \simeq \mathrm{Int}(G^*) \simeq \mathrm{Int}(\mathfrak{g}) \subset \mathrm{Aut}(\mathfrak{g})$. $\mathrm{Aut}(\mathfrak{g})$ is closed in $\mathrm{GL}(\mathfrak{g})$ since it is defined by equations, and since $\mathfrak{g}$ is semisimple, $\mathrm{Aut}(\mathfrak{g})/\operatorname{Int}(\mathfrak{g})$ is finite, and thus $\mathrm{Ad}(G^*)$ is closed as well. If $h \in K^*$, then $\mathrm{Ad}(h)$ is an isometry of B^*, i.e., $\mathrm{Ad}(K^*) \subset \mathrm{O}(\mathfrak{p}, B^*)$ and thus $K^* \simeq \mathrm{Ad}(K^*)$ is compact. The Cartan decomposition induces an involutive automorphism σ and in turn α of G^* with $d\alpha = \sigma$. Clearly $(G^*)^\alpha_0 = K^*$ and thus (G^*, K^*) is a symmetric pair of non-compact type. In particular, G^*/K^* is simply connected and K^* is maximal compact in G^*. Clearly $G^*/K^* = G/\pi^{-1}(K^*)$ and hence $G/K \to G^*/K^*$ is a finite cover and thus a diffeomorphism, i.e. G^*/K^* is the effective version of G/K. Since $\ker \pi \subset K$, it easily implies that K is maximal compact as well.

Finally, to see that K is unique up to conjugacy, we use the fact that $M = G/K$ has non-positive curvature and is simply connected. This implies that any compact group H acting on M by isometries has a fixed point on M. Indeed, a standard second variation argument shows that $d^2(p, \cdot)$ is a strictly convex proper function on M, i.e., along every geodesic, $d^2(p, \cdot)$ is convex. Fix any $p \in M$, then define a function $f : M \to M$ via

$$f(q) = \int_H d^2(q, hp)dh.$$

This is again a convex, proper function (now in q). Hence f has a unique minimum at some $p_0 \in M$, and clearly $Hp_0 = p_0$ since f is invariant under H.

Now let $H \subset G$ be a second maximal compact subgroup. By the above, H has a fixed point, say $p_0 = gK$ for some g, i.e. $hgK = gK$. Thus $g^{-1}Hg = K$, or $H = gKg^{-1}$. □

Remark 6.48 The assumption that G has finite center is essential. As an example, let $G^* = \mathrm{SL}(2, \mathbb{R})$ and G its universal cover. This is an infinite cover since $\pi_1(\mathrm{SL}(2, \mathbb{R})) = \mathbb{Z}$. $\mathrm{SO}(2)$ is a maximal compact subgroup of $\mathrm{SL}(2, \mathbb{R})$, but the maximal compact subgroup of G is $\{e\}$ since $\pi^{-1}(\mathrm{SO}(2)) = \mathbb{R}$ which has no non-trivial compact subgroups. Hence (G, K) is not a symmetric pair. On the other hand, the universal cover of $\mathrm{SL}(2,\mathbb{R})$ is diffeomorphic to $\mathbb{R}^3$.

Nevertheless, part of Proposition 6.49 remains true:

maxcompact

Proposition 6.49 *Let G be a non-compact semisimple Lie group. Then there exists a maximal compact subgroup L, unique up to conjugacy, such that G is diffeomorphic to $L \times \mathbb{R}^n$.*

Proof We proceed as in the proof of Proposition 6.49. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be a Cartan decomposition and $K \subset G$ the connected subgroup with Lie algebra $\mathfrak{k}$. Notice that (G, K) may not be a symmetric pair since K is not necessarily compact. Nevertheless, the effective version (G^*, K^*) is a symmetric pair. Thus $G/K = G^*/K^*$ is a Riemannian symmetric space such that left translation by $g \in G$ acts by isometries on G/K. If $L' \subset G$ is a compact subgroup, an argument as in Proposition 6.49 shows that there exists a fixed point, and thus an element $g \in G$ with $gL'g^{-1} \subset K$. We now claim that there exists a unique maximal compact subgroup $L \subset K$ (not just unique up to conjugacy) and hence $gL'g^{-1} \subset L$, and if L' is maximal compact in G, then $gL'g^{-1} = L$. To see this, recall that $B_{|\mathfrak{k}} < 0$ and hence $\mathfrak{k}$ is a compact Lie algebra. Recall that this implies that any Lie group with Lie algebra $\mathfrak{k}$, in particular K, is isomorphic to $R^\ell \times L$ with L compact. L is then clearly maximal compact in K. By Proposition 6.45, G is diffeomorphic to $K \times \mathbb{R}^n$ and hence diffeomorphic to $L \times \mathbb{R}^{n+\ell}$. □

Combining this with the Levi decomposition theorem one can prove:

maxcompactgen

Theorem 6.50 *Let G be a Lie group with finitely many components. Then there exists a maximal compact subgroup K, unique up to conjugacy, such that G is diffeomorphic to $K \times \mathbb{R}^n$.*

We now show that in some sense the symmetric space $\mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ is the universal symmetric space of non-compact type.

embnoncompact

Proposition 6.51 *Let (G, K) be a effective symmetric pair of non-compact type. Then there exists an isometric embedding $\phi\colon G/K \to \mathrm{SL}(n,\mathbb{R})/\mathrm{SO}(n)$ with totally geodesic image, given by $\phi(gK) = \mathrm{Ad}(g) \cdot \mathrm{SO}(n)$.*

Proof Since G/H is effective, $Z(G) = \{e\}$ and hence Ad is an embedding. To see that the image lies in $\mathrm{SL}(\mathfrak{g})$, i.e. $\det \mathrm{Ad}(g) = 1$, it is sufficient to show that $\mathrm{tr}(\mathrm{ad}\, X) = 0$ for all $X \in \mathfrak{g}$. To see why this is so, we choose a compact real form $\mathfrak{k} \subset \mathfrak{g}_{\mathbb{C}}$. By compactness, $\operatorname{tr}(\operatorname{ad} X) = 0$ holds for $X \in \mathfrak{k}$, hence also in $\mathfrak{g}_{\mathbb{C}}$ and therefore in $\mathfrak{g}$ as well.

We endow $\mathfrak{g}$ with the inner product B^* in which case $\operatorname{Ad}(K) \subset \operatorname{SO}(\mathfrak{g})$. Furthermore, $\operatorname{Ad}(G) \cap \operatorname{SO}(\mathfrak{g}) = \operatorname{Ad}(K)$ since.... Recall that the involution σ for the symmetric pair $(\operatorname{SL}(n, \mathbb{R}), \operatorname{SO}(n))$ is given by $\sigma(A) = (A^T)^{-1}$. It is sufficient to show that $\sigma(\operatorname{Ad}(G)) \subset \operatorname{Ad}(G)$ since Proposition 6.28 then implies that

$$G/H = \operatorname{Ad}(G)/\operatorname{Ad}(H) = \operatorname{Ad}(G)/(\operatorname{Ad}(G) \cap \operatorname{SO}(\mathfrak{g})) \subset \operatorname{SL}(\mathfrak{g})/\operatorname{SO}(\mathfrak{g})$$

has totally geodesic image.

Thus we need to show that $\operatorname{ad}(X)^T \in \operatorname{Ad}(G)$ for all $X \in \mathfrak{g}$. With respect to B^*, $\operatorname{ad} X$ is skew-symmetric for $X \in \mathfrak{h}$ and symmetric for $X \in \mathfrak{p}$. If $Z \in \mathfrak{g}$, split $Z = Z_1 + Z_2$ with $Z_1 \in \mathfrak{h}$, and $Z_2 \in \mathfrak{p}$. Then $(adZ)^T = (adZ_1)^T + (\operatorname{ad} Z_2)^T = -\operatorname{ad} Z_1 + \operatorname{ad} Z_2 = -\operatorname{ad}(Z_1 - Z_2) \in \operatorname{ad}(\mathfrak{g})$.

Finally, we show that ϕ is an isometric embedding. Since ϕ is clearly equivariant, we only need to check this at the base point. But on G/H the metric is given by $B^*_{|\mathfrak{p}} = B_{|\mathfrak{p}}$ and on the complement $p^* = \{A \in \operatorname{Sym}(\mathfrak{g}) \mid A = A^T\}$ for $\operatorname{SL}(n, \mathbb{R})/\operatorname{SO}(n)$ by $\operatorname{tr} AB$. Since $B(X, Y) = \operatorname{tr} \operatorname{ad}_X \operatorname{ad}_Y$ and $d\phi(X) = \operatorname{ad}_X$, the claim follows. $\square$

6.7 Hermitian Symmetric Spaces

There is an important subclass of symmetric spaces, namely those that preserve a complex structure. They have many special properties.

First some general definitions. (M, J) is called an **almost complex manifold** if J is a complex structure J_p on each tangent space T_pM. Furthermore, (M, J) is called an **complex manifold** if there are charts with image an open set in $\mathbb{C}^n$ such that the coordinate interchanges are holomorphic. The tautological complex structure on $\mathbb{C}^n$ induces an almost complex structure on M. An almost complex structure is called **integrable**, if it is induced in this fashion from local charts. It is then simply called a complex structure. There exists a tensor which measures integrability, the Nijenhuis tensor N:

$$\frac{1}{2}N(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY].$$

intcomplex

Theorem 6.52 (Newlander-Nirenberg) *If J is an almost complex structure, J is integrable iff $N = 0$.*

$(M, \langle\cdot,\cdot\rangle, J)$ is called **almost Hermitian** if the metric g and the almost complex structure J are compatible, i.e. $\langle JX, JY\rangle = \langle X, Y\rangle$. Notice that since $J^2 = -\operatorname{Id}$ this is equivalent to J being skew adjoint, i.e. $\langle JX, Y\rangle = -\langle X, JY\rangle$. It is called **Hermitian** if J is integrable. To an almost Hermitian manifold we can associate a 2-form $\omega(X, Y) = \langle JX, Y\rangle$. It is a 2-form since $\omega(X, Y) = \langle JX, Y\rangle = -\langle X, JY\rangle = -\omega(Y, X)$. Furthermore, $\omega^n \neq 0$ since we can find an orthonormal basis $u_i, v_i, i = 1, \ldots, n$ with $Ju_i = v_i$ and $Jv_i = -u_i$ and hence $\omega = \sum du_i \wedge dv_i$. M is called **almost Kähler** if (M, J) is almost Hermitian and $d\omega = 0$, and **Kähler** if in addition J is integrable. In particular an almost Kähler manifold is symplectic and hence $H^{2i}_{DR} \neq 0$ since $[\omega^i] \neq 0$.

There are some simple relationships with ∇J. Recall that $\nabla J = 0$ iff JX is parallel if X is parallel, i.e. parallel translation is complex linear.

kaehler

Proposition 6.53 *Let$J$ be an almost complex structure and g a metric.*

(a) If (M, g, J) is almost Hermitian and $\nabla J = 0$, then M is Kähler.
(b) If (M, g, J) is Hermitian, then $d\omega = 0$ iff $\nabla J = 0$.

Proof : The main ingredient is the following identity

$$4g((\nabla_X J)Y, Z) = 6d\omega(X, JY, JZ) - 6d\omega(X, Y, Z) + g(N(Y, Z), JX),$$

which is easily verified. In addition it is a general fact for differential forms that $d\omega$ is the skew symmetrization of $\nabla\omega$. Furthermore, since the metric is parallel, $\nabla J = 0$ iff $\nabla\omega = 0$. This easily implies the claims. □

It is a general fact for differential forms that the coboundary operator δ is a contraction of the covariant derivative: $\delta\omega = -\sum(\nabla_{e_i}\omega)(e_i, \ldots)$ where e_i is an orthonormal basis. Thus for a Kähler manifold, ω is also co-closed and hence harmonic, i.e. $\Delta\omega = (d\delta + \delta d)\omega = 0$. Clearly if J is a complex (resp. almost complex) structure, there exists a metric such that M is Hermitian (almost Hermitian). But being Kähler is a strong condition. E.g. the Betti numbers are all even, and cupping with $[\omega]$ is injective in DeRham cohomology up to half the dimension (strong Lefschetz theorem).

We also remark that being Kähler is equivalent to saying that the holonomy group at a point is contained in $\mathrm{U}(n) \subset \mathrm{SO}(2n)$ since by the holonomy principle this is equivalent to having a parallel complex structure.

As we will see, all of the above are equivalent for a symmetric space. Maybe the most natural definition of a Hermitian symmetric space is:

hermsymm **Definition 6.54** *A symmetric space M is called a* **Hermitian symmetric** *space of it is a Hermitian manifold and the symmetries s_p are holomorphic.* .

Here we could replace Hermitian by almost Hermitian since the Nijenhuis tensor N vanishes if s_p is complex linear. Indeed, this implies that $(s_p)_*(N) = N$, and since N has odd order, $d(s_p)_p = -\operatorname{Id}$ implies $N = 0$. There is a local characterization of being Hermitian symmetric:

hermsymmlocal **Proposition 6.55** *Let (G, K) be a symmetric pair with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. If $J : \mathfrak{p} \to \mathfrak{p}$ satisfies*

(a) J is orthogonal and $J^2 = -\operatorname{Id}$,
(b) $J \circ \operatorname{Ad}(h) = \operatorname{Ad}(h) \circ J$ for all $h \in K$.

Then M is a Hermitian symmetric space, and in fact Kähler.

Proof : Following our general principle, we define $J_{gp} = (L_g)_*(J)$, i.e., $J_{gp} = d(L_g)_p \circ J \circ d(L_{g^{-1}})_{gp}$. This is well defined since $\operatorname{Ad}(h)$ preserves $J_p = J$. Thus we obtain an almost complex structure on G/K. Furthermore, L_g preserves this almost complex structure.

We now claim that the symmetries s_p preserve J as well, i.e. $(s_p)_*(J) = J$. Recall that $s_{gp} \circ L_g = L_g \circ s_p$ which implies that $(s_p)_*(J)$ is another complex structure which is G invariant: $(L_g)_* \circ (s_p)_*(J) = (L_g \circ s_p)_*(J)) = (s_{gp} \circ L_g)_*(J) = (s_{gp})_* \circ (L_g)_*(J) = (s_{gp})_*(J)$. But J and $(s_p)_*(J)$ agree at p, and hence everywhere. As we saw above, this implies in particular that J is integrable and hence M is Hermitian symmetric.

To see that M is Kähler observe that ∇J is a tensor of odd order and is preserved by s_p and hence vanishes. □

Thus, any symmetric space whose isotropy representation is complex linear is a Hermitian symmetric space.

symmirredsimple **Corollary 6.56** *Let (G, K) be a symmetric pair. Then*

(a) (G, K) is Hermitian symmetric iff the dual is Hermitian symmetric.
(b) If (G, K) is irreducible and Hermitian symmetric, then it is Kähler Einstein.

Here are two more characterizations of being Hermitian symmetric.

hermsymmdeRham

Proposition 6.57 *Let (G,K) be an irreducible symmetric pair.*

(a) The complex structure J is unique up to sign.
(b) (G,K) is Hermitian symmetric iff K is not semisimple.
(c) If (G,K) is of compact type, it is Hermitian symmetric iff $H^2_{DR}(M) \neq 0$.

Proof : We start with (c). One direction is clear. Hermitian symmetric implies Kähler and hence $H^2_{DR}(M) \neq 0$. Now assume that $H^2_{DR}(M) \neq 0$ and let ω be a closed form whose deRham class is non-zero. We first claim that we can assume that ω is G invariant. Indeed, since G is connected, it acts trivially on cohomology and hence we can average over G: $\tilde{\omega}_p = \int_G \omega_{gp} dg$ and $\tilde{\omega}$ lies in the same deRham class as ω.

Now define $J\colon \mathfrak{p} \to \mathfrak{p}$ by $\omega_p(X,Y) = \langle JX, Y\rangle$ for all $X, Y \in \mathfrak{p}$. Then $\langle JX, Y\rangle = \omega(X,Y) = -\omega(Y,X) = -\langle JY, X\rangle$, i.e. J is skew-adjoint. Since ω is G-invariant and well defined on M, ω_p is $\mathrm{Ad}(H)$-invariant. Thus J commutes with $\mathrm{Ad}(H)$. But $\mathrm{Ad}(H)$ acts irreducibly on $\mathfrak{p}$, and so since J^2 is self-adjoint and commutes with $\mathrm{Ad}(H)$ as well, it follows that $J^2 = \lambda\,\mathrm{Id}$ for some $\lambda < 0$. Thus $J^2 = -\mu^2\,\mathrm{Id}$, for some $\mu > 0$. Now we let $J' = \frac{1}{\mu}J$, then $(J')^2 = -\,\mathrm{Id}$. Since J' is skew-adjoint and $J'^2 = -\,\mathrm{Id}$, J' is orthogonal. Now Proposition 6.55 implies that G/K is Hermitian symmetric.

(b) By duality, we can assume that (G,K) is of compact type. If G/K is Hermitian symmetric, we just showed that $H^2_{DR}(M) \neq 0$. Recall that G is connected, K has only finitely many components, and $\pi_1(G)$ is finite since G is semisimple. Thus $\pi_1(M)$ is finite as well. If $\tilde{M}$ is the (finite) universal cover, it is well know that the DeRham cohomology of M is the DeRham cohomology of $\tilde{M}$ invariant under the deck group. Thus $H^2_{DR}(\tilde{M}) \neq 0$ as well. By applying Hurewicz, we conclude that $\mathbb{Z} \subset \pi_2(\tilde{M}) = \pi_2(M)$. Now we use the fact that $\pi_2(G) = 0$ for every compact Lie groups G. Using the long homotopy sequence again, we see that $\mathbb{Z} \subset \pi_1(K)$ which means that K cannot be semisimple. This argument can clearly be reversed to prove the other direction.

(a) If J_i are two orthogonal invariant complex structures, then $\omega_i(X,Y) = \langle J_i X, Y\rangle$ defines two non-degenerate symplectic forms and as in the case of inner products, one easily shows that $\omega_1 = \lambda\omega_2$ for some $0 \neq \lambda \in \mathbb{R}$. But then $J_i^2 = -\,\mathrm{Id}$ implies that $\lambda = \pm 1$. □

We finally give a list of more detailed information similar to the ones we obtained for symmetric spaces of non-compact type.

hermsymmprops

Proposition 6.58 *Let (G, K) be an effective irreducible Hermitian symmetric space not of Euclidean type. Then*

(a) K is connected and $\pi_1(G/K) = 0$,
(b) $Z(G) = \{e\}$ and $\operatorname{rk} K = \operatorname{rk} G$.
(c) $Z(K) = S^1$ and K is the centralizer of $Z(K)$ in G.
(d) The complex structure J is given by $J = \operatorname{Ad}(i)$, for $i \in S^1$.
(e) Every isometry in $\mathrm{I}_0(M)$ is holomorphic.

Proof : We start with the claim that $Z(K) = \mathrm{S}^1$. Recall that K acts irreducibly and effectively on $\mathfrak{p}$ and that $Z(K)$ acts as intertwining operators of the isotropy representation. But the algebra of intertwining operators of a real irreducible representation is either $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Since the action is also orthogonal, and $Z(K)$ is abelian and not finite, this leaves only $Z(K) = \mathrm{S}^1$. This must act via complex multiplication on $\mathbb{R}^{2n} \simeq \mathbb{C}^n$ and hence $J = \operatorname{Ad}(i)$ satisfies $J^2 = -\operatorname{Id}$ and commutes with $\operatorname{Ad}(K)$. By uniqueness, this must be the complex structure on G/K.

Next, we claim that K is the centralizer of $Z(K)$. Let L be the the centralizer of $Z(K) = \mathrm{S}^1$. Clearly $K \subset L$ and hence $\mathfrak{k} \subset \mathfrak{l}$. As we saw, the centralizer of a circle is the union of all tori containing the circle and hence connected. Notice that if $g \in L$, then $kgk^{-1} \in L$ for all $k \in K$ and thus $\operatorname{Ad}(K)$ preserves $\mathfrak{l}$ and hence $\mathfrak{l} \cap \mathfrak{p}$. But irreducibility implies that $\mathfrak{l} \cap \mathfrak{p} = 0$ and hence $\mathfrak{l} = \mathfrak{k}$. This shows that K must be connected and hence $L = K$. Since S^1 is contained in a maximal torus T and clearly $T \subset L$, it also follows that $\operatorname{rk} K = \operatorname{rk} G$. Since $Z(G)$ is contained in every maximal torus of G, it follows that $Z(G) \subset K$ and hence effectiveness implies $Z(G) = \{e\}$.

To see that M is simply connected, let $\pi\colon \tilde{G} \to G$ be the universal cover and choose an element $z \in \tilde{G}$ such that $\pi(z) = i \in \mathrm{S}^1 = Z(K)$. The involution $\tilde{\sigma}\colon \tilde{G} \to \tilde{G}$ given by $\tilde{\sigma}(g) = zgz^{-1}$ satisfies $d\tilde{\sigma} = d\sigma$ under the identification $d\pi$ since $\sigma(g) = \operatorname{Ad}(i)$. $\tilde{G}^{\tilde{\sigma}}$ is the centralizer $C(z) := \tilde{K}$ and, as before, $C(z)$ is connected. Since $d\pi$ takes the Lie algebra of $\tilde{G}^{\tilde{\sigma}}$ to that of G^{σ}, it follows that $\pi(\tilde{G}^{\tilde{\sigma}}) = G^{\sigma}$. Also notice that $Z(\tilde{G}) \subset C(z)$ and hence $\pi^{-1}(K) = \tilde{K}$. Thus $G/K = \tilde{G}/\pi^{-1}(K) = \tilde{G}/\tilde{K}$ which is simply connected since $\tilde{G}$ is, and $\tilde{K}$ is connected.

(e) follows since Proposition 6.41 implies that $\mathrm{I}_0(M) = G$ and since L_g are holomorphic (see the proof of Proposition 6.55 and use uniqueness).

□

From the classification it follows that:

hermsymmclass

Proposition 6.59 *Let (G, K) be a simply connected irreducible Hermitian symmetric space. If G is a classical Lie group, then G/K is one of $U(n+m)/U(n)U(m)$, $\mathrm{SO}(2n)/\mathrm{U}(n)$, $\mathrm{Sp}(n)/\mathrm{U}(n)$ or $\mathrm{SO}(n+2)/\mathrm{SO}(n)\,\mathrm{SO}(2)$.*

6.8 Topology of Symmetric Spaces

We first discuss some general facts about the topology of homogeneous spaces before we specialize to the case of a symmetric space.

Let a compact Lie group G act on a compact oriented manifold M and denote by $\Omega^k_G(M)$ the space of k-forms ω on M invariant under the action of G, i.e. $g^*(\omega) = \omega$ for all $g \in G$. Since $dg^*(\omega) = g^*(d\omega)$, d induces a differential on $\Omega^k_G(M)$ and we denote by $(\Omega^k_G(M), d)$ the corresponding complex.

DeRhamInv

Proposition 6.60 *Let G be a compact Lie group acting on a manifold M. Then the cohomology of the complex $(\Omega^k_G(M), d)$ is isomorphic to the G invariant DeRham cohomology $H^*_{DR}(M)^G$. If G is connected, then $H^*_{DR}(M)^G \simeq H^*_{DR}(M)$.*

Proof : We have an **averaging operator**

$$\mathfrak{A}(\omega) = \int_G g^*(\omega) dg,$$

and clearly $\mathfrak{A}(\omega)$ is G-invariant. $\mathfrak{A}$ induces a natural homomorphism of complexes

$$\mathfrak{A} : (\Omega_k(M), d) \to (\Omega^k_G(M), d)$$

since $d \int_G g^*(\omega)dg = \int_G dg^*(\omega)dg = \int_G g^*(d\omega)$ and hence $d\mathfrak{A}(\omega) = \mathfrak{A}(d\omega)$.

$\mathfrak{A}$ thus takes closed forms to closed forms and we get an induced map $\mathfrak{A}^* \colon H^*(\Omega^k_G(M), d)) \to H^*_{DR}(M)$ in cohomology. We claim that $\mathfrak{A}^*$ is injective with image $H^*_{DR}(M)^G$. Indeed, if $\omega \in \Omega^k_G(M)$ with $\mathfrak{A}^*([\omega]) = 0$, then $\omega = d\eta$ for some $\eta \in \Omega^{k-1}(M)$. But then $\omega = d\eta'$ for some $\eta' \in \Omega^{k-1}_G(M)$ since $\omega = \mathfrak{A}(\omega) = \mathfrak{A}(d\eta) = d\mathfrak{A}(\eta)$.

If $\omega \in \Omega^k_G(M)$, then clearly $[\omega]$ is G-invariant, i.e. $[\omega] \in H^*_{DR}(M)^G$. Conversely, if $\alpha \in H^*_{DR}(M)^G$, let $\omega \in \Omega^k(M)$ be a closed form with $[\omega] = \alpha$. Then $\alpha = g^*(\alpha) = [g^*(\omega)]$ and hence $\alpha = [\mathfrak{A}^*(\omega)]$ since the integration takes place in the linear subspace of closed forms with cohomology class α. $\square$

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